LIMITATION OF THE CONCEPT OF THE CLASSICAL COLORED PARTICLE

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We argue that a clot of color charged matter placed in an external nonabelian gauge field will not, in general, propagate in any definite direction. Instead, it will tend to disperse all over the space. We also prove that if the position $\vec{x}(t)$ and the color spin vector $\vec{I}(t)$ of the classical colored particle are identified with the expectation values of the corresponding quantum operators in the state representing a wave packet, then the classical Wong's equations give wrong values of $\vec{I}(t)$ for large time t.

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1. Introduction

The classical mechanics of color charged particles has been initiated in paper [1] and developed in a number of other papers, see, e.g., [2]. It represents the very interesting example of the classical mechanics of point-like particles with internal degrees of freedom, and it has been considered in this same spirit as the very old classical mechanics of spinning particles, see, e.g., [3]. The classical mechanics of colored particles has also been considered for more practical reasons [4]. Namely, it is so simple that it provides the explicitly calculable examples of interactions with nonabelian gauge fields. Therefore, it can be a handy tool for investigating effects which are due to interactions with nonabelian gauge fields.

In the nonrelativistic limit, the equations of motion for a spinless, color charged particle are, [1],

$$m\frac{d^2\vec{x}}{dt^2} = g\vec{E}^a I^a + \frac{g}{c}\dot{\vec{x}}\times\vec{B}^a I^a, \tag{1}$$

$$\frac{dI^a}{dt} = \frac{g}{\hbar}\varepsilon^{abc}\left(A_0^b - \frac{\dot{\vec{x}}}{c}\vec{A}^b\right)I^c, \tag{2}$$

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where $\vec{I} = (I^a)$, $a = 1, 2, 3$, is the vector of the classical color spin of the particle, and

$$E^{ai} = F^a_{0i}, \quad B^{ak} = -\tfrac{1}{2}\varepsilon^{krt}F^a_{rt},$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{g}{\hbar c}\varepsilon^{abc}A^b_\mu A^c_\nu.$$

In paper [5] this set of equations was extended to include the spin of the particle.

It is natural to ask whether the classical mechanics of colored particles represents a classical limit of the first quantized theory: The rather formal derivation of equations (1), (2) from Dirac's equation with an external SU (2) gauge field, presented in paper [1], could suggest that this is just the case. In paper [5] we have tried to obtain (1), (2) as the classical limit of the Dirac equation in a more explicit manner. Namely, we have considered the wave packet of the assumed "particle-like" form in the Foldy–Wouthuysen representation for the Dirac equation (we neglect the quantummechanical spreading out of the wave packet)

$$\psi^{\alpha\eta}(\vec{x}, t) = \varphi(\vec{x} - \vec{x}(t))u^{\alpha\eta}(\vec{x}, t), \tag{3}$$

where $\varphi(\vec{x} - \vec{x}(t))$ is a localized, c — number valued wave packet moving along the trajectory $\vec{x}(t)$ with velocity $\dot{\vec{x}}(t)$, and $u^{\alpha\eta}(\vec{x}, t)$ slowly changes with $\vec{x}$. The index $\alpha = 1, 2$ refers to spin, and $\eta = 1, 2$ refers to color degrees of freedom of the Dirac particle. We have considered the expectation values of quantum operators in the state (3) and we have derived classical equations of motion for them. Unexpectedly, the equations, e.g., for the trajectory $\vec{x}(t)$, have not come out gauge invariant, in general. Only in the particular case of sufficiently large velocities $\dot{\vec{x}}(t)$ (in the sense made precise in [5]) is it possible to write approximate equations which are gauge invariant.

The obtained in [5] set of classical equations can be considered as the selfconsistent basis for the classical mechanics of colored and spinning particles for any velocity $\dot{\vec{x}}(t)$. When one neglects the spin, this set of equations reduces to (1), (2). In our approach the classical color spin vector $\vec{I}$ is identified with the expectation value of the SU(2) generators $\hat{\vec{T}}$ (the color spin operators) in the state (3). However, the relevance of this classical mechanics for the classical limit of the Dirac equation in our approach is restricted to the sufficiently large velocities, because only in this case can one neglect the troublesome gauge noninvariant terms.

In this paper we shall show that the relevance of the classical mechanics is even more restricted. Namely, in Section 2 of this paper we shall present a very simple example which shows that the particle-like Ansatz (3) itself has a rather limited region of applicability. This Ansatz approximates the exact solution of the Dirac equation only when the effects due to the external fields are small, in particular when g is small and when the time interval is not too large. The Ansatz (3) also works in some other very particular cases, which are, however, of the trivial type, $\vec{I}(t) = \text{const}$. Thus, the difficulty with obtaining (1), (2) as the classical limit of Dirac's equation becomes squared — not only the

particle-like Ansatz (3) leads to the difficulties with gauge invariance, but also it turns out that Dirac's equation does not allow for any particle-like Ansatz which would be generally correct. This suggests that the concept of the classical, color charged particle does not follow from the underlying first quantized theory. Another argument supporting this point of view is presented in Section 4. Thus, the nonabelian case is in sharp distinction from the abelian one. In the later case, the notion of a classical, electrically charged particle can easily be extracted from the first quantized theory [6].

In order to simplify the analysis, in the following we neglect the spin of the particle. Then, the nonrelativistic quantum mechanical Hamiltonian of the Dirac particle in the Foldy–Wouthuysen representation becomes essentially the Schrödinger Hamiltonian

$$H = mc^2 + \frac{1}{2m}\left(\vec{p} - \frac{g}{c}\vec{A}^a\hat{T}^a\right)^2 + gA_0^a\hat{T}^a, \tag{4}$$

where $\hat{T}^a$ are the generators of the SU(2) group. In (4) we have also neglected the Darwinian term, and we have restricted our considerations to the positive energy sector of the theory in order to avoid classical antiparticles. The particle-like Ansatz now becomes

$$\psi^\eta(\vec{x}, t) = \varphi(\vec{x} - \vec{x}(t))u^\eta(\vec{x}, t). \tag{5}$$

In the following we assume that

$$u^\eta(\vec{x}, t) = u_0^\eta(t), \tag{6}$$

because the dependence of u on $\vec{x}$ was introduced in [5] essentially for technical reasons, and it is not important here.

We shall consider the simple case of a color gauge potential of the "abelian" type

$$A_\mu^a(\vec{x}, t) = h^a A_\mu(\vec{x}, t), \tag{7}$$

where $\vec{h} = (h^a)$ is a constant vector in the color space, $\vec{h}^2 = 1$.

In the next Section we shall describe the true time evolution of a wave function forming the wave packet (5) at the moment $t = t_0$, placed in the nonabelian gauge field (7). In Section 3 we compare the time evolution of the position $\vec{x}(t)$ and the color spin $\vec{I}(t)$ of the classical particle, as calculated from the classical equations (1), (2), which were derived in [5] from the Ansatz (3) equivalent to (5) when one neglects the spin, with the true time evolution of the expectation values of the quantum operators. The conclusion of Sections 2 and 3 is that the Ansatz (5), as well as the classical equations (1), (2), give definitely wrong predictions for sufficiently large t. The true time evolution of the initial wave packet does have in general the particle-like form (5) — the initial wave packet dissociates into two separately moving wave packets with constant and opposite color spins. In Section 4 we present certain general observation concerning the behaviour of color charged matter in the external nonabelian gauge fields and we present the other argument for the lack of the particle-like classical limit of the Dirac equation.

2. The wave packet in the external nonabelian gauge field

Let us now consider the time evolution of a wave function which at the moment $t = t_0$ has the form of the wave packet (5) localized at $\vec{x} = \vec{x}_0$

$$\psi^\eta(\vec{x}, t_0) = \varphi(\vec{x}-\vec{x}_0)u_0^\eta(t_0). \tag{8}$$

In the case of the gauge potential (7), the Hamiltonian (4) can be written as

$$H = \frac{1}{2m}\left(\vec{p} - \frac{g}{c}\vec{A}h^a\hat{T}^a\right)^2 + A_0 h^a \hat{T}^a, \tag{9}$$

where we have neglected the unessential constant mc^2. Let $\vec{e}_\pm$ be the normalized eigenvectors of the hermitean matrix $h^a\hat{T}^a$, $\hat{T}^a = \frac{1}{2}\sigma^a$,

$$(h^a\hat{T}^a)\vec{e}_\pm = \pm\tfrac{1}{2}\vec{e}_\pm. \tag{10}$$

From (9), (10) it follows that the Schrödinger equation

$$i\hbar\frac{\partial\vec{\psi}}{\partial t} = H\vec{\psi}$$

decomposes into two independent scalar equations

$$i\hbar\frac{\partial}{\partial t}\varphi_\pm = H_\pm\varphi_\pm, \tag{11}$$

where

$$\varphi_\pm(\vec{x}, t) = \vec{e}_\pm\vec{\psi}(\vec{x}, t),$$

and

$$H_\pm = \frac{1}{2m}\left(\vec{p}\pm\frac{g}{2c}\vec{A}\right)^2 \pm \frac{g}{2}A_0.$$

Of course,

$$\vec{\psi} = \vec{e}_+\varphi_+ + \vec{e}_-\varphi_-. \tag{12}$$

From (11), (12) we see that any time dependent wave function can be represented as the sum of two wave functions, each evolving in time independently of the other, and according to equations (11). These equations can be regarded as the Schrödinger equations for two scalar (i.e., colorless) particles with opposite electric charges $\pm g/2$, placed in the external gauge potential A_μ.

In particular, this applies to the initial wave packet (8). Writing

$$\vec{u}_0(t_0) = c_+\vec{e}_+ + c_-\vec{e}_-, \tag{13}$$

we see that the components

$$\varphi_{\pm}(\vec{x}, t_0) = \varphi(\vec{x}-\vec{x}_0)c_{\pm} \tag{14}$$

will evolve as if they describe the scalar particles with electric charges $\pm g/2$ in the external electromagnetic field A_μ. Therefore, the time evolution of φ_+ and φ_- has the standard form of a wave packet moving in an external electromagnetic field. Hence, if the usual conditions for the classical limit of quantum mechanics are satisfied, we can write the approximation

$$\varphi_{\pm}(\vec{x}, t) = c_{\pm}\varphi(\vec{x}-\vec{x}_{\pm}(t)) \cdot \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} dt' \left[\pm\frac{g}{2}\left(-A_0 + \frac{\dot{\vec{x}}_{\pm}}{c}\vec{A}\right) + \frac{m\dot{\vec{x}}_{\pm}^2}{2}\right]\right\}, \tag{15}$$

where $\vec{x}_+(t)$ ($\vec{x}_-(t)$) is the classical trajectory of the particle with electric charge $g/2$ ($-g/2$) placed in the gauge field A_μ with the initial data $\vec{x}(t_0) = \vec{x}_0$, $\dot{\vec{x}}(t_0) = \vec{v}$. In (15) the external field is taken at the point $\vec{x}(t)$. For simplicity we have neglected the spreading out of the wave packet.

The form (15) of the wave function can be easily justified within the Feynman path integral approach to the problem, [7]. Namely, the time evolution of the wave function can be written in the form

$$\varphi_{\pm}(\vec{x}, t) = \int d^3\vec{x}' K_{\pm}(\vec{x}, t; \vec{x}', t_0)\varphi_{\pm}(\vec{x}', t_0), \tag{16}$$

where the time evolution kernel can be written as

$$K_{\pm}(\vec{x}, t; \vec{x}', t_0) = N\int [dq^i(t)] \exp\left(\frac{i}{\hbar} S_{\pm}[\vec{q}(t)]\right) \tag{17}$$

Here N is a normalization constant. The integral is over all paths $\vec{q}(t)$ such that

$$\vec{q}(t_0) = \vec{x}', \quad \vec{q}(t) = \vec{x}, \tag{18}$$

$S[\vec{q}(t)]$ is the classical action for the trajectory $\vec{q}(t)$. In our problem

$$S_{\pm}[\vec{q}(t)] = \int_{t_0}^{t} dt' \left(\frac{m}{2}\dot{\vec{q}}^2 \mp \frac{g}{2}A_0 \pm \frac{g}{2c}\dot{\vec{q}}\vec{A}\right). \tag{19}$$

In the classical limit, $h \to 0$, the kernel K can be calculated by the standard method, see [7], Ch. 3, Sect. 5 and 6, based on the expansion of $S[q]$ around the classical trajectory up to the quadratic terms,

$$S[q] \approx S[q_{cl}] + \tfrac{1}{2}\int_{t_0}^{t} dt'dt''(q^i(t') - q^i_{cl}(t'))(q^k(t'') - q^k_{cl}(t''))\frac{\delta^2 S}{\delta q^i(t')\delta q^k(t'')}\bigg|_{\vec{q}=\vec{q}_{cl}}, \tag{20}$$

where the classical trajectory $\vec{q}_{\rm cl}(t) = \vec{x}_{\pm}(t)$ is determined from the classical equations of motion

$$\left.\frac{\delta S}{\delta \vec{q}(t)}\right|_{\vec{q}=\vec{q}_{\rm cl}} = 0, \tag{21}$$

supplemented with the boundary conditions (18) for $\vec{q}_{\rm cl}(t)$. Insering (17), (19), (20) in (16) we see that the phase factor $\exp\left(\frac{i}{\hbar} S[q_{\rm cl}]\right)$ appears. The semiclassical approximation also makes it clear that the time evolution of the wave packet will follow the classical trajectory $\vec{x}_{\pm}(t)$.

From (15) and (12) we obtain that

$$\vec{\psi}(\vec{x}, t) = \vec{e}_+ c_+ \varphi(\vec{x}-\vec{x}_+(t)) \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} dt' \left[\frac{g}{2}\left(-A_0 + \frac{\dot{\vec{x}}_+}{c}\vec{A}\right) + \frac{m\dot{\vec{x}}_+^2}{2}\right]\right\}$$

$$+\vec{e}_- c_- \varphi(\vec{x}-\vec{x}_-(t)) \exp\left\{\frac{i}{\hbar}\int_{t_0}^{t} dt' \left[-\frac{g}{2}\left(-A_0 + \frac{\dot{\vec{x}}_-}{c}\vec{A}\right) + \frac{m\dot{\vec{x}}_-^2}{2}\right]\right\}. \tag{22}$$

Obviously, the r.h.s. of (22) does not have the form of the Ansatz (8), because in general the trajectories $\vec{x}_+(t)$, $\vec{x}_-(t)$ are different. The fact that the initial positions and velocities of the corresponding, electrically charged, classical particles are identical does not change this situation in general. Thus, $\vec{\psi}(\vec{x}, t)$ dissociates into two separate wave packets.

When

$$\vec{x}_+(t) \approx \vec{x}_-(t) \equiv \vec{x}(t), \tag{23}$$

which implies also that $\dot{\vec{x}}_+(t) \approx \dot{\vec{x}}_-(t)$, the solution (22) can be approximately written in the form (8). Namely,

$$\psi(\vec{x}, t) = \varphi(\vec{x}-\vec{x}(t)) \exp\left(\frac{im}{\hbar}\int_{t_0}^{t} dt' \dot{\vec{x}}^2\right) u_0(t), \tag{24}$$

where

$$u_0(t) = \vec{e}_+ c_+ \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt' \left(-\frac{g}{2}A_0 + \frac{g}{2c}\dot{\vec{x}}\vec{A}\right)\right]$$

$$+\vec{e}_- c_- \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt' \left(\frac{g}{2}A_0 - \frac{g}{2c}\dot{\vec{x}}\vec{A}\right)\right]. \tag{25}$$

Thus, the Ansatz (8) is justified only in the very particular case when condition (23) is obeyed. This condition is approximately obeyed for general external gauge fields only when the gauge coupling constant g is small and when the time interval $t-t_0$ is not too large. Only in this case solutions of Wong's equations (1), (2) give the correct value of the expectation values in the true state (22). If the size of the wave packet is of the order of the Compton wave length for the particle, $\lambda_c = \frac{\hbar}{mc}$, then (23) is certainly not true when $g|\vec{f}|(t-t_0)^2/2m \geqslant \hbar/mc$, where $g|\vec{f}|$ is a mean force along the trajectory of the particle. This gives $T \approx (\hbar/cg|\vec{f}|)^{1/2}$ as the upper value of the time interval for which (23) has a chance to be true. Thus, in general, the limit $h \to 0$ has to be associated with the limit $g \to 0$ in order to obtain T finite.

$\vec{\psi}(\vec{x}, t)$ has the form (8) also when $c_+ = 0$ or $c_- = 0$. In this trivial case there is no color spin dynamics, $\vec{I} = \text{const}$.

3. The true time evolution of the expectation values

Let us start from the discussion of the time evolution of the expectation value of $\hat{T}^a$,

$$I^a(t) = \langle \psi | \hat{T}^a | \psi \rangle .$$

Because $\varphi(\vec{x}-\vec{x}(t))$ is normalized to one, we obtain from (25) and (10) that

$$I^a(t) = \tfrac{1}{2}(|c_+|^2 - |c_-|^2)h^a$$

$$+2\,\mathrm{Re}\left[m^a c_+^* c_- \exp\left[\frac{i}{\hbar} g \int_{t_0}^{t} dt' \left(A_0 - \frac{\dot{\vec{x}}}{c}\vec{A}\right)\right]\right], \tag{26}$$

where $\vec{m} = (m^a)$ is a constant vector with the following components

$$\vec{m} = \vec{e}_+^{\dagger} \hat{T}^a \vec{e}_- = \frac{1}{2\sqrt{1-h_3^2}} \begin{pmatrix} -h_1 h_3 - i h_2 \\ -h_2 h_3 + i h_1 \\ 1 - h_3^2 \end{pmatrix}.$$

It is easy to check that $\vec{m}$ is orthogonal to $\vec{h}$.

From (26) it follows that $I^a(t)$ obeys equation (2). This can be easily seen when one notices that $\vec{h} \times \vec{m} = i\vec{m}$.

However, for sufficiently large $t-t_0$ (23) is not obeyed. Then, the formula (26) for $\vec{I}(t)$ is not true, in general. In particular, when the two wave packets in (22) become spatially separated the true color spin becomes constant,

$$\lim_{t \to \infty} \vec{I}(t) = \tfrac{1}{2}(|c_+|^2 - |c_-|^2)\vec{h}, \tag{27}$$

in contradiction to the solution of Wong's equation (2) which has the behaviour (26) for all t.

Now, let us consider Wong's equation (1) for the trajectory. The quantity $\vec{h}\,\vec{I}$, present on the r.h.s. of this equation, is equal to

$$\vec{h}\vec{I} = \tfrac{1}{2}(|c_+|^2 - |c_-|^2). \tag{28}$$

On the other hand, $\vec{x}_\pm(t)$ obey the equations

$$m\ddot{\vec{x}}_\pm(t) = \pm\frac{g}{2}\left(\vec{E} + \frac{\dot{\vec{x}}_\pm}{c} \times \vec{B}\right). \tag{29}$$

From (28), (29) it follows that

$$\vec{x}(t) = |c_+|^2\vec{x}_+(t) + |c_-|^2\vec{x}_-(t) \tag{30}$$

obeys equation (1) when (23) is obeyed. Thus, in this case Wong's equation (1) has to be interpreted as the equation for the center of mass of the two wave packets. When (23) is not satisfied, equation (1) does not have any clear interpretation in terms of the expectation values in the true state (22), except for particular cases like $c_+ \cdot c_- = 0$, or $\vec{B} = 0$, $\vec{E} = \text{const}$.

4. Remarks

From the above considerations it follows that color charged matter placed in an external nonabelian gauge field will, in general, tend to disperse all over the space. For example, let us consider the wave packet in the potential $A^a_\mu = A_\mu h^a(\vec{x})$, where $h^a(\vec{x})$ is piecewise constant i.e., the space can be divided into regions Ω_i such that $\vec{h}(\vec{x})$ is constant in each Ω_i and the direction of $\vec{h}$ in each Ω_i is different. It is easy to see that the initial wave packet will dissociated into many separate wave packets — their number depends on the number of crossed regions Ω_i.

According to certain authors [8], the vacuum in QCD is filled in with nonabelian gauge fields which have an orientation in color space fluctuating at random. In such a vacuum it is impossible to have a localized clot of colored matter propagating in a definite direction, except for the superficial case of large velocities, when the high velocity will mask the dissociation for a certain time.

Finally, we would like to present the other argument for the lack of the particle-like classical limit of the Dirac equation with an external nonabelian gauge field. Let us recall that in the abelian (i.e., electromagnetic) case the classical limit can be obtained by the substitution of

$$\psi = \exp\left(\frac{i}{\hbar}S\right)f \tag{31}$$

in the Dirac equation and letting $h \to 0$, [6]. Then, it turns out that the real-number valued function S obeys the Hamilton-Jacobi equation known from classical mechanics. In the

nonabelian case the substitution (31) leads to the following equations (in the leading order in $\hbar$)

$$\left[\gamma^\mu\left(\partial_\mu S+\frac{g}{c}A_\mu^a f^\dagger\hat{T}^a f\right)+m\right]f=0, \tag{32}$$

and

$$\gamma^\mu A_\mu^a\hat{T}^a f-\gamma^\mu A_\mu^a(f^\dagger\hat{T}^a f)f=0. \tag{33}$$

From (33) it follows that f is an eigenvector of the operator $\gamma^\mu A_\mu^a\hat{T}^a$. In the case of the SU(2) gauge potentials of the abelian type (7) we can substitute in (32) and (33)

$$f^{\alpha\eta}=f^\alpha u^\eta,$$

where $\alpha=1,2,3,4$ refers to the usual bispinor components, $\eta=1,2$ refers to color. Then, (33) will be satisfied if

$$(\vec{h}\hat{\vec{T}})\vec{u}=\lambda\vec{u}.$$

This equation is identical in form with (10): in particular $\lambda=\pm 1/2$. Then, (32) becomes identical in form with the equation obtained in the electromagnetic case. However, the "effective" electric charge takes on two values $\pm g/2$. Thus, we again obtain the situation described in Section 2; there are two independent classical motions going on.

Our considerations can be easily generalized to SU(n) gauge fields. Then, the equation (10) will have in general n different eigenvalues, and the initial wave packet, when placed in the gauge field of the abelian type (7), will dissociate into n separate wave packets after a sufficiently long time.

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