BOUNDS FOR THE LINEAR COMBINATIONS OF STATISTICAL TENSORS

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A method of estimating the physical bounds for the linear combinations of components of statistical tensors is presented. Numerical values of the bounds for tensor combinations appearing in the additive quark model predictions for resonance production in two-body processes are calculated.

Much interest has been recently paid to the angular decay distributions of the resonances produced in high-energy processes. Joint decay distributions have been studied experimentally in several processes and some theoretical models have been proposed and checked. A typical example is the additive quark model (*cf. e.g.* Ref. [1]), which predicts relations between the statistical tensors [1, 2]. When checking experimentally such relations it is interesting to know the kinematical bounds for the linear combinations of statistical tensors.

The purpose of this note is to present a convenient method of estimating these bounds and to list the numerical values of the bounds for some frequently used tensor combinations.

The method used here is essentially that of Refs [3] and [4]. A double statistical tensor is defined as the following combination of the density matrix elements:

$$T^{J_1J_2}_{M_1M_2} = \sum_{m_1m_1'm_2m_2'} (-1)^{s_1+s_2+m_1+m_2-J_1-J_2} \times$$
$$\times\, C(s_1, -m_1; s_1, m_1'|J_1, M_1)\, C(s_2, -m_2; s_2, m_2'|J_2, M_2)\, \varrho^{m_1m_1'}_{m_2m_2'}. \tag{1}$$

It may be regarded as the average value of an operator $\mathcal{O}^{J_1J_2}_{M_1M_2}$

$$T^{J_1J_2}_{M_1M_2} = \mathrm{Tr}\,(\mathcal{O}^{J_1J_2}_{M_1M_2}\varrho) = \langle \mathcal{O}^{J_1J_2}_{M_1M_2}\rangle \tag{2}$$

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TABLE I

Calculated kinematical bounds for linear combinations of statistical tensors. According to the additive quark model each combination should be equal to the value quoted in column 5. Column 6 contains references to original papers where the relations have been derived

Spins	Reaction	Combination of tensors	Calculated bounds	AQM prediction	Ref.
$1, \frac{3}{2}$	$PB \to VB^*$	$T^{20}_{00} - \sqrt{2}\, T^{02}_{00}$	$(-0.8165,\ 0.6124)$	0	[2, 6]
	$\gamma B \to VB^*$	$T^{22}_{00} - \frac{1}{2} T^{20}_{20}$	0.3750	0	[2, 6]
		$T^{22}_{02} - \frac{1}{\sqrt{2}} T^{02}_{02}$	0.4330	0	[2, 6]
		$T^{22}_{00} + \frac{1}{\sqrt{2}} T^{02}_{00}$	0.4082	$\frac{1}{2\sqrt{6}}$	[2, 6]
	$PB \to VB^*$	$T^{01}_{00} - \sqrt{2}\, T^{21}_{00}$	1.1619	0	[5]
		$T^{20}_{20} - \sqrt{2}\, T^{02}_{02}$	0.5387	0	[2]
		$T^{22}_{20} - T^{22}_{02}$	0.2887	0	[2]
		$T^{22}_{22} + T^{22}_{00} + T^{22}_{2-2}$	0.4082	$\frac{1}{2\sqrt{6}}$	[2]
$\frac{3}{2}, \frac{3}{2}$	$BB \to B^*B^*$	$T^{22}_{20} - \frac{1}{2} T^{20}_{20}$	0.2652	0	[2]
		$T^{22}_{00} + \frac{1}{2} T^{02}_{00}$	0.3750	$\frac{1}{8}$	[2]
		$T^{10}_{00} - 2\, T^{12}_{10}$	1.0062	0	[5]
		$T^{11}_{11} - \frac{5}{3} T^{22}_{11}$	0.5667	0	[5]
		$T^{1\,1}_{1-1} + \frac{5}{3} T^{2\,2}_{1-1}$	0.5667	0	[5]
		$T^{20}_{00} - T^{22}_{00} - \frac{3}{5} T^{11}_{00} - 2 T^{2\,2}_{2-2}$	$(-0.7190;\ 1.0220)$	0	[5]
		$T^{22}_{22} + T^{22}_{00} + T^{2\,2}_{2-2}$	0.5000	$\frac{1}{4}$	[2]
$1, \frac{3}{2}$	$\gamma B \to VB^*$	$T^{20}_{20} + \sqrt{2}\, T^{02}_{02}$	0.5387	0	[6]
		$T^{22}_{20} + T^{22}_{02}$	0.2887	0	[6]
		$T^{22}_{00} - T^{22}_{22} - T^{2\,2}_{2-2}$	0.4082	$\frac{1}{\sqrt{6}}$	[6]
		$T^{22}_{20} + T^{20}_{20}$	0.5000	0	[6]
		$T^{22}_{02} + \sqrt{2}\, T^{02}_{02}$	0.4339	0	[6]
		$\sqrt{2}\, T^{02}_{00} + T^{20}_{00}$	$(-0.8165,\ 0.6124)$	$-\frac{1}{2\sqrt{6}}$	[6]
		$T^{02}_{00} + 2 T^{20}_{00} + 6 T^{02}_{02}$	$(-1.1547,\ 0.8660)$	0	[6]

TABLE I (continued)

Spins	Reaction	Combination of tensors	Calculated bounds	AQM prediction	Ref.
$1, \frac{3}{2}$	$\gamma B \to VB^*$	$T^{20}_{00} - \sqrt{\frac{2}{3}}\,\mathrm{Re}\, T^{20}_{20}$	0.4082	0	[6]
		$T^{22}_{00} - \sqrt{\frac{2}{3}}\,\mathrm{Re}\, T^{22}_{20}$	0.4082	0	[6]
		$T^{22}_{22} + (T^{2\ 2}_{2-2})^* - 6T^{22}_{02}$	0.7071	0	[6]
		$T^{22}_{11} - (T^{2\ 2}_{1-1})^*$	0.3536	0	[6]

with matrix elements equal to the products of Clebsch-Gordan coefficients as seen from Eq. (1). The average value of an operator cannot exceed its extremal eigenvalues. The method of estimating the bounds for the tensors like (1) proposed in Ref. [3] consists in calculating the eigenvalues of the hermitian part of the operator $\mathcal{O}$. The values for bounds obtained in this way for single and double tensors are given in Ref. [4].

A similar method can be used to evaluate the bounds for the linear combinations of various tensor components

$$aT^{J_aJ'_a}_{M_aM'_a} + bT^{J_bJ'_b}_{M_bM'_b} + \ldots + nT^{J_nJ'_n}_{M_nM'_n} \tag{3}$$

where $a, b, \ldots n$ are known constants. This expression may be represented as the average value of the operator

$$a\mathcal{O}^{J_aJ'_a}_{M_aM'_a} + b\mathcal{O}^{J_bJ'_b}_{M_bM'_b} + \ldots + n\mathcal{O}^{J_nJ'_n}_{M_nM'_n}. \tag{4}$$

The matrix elements of $\mathcal{O}^{J_kJ'_k}_{M_kM'_k}$ are again appropriate products of Clebsch-Gordan coefficients and the problem is reduced again to finding the minimum and maximum eigenvalues of the Hermitian part of a known matrix.

In Table I we present as an example the estimated bounds for the linear combinations of tensors appearing in the predictions of the additive quark model. Each entry contains the spins of the two resonances described by the tensors, the reaction for which the prediction applies, the linear combination of tensor components, the estimated bounds, the additive quark model prediction for this combination, and finally the reference to the paper where the prediction has been derived.

The results were obtained using a computer programme. The programme calculated the matrix elements of the Hermitian part of the operator from Eq. (4) and diagonalized the matrix obtained. The numbers in the Table are the bounds for the moduli, for the real parts and for the imaginary parts of the tensor combinations, at the same time.

The method is especially effective when the linear combination contains more than two tensor components. Most of our results for combinations with two terms may also be obtained by taking the appropriate combination of bounds for tensor components from Ref. [4].

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