LETTERS TO THE EDITOR

FINITE CONFINEMENT FROM PADÉ APPROXIMANTS

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It is demonstrated that the known perturbation expansion, and its first diagonal Padé approximant, for the Fourier transform of the energy of two classical sources in QCD, agree approximately with the potential used in charmonium phenomenology for $0.5\ \mathrm{GeV} \leqslant q \leqslant 10\ \mathrm{GeV}$. It is argued that the singularity of the exact energy at $q = 0$ is stronger than q^{-2}.

The energy $E(x)$ of the two classical sources, interacting via exchange of gluons [1], is a very helpful quantity in searching for confinement. It contains all structure connected with the self-coupling of the Yang-Mills fields, while the additional complexity due to the quantum nature of fermions in abandoned. There exists in the literature a calculation of this energy up to the sixth order in coupling constant g [2]. In momentum space the result reads

$$E(x) = \frac{-g^2C_2(R)}{q^2}\left[1+\frac{g^2C_2(G)}{16\pi^2}e_1(\vec{q})+\left(\frac{g^2C_2(G)}{16\pi^2}\right)^2 e_2(\vec{q})\right], \tag{1}$$

where

$$e_1(\vec{q}) = \tfrac{11}{3}\left(\log\frac{\mu^2}{q^2}-\gamma\right)+\tfrac{31}{9},$$

$$e_2(\vec{q}) = \tfrac{121}{6}\log^2\frac{\mu^2}{q^2}+\left(\tfrac{988}{27}-\tfrac{242}{9}\gamma\right)\log\frac{\mu^2}{q^2},$$

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and $\vec{q}$ is the three-momentum conjugated to the distance of sources $\vec{x}$, μ is the mass parameter introduced through the dimensional renormalization, γ stands for the Euler constant. $C_2(R)$ and $C_2(G)$ depend on the gauge group; for SU(3) we have

$$C_2(R) = \tfrac{4}{3}, \quad C_2(G) = 3.$$

In this letter we try to answer the following question: does the confining nature of the QCD forces manifest itself already in the existing perturbative expansion? In particular, is there any connection between the formula (1) and the potential energy of the $q\bar{q}$ system

$$V(x) = \frac{-\alpha}{x}(1-x^2), \tag{2}$$

$\alpha = 0.2$, x measured in GeV^{-1}, which is so successful in charmonium phenomenology [3]? Of course, we do not expect that one can prove rigorously Eq. (2), starting from (1), for all values of the momentum/coordinate. The hope is that because the effective expansion parameter seems to be $g^2 \log \mu/q$, for momenta around μ, the lower order terms provide a sufficient approximation. Therefore, we expect that starting from a truncated expansion one should be able to prove a finite confinement (if any), i. e.; to obtain a reasonable

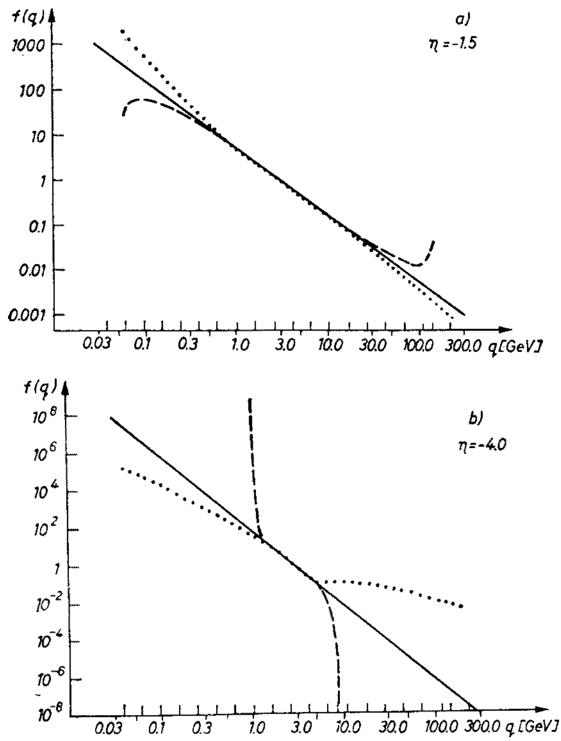

Fig. 1. The example (momentum space calculations, $\mu = 3$ GeV). Solid line — the exact function $f(q) = (q/\mu)^\eta$ and its (N, N) Padé approximants for $N_{\min} \leqslant N \leqslant 20$, $N_{\min} = 3$ for $\eta = -1.5$ and $N_{\min} = 8$ for $\eta = -4$; the dotted line — the RHS of Eq. (3) truncated to the first three terms; the dashed line — the (1,1) Padé approximant to $f(q)$

estimate for the exact function in the *finite* interval of momentum/coordinate. The larger the range of variables for which we attempt to determine the exact energy, the higher order perturbation terms are necessary. In addition, we need some technique for continuing truncated expansions of the type (1) out of the convergence circle (whose radius is probably zero in this case [4]). For the sake of simplicity we use the Padé approximants for this purpose.

Let us illustrate our method on a simple example. For $q \neq 0$ we have

$$\left(\frac{q}{\mu}\right)^{\eta} = \sum_{n=0}^{\infty} \frac{(\eta+2)^n}{n!} \frac{\log^n \frac{q}{\mu}}{q^2/\mu^2}. \tag{3}$$

We interpret the RHS of Eq. (3) as an idealized perturbation expansion for the "unknown" function $(q/\mu)^{\eta}$. For fixed q we construct the sequence of the diagonal Padé approximants to the RHS of Eq. (3). Then, we vary q and look on the dependence of the approximants on this variable. The results are shown in Fig. 1 for two values of η. Low Padés approximate the LHS in some finite interval. The range of momentum for which a given Padé has converged grows fast with the order of approximant. For example, for $\eta = -1.5$, (3,3) Padé works well through four decades of q. For larger $|\eta|$ the convergence is slower what is not surprising.

Now we turn to the real world, i. e. to the Eq. (1). Fig. 2a shows a comparison of the formula (1) and its first Padé approximant with the Fourier transform of Eq. (2). The Coulomb part of the energy has been subtracted. We have chosen $\mu = 3$ GeV and $\alpha_s(\mu) = 0.23$, $\alpha_s = g^2/4\pi$. The value for $\alpha_s(\mu)$ is taken from Ref. [5]. Indeed, theoretical

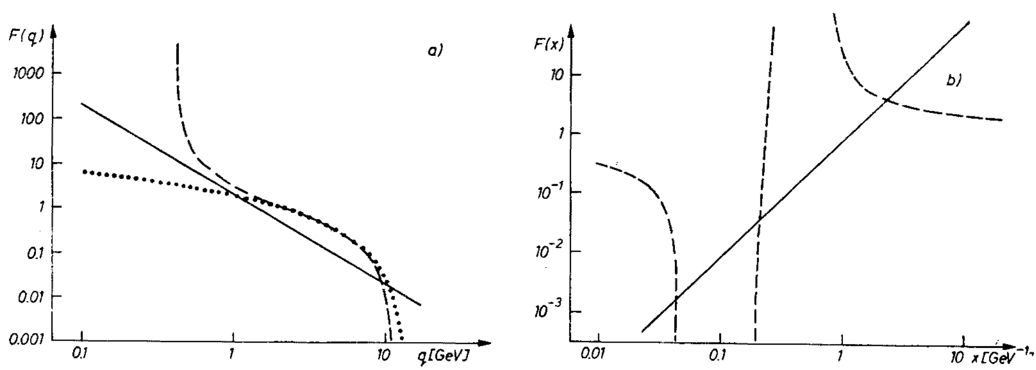

Fig. 2. Comparison of the QCD calculations and the phenomenological potential, $\alpha = 0.2$. 2a — momentum space $F(q) = \frac{-q^2 P(q)}{4\pi\alpha} - 1$. Solid line — $P(q) = \text{F. T.}^{-1}\,[V(x)](q)$, where $V(x)$ is given by Eq. (2); dotted line — $P(q) = E(q)$, where $E(q)$ is given by Eq. (1); dashed line — $P(q)$ equals the (1,1) Padé approximant to RHS of Eq. (1). 2b — configuration space, $F(x) = \frac{xP(x)}{\alpha} + 1$. Solid line — $P(x) = V(x)$; dashed line — $P(x)$ equals the (1,1) Padé approximant to the Fourier transform of the RHS of the Eq. (1)

calculations follow approximately the experimental parametrisation for q in the range $0.5\ \mathrm{GeV} \leqslant q \leqslant 10\ \mathrm{GeV}$. The first Padé is not much better than the original series, as could be expected. A power of the Padé method is seen only for higher orders. We emphasize that the parameters μ and α_s have not been fitted. They are taken from other areas of application of QCD where this theory has been compared with the experiment. Of course, higher orders are needed to confirm and extend this result.

We have repeated similar calculations in configuration space. For a better understanding what happens there we turn back to our idealized example. The Fourier transform of Eq. (3) reads

$$\frac{\mu^3}{2\pi^2}(\mu x)^{-3-\eta}\Gamma(\eta+2)\sin\left[(\eta+2)\frac{\pi}{2}\right]$$

$$= \mathrm{F.T.}\left[\left(\frac{q}{\mu}\right)^{\eta}\right](x) = \int \frac{d^3q}{(2\pi)^3}\left[\sum_{n=0}^{\infty}\frac{(\eta+2)^n}{n!}\frac{\log^n \frac{q}{\mu}}{q^2/\mu^2}\right]e^{i\vec{q}\cdot\vec{x}}$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty}\frac{(\eta+2)^n}{n!}\,\mathrm{F.T.}\left[\frac{\log^n \frac{q}{\mu}}{q^2/\mu^2}\right](x),$$

where

$$\mathrm{F.T.}\,[f(q)]\,(x) \equiv \int \frac{d^3q}{(2\pi)^3} f(q) e^{i\vec{q}\cdot\vec{x}}. \tag{4}$$

Now we try to approximate the exact function of the LHS of Eq. (4) by the series on the RHS, which is a series of the Fourier transforms of the perturbation expansion terms in Eq. (3). But the RHS of (4) is convergent to its LHS only when an interchange of summation and integration is legitimate. This is the case for $-2 \leqslant \eta \leqslant -1$. Only then the equality with a question mark is true and we may expect the first few terms of RHS to give a reasonable estimate of the LHS. For $\eta < -2$ or $\eta > -1$ the integral

$$\int \frac{d^3q}{(2\pi)^3} q^{\eta} e^{i\vec{q}\cdot\vec{x}}$$

does not exist and the Fourier transform is defined through the analytic continuation in η. The original series of the RHS of (4) *needs not* converge to the correct answer for $\eta < -2$. Fig. 3 confirms our expectations. In Fig. 3a we plotted the LHS, RHS (truncated to the first three terms) and the two diagonal Padé approximants to it for $\eta = -1.5$, versus x. The truncated series approximate the true answer quite well, as it should do for this value of η. The Padé approximants as usual work better with the growing order. For $\eta = -4$ (Fig. 3b), however, the situation is entirely different. The truncated series does not give the true answer at all. Its value is positive for all x and cannot be plotted on the logarithmic scale of Fig. 3b. The true function is negative in this case.[1] The Padé approximants, on the other hand, reproduce the proper answer quite well for high N (N being the order of the diagonal approximant). For small N they have some trouble what is understandable if we realize that we wanted them to continue the unknown function through the pole at $\eta = -3$.

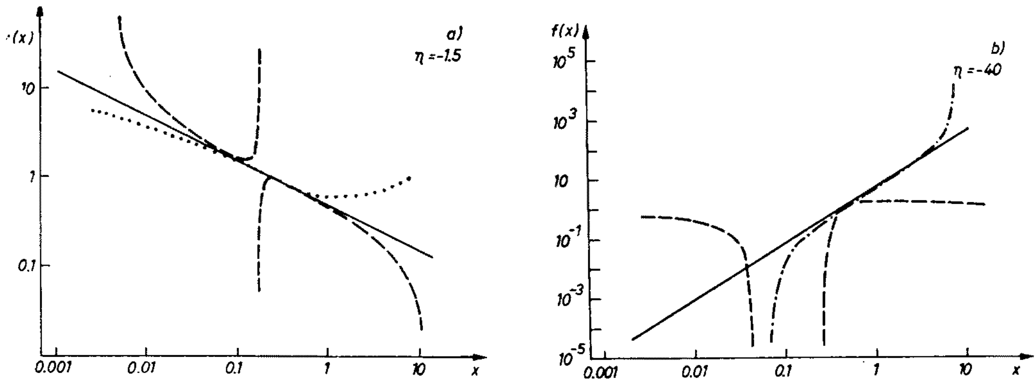

Fig. 3. The example (configuration space calculations, $\mu = 3$ GeV). Solid line — the exact function $f(x) = \frac{4\pi x}{\mu^2}$ F. T. $\left[\left(\frac{q}{\mu}\right)^\eta\right](x)$ and its (N, N) Padé approximants, for $N_{\min} \leqslant N \leqslant 20$, $N_{\min} = 3$ for $\eta = -1.5$ and $N_{\min} = 10$ for $\eta = -4$; dotted line (in Fig. 3a only) — the series of Eq. (4) truncated to the first three terms and multiplied by $\frac{4\pi x}{\mu^2}$; dotted-dashed line (in Fig. 3b only) — the (4,4) Padé approximant to $f(x)$; the dashed line $-(1,1)$ Padé to $f(x)$

Let us summarize the conclusions which emerge from this simple example. If the function to be approximated is so singular in the momentum space that its Fourier transform does not exist in the usual sense, then the Fourier transform of the perturbation series, term by term, needs not converge to the Fourier transform of this function defined by the analytic continuation.

In view of this statement we are glad to see that the Fourier transform of Eq. (1) behaves exactly in the same way which we have found in the example for $\eta < -2$. In Table I we give few values of the Fourier tansform of Eq. (1) together with the expected phenomenological potential energy, Eq. (2). They do not agree at all, what indicates that $E(q)$ is singular in momentum space. In Fig. 2b we plot the experimental potential (2) and the (1,1) Padé approximant constructed from the Fourier transform of Eq. (1). The agreement is rather poor, but we have to remember that the first Padé had also troubles in the simple example.

In conclusions, we have shown that the confining character of the QCD forces manifests itself already in the first three terms of the perturbation expansion. Present knowledge allows one to demonstrate this finite confinement in a range of momentum 0.5 GeV $\leqslant q \leqslant$ $\leqslant$ 10 GeV. Higher order calculations are necessary to confirm and extend this result.

[1] The change of sign while going from $\eta = -1.5$ to $\eta = -4$ is caused by the single pole at $\eta = -3$.

TABLE I

Comparison of the phenomenological potential, Eq. (2), with the Fourier transform of the RHS of Eq. (1) denoted by F. T. $[E(q)]\,(x)$

x [GeV^{-1}]	$\frac{\mathrm{F.T.}[E(q)](x)}{\alpha 14\pi x}-1$	$\frac{V(x)}{\alpha 14\pi x}-1$
0.01	0.575	-0.5×10^{-4}
0.10	0.325	-0.5×10^{-2}
1.07	2.98	-0.58
3.16	5.00	-5.00
11.11	8.11	-61.73

Secondly, the Fourier transform of the truncated series does not show any sign of convergence to the expected answer. We consider this fact as an indication, even stronger than that coming from the momentum space, that the Fourier transform of the energy of the two classical sources in QCD is singular at $q = 0$. This shows once more the confining nature of the $q\bar{q}$ interactions.

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