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A NEW INTERPRETATION OF FIELDS ON M $\times$SU(2)

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It is shown that free fields on some manifolds larger than the Minkowski space M (M$\times$SU(2), M$\times$SU(2)$\times$SU(2)) can be regarded as fields describing a composite system of two or three point particles.

1. Introduction

There have been numerous attempts to describe the internal structure of hadrons by ascribing to them some internal continuous degrees of freedom and also by considering fields on manifolds larger than the Minkowski space M [1].

One of the simplest possibilities is to examine fields on the seven dimensional space M$\times$SU(2). Of course, it is very tempting to interpret the additional SU(2) variables as variables of a three dimensional rigid rotator i. e. to treat hadrons as relativistic rigid bodies.

We want to present another, very unexpected and amusing, interpretation of the SU(2) variables as internal variables of a composite system of two point particles linked by a nonrelativistic classical mechanics type elastic force. The whole system would then manifest a striking analogy with an elementary particle constructed from two confined quarks.

Similarly, the internal dynamics of an object having six-dimensional internal SU(2)$\times$$\times$SU(2) configuration space can be described on this elementary level using three "quarks".

2. $SU(2)$ case

Let us consider two nonrelativistic particles at positions x_1, x_2 with their relative motion described by the Lagrangian

$$\mathscr{L} = \frac{m}{4} g_{ik}(z)\dot{z}^i\dot{z}^k, \tag{1}$$

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where $z = x_1 - x_2$, dots denote time derivatives. For simplicity, we assume that each particle has the same mass, m. $g_{ik}(z)$ is given by

$$g_{ik}(z) = \delta_{ik} + \frac{z^i z^k}{l_0^2 - z^2}, \quad g^{ik}(z) = \delta_{ik} - \frac{1}{l_0^2} z^i z^k. \tag{2}$$

The choice of such metric tensor is explained in Section 4.

The Lagrangian gives the following equation of motion for the internal variable z:

$$\ddot{z} + \frac{1}{l_0^2}\left[(\dot{z})^2 + \frac{(z\dot{z})^2}{l_0^2 - z^2}\right] z = 0. \tag{3}$$

If $|z| < l_0$ at certain moment t_0, equation (3) tells us that:

a) the distance between the particles cannot be greater than l_0. This follows from that when $|z| \sim l_0$, $|z| < l_0$ the strong attractive force between the particles causes $|z|$ to decrease. Thus, the particles always form a composite system whose maximum radius is l_0 and cannot be seen separately.

b) If the particles are at rest with respect to each other at the moment t_0, when $\dot{z} = 0$, they will always be at rest with respect to each other because the force between them vanishes. The particles are in this sense free.

The canonical momentum associated with z is

$$\Pi_i = \frac{m}{2} g_{ik}\dot{z}^k = \frac{m}{2}\left[\dot{z}^i + \frac{z^i(z\dot{z})}{l_0^2 - z^2}\right]$$

and it satisfies the equation

$$\dot{\boldsymbol{\Pi}} = \frac{2}{ml_0^2}(\boldsymbol{\Pi}z)\boldsymbol{\Pi}. \tag{4}$$

The Hamiltonian of the internal motion is

$$H = \frac{1}{m}\left[\boldsymbol{\Pi}^2 - \frac{1}{l_0^2}(\boldsymbol{\Pi}z)^2\right] \tag{5}$$

and is positive for our composite system because $|z| < l_0$.

The system has five time-independent constants of internal motion which we take as

$$\boldsymbol{\Omega}_{\mathrm{L}} = \tfrac{1}{2}(z^0\boldsymbol{\Pi} + z \times \boldsymbol{\Pi}), \quad \boldsymbol{\Omega}_{\mathrm{R}} = -\tfrac{1}{2}(z^0\boldsymbol{\Pi} - z \times \boldsymbol{\Pi}), \tag{6}$$

where $z^0 = \sqrt{l_0^2 - z^2}$.

The Hamiltonian (5) depends on $\boldsymbol{\Omega}_{\mathrm{L}}$, $\boldsymbol{\Omega}_{\mathrm{R}}$ by $H = \dfrac{4}{ml_0^2}\boldsymbol{\Omega}_{\mathrm{L}}^2 = \dfrac{4}{ml_0^2}\boldsymbol{\Omega}_{\mathrm{R}}^2$. The coefficient $\dfrac{1}{l_0^2}\left[(\dot{z})^2 + \dfrac{(z\dot{z})^2}{l^2 - z^2}\right]$ occurring in equation (3) is equal to $\dfrac{16}{m^2 l_0^4}(\boldsymbol{\Omega}_{\mathrm{L}})^2$ and is also a constant of motion. Thus, equation (3) describes a three-dimensional harmonic oscillator whose frequency depends on initial conditions at a moment t_0.

Next we introduce the Poisson brackets

$$(F, G) = \frac{\partial F}{\partial z^i}\frac{\partial G}{\partial \Pi_i} - \frac{\partial F}{\partial \Pi_i}\frac{\partial G}{\partial z^i}.$$

It is easy to check that our constants of motion $\boldsymbol{\Omega}_{\mathrm{L}}$ and $\boldsymbol{\Omega}_{\mathrm{R}}$ form two independent SU(2) algebras:

$$(\Omega^i_{\mathrm{L}}, \Omega^k_{\mathrm{L}}) = \varepsilon_{ikl}\Omega^l_{\mathrm{L}}, \quad (\Omega^i_{\mathrm{R}}, \Omega^k_{\mathrm{R}}) = \varepsilon_{ikl}\Omega^l_{\mathrm{R}}, \quad (\Omega^i_{\mathrm{L}}, \Omega^k_{\mathrm{R}}) = 0. \tag{7}$$

The internal motion can be quantized by representing the algebra (7) by differential operators of the right and left regular representation of the SU(2) group. In such a way this part of the wave function which refers to the internal motion becomes a function on the SU(2) group.

3. SU(2)×SU(2) case

Let us consider three point particles with positions $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. For simplicity we assume, as in the SU(2) case, that each particle has the same mass m. We postulate the following lagrangian to describe their internal motion

$$\mathscr{L} = \frac{m}{4} g_{ik}(z_2)\dot{z}^i_2\dot{z}^k_2 + \frac{m}{12} g_{ik}(z_1)\dot{z}^i_1\dot{z}^k_1,$$

where $z_1 = \boldsymbol{x}_1+\boldsymbol{x}_2-2\boldsymbol{x}_3$, $z_2 = \boldsymbol{x}_1-\boldsymbol{x}_2$. $g_{ik}(z_1)$, $g_{ik}(z_2)$ are given by formula (2) with the substitutions $l_0 \to l_1$, $l_0 \to l_2$ respectively.

As the Euler-Lagrange equations for the internal motion we get two independent equations of the form (3), one for each z_i, $i = 1, 2$.

Thus, we have obtained a system of three point particles whose separation can never be larger than max (l_1, l_2). Notice that "quarks" $\boldsymbol{x}_1, \boldsymbol{x}_2$ interact by a two body force and "quark" $\boldsymbol{x}_3$ interacts by a three body force.

Because the variables $\boldsymbol{x}_1, \boldsymbol{x}_2$ describing the internal motion are independent we get as an algebra of constants of motion four independent SU(2) algebras. This algebra can be realized by differential operators on the SU(2)×SU(2) group. Thus, after quantization we are led to fields with SU(2)×SU(2) internal variables. On the other hand, because SU(2)×SU(2) is the universal covering group of the SO(4) group, such fields can be interpreted as fields describing a four-dimensional spherical rigid rotator.

4. The origin of the metric tensor g_{ik}

We consider SU(2) group parametrized by SU(2) $\to u(\boldsymbol{w}) = w^0\sigma^0 + i\boldsymbol{w}\boldsymbol{\sigma}$, $\sigma^0, \boldsymbol{\sigma}$ – Pauli matrices. This formula implies $w^{02}+\boldsymbol{w}^2 = 1$, w^0 and $\boldsymbol{w}$-real, $\boldsymbol{w}^2 \leqslant 1$. Hence, $\boldsymbol{w}$ fills up the unit sphere B. The same $\boldsymbol{w}$ corresponds to two elements of SU(2) differing by sign of w_0. However, choosing the sign of w_0 by $w^0 = +\sqrt{1-\boldsymbol{w}^2} \geqslant 0$ we can regard $\boldsymbol{w}$ as coordinates in a neighbourhood of the unit element of the SU(2) group (and also as coordinates on the SO(3) group near the unit element). SU(2) acts on $\boldsymbol{w}$ by nonlinear transformations

$$\boldsymbol{w} \to \boldsymbol{w}' = v^0\boldsymbol{w} + w^0\boldsymbol{v} \pm \boldsymbol{v}\times\boldsymbol{w}, \tag{8}$$

where –, + correspond to the left group translation $u'(\boldsymbol{w}') = u_0(\boldsymbol{v})u(\boldsymbol{w})$ and the right group translation $u'(\boldsymbol{w}') = u(\boldsymbol{w})u_0(\boldsymbol{v})$ respectively.

There exists only one (up to an arbitrary constant factor) metric tensor on the SU(2) group which is invariant under both these transformations and therefore distinguished in this sense. It can be found, for example, by a simple calculation analogous to that given in [3] for the Poincaré group. As a result we obtain in a neighbourhood of the unit element the metric tensor (2) in which $z \equiv l_0 w$.

5. *Discussion*

a) The use of a nonrelativistic equation for the description of the internal motion may seem disadvantageous. However, this is not the case, because, as we have shown, the constituent particles always form a composite system and cannot be seen as separate free particles. Hence, there is no physical principle which forces us to formulate the internal dynamics in a relativistically covariant way. Only a theory of the composite system treated as a whole should be formulated in a covariant way.

So far we only know how our composite system behaves in the nonrelativistic limit when considered as a whole. In order to obtain the relativistic covariance we ought to introduce some new object which in the nonrelativistic limit behaves as our composite system. For such a generalization we can take as a guiding principle the correspondence between z and SU(2) group (Sec. 4) and introduce transformations of relativistic symmetry in the same way as for the spherical rigid body (see paper [2]). Then the boost Λ is represented on the quantum level by the Wigner rotation $u_0(\boldsymbol{p}, \Lambda)$ acting on the wave function $\psi(p^\mu, u)$ by

$$\psi(p^\mu, u) \to \psi(\Lambda^\mu{}_\nu p^\nu, u_0^{-1}(\boldsymbol{p}, \Lambda)u),$$

where $\psi(p^\mu, u)$ is the Fourier transform of $\psi(x^\varrho, u)$. Hence, the free composite system has the same wave functions and transformation properties as the relativistic spherical rigid body. In particular, the wave functions are functions on $\mathrm{M} \times \mathrm{SU}(2)$, where M is the Minkowski space.

b) There exists another way to quantize the internal motion of the constituents. Namely, we can represent the algebra

$$(z^i, \Pi_j) = \delta^i{}_j, \quad (z^i, z^j) = 0, \quad (\Pi_i, \Pi_j) = 0$$

by the usual operators $z \to z\cdot$, $\boldsymbol{\Pi} \to \dfrac{1}{i}\dfrac{\partial}{\partial z}$. The internal motion will be described in this case by a wave function using the variables z, $z \in R^3$. The whole wave function will be a function of seven variables $(t, \boldsymbol{x}, z)$, $z \in R^3$.

However, by this method of quantization we encounter some complications. First, in our Lagrangian (1) we have a velocity-dependent potential. As shown in Ref. [4], some modifications of the usual canonical formalism are needed here, e.g. the correct Hamiltonian is not equal to symmetrized Hamiltonian with $z^i \to z^i$, $\pi_j \to \dfrac{1}{i}\dfrac{\partial}{\partial z^j}$. Second, and more important, the spectrum of the correct Hamiltonian in addition to its discrete part (with states localized in the sphere $|z| < l_0$ and with positive energies) also contains the continuous part corresponding to wave functions localized outside this sphere, with energies negative and not bounded from below. This was reported also in Ref. [5] for the one dimensional version of our internal Lagrangian $\mathscr{L} = \frac{ml_0^2}{4} \frac{\dot{x}^2}{l_0^2 - x^2}$. To have a reasonable theory we must assume that the continuous part of the spectrum is a nonphysical one. Therefore, we think that this method of quantization is less adequate than quantization leading to wave functions on $\mathrm{M} \times \mathrm{SU}(2)$.

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