ORDER PARAMETERS IN SPIN-SPIN AND PLAQUETTE LATTICE THEORIES*

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We present some basic properties of the gauge theories in the lattice formulation. We discuss the possible order parameters of the theory and their usefulness from the point of view of the numerical calculations. We study the properties of the low coupling constant expansion, i.e. the continuum limit of the theory. Finally we show the results of the numerical calculations for various lattice systems.

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1. Introduction

Traditionally quantum field theory is constructed by doing perturbation theory in the coupling constant. This tradition stems from the success of quantum electrodynamics, where the smallness of the fine structure constant $\alpha \approx 1/137$ strongly suggests a perturbative approach.

With the advent of quantum chromodynamics (QCD) as a candidate for the theory of strong interactions it became clear that perturbative techniques, though very useful in describing short distance properties of the strong interactions, are inadequate to handle the long distance features of the theory, like the phenomenon of quark confinement.

In order to meet this need, Wilson invented [1] a lattice version of QCD, which leads itself quite naturally to a high coupling constant expansion and indeed produces results that can be interpreted as quark confinement. But the low coupling constant expansion is now a very heavy handed machinery, since it is lacking Lorentz-invariance and moreover

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has a growing complexity in the Feynman rules, as the order in the coupling constant increases. This is to be contrasted with the continuum version of QCD where the Feynman rules are fixed once and for all.

One obvious question is now: are continuum and lattice version of QCD the same when the lattice length becomes small? "Small" means "small with respect to some typical length in the problem".

This question is not yet answered. Yet there are encouraging results, in particular the result that the coupling constant in lattice theory renormalizes up to one loop in the same way as in continuum theory, that is, also the lattice theory is asymptotically free [3]. The renormalization group scales are substantially different (and varying with the gauge group). This is a result that will be discussed in Section 4, where we will go into the details of the small coupling constant expansion. All this means that the eventual continuum limit of the lattice theory has to be taken by letting the coupling and lattice spacing go to zero consistent with asymptotic freedom.

A second important question is: is the quark confinement that one finds for high coupling still persisting for low couplings?

In the subsequent three Sections we will phrase this question as follows: are the high coupling constant and the low coupling constant regions separated by singular points in the coupling constant? For most gauge groups this has not yet been answered by theory. In Section 3 we define order parameters that are very useful in extracting numerical information on this problem. In Section 4 their low coupling behaviour is calculated. In the last Section we give the numerical results on various gauge groups in four dimensions, and for various spin-spin theories in two dimensions. A preliminary presentation of these results has been given in a preprint (May 1980) [4].

2. Some elementary facts about lattice gauge theory

This Section serves mainly to fix notations and recall basic facts, most readers may skip this Section [5] and start with Section 3.

The basic variables in gauge theory are vector potentials $A_\mu^a(x)$, when we give the continuum formulation. Here the index μ is the Euclidean index $\mu = 1, 2, 3, 4$ and a numbers the degrees of freedom in the Lie algebra of the gauge group in question. (For example, the gauge group SU(2) has three vector potentials A_μ^a, $a = 1, 2, 3$). It is useful to introduce a matrix notation for the potentials: if λ_a ($a = 1, 2, ..., N^2-1$) are the N^2-1 generators ($N \times N$ matrices) of the SU(N) group, with normalisation

$$\text{Tr } \lambda_a \lambda_b = 2\delta_{ab} \tag{2.1}$$

and with structure constants given by

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \tag{2.2}$$

then we define $A_\mu(x) \equiv A_\mu^a(x)\lambda_a/2$, where we sum over the index a.

The gauge transformation $\Omega(x)$ are SU(N) matrices and act on the potentials as

$$A_\mu^\Omega(x) = \Omega^{-1}(x)A_\mu(x)\Omega(x) + \frac{i}{g} \Omega^{-1}(x)\partial_\mu\Omega(x). \tag{2.3}$$

The field tensor $G_{\mu\nu}(x)$ is defined by

$$G_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]. \tag{2.4}$$

It is antisymmetric in $\mu \leftrightarrow \nu$ and transforms under (2.3) as

$$G^\Omega_{\mu\nu}(x) = \Omega^{-1}(x)G_{\mu\nu}(x)\Omega(x). \tag{2.5}$$

The action S is defined as [2]

$$S = \tfrac{1}{2}\int d^4x \, \mathrm{Tr}\, G_{\mu\nu}G_{\mu\nu} \equiv S(A, g) \tag{2.6}$$

and is invariant under gauge transformations (2.3). Observe that

$$S(A, g) = \frac{1}{g^2} S(gA, 1). \tag{2.7}$$

The quantum mechanics of this system follows from the formal expression

$$Z\left(\frac{1}{g^2}\right) = \int DA_\mu \exp\left[-\frac{1}{g^2} S(A, 1)\right], \tag{2.8}$$

where the right hand side of (2.8) is an integral over all configurations of potentials and where quantity $Z(1/g^2)$ is the partition function or vacuum functional of the system.

The content of (2.8) is then defined by expanding (2.8) around $g^2 = 0$ (this involves choosing a gauge and calculating the Fadeev–Popov determinant, see Section 4) and by giving the coefficients of $(g^2)^n$ a meaning through a regulator.

As stated in the introduction it could be useful to have a formulation which does not depend on using perturbation theory. Such a formulation was provided by Wilson, and we will explain it in the next lines.

First, consider a hypercubic lattice in four dimensional space. For definiteness we will suppose periodic boundary conditions in all four directions, so that the lattice is really a four dimensional hypertorus with lengths a_μ in the μ'th direction. So in total we have $a_1 \times a_2 \times a_3 \times a_4$ lattice points.

On every link l_{ij} connecting nearest neighbour points n and $n+\mu$ (see Fig. 1) we define an SU(N) matrix $U(n, \mu)$. A natural orientation of each link is obtained by fixing
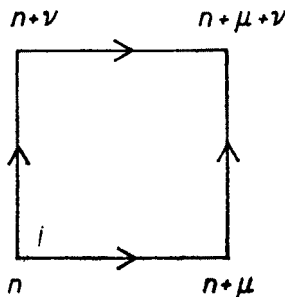

Fig. 1. Plaquette $P_{\mu\nu}(n)$. Links are oriented in positive direction; arrow gives the trace orientation in Eq. (2.10)

the positive direction on each of the four axes; then a link with positive orientation is given by $(n, n+\mu)$ and with opposite orientation by $(n+\mu, n)$. If the link $(n, n+\mu)$ carries the matrix $U(n, \mu)$, then by definition $(n+\mu, n)$ carries the matrix $U^{-1}(n, \mu)$.

The gauge transformations Ω are defined on lattice points n, and by definition

$$U^{\Omega}(n, \mu) = \Omega^{-1}(n)U(n, \mu)\Omega(n+\mu). \tag{2.9}$$

The Wilson action becomes now on every plaquette $P_{\mu\nu}(n)$ (see Fig. 1)

$$S(P_{\mu\nu}(n)) \equiv \tfrac{1}{2}\left(1 - \frac{1}{N}\,\text{Tr}\,U(n, \mu)U(n+\mu, \nu)U^{-1}(n+\nu, \mu)U^{-1}(n, \nu)\right)$$

$$+\text{complex conjugate.} \tag{2.10}$$

The factor $1/N$ renders S non-negative. The trace in (2.10) corresponds to going around the plaquette as indicated in Fig. 1. The complex conjugate corresponds to going around in the opposite way. The total action is

$$S = \sum_{n,\mu,\nu} S(P_{\mu\nu}(n)) \equiv S(U) \tag{2.11}$$

and is non-negative (to achieve this we put the factor $1/N$ in (2.10)). Then the lattice action is manifestly gauge invariant; it is also related in the limit of vanishing lattice length to the continuum action (2.6).

In order to see this we must first relate the link matrices to potentials. Calling the lattice length a, we define

$$U(n, \mu) = e^{iaA_{\mu}(n)}. \tag{2.12}$$

It follows that

$$U(n+\mu, \nu) = e^{iaA_{\nu}(n+\mu)},$$

$$U(n+\nu, \mu) = e^{iaA_{\mu}(n+\nu)},$$

$$U(n, \nu) = e^{iaA_{\nu}(n)}. \tag{2.13}$$

Introduce the lattice derivative $\Delta_{\mu}(n)$, defined by

$$\Delta_{\mu}(n)f(n) = \frac{f(n+\mu)-f(n)}{a}.$$

The three potentials appearing in (2.13) can all be expressed through the lattice derivative $\Delta_{\mu}(n)$ in terms of $A_{\mu}(n)$, e.g.:

$$A_{\nu}(n+\mu) = (a\Delta_{\mu}(n)+1)A_{\nu}(n). \tag{2.14}$$

This means that (2.10) becomes

$$S(P_{\mu\nu}(n)) = \tfrac{1}{2}\left(1 - \frac{1}{N}\,\text{Tr}\,e^{iaA_{\mu}(n)}e^{ia(a\Delta_{\mu}+1)A_{\nu}(n)}e^{-ia(a\Delta_{\nu}+1)A_{\mu}(n)}e^{-iaA_{\nu}(n)}\right)+\text{c.c.}$$

$$\equiv \tfrac{1}{2}\left(1 - \frac{1}{N}\,\text{Tr}\,e^{ia^2\Phi_{\mu\nu}(A,\Delta A)}\right)+\text{c.c.} \tag{2.10'}$$

Use now

$$\exp A \exp B = \exp \{A + B + \tfrac{1}{2}[A, B] + \text{higher order}\} \tag{2.15}$$

to see that

$$\Phi_{\mu\nu}(A, \Delta A) = \Delta_\mu A_\nu - \Delta_\nu A_\mu + i[A_\mu, A_\nu] + o(a). \tag{2.16}$$

Then (2.10′), the lattice action per plaquette equals

$$S(P_{\mu\nu}(n)) = \frac{1}{2N}[a^4 \operatorname{Tr} \Phi_{\mu\nu}^2 + o(a^6)]$$

so the total lattice action tends to

$$S_L \underset{a \to 0}{\to} \frac{1}{2N} \sum_{\mu, \nu} \int d^4x \operatorname{Tr} (\partial_{[\mu} A_{\nu]} + i[A_\mu, A_\nu])^2 = \frac{1}{2N} S(A, 1). \tag{2.10″}$$

In conclusion: lattice action and continuum action differ by a factor $1/2N$, when the lattice length goes to zero.

The statistical mechanics of the lattice system is given by

$$Z(\beta) = \int \prod_l DU(l) \exp [-\beta S_L(\{U(l)\})]. \tag{2.17}$$

The symbol $DU(l)$ stands for invariant integration over the group $SU(N)$; we suppose the volume to be normalized to 1. From what has been said below Eq. (2.10″) and comparing with Eq. (2.8) it follows that $\beta/2N$ has to be identified with $1/g^2$ in the continuum limit.

The partition function (2.17) is the corner-stone of the lattice approach. The integrand in (2.17) is the "Boltzmann factor", and is used to calculate averages of gauge invariant quantities $Q(\{U(l)\})$

3. The order parameters

In this Section we want to discuss various order parameters that are needed to characterize the eventual different phases of gauge theories, and the relations between them.

3.1. The most widely used order parameter in gauge theories is the Wilson-Wegner loop [1, 6] $A(C)$. Defined as the expectation value of the trace of the ordered product of all link variables along a closed curve C, it has the behaviour as put down in Table I.

TABLE I

Behaviour of order and disorder parameters for large perimeter and large area

Parameter	Disordered (low β)	Maxwell type (medium β)	Ordered (high β)
$-\log A(C)$	$\varrho \times$(area of C)	$\sigma \times$(perimeter of C)	$\tau \times$(perimeter of C)
$-\log B(C)$	$\tau^* \times$(perimeter of C)	$\sigma^* \times$(perimeter of C)	$\varrho^* \times$(area of C)

The disorder parameter $B(C^d)$ is defined on a loop in the dual lattice in the following way: take any surface $\Sigma^d(C^d)$ in the dual lattice with a boundary C^d. Consider all plaquettes P in the original lattice that are dual to the plaquettes in $\Sigma^d(C^d)$, and insert in the corresponding Tr $U(P)$ a fixed element $z = e^{in2\pi/N}$ of the centergroup.

The disorder parameter $B_n(C^d)$ is now defined as the ratio of the partition function calculated with the new traces $Z(n, C^d; \beta)$ and the usual partition function $Z(\beta)$

$$B_n(C^d) \equiv \frac{Z(n, C^d; \beta)}{Z(\beta)} . \tag{3.1}$$

We have put in the Table the behaviour of order and disorder parameter. If the gauge group is the centergroup itself (i.e. $Z(N)$) then the theory is selfdual and

$$A_n(C)(\beta) = B_n(C)(\beta^*) \tag{3.2}$$

for $N \leqslant 4$. (For $N \geqslant 5$ we have $[N/2]$ dimensional parameter space and (3.2) is not as simple.)

The order and the disorder parameters together do define three possible phases: the low β "confinement" phase, the intermediate β "Maxwell" type phase and the high β "ordered" phase. (For a description of these phases see Ref. [7].) The coefficient ϱ of the surface term in log $A(C)$ is called the electric string tension, the coefficient ϱ^* of the surface term in log $B(C)$ is called the magnetic string tension. The electric string tension can be calculated in a low β expansion. These series are known in some cases up to 14th order in β [18].

3.2. We will study in this section order parameters that are very useful for numerical work. The physical idea is to test the system for a phase transition by the introduction of a dislocation. We will illustrate this idea in the case of the two-dimensional Ising model.
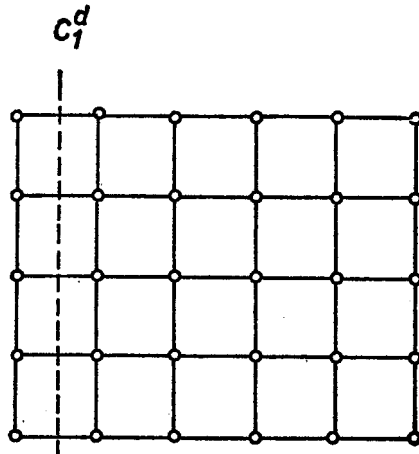

Fig. 2. A two-dimensional lattice with twist

Consider Fig. 2. We have a $a_1 \times a_2$ two-dimensional Ising system with action

$$S = \sum_{\langle i,j \rangle} (1 - Z(i)Z(j)), \quad Z(i)^2 = 1 \tag{3.3}$$

and with periodic boundary conditions, so we have put the system on a torus. Consider the circle C_1^d on the dual lattice in Fig. 2. It cuts a set of links all dual to the links constituting C_1 on which we switch the sign of the coupling. We get a new action ("twisted" action) which is

$$S_t = \sum_{\langle i,j \rangle \notin [C_1{}^d]^*} (1 - Z(i)Z(j)) + \sum_{\langle i,j \rangle \in [C_1{}^d]^*} (1 + Z(i)Z(j)). \tag{3.4}$$

We now calculate the two corresponding partition functions, keeping two independent couplings in the system: $\beta_{\parallel}$ which is parallel to the twisted links, and $\beta_{\perp}$ perpendicular to the twist. It is useful to split the (twisted) action into two parts, one containing all the parallel links and one containing all the perpendicular links:

$$S = S_{\parallel} + S_{\perp} \tag{3.5}$$

and

$$S_t = S_{t\parallel} + S_{t\perp}. \tag{3.6}$$

So we have

$$Z(\beta_{\parallel}, \beta_{\perp}) = \sum_{\{Z(i)\}} \exp\left[-\beta_{\parallel} S_{\parallel} - \beta_{\perp} S_{\perp}\right] \tag{3.7}$$

and

$$Z_t(\beta_{\parallel}, \beta_{\perp}) = \sum_{\{Z(i)\}} \exp\left[-\beta_{\parallel} S_{t\parallel} - \beta_{\perp} S_{t\perp}\right]. \tag{3.8}$$

From the Onsager solution [8] we have asymptotically, in the limit of large $a_{\parallel}$ and $a_{\perp}$

$$Z_t(\beta_{\parallel}, \beta_{\perp}) = Z(\beta_{\parallel}, \beta_{\perp}) \exp\left[-a_{\perp} f_t(\beta_{\parallel}, \beta_{\perp})\right], \tag{3.9}$$

where $Z(\beta_{\parallel}, \beta_{\perp})$ is the exactly known partition function and where

$$f_t(\beta_{\parallel}, \beta_{\perp}) = 2(\beta_{\parallel} - (\beta_{\perp})^*) \quad \text{if} \quad \beta_{\parallel} \geqslant (\beta_{\perp})^* \equiv -\tfrac{1}{2} \log \mathrm{th}\, \beta_{\perp}$$

$$= 0 \qquad \text{otherwise.} \tag{3.10}$$

The physical meaning of this result is clear: below the critical temperature $(\beta_{\parallel} > (\beta_{\perp})^*)$ we have a non-vanishing free energy per unit length associated with the twist; put slightly differently, the twist can be considered as a dislocation and below the critical temperature there is a Bloch wall building up.

Numerically (by Monte Carlo techniques, see Section 5) the free energy f_t is not easily accessible. What is easily accessible (see Groeneveld et al. [9]) is the mean twisted action, i.e. the derivative of f_t

$$\Omega(\beta) = \frac{1}{a_{\perp}} \left[\langle S_t(\beta) \rangle - \langle S(\beta) \rangle\right]$$

$$= \frac{\partial}{\partial \beta} f_t = 2\left(1 - \frac{\partial \beta^*}{\partial \beta}\right). \tag{3.11}$$

Here we put $\beta_{\parallel} = \beta_{\perp} = \beta$.

Alternatively we can define a quantity $\omega(\beta)$ by staying entirely within the twisted system

$$\omega(\beta) = -\frac{1}{a_\perp}\left(\frac{\partial}{\partial\beta_\|} - \frac{\partial}{\partial\beta_\perp}\right)\log Z_t(\beta_\|, \beta_\perp)|_{\beta_\| = \beta\perp = \beta}$$

$$= 2\left(1 + \frac{\partial\beta^*}{\partial\beta}\right). \tag{3.12}$$

The latter quantity has an advantage for computer calculations: the same precision in $\omega(\beta)$ is obtained with much less computer time than in $\Omega(\beta)$. In Fig. 3 we have plotted $\omega(\beta)$ and $\Omega(\beta)$.
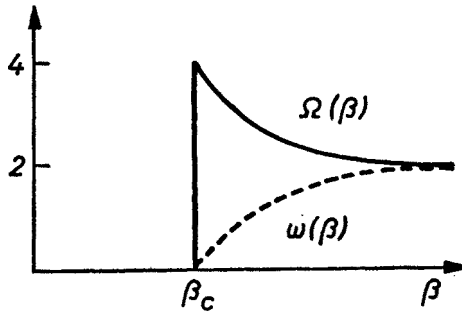

Fig. 3. $\Omega(\beta)$ (Eq. (3.11)) and $\omega(\beta)$ (Eq. (3.12))

The extension of the idea to bigger groups than $Z(2)$ is straightforward: the links are twisted with a fixed element of the group. It is important to note that the minimal value of the twisted action is always non-zero, no matter what the group is. Thus at low temperature Ω and ω will approach a constant if the group is discrete, and a value proportional to $a_\|^{-1}$ in the case of continuous group (spin wave).

3.3. Twist in four-dimensional gauge theory

In four-dimensional gauge theories the physical idea behind the twist is the same as in two-dimensional spin-spin theories. There is only a change in the formulation: instead of links coupling spin variables we have plaquettes coupling link variables. Therefore the circle C^d on the two-dimensional torus is replaced by a two-dimensional torus $T^d_{\mu\nu}$ on the four-dimensional torus. Every plaquette P dual to this torus $T^d_{\mu\nu}$ is "twisted", i.e. instead of $\mathrm{Tr}\, U(P)$ we take $z_{\mu\nu}\mathrm{Tr}\, U(P)$ in the twisted action, where $z_{\mu\nu}$ is a *fixed* element of the *center-group* of the gauge group in question. So the twisted action becomes

$$S_{t\mu\nu}(U) = \sum_{P \in [T_{\mu\nu}{}^d]^*}\left(1 - z_{\mu\nu}\frac{1}{N}\mathrm{Tr}\, U(P)\right) + \sum_{P \notin [T_{\mu\nu}{}^d]^*}\left(1 - \frac{1}{N}\mathrm{Tr}\, U(P)\right). \tag{3.13}$$

In the planes orthogonal to the $(\mu\nu)$ directions we have a twisted plaquette. We can see, by fixing the gauge on the links as in Fig. 4 that the twisted action can never become zero

if the group is abelian. This is so because only on the rungs of the two ladders in the figure we can still choose non-unit elements of the group. But in the abelian case we can never have
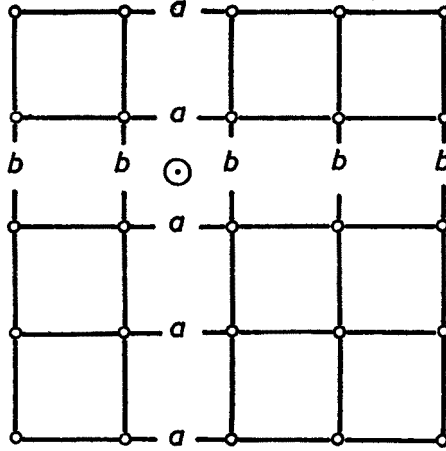
$$\text{Tr } aba^{-1}b^{-1} = z.$$

Fig. 4. An example of the "twist eating" configuration with zero twisted action for the plaquette system

In the non-abelian case we can indeed have this situation [4]. Take in SU(N) the matrix

$$a = \begin{pmatrix} 1 & & & & \\ & z_0 & & 0 & \\ & & z_0^2 & & \\ & & & \ddots & \\ & 0 & & & \\ & & & & z_0^{N-1} \end{pmatrix} z_0^{-1/2N(N-1)}, \qquad z_0 = e^{i2\pi/N}$$

and a cyclic step operator b, transforming the ith unit vector into the $(i+n)^{\text{th}}$ unit vector. Then clearly

$$abe_i = ae_{i+n} = z_0^{i+n}e_{i+n},$$

$$bae_i = z_0^i be_{i+n} = z_0^{i+n}e_{i+n}$$

so indeed $\text{Tr } aba^{-1}b^{-1} = z_0^n$.

Thus we have a zero mode of the twisted action; as a consequence, at very high β, the difference between mean twisted action and mean action will become zero. Physically it means that a non-abelian system knows how to circumvent the dislocation at very low temperature. The case where we introduce two twists in mutually orthogonal planes does not admit a zero mode [16].

4. Small coupling constant expansion

In this Section we shall try to treat in a concise but hopefully pedagogical way the small coupling constant expansion.

This expansion is not only important for the interpretation of our numerical results, but also for the continuum limit of the lattice gauge theory. In Section 4.1 we will review the main features of the expansion, in Section 4.2. we will study the relation between lattice and continuum theories.

4.1. The small coupling expansion — or high β expansion — is carried out around a (local) minimum of the action. Often this minimum is taken to be the trivial solution, sometimes a non-trivial one. The interested reader is referred to Ref. [10] for a host of details. The basic idea is to expand the field variables around the minimum configuration, and introduce suitable gauge conditions (which in general will introduce ghosts).

The gauge condition in lattice gauge field theory can be introduced in full analogy with continuum theory. Suppose we are calculating the average of some gauge invariant quantity $I(U)$ where U stands for a (periodic) configuration on the lattice

$$\langle I(U) \rangle = \frac{\int DU I(U) \exp \left[-\beta S(U) \right]}{\int DU \exp \left[-\beta S(U) \right]}. \tag{4.1}$$

It is clear that introduction of a "gauge fixing" function $G(U)$ in every lattice point n by

$$\int \prod_n D\Omega(n) \delta(G(U^\Omega)(n) - C(n)) \equiv \Delta^{-1}(U, C) \tag{4.2}$$

leads to [10]

$$\langle I(U) \rangle = \frac{\int DU I(U) \delta(G(U) - C) \Delta(U, C) \exp \left[-\beta S(U) \right]}{\int DU \delta(G(U) - C) \Delta(U, C) \exp \left[-\beta S(U) \right]} \tag{4.3}$$

$\langle I(U) \rangle$ is independent of C, and this admits the average to be written as:

$$\langle I(U) \rangle = \frac{\int DU I(U) \Delta(U, C = G(U)) \exp \left[-\beta S(U) - \frac{1}{2} G^2(U) \right]}{\int DU \Delta(U, C = G(U)) \exp \left[-\beta S(U) - \frac{1}{2} G^2(U) \right]}. \tag{4.4}$$

Formula (4.4) is generally true for any value of the coupling β. The factor $\Delta(U, C = G(U))$ is actually calculated in the large β limit in the following way:

$$\Delta^{-1}(U, G(U)) = \int D\Omega \delta(G(U^\Omega) - G(U)) \tag{4.5}$$

by supposing that the zero of the argument of the δ function for $\Omega = 1$ is the dominant one for large β. (It is known [20] that for some gauge functions many gauge transformations annull the argument of the delta function.) Accepting this one can express the $\Delta(U, G(U))$ as a determinant through the usual "Faddeev–Popov ghosts".

Now we have to become more specific in order to prepare the ground for Section 4.2. Let us assume that the expansion is around the $U = 1$ configuration. We can now write, using the results obtained in Section 2, that

$$U(n, \mu) = \exp iag A_\mu(n) \tag{4.6}$$

and perturbation theory will be in terms of the potentials $A_\mu^a(n)$ around $A_\mu^a(n) = 0$. (Note that $\beta = g^{-2}2N$ from Section 2 and (4.6).) The conversion from the measure $dU(n, \mu)$ to $dA_\mu^a(n)$ goes at the cost of introducing the curvature of the group

$$dU(n, \mu) = dA_\mu^a(n) \exp\left[-\frac{N}{24} g^2 A_\mu^a(n) A_\mu^a(n) + O(g^4 A^4) \right], \tag{4.7}$$

where we used the formula (4.10).

For most purposes the choice of the gauge fixing term $G(U)(n)$ is the Feynman gauge fixing, leading to the simplest $A_\mu(n)$ propagator

$$\langle A_\mu(n) A_\nu(n') \rangle = \delta_{\mu\nu} \Delta^{-2} \delta(n - n'). \tag{4.8}$$

The form of $G(U)$ then is

$$G(U)(n) = \Delta_\mu A_\mu(n). \tag{4.9}$$

The ghost term is now fixed, and leads — apart from usual ghost term [12] as in continuum theory (with due replacements of derivatives ∂_μ by lattice derivatives Δ_μ) — also to a local term of the same type as in (4.7) [10]. This stems from the fact that the response of $A_\mu(n)$ to infinitesimal gauge transformations determines the ghost couplings, and this response is less simple than in the continuum case, and can be easily calculated from formula (4.10). The higher order in g we want, the more vertices we can expect in the ghost couplings. Likewise, the gauge field couplings are rapidly gaining in complexity as we increase the order of g. We can calculate all the terms in the expansion of the lattice action (2.10) up to and including order g^5 [13] by using the Baker-Campbell-Hausdorf formula up to and including fourth order

$$e^A e^B = e^C, \text{ where}$$

$$C = A + B + \tfrac{1}{2}[A, B] + \tfrac{1}{12}[A, [A, B]] + \tfrac{1}{12}[B, [B, A]]$$

$$- \tfrac{1}{24}[A, [B, [A, B]]] + \text{quintuple commutators}. \tag{4.10}$$

Then we generate, through the first three terms in (4.10) the terms mentioned in (2.16), which gave the usual continuum limit. All the remaining terms are of order $g^3 a$, $g^3 a^2$ etc., and will contribute to loop calculations. A very specific and useful example of such loop calculations in lattice gauge theory will be given in the next section.

4.2. It is obvious that for anything we calculate on the lattice we want to take the limit of zero lattice length. Taking such a limit we must keep some measurable quantities fixed. In gauge theory such a quantity is the coupling constant defined as the value of the 3-point vertex at given gluon momenta M or, alternatively, the string constant.

The coupling constant $g(M)$ as defined above, is expressible in terms of the bare coupling constant g, the lattice length a and the scale M

$$g^2(M) = g^2 + g^4[b_1^1 t + b_1^0] + g^6[b_2^2 t^2 + b_2^1 t + b_2^0] + O(g^8), \tag{4.11}$$

where

$$t \equiv \log\left(\frac{\pi}{aM}\right)^2.$$

The order a terms are omitted as we are interested in the limit of zero lattice length. The coefficients b_1^1 and b_1^0 have been calculated by Hasenfratz and Hasenfratz [14] for the gauge groups SU(2) and SU(3). That there are only logarithmic divergencies is not a priori clear, and has been checked to one loop [10, 14]. The sign of b_1^1 is positive like in the analogous relation in continuum theory and leads to asymptotic freedom. The coefficient b_2^1 has been calculated in continuum gauge theory.

The interest of these quantities lies in the fact that they can define a length scale a_0. First by keeping $g_r(M)$ constant, we see from (4.11) that the bare coupling g has to decrease as a decreases, since b_1^1 is positive. More formally

$$a\frac{d}{da}g^2(a)|_{M,g_r(M)} = \Psi(g^2(a)) \tag{4.12}$$

with

$$\Psi(g^2) \equiv -\beta_0 g^4 - \beta_1 g^6 + O(g^8),$$

where

$$\beta_0 = b_1^1 \quad \text{and} \quad \beta_1 = b_2^1 - 2b_1^1 b_1^0.$$

Notice that a change of scale M to M' (or a to a') changes $t = \log(\pi/aM)^2$ to $t' = t+c$, where $c = \log(M/M')^2$ (or $c = \log(a/a')^2$). Obviously the coefficients β_0 and β_1 do not change, so if $b_1^0 = 0$ for some mass scale M_0 we can calculate β_1 from b_2^1 alone. For SU(N)

$$\frac{\beta_0}{2} = \frac{1}{16\pi^2}\frac{11}{3}N \quad \text{and} \quad \frac{\beta_1}{2} = \left(\frac{N}{16\pi^2}\right)^2\frac{34}{3}. \tag{4.12'}$$

We can integrate the differential equation (4.12) and obtain then the coupling constant as a function of a or vice versa. Working up to order g^6 in $\Psi(g^2)$ one finds

$$\left(\frac{a}{a_0}\right)^2 = \exp\left(\frac{-1}{\beta_0 g^2}\right)(\beta_0 g^2)^{-\beta_1/\beta_0^2}(1+O(g^2)). \tag{4.13}$$

Formula (4.13) is interesting since it determines the behaviour of the lattice length in terms of g^2. Formula (4.13) is only valid as $g^2 \ll 1$. This way we have a length a_0 in the system below which the coupling g^2 is small and perturbation theory is valid.

Instead of $g_r(M)$, the renormalized coupling, we could keep the string constant [15] α' fixed; if we do this

$$\frac{1}{2\pi\alpha'}a^2 = \varrho(g^2) \tag{4.14}$$

and thus

$$\frac{a_0^2}{2\pi\alpha'}\left(\frac{a}{a_0}\right)^2 = \varrho(g^2).$$

Then we would expect that

$$\varrho(g^2) = \frac{a_0^2}{2\pi\alpha'} \exp\left[\frac{-1}{\beta_0 g^2}\right] (\beta_0 g^2)^{-\beta_1/\beta_0{}^2} (1 + O(g^2)). \qquad (4.15)$$

Notice that this way of renormalizing makes the string tension of necessity non-zero for large β (small unrenormalized coupling); that is, it is non-zero for all values of β, so confinement is supposed to be realized for all values of the unrenormalized coupling.

The length a_0 can be related to a length Λ^{-1} that is already measured in deep-inelastic scattering reactions. This was done for the groups SU(2) and SU(3) [14]. We will give here a simple argument how to extend it to any group SU(N).

The length Λ^{-1} is defined analogously to the length a_0 in (4.13), but starting from a continuum formulation of QCD:

$$\Lambda^2 = M^2 \exp\left[\frac{-1}{\beta_0 g^2(M)}\right] (\beta_0 g^2(M))^{-\beta_1/\beta_0{}^2} (1 + O(g^2(M))). \qquad (4.16)$$

The left hand side of (4.16) is independent of M, the subtraction scale, just as a_0 in (4.13) was independent of a, the cutoff. Λ and a_0 are related up to order $g^2(M)$ by substituting $g_r(M)$ from Eq. (4.11) into (4.16). It gives

$$\Lambda^2 = \frac{1}{a_0^2} \pi^2 \exp\left(\frac{b_1^0}{\beta_0}\right). \qquad (4.17)$$

Now b_1^0 consists of two parts: one part proportional to N (the dimension of SU(N)) and one part inversely proportional to N. We found it easy to calculate the latter part, since only two diagrams contribute to it: one contributing to wave function renormalization
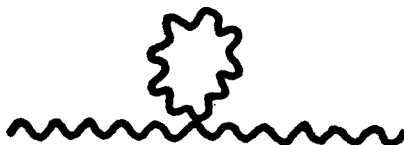

Fig. 5. Self energy diagram contributing to order $1/N$ to coupling constant renormalization

(Fig. 5) and one to vertex renormalization (Fig. 6). Moreover both these diagrams are gauge invariant and finite so their contribution is independent of the subtraction point. In fact the contribution of the vertex diagram to b_1^0 is

$$\tfrac{1}{16}\left(\frac{-6}{N}\right)$$

whereas the self energy diagram contributes
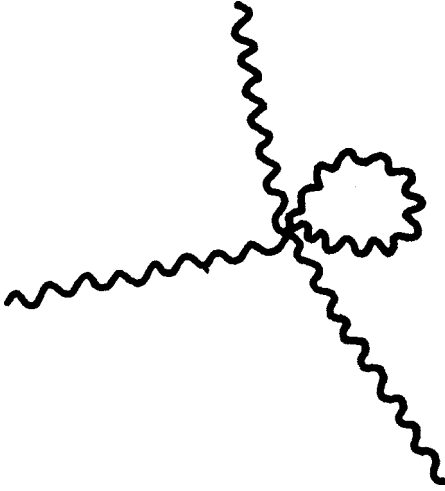
$$\tfrac{1}{16}\left(\frac{+4}{N}\right)$$

Fig. 6. Vertex diagram constributing to $1/N$ to coupling constant renormalization

to b_1^0. Therefore the value of β_0 from (4.12′) gives

$$\Lambda^2 = \frac{1}{a_0^2} C(M) \exp\left[-2\tfrac{3}{11}\frac{\pi^2}{N^2}\right]. \tag{4.18}$$

The constant $C(M)$ is N independent but subtraction-dependent, and many diagrams contribute to it. Formula (4.18) is up to less than a percent in agreement with the result quoted in Ref. [14] for $N = 2$ and $N = 3$

$$\frac{(\Lambda a_0)\,(\mathrm{SU}(3))}{(\Lambda a_0)\,(\mathrm{SU}(2))} = 1.4533 \text{ according to our formula,}$$

whereas $\qquad = 1.4521$ in Ref. [14].

Let us now discuss how it is that only Fig. 5 and Fig. 6 contribute to the $1/N$ part of b_1^0. This is a simple matter of colour counting. The ghost couplings and the measure couplings can only give contribution proportional to N, as discussed below Eq. (4.9) and in (4.7). Therefore all possible contributions come from the expansion of the original plaquette action

$$U(P_{\mu\nu}(n)) = e^{i\Phi_{\mu\nu}(n)}, \tag{4.19}$$

where $\Phi_{\mu\nu}$ is as in Eq. (2.16). The couplings come from

$$\tfrac{1}{2}(\mathrm{Tr}\, e^{i\Phi_{\mu\nu}} + \text{c.c.}) = -\tfrac{1}{2}\Phi_{\mu\nu}^a\Phi_{\mu\nu}^a + \frac{1}{4!}\mathrm{Tr}\,\Phi_{\mu\nu}^4. \tag{4.20}$$

The flux $\Phi_{\mu\nu}$ is — for one loop calculations — calculated up to and including terms of order g^4, sufficient to calculate one loop corrections to the coupling constant. Since $\Phi_{\mu\nu}$ is

a sum of multiple commutators the first term in (4.20) will only give rise to one loop diagrams proportional to N. But the second term may and indeed does give terms proportional to $1/N$. In the limit of zero lattice length the only two diagrams are those of Fig. 5 and Fig. 6, and the respective couplings are

$$g^2 a^4 \, \mathrm{Tr} \, (\Delta_{[\mu} A_{\nu]})^4 \quad \text{for the self energy diagram}$$

and

$$g^3 a^5 \, \mathrm{Tr} \, (\Delta_{[\mu} A_{\nu]})^3 \, [A_\mu, A_\nu] \quad \text{for the vertex diagram.}$$

In both diagrams the loop connects two factors $\Delta_{[\mu} A_{\nu]}$ which gives a) a gauge invariant result and b) gives in colour space a contribution

$$\delta^{ab} \, \mathrm{Tr} \, \frac{\lambda_a}{2} \, \frac{\lambda_b}{2} = \frac{N^2 - 1}{N} .$$

Notice that the factors a^4 (a^5) render the loop calculation finite. The typical integral appearing in the loop calculation is

$$\int_{-\pi/a}^{\pi/a} d^4 p \, \frac{|P_\mu(p)|^2 + |P_\nu(p)|^2}{\sum\limits_{\varrho=1}^{4} |P_\varrho(p)|^2} = \frac{1}{a^4} \frac{1}{2} \quad (\mu \neq \nu),$$

where p is the momentum, and $P_\mu(p)$ is the Fourier transform of the lattice derivative:

$$P_\mu(p) = \frac{e^{iaP_\mu} - 1}{ia} .$$

What is remarkable about the relation (4.18) is that the constant $C(M)$ turns out to be large [14]

$$\Lambda a_0 = 112.65 \exp\left[-\frac{3}{11} \frac{\pi^2}{N^2} \right].$$

It states that asymptotic freedom is already valid on lattices with lattice length roughly speaking a hundred times bigger than the length Λ^{-1} (Λ is experimentally — i.e. in a world with gluons and quarks — about 400 MeV).

5. Numerical results

We performed a number of experiments calculating the quantity $\omega(\beta)$ described in Section 3 for various two-dimensional spin-spin systems. According to the argument by Migdal [22], the position of the phase transitions of such systems should be the same as that of the four-dimensional plaquette systems. The size of the system was rather small. In all

the calculations we used a system with 400 sites in different combinations of a_1 and a_2 periods.

Fig. 7 presents our results for the U(1) system. This system is known [23] to undergo a phase transition at $\beta \approx 1.0$ from the disordered phase to the spin wave phase. For $\beta \rightarrow \infty$ we expect to approach the limiting value of $a_1 (1 - \cos \pi/a_1)$ corresponding to the lowest
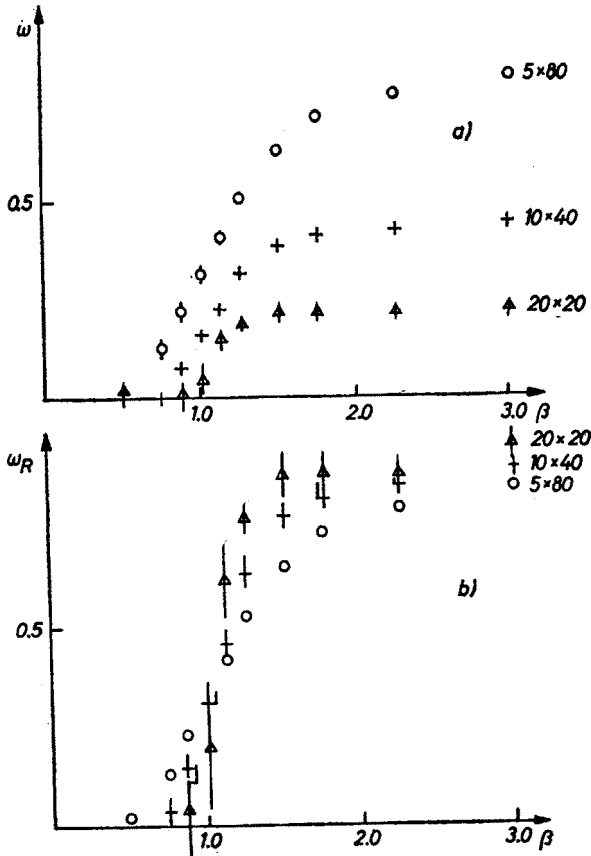

Fig. 7.a. $\omega(\beta)$ for the U(1) spin-spin system in two dimensions for 5×80, 10×40 and 20×20 systems; b. $\omega(\beta)$ scaled by the limiting high β value

energy solution with the twisted boundary conditions. Fig. 7a shows that this is really the case, and in fact the limiting value is reached very fast. Fig. 7b shows values of $\omega(\beta)$ scaled by the limiting number. The curves for growing a_1 seem to approach the limiting form close to the step function with a jump at $\beta \approx 1.0$.

Fig. 8 shows the same for the four-dimensional plaquette system. For large β we expect $\omega(\beta)$ to approach the limiting value of $a_1 a_2 (1 - \cos \pi/a_1 a_2)$, which is in good agreement with the experiment. The scaled $\omega(\beta)$ shows similar features to that of the two-dimensional case. The statistics in this experiment was much poorer (100 sweeps/point), which explains the bigger noise.
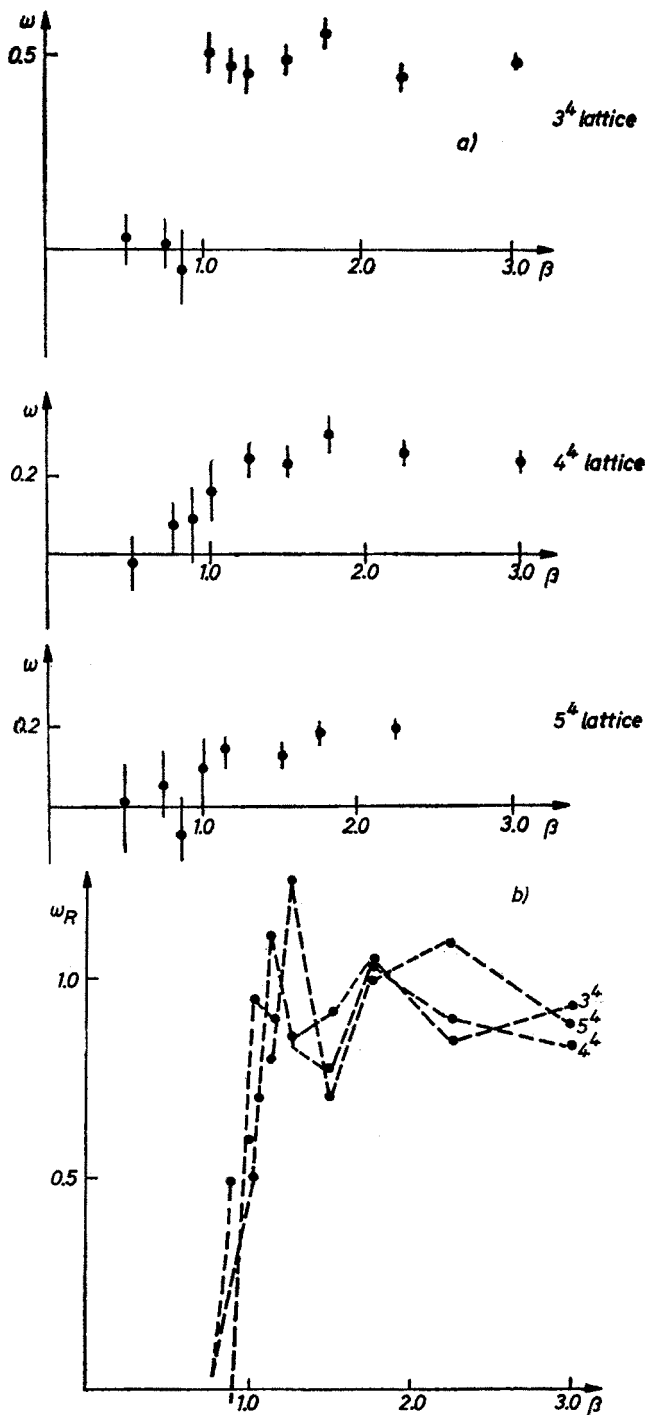
Fig. 8. a. $\omega(\beta)$ calculated for the four-dimensional U(1) plaquette system for various small sizes of the lattice; b. $\omega(\beta)$ scaled by the limiting high β value

Fig. 9 represents results for the two-dimensional SU(2) system with various combinations of the a_1 and a_2 periods. For large β we expect $\omega(\beta)$ to reach the limiting value $a_1(1-\cos \pi/a_1)$ as in the U(1) case. The scaled $\omega(\beta)$ (denoted $\omega_R(\beta)$) is shown in Fig. 9b. Again the curves seem to approach a limiting form, which however is completely different
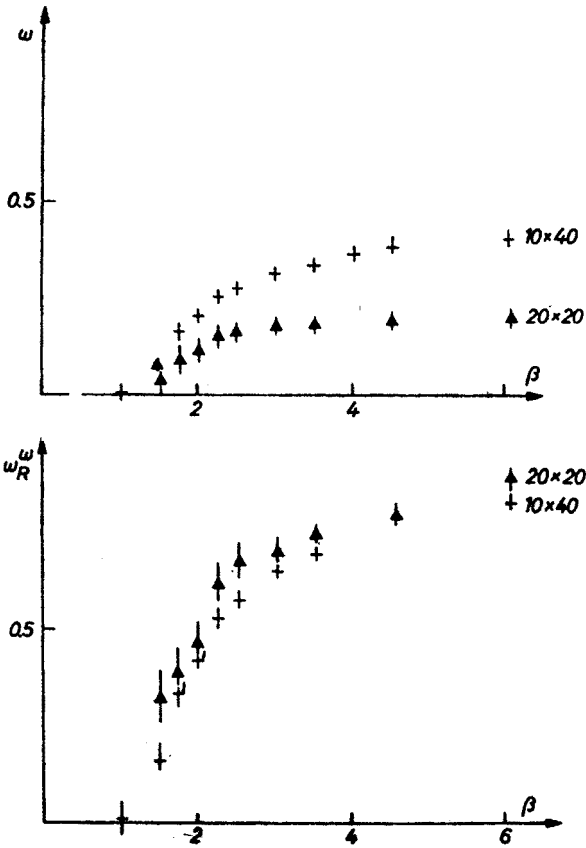

Fig. 9. a. $\omega(\beta)$ for the SU(2) spin-spin systems in two dimensions; b. $\omega_R(\beta)$ which is $\omega(\beta)$ scaled by the limiting high β value

from the U(1) case. Instead of a jump we observe a rather smooth transition from the low to the high β region. Unlike the U(1) case the limiting value seems to be reached only for $\beta \to \infty$.

Fig. 10 represents a summary of calculations for various two-dimensional systems with the gauge group chosen to be one of the subgroups of SU(2) (V*, T*, K*, and I*). Fig. 10a shows the perpendicular action $\sigma_\perp$ vs β plot for these groups together with the SU(2) points. Similarly to the results by Rebbi [21b] for the four-dimensional gauge systems, the point at which the average action plot for a given group starts to differ from the SU(2) curve moves to larger β with increasing order of the group. Fig. 10b shows $\omega(\beta)$ calculated for all these groups and two different choices for a_1 and a_2. For each of

the groups one can distinguish the ordered region, where $\omega(\beta)$ is independent of a_1 and the disordered region, where it coincides with the $\omega(\beta)$ for the SU(2) system. It seems that the K* and I* groups display an intermediate phase.

Fig. 11 shows the $\omega(\beta)$ plot for the O(3) system in two dimensions. The behaviour is qualitatively the same as that for the SU(2) system.
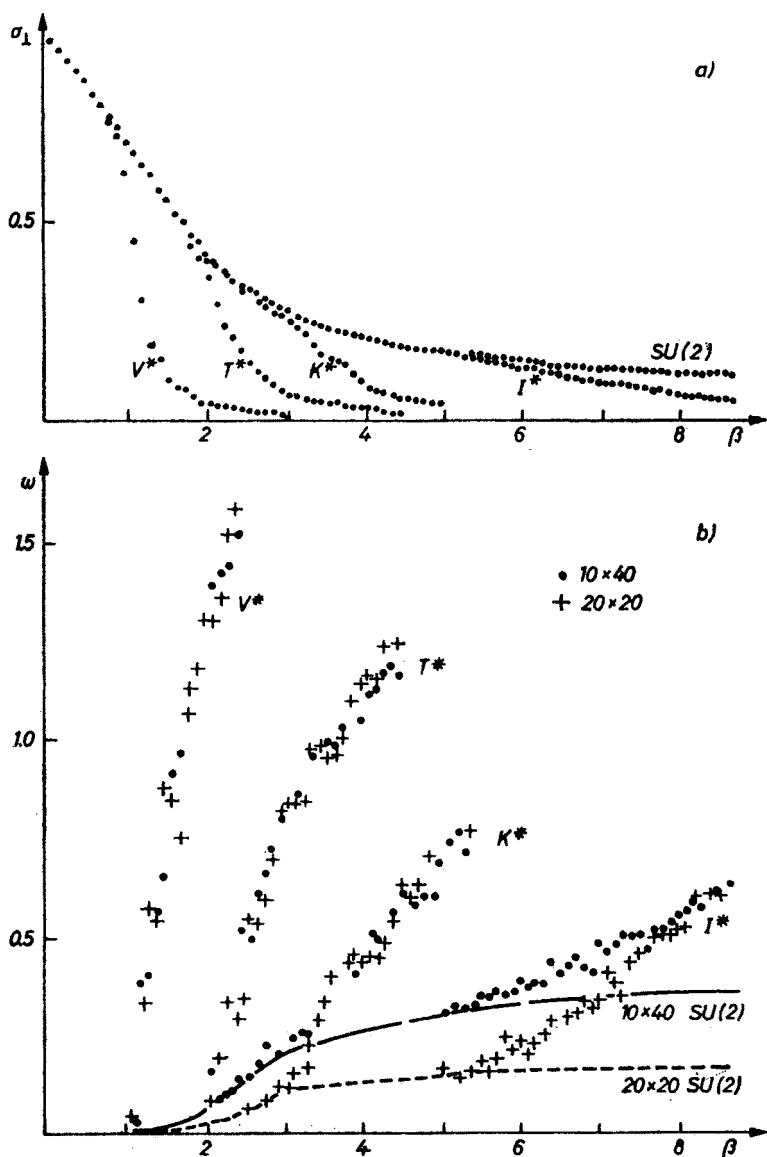

Fig. 10. a. Average perpendicular action for various subgroups of SU(2) and SU(2) for the two-dimensional spin-spin system with 20×20 sites; b. $\omega(\beta)$ calculated for various subgroups of SU(2) and SU(2) in systems with 10×40 and 20×20 sites
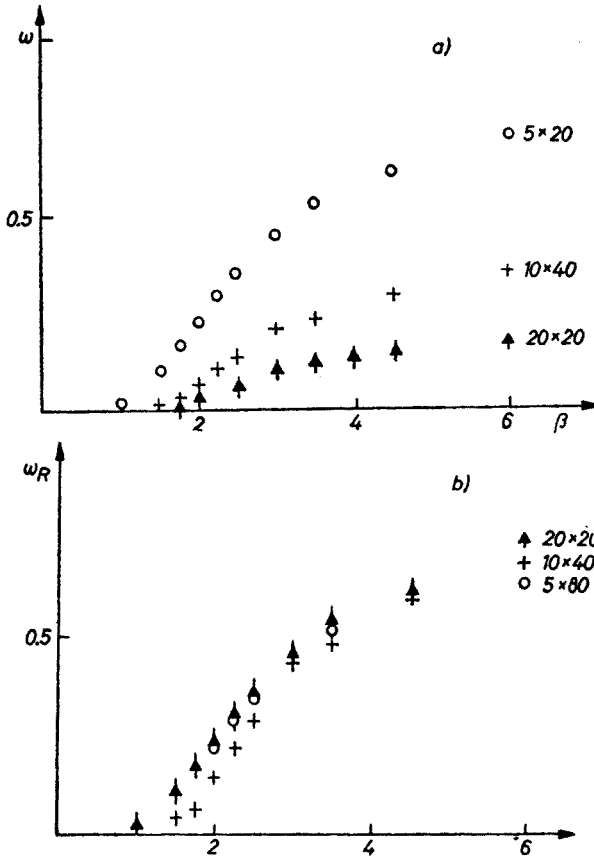
Fig. 11. $\omega(\beta)$ for the O(3) spin-spin system in two dimensions

6. Discussion

First a general remark that applies to all $d = 2$ spin-spin systems and four-dimensional plaquette systems that are examined by numerical calculations [21]. The Migdal hypothesis seems to be confirmed, i.e. the location of the phase transition seems only to depend on the group.

Next we turn to the U(1) group. Both spin-spin and plaquette version show a clear transition at $\beta \approx 1.0$ (Fig. 7, 8).

The O(3) and SU(2) systems have much less outspoken transitions to a spin wave phase at $\beta \approx 1.5$ and $\beta \approx 2.0$ respectively. The O(3) system should have no transition according to the instanton arguments of Ref. [25]. But their arguments apply only to systems with linear dimensions larger than the average distance between instantons (about $e^{8\pi\beta}$ lattice lengths). Thus our results for O(3) are not in disagreement with ideas in Ref. [25].

For the two-dimensional SU(2) system the instantons do not exist, so the spin wave phase might be genuine. But in appearance it is more alike the O(3) case than the U(1) case. The reader may notice that the "spin wave" phase starts at $\beta \approx 2.0$; it has been remarked

in Ref. [21] that the four-dimensional plaquette system of SU(2) starts to develop a "perturbative phase" equally at $\beta \approx 2.0$ [24].

The behaviour of the finite non-abelian subgroups V*, T* and K* in two-dimensional spin-spin theories is displayed in Fig. 10. The action of the V* group leads to the same statistical mechanics as the Z(8) action

$$S = \sum_{ij} [1 - \tfrac{1}{4}(z_i\bar{z}_j + (z_i\bar{z}_j)^3 + \text{c.c.})], \qquad z_i^8 = 1. \tag{6.1}$$

The observed first order transition of V* and thus of the Z(8) action (6.1) differs remarkably from the two higher order transitions for the Z(8) action without the cubic term $(z_i\bar{z}_j)^3$ as observed in Ref. [21b]. The T* group has also a first order transition, whereas the K* and I* seem to display an intermediate phase.

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seems to install itself. Making the same Ansatz for SU(3) leads to $\beta_t(3) = 9\beta_c(Z(3)) \cong 6.0$ and a reasonable value for the QCD parameter Λ emerges ($\Lambda \cong 0.1$). Because the $Z(N)$ transition in $d = 4$ is first order (if $N < 5$) the string tension remains finite at $\beta_t(N)$. But for $N > 5$ the Ansatz would lead to a vanishing string tension and therefore to a second phase which is not confining in the SU(N) sense but a) connects to the asymptotic freedom region and b) with a zero mass particle.

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