THE LONGITUDINAL PHASE-SPACE INTEGRAL WITH LEADING PARTICLES

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De Groot's method of calculating the longitudinal phase-space integrals is generalized to include leading particles. The generalization simplifies practical calculations of all quantities predicted by the uncorrelated jet model with leading particles.

It is well known that many features of the multiple production of particles at high energies can be explained by energy and momentum conservation and the transverse momentum cut-off. This is the main reason of the interest in the uncorrelated jet model [1] which is the simplest model incorporating these assumptions.

For practical calculations in this model, one is interested in the phase-space integral (called also the grand partition function), because all physical quantities predicted by the model can be derived from it. An elegant asymptotic expansion of this integral for large energies has been obtained by de Groot [2], following the ideas suggested by Lurçat and Mazur [3] and Bassetto et al. [4].

When studying high energy collisions, one gets a more realistic model by accounting for the leading particle effect (as an additional input, besides energy-momentum conservation and the transverse momentum cut-off). In this note we derive a high-energy expansion of the longitudinal phase-space integral with leading particles by an extension of de Groot's method. Our expansion proved useful in an actual calculation of the rising plateau [5].

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We discuss here the high-energy expansion of the integral

$$\Phi_\lambda(R) = \sum_n \frac{\lambda^n}{n!} \int \frac{d^3q_1}{E_1} \cdots \frac{d^3q_n}{E_n} \frac{d^3k_R}{E_R} \frac{d^3k_L}{E_L}$$

$$\delta^4(R-q_1-\ldots-q_n-k_L-k_R) f(\vec{q}_{1\perp}) \ldots f(\vec{q}_{n\perp}) E_R^+ \varphi_R(\vec{k}_{R\perp}) E_L^- \varphi_L(\vec{k}_{L\perp}), \tag{1}$$

where $k_{R,L} = (E_{R,L}, \vec{k}_{R,L})$ are four-momenta of the leading particles, $q_i = (E_i, \vec{q}_i)$ are four-momenta of the emitted particles, R is the total four-momentum of the whole system, and

$$E^\pm = E \pm k_{||}. \tag{2}$$

The subscripts $\perp$ and $||$ denote the perpendicular and parallel components of the momenta with respect to the beam direction.

The leading particle effect is described here by the factors $E_R^+ \varphi_R(\vec{k}_{R\perp})$ and $E_L^- \varphi_L(\vec{k}_{L\perp})$. With the factors E_R^+ and E_L^-, the input density for the leading particles in Eq. (1) is essentially flat in longitudinal momentum. This corresponds to the physics in the bremsstrahlung model [6] where the emission of the field quanta is not influenced by the energy loss of the leading particle except for the overall energy and momentum conservation.

De Groot proposed a way of calculating integrals as in Eq. (1), but without leading particles [2]. He used the method of stationary phase to calculate the Laplace transform of the phase-space integral [3]. In order to get the observed spectra, however, one has to integrate de Groot's result over the leading particle variables [7]. The additional multi-dimensional integration is a non-trivial problem. It is therefore of interest to develop an approach in which the leading particles are treated on equal footing with the produced ones. This is the purpose of the present note.

The Laplace transform of $\Phi_\lambda(R)$ is denoted by $\tilde{\Phi}_\lambda(x)$,

$$\tilde{\Phi}_\lambda(x) = \int e^{-R\cdot x} \Phi_\lambda(R) d^4R \tag{3}$$

and can be inverted by using the formula

$$\Phi_\lambda(R) = \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{d^4x}{(2\pi i)^4} e^{R\cdot x} \tilde{\Phi}_\lambda(x). \tag{4}$$

For $\tilde{\Phi}_\lambda(x)$ one obtains the form

$$\tilde{\Phi}_\lambda(x) = \int \frac{d^3k_R}{E_R} E_R^+ \varphi_R(\vec{k}_{R\perp}) e^{-k_R\cdot x} \int \frac{d^3k_L}{E_L} E_L^- \varphi_L(\vec{k}_{L\perp}) e^{-k_L\cdot x}$$

$$\exp\left(\lambda \int \frac{d^3q}{E} f(\vec{q}_\perp) e^{-q\cdot x}\right), \tag{5}$$

where the summation over n has been explicitly performed.

We observe now that the longitudinal integrals can be expressed by modified Bessel functions [8],

$$\int \frac{dq_{\parallel}}{E}\, e^{-Ex_0+q_{\parallel}x_{\parallel}} = 2K_0(\mu_{\perp}\bar{x}) \tag{6}$$

and

$$\int \frac{dk_{\parallel}}{E}\, E^{\pm} e^{-Ex_0+k_{\parallel}x_{\parallel}} = 2\,\frac{x^{\pm}}{\bar{x}}\, m_{\perp} K_1(m_{\perp}\bar{x}). \tag{7}$$

Our notation is $\mu_{\perp}^2 = \mu^2+q_{\perp}^2$, $m_{\perp}^2 = m^2+k_{\perp}^2$, and

$$x^{\pm} = x_0 \pm x_{\parallel}, \tag{8}$$

$$\bar{x}^2 = x_0^2 - x_{\parallel}^2 = x^+x^-. \tag{9}$$

Substituting Eqs (6) and (7) into Eqs (4) and (5) we obtain after rotating the integration contour by 90°

$$\Phi_{\lambda}(R) = \int \frac{d^2b}{(2\pi)^2}\, e^{-i\vec{R}_{\perp}\cdot\vec{b}} Z_{\lambda}(\vec{b}, \bar{R}) \tag{10}$$

with

$$Z_{\lambda}(\vec{b}, \bar{R}) = 2\int \frac{dx^+dx^-}{(2\pi i)^2}\, \bar{\Phi}_{\lambda}(\vec{b}, \bar{x}) \exp\left(\tfrac{1}{2}R^+x^- + \tfrac{1}{2}R^-x^+\right) \tag{11}$$

and

$$\bar{\Phi}_{\lambda}(\vec{b}, \bar{x}) = \int d^2k_{R\perp}\, \varphi(\vec{k}_{R\perp}) m_{R\perp} K_1(m_{R\perp}\bar{x}) e^{i\vec{k}_{R\perp}\cdot\vec{b}}$$
$$\int d^2k_{L\perp}\, \varphi(\vec{k}_{L\perp}) m_{L\perp} K_1(m_{L\perp}\bar{x}) e^{i\vec{k}_{L\perp}\cdot\vec{b}} \exp\left(2\lambda \int d^2q_{\perp} f(\vec{q}_{\perp}) K_0(\mu_{\perp}\bar{x}) e^{i\vec{q}\cdot\vec{b}}\right). \tag{12}$$

The next step is to perform the integration over x^+ and x^- in Eq. (11). This can be done by expanding the Bessel functions appearing in Eq. (12) around the point $\bar{x} = 0$. As shown by de Groot [2], the main contribution to $\Phi_{\lambda}(R)$ at high energies comes from the neighbourhood of this point. The expansion is easily achieved by inserting the known series expansions of K_0 and K_1,

$$K_0(z) = -\ln\frac{ze^{\gamma}}{2} - \frac{z^2}{4}\ln\frac{ze^{\gamma-1}}{2} + \ldots \tag{13}$$

and

$$zK_1(z) = 1 + \frac{z^2}{2}\ln\frac{ze^{\gamma-\frac{1}{2}}}{2} + \ldots \tag{14}$$

where γ is Euler's constant. In the present note we restrict ourselves to only the first two terms in these expansions. The resulting formula for $\bar{\Phi}_{\lambda}(\vec{b}, \bar{x})$ can be integrated term by term thus providing the required asymptotic expansion of the longitudinal phase-space integral (1). We obtain

$$\bar{\Phi}_\lambda(\vec{b}, \bar{x}) = \frac{\exp(-2\lambda f_{0l}(\vec{b}))}{(\bar{x}^2)^{1+\lambda f_0(\vec{b})}} \varphi_0^{\mathrm{L}}(\vec{b})\varphi_0^{\mathrm{R}}(\vec{b})$$

$$\left[1 + \frac{\bar{x}^2}{2}(\varphi_{1l}^{\mathrm{L}}(\vec{b}) + \varphi_{1l}^{\mathrm{R}}(\vec{b}) - \lambda f_{1l}(\vec{b}) + \lambda f_1(\vec{b})) + \frac{\bar{x}^2 \ln \bar{x}}{2}(\varphi_1^{\mathrm{R}}(\vec{b}) + \varphi_1^{\mathrm{L}}(\vec{b}) - \lambda f_1(\vec{b}))\right], \tag{15}$$

where we have introduced the notation

$$f_n(\vec{b}) = \int d^2 q_\perp f(\vec{q}_\perp) e^{i\vec{q}_\perp \cdot \vec{b}} \mu_\perp^{2n}, \quad n = 0, 1 \ldots, \tag{16a}$$

$$f_{nl}(\vec{b}) = \int d^2 q_\perp f(\vec{q}_\perp) e^{i\vec{q}_\perp \cdot \vec{b}} \mu_\perp^{2n} \ln \frac{\mu_\perp e^\gamma}{2}, \quad n = 0, 1 \ldots, \tag{16b}$$

$$\varphi_0^{\mathrm{L,R}}(\vec{b}) = \int d^2 k_\perp \varphi_{\mathrm{L,R}}(\vec{k}_\perp) e^{i\vec{k}_\perp \cdot \vec{b}}, \tag{16c}$$

$$\varphi_1^{\mathrm{L,R}}(\vec{b}) = \frac{1}{\varphi_0^{\mathrm{L,R}}(\vec{b})} \int d^2 k_\perp \varphi_{\mathrm{L,R}}(\vec{k}_\perp) m_{\mathrm{L,R}\perp}^2 e^{i\vec{k}_\perp \cdot \vec{b}}, \tag{16d}$$

and

$$\varphi_{1l}^{\mathrm{L,R}}(\vec{b}) = \frac{1}{\varphi_0^{\mathrm{L,R}}(\vec{b})} \int d^2 k_\perp \varphi_{\mathrm{L,R}}(\vec{k}_\perp) m_{\mathrm{L,R}\perp}^2 \ln \frac{m_{\mathrm{L,R}\perp} e^{\gamma - \frac{1}{2}}}{2} e^{i\vec{k}_\perp \cdot \vec{b}}. \tag{16e}$$

For the evaluation of $Z_\lambda(\vec{b}, \bar{R})$ we use the formulae [8]

$$\frac{1}{2\pi i} \int \frac{du}{u^\alpha} e^{su} = \frac{s^{\alpha-1}}{\Gamma(\alpha)}, \tag{17}$$

$$\frac{1}{2\pi i} \int \frac{du}{u^\alpha} e^{su} \ln u = \frac{s^{\alpha-1}}{\Gamma(\alpha)} (\psi(\alpha) - \ln s), \tag{18}$$

where $\Gamma(\alpha)$ and $\psi(\alpha)$ are Euler's gamma and digamma functions. The result is

$$Z_\lambda(\vec{b}, \bar{R}) = 2 \exp(-2\lambda f_{0l}(\vec{b})) \varphi_0^{\mathrm{L}}(\vec{b}) \varphi_0^{\mathrm{R}}(\vec{b}) \left(\frac{\bar{R}^2}{4}\right)^{\lambda f_0(\vec{b})}$$

$$\frac{1}{\Gamma(1 + \lambda f_0(\vec{b}))^2} \left[1 + \tfrac{1}{2} \lambda^2 f_0(\vec{b})^2 \frac{4}{\bar{R}^2} \left(A - \tfrac{1}{2} B \ln \frac{\bar{R}^2}{4}\right)\right] \tag{19}$$

with

$$A = \varphi_{1l}^{\mathrm{L}}(\vec{b}) + \varphi_{1l}^{\mathrm{R}}(\vec{b}) - \lambda f_{1l}(\vec{b}) + \lambda f_1(\vec{b}) + B\psi(\lambda f_0(\vec{b})) \tag{20}$$

and

$$B = \varphi_1^{\mathrm{R}}(\vec{b}) + \varphi_1^{\mathrm{L}}(\vec{b}) - \lambda f_1(\vec{b}). \tag{21}$$

To obtain the value of $\Phi_\lambda(R)$ it is now enough to substitute the formula (19) into Eq. (10) and perform the required Fourier–Bessel transform. The result will obviously depend on the specific choice of the cut-off functions $f(\vec{q}_\perp)$, $\varphi_R(\vec{k}_{R\perp})$, and $\varphi_L(\vec{k}_{L\perp})$. Because of azimuthal symmetry, the expression (10) finally reduces to a one-dimensional integral,

$$\Phi_\lambda(R) = \Phi(R_\perp, \bar{R}) = \frac{1}{2\pi}\int_0^\infty bdbJ_0(R_\perp b)Z_\lambda(b, \bar{R}). \tag{22}$$

From Eq. (19) we see that the leading term does not depend on the masses of the leading particles.

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