ON SPHERICALLY SYMMETRIC DISTRIBUTION OF MATTER WITH PRESSURE IN GENERAL RELATIVITY

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The form of the solution of Einstein's equations for a spherically symmetric distribution of matter in the co-moving coordinate system has been found. Although the pressure of matter is different from zero, the system is synchronous because of a suitable choice of the equation of state.

Attempts at obtaining an exact solution of Einstein's equations describing inhomogeneous cosmological models are especially interesting from the point of view of investigations of early stages of the Universe. From the mathematical point of view the above problem is far more complicated as compared with homogeneous models. It seems practically impossible to obtain analytical solutions in the case of inhomogeneous distribution of matter without additional assumptions. This letter presents some conclusions concerning the existence of a certain class of solutions for inhomogeneous models with the following assumptions: Einstein's equations for the ideal fluid with pressure are discussed in a coordinate system synchronous and co-moving, assuming a spherical symmetry of the distribution of matter. In the case of spherical symmetry the metric has the following form:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - e^{\mu}(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

The components of the energy-momentum tensor T_μ^ν for the ideal fluid in the co-moving coordinate system are: $T_0^0 = \varepsilon$, $T_1^1 = T_2^2 = T_3^3 = -p$, (all others vanish), and field equa-

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tions have the form:

$$\kappa p = \tfrac{1}{2} e^{-\lambda}(\tfrac{1}{2} \mu'^2 + \mu'\nu') - e^{-\nu}(\ddot{\mu} - \tfrac{1}{2} \dot{\mu}\dot{\nu} + \tfrac{3}{4} \dot{\mu}^2) - e^{-\mu}, \tag{2}$$

$$\kappa p = \tfrac{1}{4} e^{-\lambda}(2\nu'' + \nu'^2 + 2\mu'' + \mu'^2 - \mu'\lambda' - \nu'\lambda' + \mu'\nu') + \tfrac{1}{4} e^{-\nu}(\dot{\lambda}\dot{\nu} + \dot{\mu}\dot{\nu} - \dot{\lambda}\dot{\mu} - 2\ddot{\lambda} - \dot{\lambda}^2 - 2\ddot{\mu} - \dot{\mu}^2), \tag{3}$$

$$\kappa \varepsilon = -e^{-\lambda}(\mu'' + \tfrac{3}{4} \mu'^2 - \tfrac{1}{2} \mu'\lambda') + \tfrac{1}{2} e^{-\nu}(\dot{\lambda}\dot{\mu} + \tfrac{1}{2} \dot{\mu}^2) + e^{-\mu}, \tag{4}$$

$$0 = \tfrac{1}{2} e^{-\lambda}(2\dot{\mu}' + \dot{\mu}\mu' - \dot{\lambda}\mu' - \nu'\dot{\mu}), \tag{5}$$

where: $\mu = \mu(r, t)$, $\nu = \nu(r, t)$, $\lambda = \lambda(r, t)$, p — pressure, ε — energy density, κ — Einstein's gravitational constant, the dot means a differentiation with respect to time, and the comma – differentation with respect to the r-coordinate.

Using the relations $T^{\mu\nu}{}_{;\nu} = 0$ which follow from the field equations one obtains the following relations:

$$\dot{\lambda} + 2\dot{\mu} = -\frac{2\dot{\varepsilon}}{p+\varepsilon}, \qquad \nu' = -\frac{2p'}{p+\varepsilon}, \tag{6}$$

and after performing integrations:

$$\lambda + 2\mu = -2\int \frac{d\varepsilon}{p+\varepsilon} + f_1(r),$$

$$\nu = -2\int \frac{dp}{p+\varepsilon} + f_3(t), \tag{7}$$

where the functions $f_1(r)$ and $f_3(t)$ can be chosen arbitrarily because the coordinates t and r can be transformed according to the relations: $r = r(r')$ and $t = t(t')$ [1].

In order to obtain a closed system of equations we must add the equation of state — that is the relation between energy density and pressure. For example the well known Tolman solution [2] for dust matter is given for the pressure $p = 0$. Let us consider the equation of state of the form $\varepsilon(r, t) = f(r, t)p(t)$, where $f(r, t) \geqslant 1$. Although this relation has no clear physical interpretation, we use it because it makes our coordinate system synchronous. In addition one may assume that for a fixed r the function $f(r, t)$ has the form of a step function — taking for corresponding time intervals characteristic values for hadronic matter, radiation or dust respectively, as in a standard equation of state in homogeneous cosmology.

From equation (6) we obtain $p' = 0$, that means $\nu = f_3(t)$ and

$$\int \frac{d\varepsilon}{p+\varepsilon} = \ln(1+f) + \frac{f}{1+f} \ln p.$$

Thus

$$p = (e^{\lambda} e^{2\mu} e^{-f_1}(1+f)^2)^{-\frac{1+f}{2f}}, \qquad \varepsilon = fp. \tag{8}$$

From equation (5) after introducing the variable $R = e^{\frac{1}{2}\mu}$ we obtain $\dot{\lambda} = 2\dot{R}/R'$. Integration with respect to time gives $\exp\lambda = R'^2 \exp f_2(r)$, where $f_2(r)$ is an arbitrary function. After re-defining the time variable $d\bar{t} = e^{\frac{1}{2}\nu}dt$ equations (2) and (4) take the form:

$$\kappa p = e^{-f_2}\frac{1}{R^2} - 2\frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{R^2}, \tag{9}$$

$$\kappa\varepsilon = -e^{-f_2}\frac{1}{R'^2}\left\{\left(\frac{R'}{R}\right)^2 - f_2'\frac{R'}{R}\right\} + 2\frac{\dot{R}'}{R'}\frac{\dot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{1}{R^2}. \tag{10}$$

Let us re-scale the radial coordinate r assuming $f_2(r) = 0$. We obtain the following system of equations:

$$\kappa p = -\frac{2\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2, \tag{11}$$

$$\kappa\varepsilon = 2\frac{\dot{R}'}{R'}\frac{\dot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2, \tag{12}$$

$$p = \left(R^4 e^{-f_1} R'^2 (1+f)^2\right)^{-\frac{1+f}{2f}}, \tag{13}$$

$$\varepsilon = fp. \tag{14}$$

The above system is difficult to solve without additional assumptions. When $f(r, t) \gg 1$ the system is significantly simplified. In this letter an attempt has been made to answer the following question: is it possible to obtain a non-dust solution of Einstein's equations with the above assumption.

Applying the last assumption one obtains

$$p = R'^{-1}R^{-2}e^{\frac{1}{2}f_1}(1+f)^{-1} \tag{15}$$

and from equations (11) and (12):

$$\kappa(p+\varepsilon) = \kappa(1+f)R'^{-1}R^{-2}e^{\frac{1}{2}f_1}(1+f)^{-1} = 2\left(\frac{\dot{R}'}{R'}\frac{\dot{R}}{R} - \frac{\ddot{R}}{R}\right).$$

Therefore

$$\kappa e^{\frac{1}{2}f_1} = 2R(\dot{R}'\dot{R} - \ddot{R}R'). \tag{16}$$

Thus the problem of solving field equations is reduced to the solution of non-linear, second-order, partial differential equation (16).

Applying Monge's method [3] we finally obtain:

$$-t \pm \int \frac{dR}{\sqrt{\frac{F(r)}{R} - h(R)}} = g(r), \tag{17}$$

where functions $h(R)$, $g(r)$ are arbitrary and

$$F(r) = \kappa \int e^{\frac{1}{2}f_1(r)} dr.$$

It is possible to obtain a solution $R(r, t)$ after calculating the integral in (17) i.e. for a particular choice of function $h(R)$. For example the case $h(R) = 0$ is equivalent to dust solution because then equation (17) gives:

$$R = (3 \cdot 2^{-\frac{1}{2}})^{\frac{2}{3}} F^{\frac{1}{3}} (g(r)+t)^{\frac{2}{3}} \tag{18}$$

that is Tolman's solution. It is easy to test that, in general, equation (17) implies dust solution only if $h(R) = \mathrm{const}/R$.

Thus the final form of the solution of Einstein's equations for an ideal fluid with pressure which is small if compared with the energy density in the spherically symmetric and co-moving system with the universal cosmic time, is given by:

$$e^{\mu} = R^2, \qquad e^{\lambda} = R'^2$$

and

$$-t \pm \int \frac{dR}{\sqrt{\frac{F(r)}{R} - h(R)}} = g(r)$$

where functions F, g, h are arbitrary and $h(R) \neq \mathrm{const}/R$.

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