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LARGE N_C LIMIT OF THE SKYRME MODEL*

By Z. Duliński and M. Praszałowicz

Institute of Physics, Jagellonian University, Cracow**

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In Skyrme model with 3 flavors the limit $N_C \to \infty$ is explicitly performed. Mass splittings and magnetic moments for the baryonic states in such a model are calculated and compared with the data.

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1. Introduction

The Skyrme model [1, 2] describes baryons as solitons in an effective theory of Goldstone bosons and is able to give predictions for many quantities like baryon masses [3–5], magnetic moments [6], nucleon-nucleon potentials [7], phase shifts in meson-nucleon channels [8] and others. However all quantities calculated in the model are not very accurate: 20% accuracy is considered to be satisfactory.

Morover the 3 flavor model [4, 5, 9] suffers from a serious theoretical drawback; although the effective lagrangian can be obtained from QCD by large N_C expansion [10, 11], the baryon spectrum has no smooth limit for $N_C \to \infty$. The Hilbert space of the model is constrained by a requirement that $SU(3)_{FLAVOR}$ representations describing physical states should contain states of hypercharge $N_C/3$, and therefore very large representations emerge in the limit $N_C \to \infty$. The similar behaviour is exhibited by the quark model [12].

However as the authors of Ref. [12] argue, one is able to define a sensible large N_C limit for baryon spectrum by a suitable generalization of the Octet representation. The explicit calculation [12, 13] of generalized Octet magnetic moments was performed in Ref. [12] and compared with the data.

The purpose of this paper is twofold: first we investigate the large N_C limit of the Octet *and* Decuplet by the explicit calculation of the magnetic moments *as well as* the mass splittings, and second, we examine the resulting phenomenology. Comparing the ratios of the different mass splittings with the data we get the mass spectrum which shows

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** Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

a simple proportionality to the hypercharge. The electromagnetic splittings are even worse. This is rather expected since the model cannot describe accurately such tiny effects like electromagnetic mass differences [14]. As far as the magnetic moments are concerned we find that in $N_C \to \infty$ limit they are proportional to I_3(isospin) — this rule is not obeyed by the data. Also the SU(6) formula for Decuplet moments recovered in $N_C = 3$ case is broken for $N_C \to \infty$. Although the limit is well defined the phenomenology is rather poor indicating, as should be expected, that $N_C = 3$. Since the results for $N_C = 3$ and $N_C = \infty$ differ so much, the $1/N_C$ corrections seem to be important, as shown explicitly in Ref. [6a].

In the next Section we briefly introduce the model and indicate what quantities we calculate. Then we discuss our results. Notation and conventions concerning the SU(3) Clebsch-Gordan coefficients are presented in the Appendix.

2. *The model*

The Skyrme model has been described in many papers (for a review see Refs. [15]); here we only specify the most important points which we shall use afterwards. The link between QCD and the model is the chiral symmetry group G (in our case G = SU(3) × SU(3)) broken by the QCD vacuum to $SU(3)_V$ (diagonal subgroup of G). An object on which the symmetry acts in an effective theory is a matrix $U(x, t) \in SU(3)$ transforming as $g_L U g_R^+$, parametrized by 8 fields associated with Goldstone bosons. Lagrangian symmetric under G can be constructed [1, 3], then G is broken to $SU(3)_V$ ($g_L = g_R$) by the vacuum $U(\infty, t) = 1$.

Now U as mapping from x-space to SU(3) can be characterized by an integer n_W which counts how many times the x-space wraps the SU(3) manifold; if $n_W > 0$ then we call such U a soliton. The $n_W = 1$ static soliton (Skyrmion) has been found in a form

$$U_0(x) = \begin{bmatrix} \tilde{U}_0(x) & \\ & 1 \end{bmatrix}, \tag{1}$$

where $\tilde{U}_0(x)$ is topologically nontrivial SU(2) matrix [1, 3].

The quantization of the model proceeds by rotating $U_0(x)$ by a time-dependent matrix $A(t)$:

$$U(x, t) = A(t) U_0(x) A(t)^+ \tag{2}$$

introducing 8 collective coordinates (parametrizing A). The 8-th coordinate does not appear in the kinetic energy since λ_8 commutes with U_0. Therefore the system is constrained [4]. Quantization procedure allows us to construct a Hamiltonian which describes the baryon energy levels as well as the wave functions. The wave function for a baryon $B = (Y, I, I_3)$ of spin (J, J_3) is given by [4]:

$$\psi_B = \mathcal{N} \left\langle Y, I, I_3 \middle| D^{(\varrho)}(A) \middle| \frac{N_C}{3} J, -J_3 \right\rangle, \tag{3}$$

where ϱ labels $SU(3)_{FLAVOR}$ representations. Because of the constraint mentioned above the right hypercharge in Eq. (3) is $N_C/3$. Therefore ϱ has to include $Y = N_C/3$:

$$\varrho = (N_C, 0), (N_C-2,1) \ldots \left(1, \frac{N_C-1}{2}\right) \tag{4}$$

and higher. For $N_C = 3$ we have **8** and **10**. For higher N_C we have to decide which representations from Eq. (4) are physical i.e. generalizations of **8** and **10**, and which are unphysical. As in Ref. [12] we assume that the physical representations are those which have physical spin (1/2 and 3/2), namely:

$$\mathbf{8} \Rightarrow \text{“8”} = (1, n);\ n = \frac{N_C-1}{2} \qquad \mathbf{10} \Rightarrow \text{“10”} = (3, k); \quad k = \frac{N_C-3}{2}. \tag{5}$$

Other representations are spurious. Note that according to Eq. (3) spin carried by the entire representation is equal to the isospin of the highest weight.

Representations (5) contain many more states than needed. We select the Octet- and Decuplet-like structures in ϱ requiring that generalizations of the physical states have correct physical isospin. All other states are spurious.

The baryon mass formula reads:

$$M_B = N_C M_{CL} + M_{COLL} + \frac{1}{N_C} M_J J(J+1) - N_C M_{BR} d_B - N_C M_{ELM} I_3 f_B, \tag{6}$$

where

$$d_B = \sum_{\pm} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \eta & B & B \end{pmatrix} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \eta & n & n \end{pmatrix}, \tag{7}$$

$$f_B = \sum_{\pm} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \pi^0 & B & B \end{pmatrix} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \eta & n & n \end{pmatrix}. \tag{8}$$

To obtain Eq. (6) we enriched the SU(3) symmetric lagrangian by the breaking terms $\mathrm{Tr}\,(\lambda_8[U+U^+-2])$ and $\mathrm{Tr}\,(\lambda_3[U+U^+-2])$, which after quantization lead to the operators:

$$N_C M_{BR} D^{(8)}_{88} = N_C M_{BR} \langle 000 | D^{(8)} | 000 \rangle, \tag{9}$$

$$N_C M_{BR} D^{(8)}_{38} = N_C M_{BR} \langle 010 | D^{(8)} | 000 \rangle. \tag{10}$$

Numbers d_B and f_B in Eqs. (6)–(8) are just the matrix elements of the operators (9), (10) between the baryon states (3) with spin “up”.

Similarly the magnetic moment operator takes a form [6]:

$$\mu_B = N_C \zeta h_B, \tag{11}$$

where

$$h_B = \sum_{\pm} \left\{ \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \pi^0 & B & B \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \eta & B & B \end{pmatrix} \right\} \begin{pmatrix} 8 & \varrho & \varrho_\pm \\ \pi^0 & n & n \end{pmatrix}. \tag{12}$$

As explained in the Appendix in the direct product of the Octet (operator) and $\varrho = (p, q)$ (baryon state) representation ϱ appears twice (if $p, q \neq 0$); the two different ϱ are labeled $\varrho_\pm$. This is the source of the summation over "$\pm$" in Eqs. (7), (8), (12).

Our aim is to calculate d_B, f_B and h_B for representations (5) and then evaluate baryon masses and magnetic moments. The details of the calculations are presented in the Appendix. In the next Section we summarize our results.

3. *Results*

Mass splittings

Our results for mass splittings i.e. for d_B and f_B are given in Table I. For $N_C = 3$ these numbers agree with the ones known from the literature [4, 14]. Disregarding for the moment the electromagnetic splittings one can fit 8 baryon masses with the help of the following formula (see Eq. (6)):

$$M_B = N_C M_\varrho - N_C M_{BR} d_B \tag{13}$$

fitting three parameters $M_{8,10}$, and M_{BR} ($N_C = 3$). Then the predicted mass spectrum agrees with the data with an accuracy of 6% [4, 5]. In the limit of $N_C \to \infty$ all masses

TABLE I

Mass splittings (see Eq. (6) in the text)

ϱ	B	d_B		f_B	
		$N_C = 2n+1$	$N_C = 3$	$N_C = 2n+1$	$N_C = 3$
(1, n)	N	$\frac{2n^2+6n+1}{2(n+2)(n+4)}$	$\frac{3}{10}$	$\frac{\sqrt{3}}{(n+2)(n+4)}$	$\frac{1}{5\sqrt{3}}$
	Λ	$\frac{(n+2)(2n-1)}{2(n+2)(n+4)}$	$\frac{1}{10}$	0	0
	Σ	$\frac{2n^2+3n-8}{2(n+2)(n+4)}$	$-\frac{1}{10}$	$\frac{\sqrt{3}}{(n+2)}$	$\frac{1}{\sqrt{3}}$
	Ξ	$\frac{n-2}{n+4}$	$-\frac{1}{5}$	$\frac{1}{\sqrt{3}(n+4)}$	$\frac{1}{5\sqrt{3}}$
(3, $n-1$)	Δ	$\frac{2n^2+6n-5}{2(n+1)(n+5)}$	$\frac{1}{8}$	$\frac{\sqrt{3}}{(n+1)(n+5)}$	$\frac{1}{4\sqrt{3}}$
	Σ*	$\frac{(n-1)(2n+5)}{2(n+1)(n+5)}$	0	$\frac{-\sqrt{3}(n-5)}{4(n+1)(n+5)}$	$\frac{1}{4\sqrt{3}}$
	Ξ*	$\frac{2n^2-5}{2(n+1)(n+5)}$	$-\frac{1}{8}$	$\frac{-2n+5}{\sqrt{3}(n+1)(n+5)}$	$\frac{1}{4\sqrt{3}}$
	Ω	$\frac{2n^2-3n-5}{2(n+1)(n+5)}$	$-\frac{1}{4}$	0	0

become infinite. However the ratios of mass differences are finite and can be compared with the data. As an input we take $N-\Xi$, Σ^* and N. Our results can be parametrized in a following way:

$$M_B = \tilde{M}_Q - \tilde{m}_{BR} Y_{SU(3)} \tag{14}$$

with $\tilde{m}_{BR} = 189$ MeV, $\tilde{M}_8 = 1129$ and $\tilde{M}_{10} = 1385$, $Y_{SU(3)} = Y_B - Y_\Lambda$. All experimental values lie between the predictions for $N_C = 3$ and $N_C = \infty$, however for $N_C = \infty$ there is no splitting between Λ and Σ. The serious drawback of formula (14) is that in fact $\tilde{M}_8 = \tilde{M}_{10}$ for $N_C = \infty$ (as can be seen from Eq. (6)). So the mass splittings within the multiplets can be described in large N_C limit, however the multiplets themselves are degenerate.

Magnetic moments

Octet (i.e. $(1, n)$) magnetic moments were discussed in Ref. [12]. Here we only note that (as can be seen from Tabl. II):

$$\sum_{I_3} \mu_B = 0 \tag{15}$$

for each isospin multiplet. This rule is not obeyed by the data.

Let us point that there is one N_C independent ratio:

$$\mu_p/\mu_{\Sigma^+} = 1 \quad (= 1.17 \text{ exp.}). \tag{16}$$

The sign of μ_{Ξ^-} is changed with respect to $N_C = 3$ case (and experiment).

For Decuplet (i.e. $(3, k)$) the SU(6) formula $\mu_B \sim Q_B$ (electric charge) is badly broken (Eq. (15) remains valid). Drastic differences between $N_C = 3$ and $N_C = \infty$ can be observed for Δ^0, Ξ^{*0} and Ω. However the lack of the data prevents us from drawing any definite conclusion.

To summarize: the magnetic moments and the mass splittings seem to be phenomenologically better described by the $N_C = 3$ quantization. Still the real $N_C = 3$ world requires more care; $1/N_C$ corrections to the magnetic moments [6a] provide good example of substantial improvement of theoretical result.

We think however that it is not trivial that the large N_C limit can be defined for the baryon spectrum. The fact that the phenomenology of the model fails in this limit is merely the reflection of the very fact that QCD has only 3 colors. It seems that both the Skyrme model and the naive quark model give the same results in this limit [16].

It should be observed that other choices of the generalization of the Octet and Decuplet are possible. For example one could take (n, n) as a generalized Octet. In that case the phenomenology of the different choices may be considered as an additional criterion for defining the large N_C-limit of the Skyrme model [18].

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TABLE II

Magnetic moments (see Eq. (11) in the text)

ϱ	B	h_B: $N_C = 2n+1$ or $N_C = 2k+3$	h_B: $N_C = 3$	h_B: $N_C = \infty$
(1, n)	p	$-\frac{1}{3}\frac{n+3}{n+4}$	$-\frac{4}{15}$	$-\frac{1}{3}$
	n	$\frac{1}{3}\frac{n^2+5n+3}{(n+2)(n+4)}$	$\frac{3}{15}$	$\frac{1}{3}$
	Σ^+	$-\frac{1}{3}\frac{n+3}{n+4}$	$-\frac{4}{15}$	$-\frac{1}{3}$
	Σ^0	$-\frac{1}{6}\frac{n+8}{(n+2)(n+4)}$	$-\frac{1}{10}$	0
	Σ^-	$\frac{1}{3}\frac{n^2+4n-2}{(n+2)(n+4)}$	$\frac{1}{15}$	$\frac{1}{3}$
	Λ^0	$\frac{1}{2}\frac{1}{n+4}$	$\frac{1}{10}$	0
	Ξ^0	$\frac{1}{9}\frac{n+8}{n+4}$	$\frac{1}{5}$	$\frac{1}{9}$
	Ξ^-	$-\frac{1}{9}\frac{n-4}{n+4}$	$\frac{1}{15}$	$-\frac{1}{9}$
(3, k)	Δ^{++}	$-\frac{3}{5}\frac{k+5}{k+6}$	$-\frac{1}{2}$	$-\frac{3}{5}$
	Δ^+	$-\frac{1}{5}\frac{k^2+7k+15}{(k+2)(k+6)}$	$-\frac{1}{4}$	$-\frac{1}{5}$
	Δ^0	$\frac{1}{5}\frac{k(k+7)}{(k+2)(k+6)}$	0	$\frac{1}{5}$
	Δ^-	$\frac{3}{5}\frac{k^2+7k+5}{(k+2)(k+6)}$	$\frac{1}{4}$	$\frac{3}{5}$
	Σ^{*+}	$-\frac{1}{4}\frac{(k+4)(2k+3)}{(k+2)(k+6)}$	$-\frac{1}{4}$	$-\frac{1}{2}$
	Σ^{*0}	$\frac{k}{(k+4)(k+6)}$	0	0
	Σ^{*-}	$\frac{2k^3+23k^2+72k+48}{4(k+2)(k+4)(k+6)}$	$\frac{1}{4}$	$\frac{1}{2}$
	Ξ^{*0}	$-\frac{1}{3}\frac{k(k+3)}{(k+2)(k+6)}$	0	$-\frac{1}{3}$
	Ξ^{*-}	$\frac{1}{3}\frac{k^2+9k+9}{(k+2)(k+6)}$	$\frac{1}{4}$	$\frac{1}{3}$
	Ω^-	$\frac{3}{2}\frac{1}{k+6}$	$\frac{1}{4}$	0

APPENDIX

In this Appendix we collect our conventions for calculating Clebsch-Gordan coefficients used in the text. Tensorial methods of calculating C–G coefficients are described in Ref. [13]. Representation (p, q) of SU(3) acts on a space of symmetric and traceless tensors $T^{\{q\}}_{\{p\}}$ labeled by q upper and p lower indices. Tensor $T^{\{q\}}_{\{p\}}$ has q upper (antiquark) and p lower (quark) indices. In what follows we omit labels $\{p\}$ and $\{q\}$. In the direct product of the Octet $T^{[8]} = T^{\{\alpha\}}_{\{\beta\}}$, $\alpha, \beta = 1, 2, 3$ and (p, q) representation (p, q) appears twice:

$$T^{[\varrho_1]i_1 i_2 \cdots i_q}_{\quad j_1 j_2 \cdots j_p} = \sum T^{[8]n}_{\quad j_1} T^{i_1 i_2 \cdots i_q}_{n\, j_2 \cdots j_p} \tag{A.1}$$

$$T^{[\varrho_2]i_1 i_2 \cdots i_q}_{\quad j_1 j_2 \cdots j_p} = \sum T^{[8]i_1}_{\quad n} T^{n\, i_2 \cdots i_q}_{j_1 j_2 \cdots j_p}, \tag{A.2}$$

where the sum in (A.1), (A.2) extends over all permutations $\{n, i_1, i_2, \ldots\}$. We define

$$\varrho_\pm = \varrho_2 \pm \alpha \varrho_1, \tag{A.3}$$

where α is chosen in such a way that $\varrho_\pm$ are orthogonal:

$$\alpha^2 = \frac{q(6+3p+8q+2pq+2q^2)}{p(6+3q+8p+2qp+2p^2)}. \tag{A.4}$$

The Clebsch-Gordan series for the highest weight of (p, q) called "p" (with obvious notation in quotation marks for other states in (p, q)) reads:

$$\text{“p”} = N_\pm \Bigg[\pm \frac{\alpha p}{\sqrt{2}} \Sigma^0 \times \text{“p”} + \frac{-2q \pm \alpha p}{\sqrt{6}} \Lambda \times \text{“p”} - (\pm\alpha)\sqrt{p}\, \Sigma^+ \times \text{“n”}$$

$$+ \sqrt{\frac{q}{p+1}}\, \text{p} \times \text{“}\Sigma^0\text{”} - \sqrt{\frac{p}{(p+1)(1+p+q)}} (\pm\alpha(1+p) - q)\text{p} \times \text{“}\Lambda\text{”} - \sqrt{q}\, \text{n} \times \text{“}\Sigma^+\text{”} \Bigg], \tag{A.5}$$

where

$$N_\pm^2 = \frac{3(1+p+q)}{2q[6+3p+8q+2pq+2q^2 \pm (-\alpha)p(4+p+q)]}. \tag{A.6}$$

Our phase conventions are such that the C–G coefficients for $\mathbf{8} \times \mathbf{8}$ $(p = q = 1)$ agree with the standard ones [17].

$$\mathbf{8}_{\mathrm{S,A}} = (1, 1)_\pm. \tag{A.7}$$

The limiting procedure for **10** requires more care since for $q = 0$ we have only one representation $(p, 0)$ in $(1, 1) \times (p, 0)$ (see Eq. (A.2)). Therefore we have:

$$\mathbf{10} = \frac{1}{\sqrt{2}} [(3, 0)_- - (3, 0)_+]. \tag{A.8}$$

Fortunately all formulae for magnetic moments and mass splittings for $(3, k)$ have smooth limit to **10** as $k \Rightarrow 0$.

REFERENCES

[1] T. H. R. Skyrme, *Proc. R. Soc.* **A260**, 127 (1961); *Nucl. Phys.* **31**, 556 (1962).
[2] E. Witten, *Nucl. Phys.* **B223**, 422, 433 (1983).
[3] G. S. Adkins, C. R. Nappi, E. Witten, *Nucl. Phys.* **B228**, 552 (1984); G. S. Adkins, C. R. Nappi, *Phys. Lett.* **137B**, 251 (1984); *Nucl. Phys.* **B233**, 109 (1984).
[4] E. Guadagnini, *Nucl. Phys.* **B236**, 35 (1984); A. V. Manohar, *Nucl. Phys.* **B248**, 19 (1984).
[5] M. Praszałowicz, *Phys. Lett.* **158B**, 264 (1985).
[6] a) G. S. Adkins, C. R. Nappi, *Nucl. Phys.* **B249**, 507 (1985); b) E. Gaudagnini, Pisa preprint, IFUP TH 10/84.
[7] M. Lacombe, B. Loiseau, R. Vinh Mau, W. N. Cottingham, IPNO/TH 85-68, Orsay preprint 1985.
[8] M. Karliner, M. P. Mattis, SLAC-PUB-3901, April 1986; B. Schwesinger, C. Hajduk, Lewes 1984, Proceedings, Solitons in Nuclear and Elementary Particle Physics, pp. 129–141.
[9] S. Jain, S. R. Wadia, Tata Inst. preprint, TIFR-TH-84-7, 1984; M. A. Nowak, P. O. Mazur, M. Praszałowicz, *Phys. Lett.* **147B**, 137 (1984).
[10] G.'t Hooft, *Nucl. Phys.* **B72**, 461 (1974).
[11] E. Witten, *Nucl. Phys.* **B160**, 57 (1979).
[12] G. Karl, J. Patera, S. Perantonis, *Phys. Lett.* **172B**, 49 (1986).
[13] J. Bijnens, H. Sonoda, M. B. Wise, *Can. J. Phys.* **64**, 1 (1986).
[14] M. S. Sriram, H. S. Mani, R. Ramachandran, *Phys. Rev.* **D30**, 1141 (1984).
[15] I. Zahed, G. E. Brown, *Phys. Rep.* **142**, 1 (1986); U.-G. Meissner, I. Zahed, Stony Brook preprint 1985, *Advan. Nucl. Phys.* to appear.
[16] G. Karl, J. E. Paton, *Phys. Rev.* **D30**, 238 (1984).
[17] J. J. de Swart, *Rev. Mod. Phys.* **35**, 916 (1963); P. McNamee, S. J. Chilton, F. Chilton, *Rev. Mod. Phys.* **9**, 1005 (1964).
[18] Z. Dulinski, in preparation.