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KALUZA-KLEIN REDUCTION SCHEME ON SUPERMANIFOLDS

By J. Karkowski

Institute of Physics, Jagellonian University, Cracow*

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Kaluza-Klein reduction scheme is generalized to a supermanifold which is a product of some basic supermanifold and a classical simple Lie supergroup with nondegenerate Killing form. The resulting theory and its invariances are discussed in detail.

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1. *Introduction*

Kaluza-Klein type theories on ordinary manifolds have been widely discussed for a long time (for a review see [1] and references therein). They are expected to be useful, for example, in supergravity [2]. This Kaluza-Klein scheme was also applied to some special manifolds with Bose and Fermi coordinates [3]. In the present work we generalize this scheme to an arbitrary supermanifold which is (locally) a product of some other (basic) supermanifold and a simple Lie supergroup with nondegenerate Killing form. Another new feature of our analysis (in comparison to [3]) is the inclusion of (even and odd) Brans-Dicke-like scalars. The generalization we present here is done mainly from the mathematical point of view and remains in a close analogy with the geometric treatment of ordinary Kaluza-Klein reduction scheme given, e.g. in [4]. However, it turns out that odd coordinates which we must use to describe our supermanifolds cause not only technical troubles but also imply some limitations on possible invariances of the final theory, as it is discussed in Sect. 3.

Our paper is organized as follows. In the first part some mathematical preliminaries concerning differential geometry on supermanifolds are reviewed, and generalized Kaluza-Klein Ansatz imposed on metric is formulated. In the second part we show that starting from the product supermanifold described above, and taking as a lagrangian the supercurvature density, we obtain, after reducing it in Kaluza-Klein manner, the sum of supercurvature scalar of the basic supermanifold, a term describing Yang-Mills type field valued

* Address: Instytut Fizyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland.

in Lie superalgebra of the chosen supergroup, and a lagrangian of scalar fields. In the Appendix some mathematical theorems concerning Lie superalgebras and their (invariant) metrics are given, and possible invariances of our final lagrangian are obtained in detail.

2. Riemannian supergeometry and the Kaluza-Klein Ansatz

2.1. The metric

Let M be an (m, n)-dimensional supermanifold (mathematical details can be found in [5] and [6]). We can locally describe it by m even $\{x^\mu\}$ and n odd $\{\theta^a\}$ coordinates (shortly $\{x^\mu, \theta^a\} \equiv \{x^A\}$). The coordinate basis (associated with $\{x^A\}$) of the bundle tangent to this supermanifold is denoted by $\{\partial_A\} = \{\partial_\mu, \partial_a\}$. The basic object of metrical supergeometry — the metric g — is a covariant, second rank tensor such that the supermatrix $[g_{AB}]$ is even, invertible (even and invertible supermatrices form a supergroup denoted by $\mathrm{GL}(m|n)$; for details see [5]) and supersymmetric, that is

$$g_{BA} = (-)^{p(A)\cdot p(B)} g_{AB}, \tag{2.1.1}$$

where $p(\alpha) = 0$ and $p(a) = 1$. Let $\{e_A\} \equiv \{e_\mu, e_a\}$ be some other (may be non-coordinate) basis of the tangent bundle, related to $\{\partial_A\}$ by

$$e_{A'} = L_{A'}{}^A \partial_A, \tag{2.1.2}$$

where the supermatrix $[L_{A'}{}^A]$ is an element of the supergroup $\mathrm{GL}(m|n)$ described above. Using (2.1.2) we obtain

$$g_{A'B'} = g(e_{A'}, e_{B'}) = L_{A'}{}^A g_{AB} (L^{\mathrm{st}})^B{}_{B'}, \tag{2.1.3}$$

where

$$(L^{\mathrm{st}})^B{}_{B'} = (-)^{p(B)(1+p(B'))} L_{B'}{}^B \tag{2.1.4}$$

is the supertranspose to $[L_{B'}{}^B]$. The formula (2.1.3) describes the transformation rule of the metric under the basis change (2.1.2). The contravariant metric tensor is defined by

$$g^{AB} g_{BC} = g_{CB} g^{BA} = \delta_C{}^A \tag{2.1.5}$$

and satisfies from (2.1.3) and (2.1.5):

$$\begin{aligned} g^{A'B'} &= ((L^{-1})^{\mathrm{st}})^{A'}{}_A g^{AB} (L^{-1})_B{}^{B'}, \\ g^{AB} &= (-)^{p(A)+p(B)+p(A)p(B)} g^{BA}, \end{aligned} \tag{2.1.6}$$

where $[(L^{-1})_B{}^{B'}]$ denotes the supermatrix inverse to $[L_{B'}{}^B]$.

2.2. The connection

The connection coefficients Γ^C_{AB} are defined by:

$$\nabla_{e_A} e_B = (-)^{p(C)(1+p(A)+p(B))} \Gamma^C_{AB} e_C = e_C \Gamma^C_{AB} \tag{2.2.1}$$

(e_C does not act on Γ^C_{AB}), where ∇ denotes the supercovariant derivative, which satisfies [6]:

$$\nabla_{c^A e_A} = c^A \nabla_{e_A},$$
$$\nabla_{e_A} f = e_A f,$$
$$\nabla_{e_A}(S\otimes T) = (\nabla_{e_A} S)\otimes T + (-)^{p(S)p(T)}(\nabla_{e_A} T)\otimes S \tag{2.2.2}$$

(S, T arbitrary tensors; $p(S)$, $p(T)$ their parities). Assuming the connection to be metrical, that is $\nabla g = 0$, and solving this equation, we obtain the following formulas valid in arbitrary non-coordinate basis:

$$\Gamma^C_{AB} = \tfrac{1}{2} g^{CD}\{(-)^{p(D)(p(A)+p(B))}(\partial_A g_{BD} - c_{A,BD})$$
$$+(-)^{p(A)\cdot p(B)+p(A)\cdot p(D)+p(B)p(D)}(\partial_B g_{AD} - c_{B,AD}) - (\partial_D g_{AB} - c_{D,AB})\} \tag{2.2.3}$$

(please note different conventions than in [6]), where

$$[e_A, e_B\} \equiv e_A e_B - (-)^{p(A)p(B)} e_B e_A = c^D_{AB} e_D,$$
$$c_{D,AB} = g_{DE} c^E_{AB}. \tag{2.2.4}$$

2.3. The supercurvature tensor

The supercurvature tensor is defined by

$$R(e_B, e_C)e_D = (-)^{p(A)(1+p(B)+p(C)+p(D))} R^A{}_{BCD} e_A = e_A R^A{}_{BCD} \tag{2.3.1}$$

(e_A also does not act on $R^A{}_{BCD}$), where

$$R(e_B, e_C) = [\nabla_{e_B}, \nabla_{e_C}\} - \nabla_{[e_B, e_C\}}. \tag{2.3.2}$$

Ricci supertensor equals to

$$R_{CD} = (-)^{p(A)} R^A{}_{ACD} = \partial_A \Gamma^A_{CD} - (-)^{p(A)} \partial_C \Gamma^A_{AD}$$
$$+(-)^{p(A)} \Gamma^A_{AE} \Gamma^E_{CD} - (-)^{p(A)(1+p(C))} \Gamma^A_{CE} \Gamma^E_{AD}$$
$$-(-)^{p(E)+p(C)(p(A)+p(E))} c^E_{AC} \Gamma^A_{ED} \tag{2.3.3}$$

and we define Ricci superscalar as:

$$R = (-)^{p(A)} g^{AB} R_{BA} = R_{BA} g^{AB} (-)^{p(B)}. \tag{2.3.4}$$

Under the basis change (2.1.2) R_{CD} transforms like the covariant metric tensor (this is due to $(-)^{p(A)}$ in the contraction of supercurvature tensor).

2.4. The Kaluza-Klein Ansatz

We start from a supermanifold P which is assumed to be a bundle space of some principal fiber superbundle with superconnection. The precise mathematical meaning of these notions is not important for our purposes because Kaluza-Klein reduction scheme is based only on local properties of (super) manifolds. This means that P looks locally like a (direct) product of some basic (physical) (m, n)-dimensional supermanifold M and (p, q)-dimensional Lie supergroup G, and a supermatrix $V_A{}^Z(x^B, y^W)$ is given, which (also locally) describes the connection.

(Conventions: $x^A, x^B, \ldots$ are coordinates of M; $y^Z, y^W, \ldots$ coordinates of G; $e_Z, e_W, \ldots$ (which will be used later on) are left-invariant vector fields on G — the set $\{e_Z\}$ is isomorphic to Lie superalgebra $\bar{G}$ of G and satisfies

$$[e_Z, e_W\} = f^U_{ZW} e_U. \tag{2.4.1}$$

f^U_{ZW} — structure constants of $\bar{G}$; $e_Z, e_W \ldots$ form also a basis of the bundle tangent to G).

Let $e_A = \partial_A - V_A{}^Z e_Z$ be the horizontal lift of ∂_A; we must have [4]

$$[e_A, e_W\} = 0 \tag{2.4.2}$$

(as $V_A{}^Z$ describes the superconnection). From (2.4.2) we obtain

$$(-)^{p(A)p(Z)} e_Z V_A{}^W = V_A{}^U f^W_{UZ} \tag{2.4.3}$$

and this formula, which can be integrated to:

$$V_A{}^Z(x, y) = (-)^{p(Z)\,(1+p(W))} V_A{}^W(x, 0)\, \mathrm{Ad}^Z{}_W(\gamma^{-1}(y)) \tag{2.4.4}$$

($[\mathrm{Ad}^Z{}_W(\gamma^{-1}(y))]$ is the representative of $\gamma^{-1}(y)$ in the adjoint representation — $\gamma(y)$ is the element of G corresponding to $\{y^Z\}$; we assume $\gamma(0) = 1$) gives us the y-dependence of $V_A{}^Z(x, y)$, while its x-dependence is arbitrary.

Having assumed the supermanifold M to have the metric $g_{AB}(x)$ we organize $P = M \times G$ into Riemannian superspace giving its metric (in the basis $\{\partial_A, e_Z\}$) by the matrix

$$[\bar{g}_{IJ}(x, y)] = \left[\begin{array}{c|c} g_{AB}(x) - V_A{}^Z(x, y) g'_{ZW}(x)\, (V^{st})^W{}_B(x, y) & -V_A{}^Z(x, y) g'_{ZQ}(x) \\ \hline -g'_{PZ}(x)\, (V^{st})^Z{}_B(x, y) & g'_{PQ}(x) \end{array}\right] \tag{2.4.5}$$

($I, J \ldots$ denote both $A, B \ldots$ and $Z, W \ldots$; note the mixed order of even and odd rows and columns in the supermatrix (2.4.5) as a whole).

The formula (2.4.5) is a straightforward generalization of Kaluza-Klein Ansatz (with scalar fields) to a supermanifold case. $g'_{PQ}(x)$ is an arbitrary left-invariant metric on G. In the basis of left-invariant vector fields $\{e_Z\}$ it does not depend on y, but of course may depend on x. Let us note that left invariant metrics on G are in one-to-one correspondence with metrics on its Lie superalgebra $\bar{G}$ (see for instance [7] for analogous result concerning ordinary Lie groups; the generalization of it to supergroup case is simple).

3. Dimensional reduction

As a lagrangian on the supermanifold $P = M \times G$ we take the superscalar density

$$\mathscr{L} = ((-)^{p(M)+p(G)} \operatorname{sdet} [\bar{g}_{IJ}])^{1/2} R_P, \tag{3.1}$$

where sdet denotes superdeterminant [5].

(*Remark.* It is easy to show that for arbitrary (m, n)-dimensional supermanifold (we assume n to be even) with the metric g_{AB} we can find a basis in which g_{AB} is given by

$$[g_{AB}] = \left[\begin{array}{c|cc} I_{m_1, m_2} & & \\ \hline & & I_r \\ & -I_r & \end{array}\right] \text{ where } \begin{cases} I_{m_1, m_2} = \left[\begin{array}{c|c} I_{m_1} & \\ \hline & -I_{m_2} \end{array}\right] \\ m = m_1 + m_2;\ n = 2r \end{cases} \tag{3.2}$$

we define $(-)^{p(M)} = \operatorname{sdet}[g_{AB}]$ (in this basis) and $(-)^{p(G)}$ is defined analogously.)

Our aim is to calculate (3.1) and to integrate it over G. We shall perform the calculations in the horizontal lift basis $\{e_A, e_Z\}$ given by (2.4.1), (2.4.2) and satisfying:

$$\begin{aligned} [e_A, e_B\} &= -F_{AB}{}^Z e_Z, \\ [e_A, e_Z\} &= 0, \\ [e_W, e_Z\} &= f^U_{WZ} e_U, \end{aligned} \tag{3.3}$$

where (as it is easy to check from (2.4.1))

$$F_{AB}{}^Z(x, y) = \partial_A V_B{}^Z - (-)^{p(A)p(B)} \partial_B V_A{}^Z - (-)^{p(R)p(B)} V_A{}^R V_B{}^S f^W_{SR}. \tag{3.4}$$

The horizontal lift basis is very suitable for calculations because $[g_{IJ}]$ has in this basis a block-diagonal form

$$[\bar{g}_{IJ}] = \left[\begin{array}{c|c} g_{AB}(x) & \\ \hline & g'_{PQ}(x) \end{array}\right]. \tag{3.5}$$

From (2.4.5) we obtain (in the coordinate basis $\{\partial_A, \partial_Z\}$)

$$\operatorname{sdet}[\bar{g}_{IJ}] = \operatorname{sdet}[g_{AB}] \cdot \operatorname{sdet}[g'_{PQ}] \cdot (\operatorname{sdet}[E_Z{}^{Z'}])^2, \tag{3.6}$$

where $[E_Z{}^{Z'}(y)]$ is the inverse of $[K_{Z'}{}^Z(y)]$ defined by

$$e_{Z'}(y) = K_{Z'}{}^Z(y)\partial_Z. \tag{3.7}$$

R_p in the horizontal lift basis equals to

$$R_P = (-)^{p(A)} g^{AB} R_{BA} + (-)^{p(P)} g'^{PQ} R_{QP}. \tag{3.8}$$

After long and tedious calculations we obtain (discarding the total divergence) the following final form of our lagrangian (3.1)

$$\mathscr{L} = ((-)^{p(M)} \operatorname{sdet}[g_{AB}])^{1/2} \cdot \operatorname{sdet}[E_Z{}^{Z'}] \cdot [\mathscr{L}_{\text{grav}} + \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{scal}}], \tag{3.9}$$

where

$$\begin{aligned} \mathscr{L}_{\text{grav}} &= \phi^{\frac{p+q}{2}} R_M(g_{AB}), \\ \mathscr{L}_{\text{gauge}} &= \tfrac{1}{4} \phi^{\frac{p+q}{2}+1} \mathscr{G}((-)^{p(A)} g^{AB} F_{BC}{}^W e_W, g^{CD} F_{DA}{}^Z e_Z) \end{aligned} \tag{3.10}$$

and $\mathscr{L}_{\text{scal}}$ is the sum of two terms (kinetic and potential) of the following form

$$\mathscr{L}_{\text{scal-kin}} = -\tfrac{1}{4}\,\phi^{\frac{p+q}{2}}(-)^{p(P)}(\partial_A\mathscr{G}^{PQ})\mathscr{G}_{QP}g^{AB}(-)^{p(x)}(\partial_B\mathscr{G}^{XY})\mathscr{G}_{YX}$$

$$-\tfrac{1}{4}\,\phi^{\frac{p+q}{2}}(-)^{p(A)\,(1+p(P)+p(Q))+p(P)}g^{AB}(\partial_B\mathscr{G}^{PQ})\,(\partial_A\mathscr{G}_{QP})$$

$$-\frac{2q+1}{2}\,\phi^{\frac{p+q}{2}-1}\,\partial_A\phi g^{AB}(-)^{p(P)}(\partial_B\mathscr{G}^{PQ})\mathscr{G}_{QP}$$

$$+\frac{(p-q)\,(3q-1)}{4}\,\phi^{\frac{p+q}{2}-2}\,\partial_A\phi g^{AB}\partial_B\phi,$$

$$\mathscr{L}_{\text{scal-pot}} = \phi^{\frac{p+q}{2}}R_G. \tag{3.11}$$

We have also substituted

$$g'_{PQ} = \phi\cdot\mathscr{G}_{PQ};\ \text{sdet}\,[\mathscr{G}_{PQ}] = (-)^{p(G)}, \tag{3.12}$$

where ϕ is the scalar (Brans-Dicke-like) field. R_G is the Ricci superscalar of G and it equals to

$$R_G = (-)^{p(P)}\mathscr{G}^{PQ}\big[\tfrac{1}{2}f^Y_{QX}f^X_{YP}(-)^{p(Y)+p(Y)p(Q)}$$

$$+(-)^{p(Y)\,(1+p(Q)+p(Z))}\tfrac{1}{4}f^Y_{QX}\mathscr{G}^{XZ}\mathscr{G}_{YU}f^U_{ZP}$$

$$-(-)^{p(X)}f^X_{XQ}(-)^{p(Y)}f^Y_{YP}\big]. \tag{3.13}$$

It plays the role of the scalar superpotential. In the case when g'_{PQ} does not depend on x, the scalar superpotential reduces to a (cosmological-like) constant. To obtaining the above results we have used the following formula derived from properties of superdeterminant and metrical superconnection:

$$(-)^{p(A)}g^{AB}\nabla_BW_A = ((-)^{p(M)}\text{sdet}\,[g_{AB}])^{-1/2}\partial_B[((-)^{p(M)}\text{sdet}\,[g_{AB}])^{1/2}g^{BA}W_A]. \tag{3.14}$$

The last step in the Kaluza-Klein program is to integrate (3.9) over the supergroup manifold G. We shall limit ourselves to those Lie supergroups whose maximal Lie subgroups are compact and whose Lie superalgebras are simple (over complex numbers.) and have non-degenerate Killing forms (for details concerning Lie superalgebras see [8]), Let $\bar{G}$ be one of such superalgebras with p even and q odd generators. By Lie supergroup corresponding to this Lie superalgebra we mean (formally) the set of elements of the type

$$\gamma(y) = \exp\{i(y^aT_a+y^\alpha Q_\alpha)\}, \tag{3.15}$$

where T_a are generators of some compact real form of maximal Lie subalgebra $\bar{G}_0$ of $\bar{G}$ ($\bar{G}_0$ for our class of superalgebras is semisimple or is the direct sum of semisimple and U(1) Lie algebras, and a compact real form of it always exists, see [9]), Q_α are odd generators of $\bar{G}$, y^a and y^α are respectively even and odd elements of $R\otimes\Lambda(q)$ where $\Lambda(q)$ is appropriately chosen Grassmann algebra with q generators. It is known that the volume of supergroup whose maximal Lie subgroup is compact equals to zero (see [10] and references therein). This fact makes the integration over such Lie supergroup non-trivial as $\mathscr{L}_{\text{grav}}$ and $\mathscr{L}_{\text{gauge}}$ in (3.9) do not depend on y. The measure $d\mu = dy \cdot \text{sdet}\,[E_Z{}^{Z'}]$ is invariant. Therefore, by integrating these two terms over the whole supergroup manifold we obtain these terms multiplied by the volume of the supergroup, that is zero. As the way out of this difficulty we propose to integrate $\mathscr{L}_{\text{grav}}$ and $\mathscr{L}_{\text{scal}}$ over maximal Lie subgroup $\tilde{G}$ of G. The $\mathscr{L}_{\text{gauge}}$ term may depend on y because of the y-dependence of $F_{AB}{}^Z$ given by

$$F_{AB}{}^W(x, y) = F_{AB}{}^Z(x, 0)\,\text{Ad}\,(\gamma^{-1}(y))^W{}_Z(-)^{p(W)(1+p(Z))} \tag{3.16}$$

and obtained from (2.4.4) and (3.4). We can now distinguish two cases. In the first we assume the metric $\mathscr{G}_{PQ}$ to be Ad(G)-invariant. This means (for our class of superalgebras) that $\mathscr{G}_{PQ}$ is proportional to the Killing form of $\bar{G}$ and the $\mathscr{L}_{\text{gauge}}$ term does not depend on y. In this case we integrate it like $\mathscr{L}_{\text{grav}}$ and $\mathscr{L}_{\text{scal}}$ over $\tilde{G}$ described by coordinates $\tilde{y}$ only and we obtain finally

$$\int d\tilde{y}\,V(\tilde{y})\mathscr{L} = V_{\tilde{G}}\phi^{\frac{p+q}{2}}\Bigg\{R_M + \Lambda + \frac{(p-q)(3q-1)}{4}\,\frac{\partial_A\phi g^{AB}\partial_B\phi}{\phi^2}$$
$$+\tfrac{1}{4}\phi^{\frac{p+q}{2}+1}K((-)^{p(A)}g^{AB}F_{BC}{}^W(x, 0)e_W,\ g^{CD}F_{DA}{}^Z(x, 0)e_Z)\Bigg\}, \tag{3.17}$$

where $\Lambda = R_G$ is a constant, K denotes the Killing form on $\bar{G}$. The last term on the r.h.s. of (3.17) (the gauge term) is invariant under the action of the whole supergroup G, that is we can replace $F_{AB}{}^W(x, 0)$ in it by $F_{AB}{}^W(x, y(x))$ and the x-dependence of y is arbitrary. Let us note that the scalar term in (3.17) is of Brans-Dicke type.

The second possibility arises when we do not impose any constraints on $\mathscr{G}_{PQ}$. Then the gauge term must be integrated over the whole supergroup G and we obtain from it after reduction a Yang-Mills type lagrangian whose internal metric $\bar{K}$ is the average of $\mathscr{G}$ over G

$$\bar{K}_{XY}(x) \equiv \frac{1}{V_{\tilde{G}}}\int dy V(y)\,(-)^{p(Z)(1+p(X))}\,\text{Ad}\,(\gamma^{-1}(y))^Z{}_X\mathscr{G}_{ZW}\,\text{Ad}\,(\gamma^{-1}(y))^W{}_Y. \tag{3.18}$$

As it is discussed in the Appendix, this metric is invariant under the action of $\tilde{G}$ and in general it is not invariant under the whole supergroup G. It depends on x via one or two scalar fields which are linear combinations of $\mathscr{G}_{PQ}(x)$ and can be calculated from (3.18) (for the general form of $\bar{K}$ and its implications for the invariance of the gauge term see Appendix). The whole reduced lagrangian in this case has the form:

$$\mathscr{L} = V_{\tilde{G}}\{\phi^{\frac{p+q}{2}}R_M + \mathscr{L}_{\text{scal}} + \mathscr{L}_{\text{gauge}}\}, \tag{3.19}$$

where $\mathscr{L}_{\text{scal}}$ is the same as in (3.11) and

$$\mathscr{L}_{\text{gauge}} = \tfrac{1}{4}\phi^{\frac{p+q}{2}+1}\bar{K}(x)\,((-)^{p(A)}g^{AB}F_{BC}{}^W(x, 0)e_W,\ g^{CD}F_{DA}{}^Z(x, 0)e_Z). \tag{3.20}$$

As it was mentioned above, $\mathscr{L}_{\text{gauge}}$ does not change when we replace $F_{AB}{}^{Z}(x, 0)$ by $F_{AB}{}^{Z}(x, \tilde{y}(x))$, where $\tilde{y}$ are coordinates of $\tilde{G}$ only. However, that term is usually not invariant under the action of the whole G. To interpret the scalar term in (3.19), we write g' in the form:

$$g'_{PQ} = M_P{}^X K_{XY} (M^{\text{st}})^Y{}_Q, \tag{3.21}$$

where K is Killing form of $\bar{G}$ and $M_P{}^X(x)$ is an element of $\text{GL}(p|q) \cdot g'_{PQ}$ is invariant under the replacement

$$M_P{}^X \to M'_P{}^X = M_P{}^Y B_Y{}^X(x) \tag{3.22}$$

if $[B_Y{}^X(x)]$ leaves K unchanged. So $[B_Y{}^X]$ is an element of Osp $(\bar{G}, K)$ — the orthosymplectic supergroup leaving the Killing form K on $\bar{G}$ invariant. Thus the scalar term may be interpreted as some $\text{GL}(p|q)/\text{Osp}(\bar{G}, K)$ super σ-model (for the discussion of similar problems in ordinary group case see [11] and references therein).

4. *Final remarks*

Although we do not discuss any possible applications of our scheme, let us note at the end that taking ordinary or extended superspace as M, the gravity term in our lagrangian (3.17) or (3.19) is simply the starting lagrangian of the so-called gauge-supersymmetry approach to supergravity (for review of this approach see [12] and references therein). Thus one of possible applications of our results may be the obtaining of a superspace version of supergravity (simple or extended) coupled to matter represented by Yang-Mills field valued in some Lie superalgebra and Brans-Dicke-like scalar fields. So Kaluza-Klein type theories on supermanifolds may be useful for superunifications.

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APPENDIX

In the Appendix we discuss the averaged metric on $\bar{G}$ given by (3.18), and particularly its invariance properties. Let us remind that our analysis is limited to simple (over **C**) superalgebras with nondegenerate Killing form. Any Lie superalgebra with these properties is isomorphic to one of the following

$$\begin{cases} \text{Spl}\,(n, m) & \text{with} \quad n, m \geqslant 1;\ n \neq m, \\ \text{Osp}\,(n, 2r) & \text{with} \quad n, r \geqslant 1;\ n \neq 2r+2, \\ \Gamma_2, \Gamma_3 & \end{cases} \tag{A.1}$$

(for details and notations see [8] and [13]). It is also true that any invariant bilinear (even) form on such superalgebra is proportional to its Killing form. It is easy to show that the metric $\bar{K}_{XY}$ given by (3.18) is invariant under $\tilde{G}$. In fact

$$(-)^{p(Z)(1+p(X))}\,\mathrm{Ad}^Z{}_X(\gamma^{-1}(\tilde{y}))\bar{K}_{ZU}\,\mathrm{Ad}^U{}_Y(\gamma^{-1}(y))$$
$$= \frac{1}{V_{\tilde{G}}}\int dy' V(y')\,(-)^{p(P)(1+p(X))}\,\mathrm{Ad}^P{}_Z(\gamma^{-1}(y'))\,\mathrm{Ad}^Z{}_X(\gamma^{-1}(\tilde{y}))$$
$$\mathscr{G}_{PQ}\,\mathrm{Ad}^Q{}_U(\gamma^{-1}(y'))\,\mathrm{Ad}^U{}_Y(\gamma^{-1}(\tilde{y})) = \bar{K}_{XY} \tag{A.2}$$

because of the invariance of the measure. It is impossible to do the same for the whole supergroup G, because we can not interchange the order of integration and multiplication by odd Grassmann variables. The most general Ad($\tilde{G}$)-invariant metric on $\bar{G}$ belonging to (A.1) has the form

$$[K] = \left[\begin{array}{c|c} \bar{K}_0 & 0 \\ \hline 0 & \bar{K}_1 \end{array}\right], \tag{A.3}$$

where $\bar{K}_0$ is symmetric, Ad($\bar{G}_0$) — invariant metric on $\bar{G}_0$ and $\bar{K}_1$ (skew-symmetric), Ad($\bar{G}_0$) — invariant metric on $\bar{G}_0$-module $\bar{G}_1$($\bar{G} = \bar{G}_0 \oplus \bar{G}_1$; $\bar{G}_0$, $\bar{G}_1$ are even and odd parts of $\bar{G}$). To prove this we use the infinitesimal form of (A.2)

$$\bar{K}([e_W, e\}; e_Z) = \bar{K}(e_W; [e, e_Z\}), \tag{A.4}$$

where e belongs to $\bar{G}_0$. Using the basis $\{e_Z\}$ consistent with the root-space decomposition of $\bar{G}$, and assuming e to be an element of Cartan subalgebra of $\bar{G}_0$ we obtain in particular

$$-\lambda_W(e\bar{K})_{WZ} = \lambda_Z(e)\bar{K}_{WZ} \quad \text{(for all } e\text{)}, \tag{A.5}$$

where $\{e, e_W\} = \lambda_W(e)e_W$. Let $e_W \in \bar{G}_0$ and $e_Z \in \bar{G}_1$. Then $K_{WZ} \neq 0$ implies $+\lambda_W(e) = -\lambda_Z(e)$ (for all e). But this is a contradiction because λ_W is an even root and $(-\lambda_Z)$ is odd and for our class of superalgebras they have to be different. This proves (A.3).

(Skew)-symmetry of $\bar{K}_0(\bar{K}_1)$ follows from (3.18) and the fact that $\mathscr{G}$ is supersymmetric. Using similar arguments it is not difficult to prove that $\bar{K}_1$ is nondegenerate and unique up to a constant. (It is true for both possible cases:

1. $\bar{G}_0$-module $\bar{G}_1$ is irreducible — then is obvious.
2. G_0-module $\bar{G}_1$ is a direct sum of two irreducible pieces — then it is necessary to use skew-symmetry of $\bar{K}_1$ in the proof.) For Lie superalgebras (A.1) $\bar{G}_0$ is either simple (then $\bar{K}_0$ is unique up to a constant) or it is a direct sum of two algebras (which are both semisimple or one of them is U(1) algebra) $\bar{G}_0 = \bar{G}_{0(1)} + \bar{G}_{0(2)}$ — then $\bar{K}_0$ depends on (at least) two arbitrary constants). So the most general form of $\bar{K}$ is

$$[\bar{K}] = \left[\begin{array}{c|c} a(x)\bar{K}_0 & 0 \\ \hline 0 & b(x)\bar{K}_1 \end{array}\right] \quad \text{(in the first case)}$$

or

$$[\bar{K}] = \left[\begin{array}{c|c|c} a_1(x)\bar{K}_{0(1)} & 0 & \\ \cline{1-2} 0 & a_2(x)\bar{K}_{0(2)} & 0 \\ \hline \multicolumn{2}{c|}{0} & b(x)\bar{K}_1(x) \end{array}\right] \quad \text{(in the second case)} \tag{A.6}$$

where $\bar{K}_0$, $\bar{K}_{0(1)}$, $\bar{K}_{0(2)}$, $\bar{K}_1$ do not depend on x and $a(x)$, $a_1(x)$, $a_2(x)$, $b(x)$ are normalized in such a way that for $a = a_1 = a_2 = b = 1$ $\bar{K}$ is simply the appropriate Killing form K. However from (3.18) using the invariance of the measure we obtain

$$\mathrm{str}\,(K^{-1}\bar{K}) = \int dy V(y)\,\mathrm{str}\,(K^{-1}\mathscr{G}) = 0 \tag{A.7}$$

(str denotes supertrace), where K^{-1} is the (Ad(G)-invariant) inverse of K. (A.6) and (A.7) imply

$$\begin{cases} a(x)p - b(x)q = 0 & \text{(for simple } G_0\text{)}, \\ a_1(x)p_1 + a_2(x)p_2 - b(x)q = 0 & \text{(in the second case)}, \end{cases} \tag{A.8}$$

where p, p_1, p_2, q are dimensions of $\bar{G}_0$, $\bar{G}_{0(1)}$, $\bar{G}_{0(2)}$, $\bar{G}_1$ respectively. It is easily seen from (A.8) that $\bar{K}$ in general can not be Ad(G)-invariant. Ad(G)-invariance of $\bar{K}$ is possible only for Lie superalgebras for which $p_1 + p_2 - q = 0$. ($p - q = 0$ never happens for Lie superalgebras belonging to (A.1)).

REFERENCES

[1] A. Salam, J. Strathdee, *Ann. Phys.* **141**, 316 (1982); J. Rayski, *Acta Phys. Pol.* **27**, 89 (1965).

[2] E. Witten, *Nucl. Phys.* **186B**, 412 (1981); M. J. Duff, CERN preprint Th. 3394 (1982); J. Rayski, *Lett. Nuovo Cimento* **34**, 142 (1982); J. Rayski, *Nuovo Cimento* **70A**, 49 (1982).

[3] R. Delbourgo, P. D. Jarvis, G. Thompson, *J. Phys. A* **15**, 2813 (1982); R. Delbourgo, R. J. Farmer, P. D. Jarvis, Hobart preprint, October 1982.

[4] Y. Cho, *J. Math. Phys.* **16**, 2029 (1975).

[5] D. A. Leites, *Usp. Mat. Nauk* **35**:**1**, 3 (1980).

[6] F. Mansouri, *J. Math. Phys.* **18**, 52 (1977).

[7] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. II, Interscience Publishers, New York 1969.

[8] M. Scheunert, *The Theory of Lie Superalgebras*, Lecture Notes in Mathematics **716**, Springer-Verlag, Berlin 1979.

[9] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York 1962, p. 147.

[10] F. A. Berezin. *Yad. Fiz.* **29**:**6**, 1670 (1979).

[11] R. Percacci, S. Randjbar-Daemi, I. C. T. P. preprint IC/82/18 (1982).

[12] P. van Nieuwenhuizen, *Phys. Rep.* **68**, 189 (1981), Sec. 5.7.

[13] V. G. Kac, *Representations of Classical Lie Superalgebras*, in Lecture Notes in Mathematics **676**, ed. A. Dold and R. Eckmann, Springer-Verlag, Berlin 1978.