COSMIC TIME AND CHAOS

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It is shown that the Friedman cosmological models (1) with bulk viscosity dissipation, (2) with Weyssenhoff fluid (perfect fluid with macroscopic spin), (3) with a phase transition in a very early stage of the evolution, all possessing negative space-curvature, after being compactified, exhibit chaotic behaviour in asymptotic states. Geodesic flows in such models are characterized by an exponential instability; they are mixing ergodic, and have non-zero metric entropy. In fact these world models are special cases of a "chaotic evolution" described by Lockhart, Misra and Prigogine. In particular, Prigogine's "internal time" may be defined in them. Some remarks, concerning a predictability in cosmological models with the geodesic instability, are made.

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Introduction

Time problem belongs to the most discussed questions in foundations of cosmology (see, for instance, Ref. [1-4]). An interesting contribution to this question has been recently given by Lockhart, Misra and Prigogine (LMP) [5]. These authors, basing on earlier results obtained by Prigogine and his co-workers [6, 7, 8], attempted at introducing into Friedman cosmological models intrinsic randomness and irreversibility which would result in the existence of an internal time operator T. It turns out that exponential instability of geodesics on compact surfaces with negative constant curvature, being an example of the so-called $\mathscr{K}$-flow, leads to a stochastic behaviour of geodesics in some asymptotic regions [8, 9]. Such geodesic flows enjoy many interesting features such as mixing and ergodic property, structural stability, etc. LMP show that non-spacelike geodesics in a Robertson-Walker space-time, when suitably "projected" into a 3-dimensional compactified hypersurface

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of a constant negative curvature, change into geodesics parametrized by a new affine parameter $\lambda(t)$ satisfying the following equations

$$\frac{d\lambda}{dt} = \frac{A}{R(t)}, \quad \text{for zero geodesics,} \tag{Ia}$$

$$\frac{d\lambda}{dt} = \frac{A}{R(t)\,[\alpha^2 + R^2(t)]^{1/2}}, \quad \text{for timelike geodesics,} \tag{Ib}$$

where t is the usual cosmic time of Friedman's models; A and α are constants. These equations are valid for *any* Friedman world model but only for those with the *negative* space curvature geodesics exhibit the properties of a $\mathscr{K}$-flow.

The authors claim that for such cosmological models, an internal time operator T may be introduced, acting on distribution functions of the phase space and permits one to define an internal time in such a manner that its arrow corresponds to the increase of entropy [7]. The point being, however, that the internal time as defined by T "keeps step not with t but with $\lambda(t)$" [5]. Consequently, it is $\lambda(t)$ which should be thought of as a suitably averaged internal time. Let us notice that our further analysis remains valid with no respect of whether $\lambda(t)$ is interpreted as an averaged internal time or simply as a new variable. In the following we shall speak of λ-time or of Prigogine's time.

LMP procedure to define λ-time removes the initial singularity to minus infinity and implies the absence of particle horizons which is a desired property liquidating the known causality problems in cosmology. However, all these attractive possibilities are spoiled by the fact that the above procedure requires negative pressures in the energy-momentum tensor. In connection with this LMP write: "Nevertheless, it should be stressed that the concept of pressure near the singularity is far from being unambiguous and the meaning of negative pressure needs to be examined further." [5].

In the present work we give three examples of cosmological models satisfying all necessary conditions to implement LMP procedure and to realize "deterministic chaos" in geodesics. The models provide reasonable physical mechanisms producing negative pressure, namely (1) bulk viscosity dissipation, (2) intrinsic angular momentum connected with the macroscopic spin present in the fluid, (3) vacuum energy density related to a phase transition in the early universe. We do not claim that these mechanisms acted in the real world; we think that they might serve as good examples to study the idea of a "deterministic chaos" in the framework of general relativity (for a general discussion of this subject see Ref. [9]). A mechanism, similar to mechanism (3) proposed by us, was discussed by Kandrup in a short note [36].

The organization of the material is the following. Sections 1–3 present our three classes of world models which after being spatially compactified satisfy LMP conditions. The chaotic behaviour of geodesics, predictability, LMP time problem, and the initial singularity of the above models are discussed in Sec. 4. In Sec. 5, by reversing the argument, it is shown that if one postulates a degree of predictability, one can estimate the value of the *negative* space curvature: almost predictable universe must be almost flat. This might be thought of as a kind of anthropic explanation why the spacial curvature of the observed universe is so nearly zero.

1. An imperfect fluid world model without horizons

The Friedman equations

$$\frac{\varepsilon}{3} = \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}, \tag{1.1a}$$

$$-p = 2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}, \tag{1.1b}$$

where R is the scale factor, $k = 0, \pm 1$ — the space curvature constant, ε — the energy density, and p — the pressure, may be reduced to the form of the dynamical system [15, 16]

$$\frac{dH}{dt} \equiv \dot{H} = -H^2 - \tfrac{1}{6}(\varepsilon+3p), \tag{1.2a}$$

$$\frac{d\varepsilon}{dt} \equiv \dot{\varepsilon} = -3H(\varepsilon+p), \tag{1.2b}$$

where $H = \dot{R}/R$ is the Hubble parameter. Bulk viscosity dissipation appears as a dependence $p = p(H, \varepsilon)$, the quantity $\zeta(\varepsilon) = -\dfrac{1}{3}\dfrac{\partial p}{\partial H}$ being interpreted as the bulk viscosity coefficient. The "effective pressure" P is defined as $P = p(\varepsilon) - \zeta(\varepsilon)H$ and may be negative. Equation (1.1a) readily gives the equation describing model trajectories on the phase plane

$$\varepsilon - 3H^2 = \frac{3k}{R^2}.$$

This equation tells us that if $R \to 0$ $(t \to 0)$ and the function $\varepsilon(R)$ changes as $\varepsilon \sim R^{-2+\beta}$ with $\beta \geqslant 0$, the curvature effects cannot be neglected in a vicinity of the initial singularity. However, if $\beta < 0$, then $\varepsilon/3H^2 \to 1$ as $R \to 0$. A cosmological model has a particle horizon if $\displaystyle\int_{t_0}^{t} \frac{dt}{R}$ converges as $t_0 \to 0$, for every t. On the other hand, it can be shown that if $\varepsilon/3H^2 \to C \neq 1$, as $R \to 0$, then the corresponding world models have no particle horizons. Indeed from Eq. (1.1a) one gets

$$\dot{R} = \left[k\Big/\left(\frac{\varepsilon}{3H^2} - 1\right)\right]^{1/2}$$

and if $\lim_{\varepsilon\to\infty} \varepsilon/3H^2 \neq 1$, $\dot{R}$ goes to a finite limit, i.e. there exists a constant a such that for sufficiently large ε, $dR/dt < a$. Consequently

$$\int_0^{R_0} \frac{dR}{R} < a\int_0^{t_0} \frac{dt}{R} \equiv a(\eta_0 - \eta_{\text{singularity}}).$$

The integral on the left hand side of this formula diverges which means that the η-time goes to $-\infty$, and that there are no causally disconnected regions.

Dynamical system (1.2) was extensively studied [17, 19, 22, 23, 27] and its phase space is very well known. From our present point of view, one solution of (1.1) catches our attention. This is the solution $R \sim t$, $\varepsilon \sim t^{-2}$, $k = -1$ with the bulk viscosity coefficient $\zeta \sim \varepsilon^{1/2}$ which makes the effective pressure negative, and its absolute value equal to one third of the energy density. Therefore, the model satisfies the strong energy condition $\varepsilon + 3P = 0$; in the phase plane, it is situated at the boundary of the region on which the strong energy condition is satisfied [16, 17]. Since within this model there are no particle horizons, it is a good candidate, after being compactified, to serve as an example of the LMP procedure. The model was studied in references [19] and [22] (in Ref. [19], pay attention to some misprints).

2. *A cosmological model with macroscopic spin*

Let us consider a world model having Robertson-Walker symmetries filled with "perfect fluid with spin". As it is known [28], the macroscopic spin tensor $\tau^{\lambda}_{\mu\nu}$ may be expressed in terms of the spin density tensor $s_{\mu\nu}$ and four-velocity u^{μ} of the fluid

$$\tau^{\lambda}_{\mu\nu} = s^{\lambda} s_{\mu\nu}, \qquad s_{\mu\nu} s^{\mu} = 0. \tag{2.1}$$

Matter described by these quantities is sometimes called Weyssenhoff fluid [28, 29]. The influence of the macroscopic spin present in the fluid on the dynamics of the universe is described by a "correction" to the energy density and pressure

$$\varepsilon_{ef} = \varepsilon - \tfrac{1}{4} s^2, \qquad p_{ef} = p - \tfrac{1}{4} s^2, \tag{2.2}$$

where $s^2 = \frac{1}{2} s_{\mu\nu} s^{\mu\nu}$, $s_{\mu\nu}$ being the spin density tensor [30]. s describes "repulsive forces" and is always positive. Let us consider a pressureless fluid ($p = 0$) of particles with spin 1/2 and mass m. Energy density ε and absolute value of spin density s depend on the number n of particles in the volume unit: $\varepsilon = n \cdot m$, $s = \dfrac{\hbar}{2} \cdot n$. Hence, we have

$$\varepsilon_{ef} = \varepsilon - \left(\frac{\hbar}{n \cdot m}\right)^2 \varepsilon^2, \qquad p_{ef} = -\left(\frac{\hbar}{4m}\right)^2 \varepsilon^2. \tag{2.3}$$

Effective pressure is negative and may be expressed in the following form

$$p_{ef} = -\left(\frac{2m}{\hbar}\right)^2 \left[1 - \sqrt{1 - \left(\frac{\hbar}{2m}\right)^2 \varepsilon_{ef}}\,\right]. \tag{2.4}$$

Energy density is never greater than $\varepsilon^* = 1.8 \times 10^{28}\ \text{cm}^{-1}$. By inserting ε_{ef} and p_{ef} into dynamical system (2.1), the evolution of the model may be easily computed [35]. For $k = +1$, we have oscillating world models without singularities. Models with $k = -1$ begin with the initial singularity ($H \to \infty$, $\varepsilon \to \varepsilon^*$) and, as it can be easily checked, have no horizons. These models are another candidates for Prigogine's procedure.

3. *World models with a phase transition*

The third example is supplied by considerations closely related to a phase transition in the very early universe postulated by the Grand Unifying Theories. During such a phase transition the dynamics of the universe may be controlled by the energy density $\varepsilon_v(T)$, which is a function of temperature T, associated with the vacuum. If $\varepsilon_v(T)$ is to be Lorentz invariant the corresponding pressure p_v should be negative and equal $p_v = -\varepsilon_v(T)$. In a simplified model, constructed by Kazanas [31], the following dependence of the vacuum energy density on temperature has been assumed

$$\varepsilon_v(T) = \begin{cases} \varepsilon_0 = \text{const}, & \text{for} \quad T > T_c, \\ \varepsilon_0 \left(\dfrac{T}{T_c}\right)^\beta, & \text{for} \quad T < T_c, \end{cases} \tag{3.1}$$

where T_c is the temperature characteristic for the phase transition. Dependence (3.1) may be written in the form

$$\varepsilon_v(T) = \varepsilon_0 \tau^{\beta\theta(1-\tau)}, \tag{3.2}$$

where $\tau \equiv T/T_c$ and θ is the Heaviside function. The effective energy density and pressure are

$$\varepsilon_{ef} = \varepsilon_v + \varepsilon_r, \qquad p_{ef} = -\varepsilon_v + \tfrac{1}{3}\varepsilon_r \tag{3.3}$$

where ε_r is the radiation energy density. With these quantities dynamical system (2.1) assumes the form [32]

$$\dot{H} = -H^2 - \tfrac{1}{3}\left[\varepsilon_c \tau^4 - \varepsilon_0 \tau^{\beta\theta(1-\tau)}\right],$$
$$\dot{\tau} = -\frac{H\tau}{1 + \frac{1}{4}\alpha\beta\theta(1-\tau)\tau^{\beta\theta-4}}, \tag{3.4}$$

where $\alpha \equiv \varepsilon_0/\varepsilon_c$ and ε_c comes from the formula for relativistic matter: $\varepsilon_r(T) = \varepsilon_c \tau^4$. For $\beta = 4$, one has $\varepsilon_{ef} + 3p_{ef} = 0$ which implies $R = R_0 t$, and there are no horizons provided that the phase transition occurred sufficiently early. It can be seen that some models with $k = +1$ never attain the phase transition temperature ($\tau > 1$) whereas within open models ($k = 0, -1$) the phase transition may occur.

Let us notice that some field theoretical considerations seem to distinguish $\beta = 2$ value which makes the above model physically less interesting.

4. *Some comments*

Rescaling to Prigogine's time requires negative pressure. As we have seen, this is naturally provided by models discussed in Secs. 1–3. For world models with bulk viscosity $\sim \varepsilon^{1/2}$ and for models with the phase transition ($\beta = 4$), $R = R_0 t$, Eqs. (Ja, b) can be easily integrated to obtain

$$\lambda = d \ln t, \quad \text{along zero geodesics,} \tag{4.1a}$$

$$\lambda = D \ln \frac{R_0 t}{\alpha + [R_0^2 t^2 + \alpha^2]^{1/2}}, \quad \text{along timelike geodesics,} \tag{4.1b}$$

where d and $D \equiv A/R_0\alpha$ are constants. Having in mind that in all these models there are no particle horizons, the above results might be interpreted by stating that "the mixing rate (i.e. rate of approach to equilibrium) with respect to change in t approaches infinity as one nears the singularity." [5].

LMP procedure produces the desired effect ($\mathscr{K}$-flow in geodesics) only within models having compact 3-spaces of negative curvature. 3-spaces of negative curvature can always be compactified [24, 33], but their topology may be very complex, and in general they are not simply connected. The usual Friedman topology could be recovered by changing to the universal covering manifold [33].

It is worth noticing that our models are what has been called by Ellis [34] small universes: (a) they are expanding, (b) have compact spacial sections, (c) the world line of each test particle intersects the past light cone of every other test particle multiple times. Such universes are quite exceptional as far as their causal and observational features are concerned. A small universe contains only a finite number of galaxies and every observer in it has had enough causal contacts with all of them to predict the future of his universe [36]. An observer, living in a small universe, having flat or positively curved space sections, may hope to do cosmology in such an easy way. His colleague, sentenced to make his explorations within a small universe of compact negatively curved space sections and satisfying other conditions for LMP procedure to be applied, faces a hopeless situation. Geodesic flows, in such a universe, exhibit chaotic behaviour as $\lambda \to -\infty$. They are mixing, ergodic, and have a non-zero metric entropy. Moreover, being $\mathscr{K}$-flows they satisfy the so called 0–1 law which asserts that if a certain event can be predicted within the model, by using information from arbitrary distant past, then this event must be of probability either zero or one [25]. Processes satisfying this law are strongly unstable against small "perturbations" of the initial data; their behaviour is fully stochastic (see [26] pp. 285–291).

Geodesic flows on compact spaces of negative curvature have another interesting proparty: they are structurally stable in the set of all dynamical systems (they form in it non-empty open subsets) [26].

5. Negative space curvature and predictability

Let us assume that the universe is well described by a Friedman dynamical system with compactified space sections of negative curvature in which geodesics form a $\mathscr{K}$-flow, and in which Prigogine's time can be defined. Let further K be a negative space curvature which differs from zero by b^2 or by less than b^2, R_0 — the present value of the scale factor, and s — the characteristic distance, i.e. a distance along which small perturbations of the initial conditions increase e times. Then

$$(K \leqslant -b^2) \Rightarrow (s \leqslant b^{-1}) \tag{5.1}$$

(see, Ref. [25], p. 287). If an investigator, at the present epoch, knows the initial conditions with an error ε, then after a distance l, as measured along geodesics, the error will change to

ε. We have

$$\tilde{\varepsilon} = \varepsilon e^{l/R_0} = \varepsilon^{l\sqrt{-K}}. \tag{5.2}$$

Basing on these assumptions we can estimate the value of K. From actually available astronomical data we confidently retrodict the history of the universe back to the last scattering surface, i.e. roughly to $R = R_0/100$. The error increases as

$$\tilde{\varepsilon} = \varepsilon \exp\left[\left(\int_{R_0/100}^{R_0} \frac{U(R)dR}{R}\right)\sqrt{-K}\right], \tag{5.3}$$

where $U(R) = dt/dR$. By assuming that $\tilde{\varepsilon}$ and ε are of the same order of magnitude we obtain

$$\int_{R_0/100}^{R_0} \frac{U(R)dR}{R} \sqrt{-K} \leqslant 1. \tag{5.4}$$

From Eq. (5.4), if $u(R) \leqslant C$, one gets

$$C\sqrt{-K} \ln 100 < 1$$

or, more generally, for $R = R_0/n$

$$C\sqrt{-K} \ln (n) < 1. \tag{5.5}$$

Our conclusion is: if God wanted to create a universe predictable for its inhabitants (and being "sophisticated but not malicious" he certainly did), and if he decided to begin with the model having compact space sections of negative curvature, he had to create a universe with sufficiently small space curvature. Or to put this in another way, if a human observer, placed within the universe belonging to the class of worlds discussed in this paper, is supposed to have a possibility to make retrodictions deep into the past, there must be: $K \to 0$ as $n \to \infty$. The same is valid for predictability to the future with $R = R_0/n$ replaced by $R = R_0 n$. This might be also viewed as a kind of anthropic explanation of why our universe is so close to spatial flatness.

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