Contributions to First Passage Time Problems of Brownian Motion

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Abstract

Let W be a standard Brownian motion with $W_0 = 0$ and let $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with $b(0) > 0$. The first passage time of b (from below) is then defined as

$$\tau \coloneqq \inf\{t \geq 0 \,|\, W_t \geq b(t)\}.$$

A well-known method to determine the distribution of τ is the method of images. It gives an explicit expression for the the first hitting time distribution for all boundaries b that are the implicit solution to the integral equation

$$1 = \int_{(0,\infty)} r_\theta(t,x)\mu(d\theta), \qquad t > 0,$$

for some σ-finite measure μ. However, the boundary b cannot be determined explicitly for most choices of μ. Therefore, the inverse method of images has emerged – at least as a numerical approach – to determine the measure μ and thus the distribution of τ for a given boundary b. If a measure μ exists, the boundary b will be called representable. Until now, it remained an open question for which boundaries b a measure μ exists such that b is representable. In this thesis, we present a new duality approach to the inverse method of images which enables us to give sufficient conditions for the existence of representing measures μ for concave, analytic boundaries b. Based on that, we put forward a new algorithm that improves upon existing algorithms for the inverse method of images by requiring less discretisations and giving precise approximations with very fast convergence. We also give convergence guarantees for this algorithm. As an application, we establish a connection between the existence of first passage time distributions and the representability of American options via European options. This thesis concludes by approaching the first passage time problem via Fredholm type integral equations for which we show that these uniquely determine the distribution of the first passage time τ.

Zusammenfasssung

Sei W eine Standard Brownsche Bewegung mit $W_0 = 0$ und sei $b : \mathbb{R}_+ \to \mathbb{R}$ eine stetige Funktion mit $b(0) > 0$. Die Erstauftreffszeit auf b (von unten) ist dann definiert als

$$\tau \coloneqq \inf\{t \geq 0 \,|\, W_t \geq b(t)\}.$$

Eine bekannte Methode, um die Verteilung von τ zu bestimmen ist die *method of images*. Diese gibt einen expliziten Ausdruck für die Verteilung der Erstauftreffszeit für alle Grenzen b, die sich als implizite Lösung der Integralgleichung

$$1 = \int_{(0,\infty)} r_\theta(t,x)\mu(d\theta), \qquad t > 0,$$

für ein σ-endliches Maß μ ergeben. Allerdings kann die Grenze b für die meisten Wahlen von μ nicht explizit bestimmt werden. Daher ist die inverse method of images entstanden – zumindest als numerischer Ansatz – , um für eine gegebene Grenze b das Maß μ und damit die Verteilung von τ zu bestimmen. Wenn ein solches maß μ existiert, heißt die Grenze b darstellbar. Bis jetzt war es eine offene Frage, für welchen Grenzen b ein Maß μ existiert, sodass b darstellbar ist. In dieser Arbeit präsentieren wir einen neuen Dualitätsansatz für die inverse method of images, der es uns ermöglicht, hinreichende Bedingungen für die Existenz von darstellenden Maßen μ für konkave, analytische Grenzen b anzugeben. Basierend darauf schlagen wir einen neuen Algorithmus vor, der bestehende Algorithmen für die inverse method of images verbessert, indem er bei sehr schneller Konvergenz weniger Diskretisierungen benötigt und präzisere Annäherungen möglich macht. Wir beweisen außerdem, dass der Algorithmus konvergiert. Als Anwendung zeigen wir eine Verbindung zwischen der Existenz von Verteilungen von Erstauftreffszeiten und der Darstellbarkeit von amerikanischen Optionen durch europäische Optionen. Die Arbeit schließt mit einem Ansatz für das Erstauftreffszeit-Problem über Fredholm Integralgleichungen, für die wir beweisen, dass diese die Verteilung der Erstauftreffszeit eindeutig bestimmen.

Contents

Chapter 1

Introduction

1.1 Boundary hitting of Brownian motion

Just as the normal distribution is in a certain way at the heart of classical probability theory, the Brownian motion is at the heart of the theory of stochastic processes. Like the central limit theorem, Donsker's theorem provides the reason why the Brownian motion plays such a big role for stochastic processes. Moreover, the Brownian motion is the driver for many stochastic processes via stochastic differential equations.

It is therefore not surprising that the Brownian motion and its properties form the basis of many textbooks and have been studied extensively for a long time. The (standard) Brownian motion or Wiener process is defined as the stochastic process $W = (W_t)_{t\geq 0}$ such that $W_0 = 0$, W has independent stationary increments, $W_t \sim \mathcal{N}(0, t)$ for all $t > 0$ and $t \mapsto W_t$ is almost surely continuous (cf. [Kle14], Definition 21.8). Alternatively, Brownian motion can also be defined as the Gaussian process $(B_t)_t$ with expectation function $E(B_t) = 0$ for $t > 0$ and covariance function $\operatorname{Cov}(B_s, B_t) = \min(s, t)$. The Brownian motion has the strong Markov property (cf. [Kle14], Theorem 21.18) and it satisfies the Law of the iterated logarithm (cf. [Kle14], Theorem 22.1), i.e.,

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log(\log(t))}} = 1 \qquad \text{almost surely.}$$

Heuristically, this means that the Brownian motion stays within certain boundaries in the long term. One more important result is Donsker's theorem (also known as functional central limit theorem) which states the following: let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with $E(X_1) = 0$ and $Var(X_1) = 1$. Donsker's theorem (cf. [Kle14], Theorem 21.43) now states that the rescaled process $(S_t^n)_{t\geq 0}$ given by

$$S_t^n = \frac{\sum_{i=1}^{\lfloor nt \rfloor} X_i}{\sqrt{n}}$$

converges in distribution (i.e., in the sense of weak convergence on $C[0,\infty)$) to the Brownian motion.

One of the most intuitive questions to ask is when a Brownian motion crosses a given boundary. This problem is widely known as the First Passage Time (or FPT) problem. Let W be a standard Brownian motion with $W_0 = 0$ and let $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with $b(0) > 0$. The first passage time of b (from below) is then defined as

$$\tau := \inf\{t \geq 0 \,|\, W_t \geq b(t)\}.$$

Now, the goal of the FPT problem is to determine the cumulative distribution function (c.d.f.) F of τ under $P_{(0,0)}$, i.e., $F(t) = P_{(0,0)}(\tau \leq t)$ where $P_{(0,x)}$ denotes the probability distribution if W is started in x at time 0. Or, more generally, the goal is to determine the distribution of τ conditional on the Brownian motion running from 0 at time 0 to some x_0 at time t_0, i.e.,

$$P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0), \quad t_0 > 0,\ x_0 < b(t_0). \tag{1.1}$$

This is then of course a Brownian bridge anchored at the time-space points $(0,0)$ and (t_0, x_0). It is known that F has a continuous density f if b is continuously differentiable (cf. Theorem 6.1, [Fer82b]).

The FPT problem has been studied extensively and many applications have been developed: In statistics, FPT problems arise for problems in testing (cf. [RS74], [Fer82a] or the surveys [Sie86], [Lai01]). In finance, the problem emerges in the valuation of barrier options (cf. [KI92], [GY96], [RS97]) as well as in default models (cf. [CGJ08], [HW01]) and in the evaluation of credit risks (cf. [Che+06]). In [SZ13], the authors formulate an inventory-control problem which can be seen as an FPT problem in certain cases. There are also applications in biology for the modelling of neuronal activity (cf. [SVZ06]). For an overview of the FPT problem and some applications in the field of physics, including a connection to electrostatics, see the monograph [Red01] or the more recent and shorter paper [Red22].

We do not attempt to give a complete history of the FPT problem but rather highlight some results and approaches that have been developed over the years. In Section 1.2 below we go into detail on the method of images which will therefore be omitted in this section.

The FPT problem has a long history and can be traced back as far as the thesis "Théorie de la spéculation" by Bachelier (cf. [Bac00]), where the problem is first formulated (though not as rigorously as by later authors) but only for constant b. Other early formulations of the problem include a paper by Schrödinger ([Sch15]) and joint work by Kolmogorov and Khintchine which was later published in [Khi33]. Khintchine formulates the FPT problem in continuous time but then has to discretise the time axis as he has no concept available for the event in question ("[...] the general principles in probability theory give, even in their most modern formulation, no indication for the general definition of such a probability [...]"[1]).

Even though the problem has been studied for a long time, closed form solutions for F (given b) are rare. There is the well known Bachelier-Levy formula for linear boundaries $b(t) = a + mt$ for some $a > 0$, $m \in \mathbb{R}$ where the density of τ is given by

$$\frac{a}{t^{3/2}}\phi\left(\frac{a+mt}{\sqrt{t}}\right), \qquad t > 0 \tag{1.2}$$

where ϕ is the density of a standard normal distribution. The formula is given in [Lév65] (p. 82ff) but as the book is written in French, non-French speaking readers may want to refer to [RY99], Chapter III, §3 for the case $m = 0$ or [Ler86], Example 1, p.27 for the more general case $m \in \mathbb{R}$.

Apart from the case of linear b, there are results for the square-root boundary and for the quadratic boundary. For the square-root boundary, [Bre67] investigates boundaries of the form $b(t) = c(\sqrt{t}+1)$ for some $c > 0$ and the stopping time $T_c = \inf\{t \geq 0 | W_t \geq b(t)\}$ (notation slightly adapted). Using a transformation of the Brownian motion into the Ornstein-Uhlenbeck process he shows that for large t it holds that $P(T_c > t) \sim \alpha t^{-\beta(c)}$, where $\beta(c) \to 0$ for $c \to \infty$ and $\beta(c) \to \infty$ for $c \to 0$. [RSS84] uses a Volterra type integral equation and a transformation of the Brownian motion to an Ornstein-Uhlenbeck process to derive a rather lengthy but explicit formula for the square-root boundary (cf. Theorem 3.4 and the following Corollary in [RSS84]).

[1] translation by the author from the original German: "[...] die allgemeinen Prinzipien der Wahrscheinlichkeitsrechnung geben, auch in ihrer modernsten Fassung, keinen Anhaltspunkt für die allgemeine Definition einer derartigen Wahrscheinlichkeit [...]", [Khi33], p.69

Moreover, [NFK99] uses piecewise linear approximations to derive an explicit formula for the square-root boundary as an infinite power series (cf. Theorem 4 in Section 5, [NFK99]).

The quadratic boundary was investigated among others in [Sal88], where a formula for the c.d.f. F of τ is derived. The formula depends on the expected value of a Brownian functional but it is shown that this can be computed in some cases. In particular, for a quadratic boundary, this can be expressed more explicitly in terms of Airy functions (cf. Proposition 3.10, [Sal88]). [Gro89] derives a similar result by first expressing the c.d.f. F in terms of a Bessel process and then reducing that expression to one depending on Airy functions (cf. Theorem 2.1, [Gro89]).

Another approach to the FPT problem is by integral equations (typically Volterra or Fredholm type integral equations) connecting b and F. While there have been many approaches involving Volterra type integral equations for many years previously, [Pes02] presents a unifying approach to derive these equations. Peskir derives an integral equation he calls the "master equation" which is given as (cf. Theorem 3.1, [Pes02], notation is slightly adapted)

$$1 - \Phi\left(\frac{z}{\sqrt{t}}\right) = \int_0^{\sqrt{t}} \left(1 - \Phi\left(\frac{z - b(s)}{\sqrt{t-s}}\right)\right) F(ds)$$

for all $z \geq b(t)$, where $T > 0$ and where Φ is the c.d.f. of the standard normal distribution. [Pes02] goes on to derive several known equations studied by other authors from the master equation as well as a set of new equations (cf. Theorem 6.1, [Pes02])

$$t^{n/2} H_n\left(\frac{b(t)}{\sqrt{t}}\right) = \int_0^t (t-s)^{n/2} H_n\left(\frac{b(t) - b(s)}{\sqrt{t-s}}\right) F(ds)$$

for $t > 0$ and $n = -1, 0, 1, \ldots$ and where $H_n(x) = \int_x^\infty H_{n-1}(z)dz$ and $H_{-1} = \phi$ (with ϕ the standard normal density).

These integral equations are in most cases difficult to solve analytically but there are numerical approaches. In [Dur71], the boundary b is approximated by linear functions in subintervals and then the integral equations are solved with linear recursions. The paper [Dur85] builds upon this work by introducing a series expansion of the first passage time density of a continuous Gaussian process using integral equations and [Dur92] makes this series more computable by only considering the case of Brownian motion. [Smi72] suggests two alternative methods which improve upon [Dur71] by handling singularities in the kernel of the integral equation. In [PP74], semi-closed forms for the solution are proposed, which are more explicit than in [Dur71] and [Smi72], but these are still hard to compute explicitly except in special cases (i.e., in the cases b is constant or linear). In [Pes02], the author discretises the given master equation in time and solves the resulting system of linear equations. In [Di +01], the authors prove that the first passage time density satisfies a simpler Volterra integral equation of the second kind but the proposed algorithm again requires numerical integration which slows down the procedure. An important question for determining first passage time distributions from integral equations is whether the distribution is uniquely determined by these equations. Only in this case, numerical approaches are useful. Uniqueness results for Volterra integral equations can for example be found in [JKV09], Theorem 3.

There has also been use of Fredholm type integral equations but more rarely. In [She67] and [Nov81], Fredholm type integral equations help to formulate known integral transformations of F and allow to determine moments of τ as well as the asymptotic behaviour of F. In [Dan00], linear boundaries with perturbations are considered and a Fredholm type integral equation is used to derive a density of the FPT distribution. Moreover, [JKV09] generalises the approach from [Pes02] and links the Volterra type integral equations to Fredholm type equations of the form (cf. Theorem 5, [JKV09])

$$\int_0^\infty e^{-\alpha b(s)-\frac{\alpha^2}{2}s}F(ds)=1$$

for all $\alpha \in \mathbb{C}$ with $|\arg(\alpha)| \leq \pi/2$. In [CMW19], an algorithm for solving Fredholm integral equations is presented and applied to the FPT problem. To the best of the author's knowledge, the important question whether Fredholm integral equations uniquely determine the first passage time distribution has not been treated.

In recent years, numerical approaches have rather concentrated on Monte Carlo methods. In [WP97], an explicit formula for the probability of a Brownian motion crossing a piecewise linear boundary is obtained. This formula is then used to approximate the probability that a Brownian motion crosses curves that are uniform limits of piecewise linear functions. The formula for the latter probability entails integrals, which the authors chose to approximate using Monte Carlo simulation. This method was improved in [PW01], where the authors provide a better upper bound for the approximation error and extend the method to two-sided boundaries. In [BN05] the method is generalised by approximating not only via piecewise linear boundaries but also more general functions and by giving an exact upper bound for the approximation error. The paper [Pöt12] again uses piecewise linear boundaries as an approximation but bases the Monte Carlo estimators on an m-dimensional Brownian motion. In [JW17], the method from [WP97] is extended to piecewise linear boundaries which can be discontinuous.

Other numerical methods include Monte Carlo methods relying on nested algorithms to account for undetected crossing in between discretisation steps ([GSZ01]) or Monte Carlo methods using simulation of three-dimensional Brownian bridges ([IK11]). The most important drawbacks of Monte Carlo methods are the extensive computation time and the problem of undetected crossings in between discretisation steps. There are also numerical methods relying on crossings of horizontal lines ([HT16]), a connection between Brownian motion and Bessel processes ([DHM17]), acceptance-rejection methods ([HZ20]) and eigenfunction expansions ([Nil20]). But all of these approaches either require extensive computation time or approximate the crossing boundary.

Although there has been research into the FPT problem for decades, general solutions or solution methods have evaded discovery or invention and there remains a veritable zoo of approaches.

1.2 The method of images

The method of images was first introduced by [Dan82]. The following sections, however, for the most part closely follow Lerche's "Boundary Hitting of Brownian Motion" ([Ler86]) which still represents the most complete introduction to the method of images. The notation has been adapted to a modernised version.

The method of images is based on the following idea (cf. [Ler86], p.18): imagine a standard Brownian motion endowed with unit mass and more Brownian motions with starting points greater than 0. These starting points are chosen according to some positive measure μ and the corresponding Brownian motions are assigned negative mass according to μ. Then, we observe the "superposition" of the Brownian motion with unit mass and the Brownian motions with negative mass, i.e., we observe the points in space-time, where the Brownian motions "cancel out". The superposition can then be represented as a boundary b for which the first hitting time distribution F can be given in terms of μ (see Proposition 1.3 below).

In order to make this more rigorous, assume that μ is a positive, σ-finite measure such that for all $\varepsilon > 0$

$$\int_0^\infty \phi(\sqrt{\varepsilon}\theta)\mu(d\theta) < \infty,$$

where ϕ is the density of the standard normal distribution. Moreover, define

$$h(t,x) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - \int_0^\infty \frac{1}{\sqrt{t}}\phi\left(\frac{x-\theta}{\sqrt{t}}\right)\mu(d\theta)$$

and

$$r_\mu(t,x) = r(t,x) = \int_0^\infty r_\theta(t,x)\mu(d\theta)$$

where $\int_0^\infty = \int_{(0,\infty)}$ and

$$r_\theta(t,x) = \exp\left(-\frac{\theta^2}{2t} + \frac{\theta x}{t}\right).$$

We will in most cases write r instead of r_μ whenever it is clear which measure μ we are referring to. Then, we have

$$h(t,x) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)(1 - r(t,x)).$$

Now denote by $b : \mathbb{R}_+ \to \mathbb{R}$ the solution to $h(t,x) = 0$ or equivalently to

$$r(t,x) = 1,$$

i.e., b is such that $r(t,b(t)) = 1$. It can be shown that a unique solution b exists to this equation: Note that for all $t > 0$ the function $x \mapsto r(t,x)$ is continuous and monotone increasing. Moreover, we find that $r(t,x) \to 0$ as $x \to -\infty$ and $r(t,x) \to \infty$ as $x \to \infty$. Then, the intermediate value theorem tells us that there exists a $b(t)$ such that $r(t,b(t)) = 1$. Since $x \to r(t,x)$ is strictly monotone, we find that $b(t)$ is unique.

All b satisfying this equation for some μ can be shown to have certain properties. The following lemma is a slight extension of Lemma 1.1, [Ler86].

Lemma 1.1. *Let b satisfy $r(t,b(t)) = 1$ for all $t > 0$. Then,*

(i) b is analytic.

(ii) $b(t)/t$ is monotone decreasing.

(iii) b is concave.

Proof. (i) We have that $(t,x) \to r(t,x)$ is analytic, $\frac{\partial r}{\partial x} > 0$ and b fulfils $r(t,b(t)) = 1$.

Then, the claim follows directly from the theorem on implicit functions in the version for analytic functions (e.g., cf. [KK83], Chapter 0, Theorem 8.6).

(ii) The claim follows again directly from $r(t, b(t)) = 1$, since $\exp(-\theta^2/(2t))$ is increasing in t, so $\exp(\theta b(t)/t)$ has to be decreasing in t.

(iii) Consider the the derivative of $r(t, b(t)) = 1$ with respect to t. Then, we find

$$0 = \int_0^\infty \left(\frac{\theta^2}{2t^2} + \theta \frac{b'(t)t - b(t)}{t} \right) \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t} \right) \mu(d\theta).$$

Multiplying by t^2 and taking the derivative with respect to t again, we find that

$$\begin{aligned} 0 &= \int_0^\infty \left(\theta b''(t)t + \frac{\theta^2}{t^2} \left(\frac{\theta}{2} + b'(t)t - b(t) \right)^2 \right) \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t} \right) \mu(d\theta) \\ &= b''(t) \cdot \underbrace{\int_0^\infty \theta t \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t} \right) \mu(d\theta)}_{\geq 0} \\ &\quad + \underbrace{\int_0^\infty \left(\frac{\theta^2}{t^2} \left(\frac{\theta}{2} + b'(t)t - b(t) \right)^2 \right) \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t} \right) \mu(d\theta)}_{\geq 0} \end{aligned}$$

and therefore $b''(t) \leq 0$ which means that b is concave.

□

The following result (which is due to Lemma 1.2, [Ler86]) tells us how b behaves for $t \searrow 0$.

Lemma 1.2. *Let b satisfy $r(t, b(t)) = 1$ for all $t > 0$. Let $\theta^* = \inf\{\theta \,|\, \mu(0, \theta] > 0\} \geq 0$. Then,*

$$\lim_{t \searrow 0} b(t) = \frac{\theta^*}{2}.$$

Proof. We show that

$$\liminf_{t \searrow 0} b(t) \geq \frac{\theta^*}{2} \qquad \text{and} \qquad \limsup_{t \searrow 0} b(t) \leq \frac{\theta^*}{2}$$

from which the claim follows. The proof is by contradiction. So, first assume that there exists $0 < \varepsilon < 1$ such that

$$\liminf_{t \searrow 0} b(t) \leq \frac{\theta^* - \varepsilon}{2}.$$

Then, there exists a sequence $t_i \searrow 0$ such that

$$\begin{aligned}
1 &= r(t_i, b(t_i)) \\
&= \int_{\theta^*}^{\infty} \exp\left(-\frac{\theta^2}{2t_i} + \frac{\theta b(t_i)}{t_i}\right) \mu(d\theta) \\
&\leq \int_{\theta^*}^{\infty} \exp\left(-\frac{\theta\theta^*(t_i^{-1} - \varepsilon)}{2} + \frac{\theta(\theta^* - \varepsilon)}{2t_i}\right) \exp\left(-\frac{\varepsilon\theta^2}{2}\right) \mu(d\theta) \\
&\leq \int_{\theta^*}^{\infty} \exp\left(-\frac{\varepsilon\theta}{2}(t_i^{-1} - \theta^*)\right) \mu(d\theta) \xrightarrow[i \to \infty]{} 0.
\end{aligned}$$

This is a contradiction and therefore $\liminf_{t \searrow 0} b(t) \geq \frac{\theta^*}{2}$. In a similar way, one can show that $\limsup_{t \searrow 0} b(t) \leq \frac{\theta^*}{2}$. □

Now assume that W is a standard Brownian motion starting in 0. As before, let τ be the first passage time of W to b, i.e.,

$$\tau := \inf\{t \geq 0 \,|\, W_t \geq b(t)\}.$$

Moreover, let us fix $t_0 > 0$, and $x_0 < b(t_0)$. If μ is chosen in a suitable way, the function r can be used to approximate the hitting probability of W to b as shown in the following proposition that is a slight reformulation and extension of the well-known results which are presented in Theorem 1.1, [Ler86]. We will from now on not necessarily assume that b fulfils $1 = r(t, b(t))$ for a given measure μ.

Proposition 1.3. *Let $b : \mathbb{R}_+ \to \mathbb{R}$ be a boundary and μ a measure on $\mathbb{R}_+$. Assume that for certain $\delta_1 \in [0, 1)$, $\delta_2 \geq 0$ we have*

$$1 - \delta_1 \leq r(t, b(t))^{-1} \leq 1 + \delta_2, \qquad t \in (0, t_0]. \tag{1.3}$$

Then, for all $s \in (0, t_0]$ and all $x < b(s)$ it holds

$$(1 - \delta_1) r(s, x) \leq P_{(0,0)}(\tau \leq s | W_s = x) \leq (1 + \delta_2) r(s, x).$$

In particular, if $r(t, b(t)) = 1$ for all $t \in (0, t_0]$, it holds for all $s \in (0, t_0]$ and all $x < b(s)$

that

$$r(s,x) = P_{(0,0)}(\tau \le s | W_s = x).$$

Proof. This proof is based on the alternative proof of Theorem 1.1 in [Ler86] on pp. 40 f. Let $s \in (0, t_0]$ and $y < b(s)$. To make the above notation precise, denote by $P^{(s,x)}_{(0,y)}$ the law of a Brownian bridge from (s,x) to $(0,y)$ and by σ the first hitting time of the Brownian bridge of b. Here, we consider W as a process in reversed time. Then,

$$P_{(0,0)}(\tau \le s | W_s = x) = P^{(s,x)}_{(0,0)}(\sigma > 0).$$

Then, the process $M_t := r(t, W_t)$, $t \in (0,s]$, is a martingale under $P^{(s,x)}_{(0,0)}$ (but not on $[0,s]$ as $M_0 = \lim_{t \searrow 0} r(t, W_t)) = 0$). If we assume that the function r fulfils

$$1 - \delta_1 \le r(t, b(t))^{-1}, \quad t \in (0, t_0], \tag{1.4}$$

we obtain the estimate

$$P_{(0,0)}(\tau \le s | W_s = x) = \int_{\{\sigma > 0\}} \frac{M_\sigma}{r(\sigma, W_\sigma)} dP^{(s,x)}_{(0,0)} \ge (1 - \delta_1) \int_{\{\sigma > 0\}} M_\sigma dP^{(s,x)}_{(0,0)}.$$

Now, note that for all $n \in \mathbb{N}$ we have by the optimal stopping theorem (e.g., cf. [RY99], Chapter II, Theorem 3.2) that

$$E(M_{\sigma \vee \frac{1}{n}}) = E(M_{t_0}).$$

Moreover, we have that $\max_{t \in (0,s]} r(t, b(t))$ is an integrable majorant of $M_{\sigma \vee \frac{1}{n}}$. Recalling $M_0 = 0$, we find

$$\begin{aligned} \int_{\{\sigma > 0\}} M_\sigma dP^{(s,x)}_{(0,0)} &= E\left(M_\sigma 1_{\{\sigma > 0\}} + M_0 1_{\{\sigma = 0\}}\right) = E\left(\lim_{n \to \infty} M_{\sigma \vee \frac{1}{n}}\right) \\ &= \lim_{n \to \infty} E\left(M_{\sigma \vee \frac{1}{n}}\right) = E(M_s) = r(s,x), \end{aligned}$$

so that

$$P_{(0,0)}(\tau \le s | W_s = x) \ge (1 - \delta_1) r(s,x).$$

The other inequality follows the same way. ∎

If b fulfils $r(t, b(t)) = 1$, we can now recover the c.d.f. $F(t) = P(\tau \leq t)$ from $r(t, x) = P(\tau \leq t | W_t = x)$ by integration. We obtain

$$\begin{aligned} F(t) &= P(W_t \geq b(t)) + \int_{-\infty}^{b(t)} P(\tau \leq t | W_t = x) p_t(0, x) dx \\ &= 1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \int_0^\infty 1 - \Phi\left(\frac{\theta - b(t)}{\sqrt{t}}\right) \mu(d\theta). \end{aligned}$$

Then, the density can be recovered from this as

$$f(t) = \frac{1}{2t^{3/2}} \int_0^\infty \theta \phi\left(\frac{\theta - b(t)}{\sqrt{t}}\right) \mu(d\theta).$$

The method of images is traditionally applied as follows: One starts with a measure μ with associated function $r = r_\mu$ and then considers a curve b that is the implicit solution to the equation

$$r(t, x) = 1,$$

i.e., b is chosen such that $r(t, b(t)) = 1$ for all $t \in (0, t_0]$ for some $t_0 > 0$. Proposition 1.3 then yields that, under certain assumptions, r describes the hitting probability of the curve b.

Remark 1.4. Using this approach, one can generate curves with explicit hitting probabilities. The easiest examples are linear boundaries $b(t) = a + mt$, where $a > 0, b \in \mathbb{R}$, which are generated by $\mu = \exp(-2am)\delta_{2a}$ (cf. [Ler86], Example 1, p. 27).

Therefore, the method of images can be seen as a method to generate certain curves with explicitly given hitting distribution which is why the method has rightly been celebrated. However, explicit solutions remain scarce apart from the linear boundary (cf. [Ler86], pp. 27 ff. for more explicit examples).

In practice, therefore, the more relevant problem seems to be the inverse: given a curve b, does there exist a measure μ such that the method of images applied to μ yields b? This question was already asked in [Ler86], p.40, and was more recently raised again in [Kah08], p.1439 (in a slightly modified setting). This is also called the *inverse method of images*. To the best of our knowledge, there is no answer yet.

We approach this question in this thesis by methods of linear program. In order to do this, we first make the following definition.

Definition 1.5. A function $b : \mathbb{R}_+ \to \mathbb{R}$ is called *representable (in the sense of the* *method of images)* if there exists a positive, σ-finite measure μ on $\mathbb{R}_+$ such that b is the solution of

$$\int_0^\infty r_\theta(t,x)\mu(d\theta) = r(t,x) = 1,$$

i.e., if $r(t,b(t)) = 1$. In that case, we call μ the *representing measure* of b and say μ *represents* b.

With this definition in mind, we can formulate the following theorem on the existence of representing measures, where we use the analyticity of b. Denote by $\mathcal{M}^+[2b(0),\infty)$ the set of all (positive) regular, σ-finite Borel measures on $[2b(0),\infty)$.

Theorem 1.6. *Let $\mu \in \mathcal{M}^+(\mathbb{R}_+)$. Let b be analytic and let $b(0) > 0$. Assume one of the following conditions holds:*

(i) There exists a sequence $t_1, t_2, \ldots$ with accumulation point $\tilde{t} \in (0, t_0)$ such that

$$\int_0^\infty r_\theta(t_n, b(t_n))\mu(d\theta) = 1 \qquad \text{for all } n \in \mathbb{N}.$$

(ii) There exists a strictly increasing sequence $t_1, t_2, \ldots \nearrow t_0$ such that

$$\int_0^\infty r_\theta(t_n, b(t_n))\mu(d\theta) = 1 \qquad \text{for all } n \in \mathbb{N}$$

and there exists some $t^* > t_0$ such that

$$\int_{[0,\infty)} \exp\left(-\frac{\theta^2}{2t^*}\right)\mu(d\theta) < \infty.$$

Then, b is represented by μ.

Proof. Note that b is analytic on $(0, t_0]$ (cf. Lemma 1.1 above). Then, the claim follows directly from assertion (i) by use of the identity theorem for analytic functions (e.g., cf. Theorem III.3.2, [FB09]). In order to prove that the claim follows from assertion (ii), consider for $t > t_0$

$$\begin{aligned} r(t,b(t)) &= \int_{[2b(0),\infty)} \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t}\right)\mu(d\theta) \\ &= \int_{[2b(0),K]} \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t}\right)\mu(d\theta) + \int_{(K,\infty)} \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t}\right)\mu(d\theta) \end{aligned}$$

where we choose $K := \sup_{t\in(t_0,\tilde{t}]} \frac{2t^*b(t)}{t^*-t}$ for some $\tilde{t} \in (t_0, t^*)$. Then, K is finite and μ is σ-finite by assumption and therefore the first integral is finite. For the second integral, we note that a short calculation shows that for $\theta > K$ we have that for all $t \in (0, \tilde{t})$

$$\exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t}\right) \leq \exp\left(-\frac{\theta^2}{2t^*}\right)$$

and so the second integral is finite by assumption. Thus, $t \mapsto r(t, b(t))$ is an analytic function on $(0, \tilde{t})$. Now, claim (ii) follows directly from the identity theorem for analytic functions. □

Another question is how to find the representing measure numerically. There were some attempts over the last years to find methods for a numerical approximation of a representing measure μ. We will look at these results and present a new method in Chapter 3.

1.3 Extensions of the method of images

In [Dan96], the method of images is heuristically extended to include signed measures μ, i.e., to allow negative mass on $[2b(0), \infty)$. However, if signed measures are allowed, the implicit boundary b does not need to be unique anymore and therefore signed measures need to be treated with care. [Dan96] investigates a square-root boundary numerically but runs into the problem of either having a bad fit of the approximation due to a too coarse discretisation or numerical instabilities due to a too fine discretisation.

In [Kah08], an extension of the method of images is considered. Before the author gives the extension, another approach to obtain first hitting probabilities using Schwartz distributions was given. Consider the first hitting time τ_u of a standard Brownian motion started in $u \in \mathbb{R}$ to some boundary b. Let μ be a Schwartz distribution or a σ-finite measure, and let

$$U(u, t; \mu) = \begin{cases} \int \Phi\left(\frac{u-v}{\sqrt{t_0 - t}}\right) \mu(dv), & \text{for } 0 \leq t < t_0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, [Kah08] shows (cf. Theorem 2.5) that if μ_n is a sequence of Schwartz distributions with $U(b(t), t; \mu_n) \to 1$ as $n \to \infty$ (and if some other regularity assumptions hold), then, $U(u, 0; \mu_n) \to P(\tau_u < t_0)$ as $n \to \infty$. Then, the author gives an extension of the method of images to Schwartz distributions. In particular, if μ is a Schwartz distribution such that

$$\int \exp\left(-\frac{\theta^2}{2t} + \frac{\theta b(t)}{t}\right)\mu(d\theta) = 1$$

for all $t \in (0, t_0]$, then, (cf. [Kah08], Theorem 6.2)

$$P(\tau < t_0) = \Phi\left(\frac{-b(t)}{\sqrt{t_0}}\right) + U(b(t_0), 0; \mu).$$

In [AP10], the authors introduce the mapping $S^{(\beta)}$ as

$$S^{(\beta)} : f \mapsto (1 + \beta\cdot) f\left(\frac{\cdot}{1 + \beta\cdot}\right)$$

for $\beta \in \mathbb{R}$. They show (Theorem 3.1, [AP10]) that the first crossing density of a boundary b and the first crossing density of a boundary $S^{(\beta)}(b)$ are functional transforms of each other. The authors connect this mapping to the method of images and show that if μ is the measure representing b then the boundary $S^{(\beta)}(b)$ is represented by $\mu^{(\beta)}$ where $\mu^{(\beta)}(d\theta) := \exp(-\beta\theta^2/2) \cdot \mu(d\theta)$.

Finally, we present some results by [Zip16] (note that the paper comes chronologically before the paper [Zip13] which we will quote later on and which builds on [Zip16]). The paper investigates boundaries b generated by the method of images by linking its behaviour for large t to the Laplace transform $\tilde{\mu}$ of the representing measure μ. To make this more precise, define

$$\beta_\infty = \lim_{t\to\infty} b'(t),$$
$$\alpha_\infty = \lim_{t\to\infty} b(t) - tb'(t),$$

where we call β_∞ the *limiting slope* and α_∞ the *limiting intercept* of b. Note that $\beta_\infty \in [-\infty, \infty)$ and $\alpha_\infty \in (-\infty, \infty]$ since b is concave. Then, [Zip16] shows two main results for the limit behaviour of b:

- (cf. Theorem 1, [Zip16]) $\beta_\infty < \infty$ if and only if $\tilde{\mu}(s)$ is convergent for $s \geq -\beta_\infty$ and $\tilde{\mu}(-\beta_\infty) = 1$.

- (cf. Theorem 2, [Zip16]) b is asymptotically linear with limiting slope β_∞ and limiting intercept α_∞ if and only if $\tilde{\mu}(-\beta_\infty) = 1$, $\tilde{\mu}(s)$ is twice differentiable for

$s \geq -\beta_\infty$ and

$$\alpha_\infty = -\frac{1}{2}\frac{\tilde{\mu}''(-\beta_\infty)}{\tilde{\mu}'(-\beta_\infty)}.$$

Consequently, [Zip16] summarises that the limiting behaviour falls into one of three categories: the limiting slope β_∞ goes to $-\infty$ if and only if $\tilde{\mu}$ is not convergent, $\beta_\infty < \infty$ if and only if $\tilde{\mu}$ is convergent and b is even asymptotically linear if and only if $\tilde{\mu}$ is convergent and twice differentiable (cf. Table 1, [Zip16]).

Finally, let us give a short outlook on the method of images for two-sided boundaries. In [Ler86], Lerche notes that the method indeed extends to two-sided boundaries. Following this lead, we denote by b_1 the lower bound and by b_2 the upper bound for which we are interested in the hitting distribution of a Brownian motion W. In particular, we assume $b_1(t) \leq b_2(t)$ for all $t > 0$, $b_1(0) < W_0 < b_2(0)$ and define in this case $\tau := \inf\{t \geq 0 : W_t \notin (b_1(t), b_2(t))\}$. Let μ be a measure on $\mathbb{R}$ with $\mu(\{0\}) = 0$. If we have both $r(t, b_1(t)) = 1$ and $r(t, b_2(t)) = 1$ for all $t \leq t_0$, where

$$r(t,x) = \int_{-\infty}^{\infty} r_\theta(t,x)\mu(d\theta),$$

then, we can again obtain the conditional distribution of τ or at least bounds on it. The following proposition is the analogue of Proposition 1.3 above.

Proposition 1.7. *Assume the above set-up with two boundaries b_1 and b_2 and assume that there exist $\delta_1 \in [0,1)$, $\delta_2 \geq 0$ such that for $t \in (0, t_0]$ we have*

$$1 - \delta_1 \leq r(t, b_1(t))^{-1} \leq 1 + \delta_2 \quad \text{and} \quad 1 - \delta_1 \leq r(t, b_2(t))^{-1} \leq 1 + \delta_2.$$

Then, for all x_0 with $b_1(t_0) < x_0 < b_2(t_0)$ we find

$$(1-\delta_1)r(t_0,x_0) \leq P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0) \leq (1+\delta_2)r(t_0,x_0).$$

In particular, if $r(t, b_1(t)) = 1$ and $r(t, b_2(t)) = 1$ for all $t \in (0, t_0]$, then it holds that

$$r(t_0,x_0) = P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0).$$

Proof. Denote by $P_{(0,\zeta)}^{(t_0,x_0)}$ the measure of a Brownian bridge from (t_0, x_0) to $(0, \zeta)$ with first hitting time σ of b_1 or b_2. As before, we consider W as a process in reversed time. Then, the proof is analogous to the proof of Proposition 1.3. $\square$

With this proposition in mind, we can again compute the distribution function F of τ as

$$F(t) = 1 - \Phi\left(\frac{b_2(t)}{\sqrt{t}}\right) + \Phi\left(\frac{b_1(t)}{\sqrt{t}}\right) + \int_{-\infty}^{\infty} \Phi\left(\frac{\theta - b_1(t)}{\sqrt{t}}\right) - \Phi\left(\frac{\theta - b_2(t)}{\sqrt{t}}\right) \mu(d\theta).$$

See Lemma A.2 in the appendix for a detailed derivation of the cumulative distribution function. Moreover, [Ler86] notes that the results for the method of images carry over to the two-sided case. In particular, we have that μ only puts mass on $(-\infty, 2b_1(0)] \cup [2b_2(0), \infty)$ (analogous to Lemma 1.2).

1.4 Related methods

There are two prominent examples of methods related to the method of images: the method of weighted likelihoods and the tangent approximation. We will introduce these methods and show their connection to the method of images. This section is again based on [Ler86].

The method of weighted likelihoods was first introduced by [RS70]. Its close connection to the method of images was shown by [Ler86]. Consider a positive, σ-finite measure μ on $\mathbb{R}_+$ and let

$$q(t, x) = \int_0^{\infty} \exp\left(\theta x - \frac{1}{2}\theta^2 t\right) \mu(d\theta).$$

Consider the equation

$$q(t, x) = 1$$

for $t \geq t_0$ for some $t_0 > 0$ and denote its implicit solution by $\beta(t)$, i.e., β fulfils $q(t, \beta(t)) = 1$. It can be shown that β has similar properties as the boundary b in the method of images. In particular, β can be shown to be monotone increasing, concave, and infinitely often continuously differentiable (cf. [Ler86], p.34). Define now T as the first hitting time of a standard Brownian motion W to β after t_0, i.e.,

$$T = \inf\{t > t_0 \,|\, W_t \geq \beta(t)\}.$$

Then, it can be shown for $x_0 < \beta(t_0)$ that (cf. Theorem 2.1, [Ler86])

$$P(t_0 \leq T < \infty | W_{t_0} = x_0) = q(t_0, x_0).$$

Integration yields that (cf. Corollary 2.1, [Ler86])

$$P(t_0 \leq T < \infty) = 1 - \Phi\left(\frac{\beta(t_0)}{\sqrt{t_0}}\right) + \int_0^\infty \Phi\left(\frac{\beta(t_0) - \theta t_0}{\sqrt{t_0}}\right) \mu(d\theta).$$

Now, recall that the boundary b from the method of images is the implicit solution of the equation

$$1 = r(s, y) = \int_0^\infty \exp\left(\frac{\theta y}{s} - \frac{\theta^2}{2s}\right) \mu(d\theta)$$

whereas β is the implicit solution of

$$1 = q(t, x) = \int_0^\infty \exp\left(\theta x - \frac{1}{2}\theta^2 t\right) \mu(d\theta).$$

Using the time-inversion transformation

$$x = \frac{y}{s}, t = \frac{1}{s}$$

we find that (cf. [Ler86], (2.11)-(2.14))

$$\frac{b(s)}{s} = \beta(t) \qquad \text{and} \qquad b(s) = \frac{\beta(t)}{t}$$

as well as

$$q\left(\frac{1}{t}, \frac{x}{t}\right) = r(t, x).$$

The methods are therefore equivalent up to time inversion. Moreover, it can be shown that the method of images and the method of weighted likelihoods use harmonic functions for the forward and backward diffusion equation, i.e., different characteristics of the same objects (cf. Theorems 1.3 and 2.2 in [Ler86]).

The close connection between these two methods means that they are essentially the same. Thus, both have the same advantages and disadvantages and while they in principle offer a way to find the hitting time distribution for a Brownian motion crossing a boundary, few explicit solutions are available.

The second method that is connected to the method of images is the tangent approximation. A formal introduction of the method requires a lot of technical considerations (for a detailed introduction and a derivation from the method of images, see [Ler86], Chapters 3 and 4). Here, we will limit ourselves to a shorter, more intuitive account. The idea behind the tangent approximation is the following: Consider for some point t the tangent d_t to the concave boundary b in t, i.e., consider $d_t(s) = b(t) + (s - t)b'(t)$. Then, it is intuitively clear that for every t the density of the hitting time for b can be approximated from above with the density of the hitting time for d. Or, more precisely, it holds

$$f_b(t) \leq f_{d_t}(t)$$

where f_b and f_d denote the first hitting time densities for b and d, respectively. The argument for this is that d lies above b since b is concave. But then, a standard Brownian motion started in 0 has a higher probability of hitting b before t than hitting d before t and has therefore "used" more of its hitting probability for b than it has for d. This can also be formally proven (cf. the deduction of formula (3.2) in [Ler86]).

The advantage of this approach is that the density of the hitting time for a linear function (in this case, the tangent) is well-known and is given by the Bachelier-Lévy formula (cf. Formula (1.2) above). Now, it can be shown that (cf. Theorem 4.1, [Ler86]) $f_{d_t}(t)$ converges uniformly to $f_b(t)$ on intervals of the form $(0, t_1]$ if b increases to infinity. The same result was previously shown for some class of boundaries b as $t \to 0$ by [Str67]. Note however that the tangent approximation is a purely local method, as approximating the first hitting time density of b for each t with the first hitting time density of the tangent d in the same point t will not yield a probability density except for the case when b is a linear function (cf. [Ler86], page 61) in which case we can revert to the Bachelier-Lévy formula.

The tangent approximation was improved upon by several authors. In [Dur85] and [Dur92], the first-passage density is given as an infinite series whose first term is the tangent approximation. The author gives some examples where he shows higher accuracy of his improved method compared to the classical tangent approximation. In [RS95], the hazard rate tangent approximation is introduced. This method makes use of the hazard rate

$$r_\tau(t) := \lim_{\varepsilon \searrow 0} \frac{P(\tau \leq t + \varepsilon \,|\, \tau > t)}{\varepsilon}$$

and approximates the hazard rate of the boundary b at some time t with the hazard rate of the tangent of b in t. The method from [RS95] shows faster computation time than the method set out in [Dur92] but [RS95] notes that his method is less accurate if [Dur92] uses sufficiently many terms of his series. Finally, [Dan96] suggests a change to the tangent approximation by replacing the linear tangent with a tangential curve of higher order which cannot be given explicitly but for which an explicit first crossing density can be derived. The author provides numerical approximations showing better performance than the standard tangent approximation.

It is clear from what we have seen above that there are several open questions concerning the first passage time problem. Most prominent among these is probably: For a given curve b does there exist a measure μ such that b is representable in the sense of the method of images? And how to obtain μ numerically in an efficient way? And while there are different algorithms to obtain first hitting time distributions via Fredholm equations, uniqueness of the distribution obtained via the Fredholm equation remains an open question.

1.5 Structure and contribution of this thesis

Let us set out the structure and contribution of this thesis. In Chapter 2 we present a new infinite linear programming approach to the inverse method of images. We give two different linear programs and quickly find that these provide upper and lower bounds for the hitting time distribution (Theorem 2.1). We continue to investigate these programs by formulating the according dual programs and find strong duality results for both set-ups (cf. Theorems 2.7 and 2.13). The proof requires compactification of the underlying spaces.

Then, we use these results to obtain sufficient conditions for representability of a given concave, analytical boundary b. One of the sufficient conditions makes use of the distribution $\bar{\lambda}$ of the last hitting time to b of a Brownian bridge W running from 0 at time 0 to x_0 at time t_0 (cf. Theorem 2.15). We will discover a possible connection between this (conditional) distribution of the last hitting time and the (conditional) distribution of the first hitting time as the solutions of the dual programs introduced before. Another sufficient condition exploits the analyticity of b and general properties of attainable measures in the respective dual problems (cf. Theorem 2.19). For both theorems, the sufficient conditions can be checked numerically.

In Chapter 3, we first look at existing computational methods for the inverse method of images and see that while the methods are easy to implement, they require discretisation of both time and space and no convergence results are available. We give convergence results for our linear programs, where we only discretise the space axis (cf. Propositions 3.1 and 3.2). We formulate a new algorithm based on these discretised versions of the linear programs and give error bounds for the approximation.

In Chapter 4, we give a short introduction of the concept of European options representing American options (in particular, the American put). The question of existence of such representing European options is still open. We show that if a European pay-off f represents some American pay-off g, then the stopping boundary b_g associated with g is representable in the sense of the method of images (cf. Theorem 4.1). In particular, this gives rise to a new proof idea for showing existence of a representing European option given an American option.

Finally, Chapter 5 deals with a well-known set of Fredholm integral equations that characterise the first hitting time distribution. We show that this characterisation is indeed unique (cf. Theorem 5.3).

Chapter 2

A Linear Programming Approach for the Inverse Method of Images

We investigate two linear programs that give upper and lower bounds for the first hitting time distribution of a boundary b. Then, we show classical duality results for these programs and finally use these duality results to give sufficient conditions under which a representing measure μ for b exists.

2.1 Two set-ups of linear programs

We fix a real-valued, analytic, concave boundary $b : [0, \infty) \to \mathbb{R}$ with finite slope at 0, i.e., $|b'(0)| < \infty$, and $b(0) > 0$. Note that we do not assume that $t \mapsto b(t)/t$ is monotone decreasing (the second property from Lemma 1.1 above). Indeed, this property already follows from our assumptions: Since b is concave, we have that $b''(t) \leq 0$ for all t. Using this and the fact that $b(0) > 0$ we find with the help of the Taylor expansion that

$$0 < b(0) \leq b(t) + b'(t)(0 - t) = b(t) - b'(t)t \qquad \Leftrightarrow \qquad b'(t)t - b(t) \leq 0.$$

Hence, we find that

$$\frac{d}{dt}\frac{b(t)}{t} = \frac{b'(t)t - b(t)}{t^2} \leq 0$$

which means that $t \mapsto b(t)/t$ is decreasing. Moreover, we fix a point (t_0, x_0) with $t_0 > 0$ and $x_0 < b(t_0)$. In the light of Proposition 1.3, it seems natural to consider either the following linear problem

$$\begin{aligned} &\text{maximise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in \mathcal{M}^+[2b(0), \infty), \\ &&& \int r_\theta(t, b(t))\mu(d\theta) \leq 1 \text{ for any } t \in (0, t_0] \end{aligned} \tag{D_1}$$

where we approximate the measure μ representing b "from below". Alternatively, we could also consider the following linear program

$$\begin{aligned} &\text{minimise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in \mathcal{M}^+[2b(0), \infty), \\ &&& \int r_\theta(t, b(t))\mu(d\theta) \geq 1 \text{ for any } t \in (0, t_0] \end{aligned} \tag{P_2}$$

where we approximate the measure μ representing b "from above". We omitted the bounds of integration as we integrate over the whole space in each case. Note that due to Lemma 1.2 it is enough to consider measures μ on $[2b(0), \infty)$ for a given boundary b. The programs are linear as the objective functions as well as the constraints are linear in the optimisation variable μ. However, note that both programs are infinite-dimensional.

The labelling of the first program as (D_1) and the second as (P_2) may at first glance be confusing. The notation will make more sense down the line when we consider the associated formal dual problems and weak and strong duality.

The reader may rightly ask herself which program to prefer over the other, i.e., whether there is a "natural" or "better" choice for one or the other. While we need both programs in order to find sufficient conditions for a representing measure μ to exist, the most immediate use is that admissible solutions to these programs lead to lower and upper bounds for the probability $P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0)$ as the following theorem shows.

Theorem 2.1. *For each (D_1)-admissible μ_1, we find that*

$$r_{\mu_1}(t_0, x_0) := \int r_\theta(t_0, x_0)\mu_1(d\theta) \leq P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0)$$

and for each (P_2)-admissible solution μ_2 that

$$r_{\mu_2}(t_0, x_0) := \int r_\theta(t_0, x_0)\mu_2(d\theta) \geq P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0)$$

Moreover, if there exists a representing measure μ^*, i.e.,

$$\int r_\theta(t, b(t))\mu^*(d\theta) = 1 \text{ for any } t \in (0, t_0],$$

then μ^* is a maximiser in (D_1) and a minimiser in (P_2).

Proof. We will use Proposition 1.3 to prove this statement. Let μ_1 be a (D_1)-admissible solution. Then,

$$r_{\mu_1}(t, b(t)) = \int r_\theta(t, b(t))\mu_1(d\theta) \leq 1 \text{ for any } t \in (0, t_0].$$

So, we have that $r(t, b(t))^{-1} \geq 1$ for all $t \in (0, t_0]$ and so we can choose the constant δ_1 in Proposition 1.3 to be 0. We then find that $r(t_0, x_0) \leq P_{(0,0)}(\tau \leq t | W_{t_0} = x_0)$. The other inequality follows the same way.
If now μ^* is indeed a representing measure, i.e.,

$$\int r_\theta(t, b(t))\mu^*(d\theta) = 1 \text{ for any } t \in (0, t_0],$$

then we find again by Proposition 1.3 that

$$r(t_0, x_0) = \int r_\theta(t_0, x_0)\mu^*(d\theta) = P_{(0,0)}(\tau \leq t | W_{t_0} = x_0)$$

and so μ^* must be a maximiser in (D_1) and a minimiser in (P_2). □

This theorem tells us that the optimal solutions to these programs are candidates for a measure μ representing b. We now need to investigate the existence of optimal solutions to these problems and whether these fulfil $r(t, b(t)) = 1$ for any $t \in (0, t_0]$.

To further analyse the problems (D_1) and (P_2), we consider the associated formal dual problems. If the reader is unfamiliar with linear programming (especially in the infinite-dimensional context), she is kindly referred to [Roc70] or the more concise [Roc74]. Alternatively, we have provided the most important notions in Appendix A.4. For (D_1), the dual program is

$$\begin{aligned} &\text{minimise} && \|\lambda\| \\ &\text{subject to} && \lambda \in \mathcal{M}^+(0, t_0], \\ &&& \int r_\theta(t, b(t))\lambda(dt) \geq r_\theta(t_0, x_0) \text{ for any } \theta \in [2b(0), \infty) \end{aligned} \qquad (P_1)$$

and for (P_2), the dual program is

$$\begin{aligned} &\text{maximise} \quad \|\lambda\| \\ &\text{subject to} \quad \lambda \in \mathcal{M}^+(0, t_0], \\ &\qquad\qquad \int r_\theta(t, b(t))\lambda(dt) \le r_\theta(t_0, x_0) \text{ for any } \theta \in [2b(0), \infty), \end{aligned} \tag{D_2}$$

where $\|\lambda\|$ denotes the total variation norm of a measure λ. We now have two set-ups: either (D_1) and its dual (P_1) or (P_2) and its dual (D_2). The reader will note that in both set-ups the maximising problem gets tagged with a "D" and the minimising problem gets tagged with a "P". This is in line with the "usual" notation for linear programs where the minimising problem is often regarded as the canonical primal problem and the maximising problem is often the canonical dual problem. In particular, with this notation we will see that the canonical version of weak duality holds. Let d_1, p_1, d_2 and p_2 denote the optimal values of (D_1), (P_1), (D_2) and (P_2), respectively. Then, we will show $d_1 \le p_1$ and $d_2 \le p_2$, respectively.

In the following sections we first establish strong duality for (D_1) and (P_1) (Section 2.2), then for (P_2) and (D_2) (Section 2.3) and finally use these results to establish sufficient conditions for the existence of a representing measure μ for a given boundary b exists (Section 2.4).

2.2 Strong duality in the first set-up

In this and the following section, we rely on the method set out in [CKL22]. The reader will note the parallel structures we use but also the divergences in some proof methods, where the structure of the current problem and the structure of the problem in [CKL22] do not align. Throughout, we will use analogous notation to [CKL22] to make the comparison easier for the reader. For a more detailed comparison of the problems, see Section 4.1.

We begin by defining $\Omega := (0, t_0]$ and

$$\begin{aligned} T &: \mathcal{M}[2b(0), \infty) \to C(\Omega), & T\mu(t) &:= \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu(d\theta), \\ T' &: \mathcal{M}(\Omega) \to C_0[2b(0), \infty), & T'\lambda(\theta) &:= \int_\Omega r_\theta(t, b(t))\lambda(dt), \end{aligned}$$

where C and C_0 denote the spaces of continuous functions and continuous functions vanishing at infinity, respectively. Also, let $\langle f, \nu \rangle := \int f d\nu$ on the Cartesian products $C_0(\Omega) \times \mathcal{M}^+(\Omega)$ and $C_0[2b(0), \infty) \times \mathcal{M}^+[2b(0), \infty)$, respectively. Moreover, define $g(\theta) := r_\theta(t_0, x_0)$. We can now reformulate our programs as follows:

$$\begin{aligned} \text{maximise} \quad & \langle g, \mu \rangle \\ \text{subject to} \quad & \mu \in \mathcal{M}^+[2b(0), \infty), \\ & 1 - T\mu \in C^+(\Omega) \end{aligned} \tag{D_1}$$

and

$$\begin{aligned} \text{minimise} \quad & \|\lambda\| \\ \text{subject to} \quad & \lambda \in \mathcal{M}^+(\Omega), \\ & T'\lambda - g \in C_0^+[2b(0), \infty), \end{aligned} \tag{P_1}$$

where $\mathcal{M}^+$, C^+ and C_0^+ denote the cones of non-negative elements in the spaces $\mathcal{M}$, C and C_0. We now restrict our problems to compact sets. This enables us to show existence of optimal solutions to these restricted problems as well as show strong duality. Then, we can extend our findings and investigate strong duality between (D_1) and (P_1).

2.2.1 Strong duality of the restricted linear programs

Let the operator T and the space Ω be defined as above. Set for any $\varepsilon \in \Omega$

$$\Omega_\varepsilon := [\varepsilon, t_0].$$

We will consider this compact subset of Ω as it allows us to obtain existence of optimal measures of the programs restricted to Ω_ε. Now, we define

$$T^* : \mathcal{M}(\Omega_\epsilon) \to C_0[2b(0), \infty), \qquad T^*\lambda(\theta) := \int_{\Omega_\varepsilon} r_\theta(t, b(t))\lambda(dt).$$

Note that T^* is defined analogously to T' with the difference between the two being the areas of integration, i.e., Ω for T' and Ω_ε for T^*. Consider now the Cartesian products from before, but restricted to Ω_ε, i.e., $C(\Omega_\varepsilon) \times \mathcal{M}(\Omega_\varepsilon)$ and $C_0[2b(0), \infty) \times \mathcal{M}[2b(0), \infty)$, respectively. We will use the algebraic pairing

$$\langle f, \nu \rangle = \int f d\nu$$

as before. This algebraic pairing is finitely valued, bilinear and point separating. Moreover, if $C_0(\Omega_\varepsilon)$ and $C_0^+[2b(0),\infty)$ are endowed with the weak topologies $\sigma(C_0,\mathcal{M})$ and the spaces $\mathcal{M}(\Omega_\varepsilon)$ and $\mathcal{M}[2b(0),\infty)$ with the vague topologies $\sigma(\mathcal{M},C_0)$ induced by the algebraic pairing, then all four spaces are locally convex Hausdorff spaces and $C_0(\Omega_\varepsilon)$ is the continuous dual of $\mathcal{M}(\Omega_\varepsilon)$ and vice versa. The same holds for $C_0[2b(0),\infty)$ and $\mathcal{M}[2b(0),\infty)$. If the reader is unfamiliar with locally convex spaces, we refer her to Section A.3.

Using Fubini's theorem, it can be shown that we have

$$\langle T\mu,\lambda\rangle = \langle \mu, T^*\lambda\rangle$$

for every $\mu\in\mathcal{M}[2b(0),\infty)$ and $\lambda\in\mathcal{M}(\Omega_\varepsilon)$ and so the operators T and T^* are indeed adjoint operators. Moreover, the operators T and T^* are $\sigma(\mathcal{M},C_0)-\sigma(C_0,\mathcal{M})$-continuous (with respect to the corresponding spaces) due to Lemma A.6.

Recall that our ultimate goal is to find a measure $\mu_1\in\mathcal{M}^+[2b(0),\infty)$ solving (D_1). The approach now is to approximate this measure with a series of measures that are the solutions to problems with weaker constraints. To this end, consider

$$\begin{aligned} &\text{maximise} && \langle g,\mu\rangle \\ &\text{subject to} && \mu\in\mathcal{M}^+[2b(0),\infty), \\ & && 1-T\mu\in C^+(\Omega_\varepsilon). \end{aligned} \tag{$D_{1,\varepsilon}$}$$

Later on, we show that the ($D_{1,\varepsilon}$)-optimal elements yield a sequence whose limit is the solution of (D_1). For now, we start by considering the formal Lagrange dual problem associated to ($D_{1,\varepsilon}$)

$$\begin{aligned} &\text{minimise} && \|\lambda\| \\ &\text{subject to} && \lambda\in\mathcal{M}^+(\Omega_\varepsilon), \\ & && T^*\lambda-g\in C^+[2b(0),\infty). \end{aligned} \tag{$P_{1,\varepsilon}$}$$

Denote the optimal values of ($D_{1,\varepsilon}$) and ($P_{1,\varepsilon}$) by $d_{1,\varepsilon}$ and $p_{1,\varepsilon}$, respectively. Note that by construction we have that $0\le d_{1,\varepsilon}\le p_{1,\varepsilon}$, i.e., weak duality holds. We can confirm this with the following short calculation: Recall that T and T^* are adjoint operators. Using the constraints of the linear programs, we obtain that for any dual admissible

$\mu \in \mathcal{M}^+[2b(0), \infty)$ and any primal feasible $\lambda \in \mathcal{M}^+(\Omega_\varepsilon)$ we have

$$0 \leq \langle g, \mu \rangle \leq \langle T^*\lambda, \mu \rangle = \langle \lambda, T\mu \rangle \leq \langle \lambda, 1 \rangle = \|\lambda\|.$$

Then, we obtain by taking the supremum over all $(D_{1,\varepsilon})$-feasible μ and all $(P_{1,\varepsilon})$-feasible λ

$$d_{1,\varepsilon} = \sup_\mu \langle g, \mu \rangle \leq \inf_\lambda \|\lambda\| = p_{1,\varepsilon}$$

The following lemma confirms the existence of primal and dual attainable and optimal measures.

Lemma 2.2. *There exist a $(D_{1,\varepsilon})$-admissible $\mu_{1,\varepsilon}$ such that $d_{1,\varepsilon} = \langle g, \mu_{1,\varepsilon} \rangle$ and a $(P_{1,\varepsilon})$-admissible $\lambda_{1,\varepsilon}$ such that $p_{1,\varepsilon} = \|\lambda_{1,\varepsilon}\|$.*

Proof. Consider the following modified version of $(D_{1,\varepsilon})$ where we absorb $g(\theta) = r_\theta(t_0, x_0)$ into the measure μ

$$\begin{aligned} \text{maximise} \quad & \|\mu\| \\ \text{subject to} \quad & \mu \in \mathcal{M}^+[2b(0), \infty), \\ & 1 - T_{mod}\mu \in C^+(\Omega_\varepsilon) \end{aligned} \tag{$D_{1,\varepsilon,mod}$}$$

where

$$T_{mod}\mu(t) = \int_{[2b(0),\infty)} \frac{r_\theta(t, b(t))}{r_\theta(t_0, x_0)} \mu(d\theta)$$

for all $t \in \Omega_\varepsilon$. Note that T_{mod} is continuous. Moreover, note that this program is equivalent to $(D_{1,\varepsilon})$. Now, consider the constraint for $t = t_0$ to obtain that for any $(D_{1,\varepsilon,mod})$-admissible μ we have

$$\begin{aligned} 1 &\geq \int_{[2b(0),\infty)} \exp\left(-\frac{\theta^2}{2t_0} + \frac{\theta b(t_0)}{t_0} + \frac{\theta^2}{2t_0} - \frac{\theta x_0}{t_0} \right) \mu(d\theta) \\ &= \int_{[2b(0),\infty)} \exp\Bigg(\underbrace{\frac{\theta(b(t_0) - x_0)}{t_0}}_{\geq 0} \Bigg) \mu(d\theta) \geq \int_{[2b(0),\infty)} \mu(d\theta) = \|\mu\|. \end{aligned}$$

In particular, we find that $\|\mu\| \leq 1$ for all $(D_{1,\varepsilon,mod})$-admissible μ. Instead of solving the maximisation problem $(D_{1,\varepsilon,mod})$ we can equivalently maximise the $\sigma(\mathcal{M}, C)$-continuous mapping $\mu \mapsto \|\mu\|$ over the set

$$C_d^\varepsilon := T_{mod}^{-1}\left(1 - C^+(\Omega_\varepsilon)\right) \cap \mathcal{M}^+[2b(0),\infty) \cap B_{\mathcal{M}^+[2b(0),\infty)}(1)$$

where $B_{\mathcal{M}(\mathbb{R})}(1)$ denotes the $\sigma(\mathcal{M}, C)$-closed ball of radius 1 around 0 on $\mathcal{M}(\mathbb{R})$. We now show that C_d^ε is indeed compact. First, $1 - C^+(\Omega_\varepsilon)$ is closed as it is homeomorphic to the $\sigma(C, \mathcal{M})$-closed cone

$$C^+(\Omega_\varepsilon) = \bigcap_{\lambda \in \mathcal{M}^+(\Omega_\varepsilon)} \left\{f \in C(\Omega_\varepsilon) | \langle f, \lambda\rangle \geq 0\right\}.$$

The continuity of T_{mod} then yields that $T_{mod}^{-1}(1 - C^+(\Omega_\varepsilon))$ is $\sigma(\mathcal{M}, C)$-closed as well. Then, we can rewrite

$$\mathcal{M}^+[2b(0),\infty) = \bigcap_{f \in C_0^+[2b(0),\infty)} \{\mu \in \mathcal{M}[2b(0),\infty) | \langle f, \mu\rangle \geq 0\}$$

and so $\mathcal{M}^+[2b(0),\infty)$ is $\sigma(\mathcal{M}, C)$-closed, too. Finally, we find that $B_{\mathcal{M}^+[2b(0),\infty)}(1)$ is $\sigma(\mathcal{M}, C)$-compact due to Theorem A.7. Now, $\mu \mapsto \|\mu\|$ is upper-semi-continuous with respect to the topology $\sigma(\mathcal{M}, C)$ and thus Lemma A.5 tells us that it attains its maximal value $d_{1,\varepsilon}$ at some measure $\mu_{\varepsilon,mod} \in C_d^\varepsilon$. Finally, the optimal value $d_{1,\varepsilon}$ of ($D_{1,\varepsilon}$) is obtained at $\mu_{1,\varepsilon}$ where

$$\frac{d\mu_{1,\varepsilon}}{d\mu_{\varepsilon,mod}} = g^{-1}.$$

Now, turning our attention to ($P_{1,\varepsilon}$), we define $\tilde{\lambda} = \delta_{t_0}$ which is admissible in ($P_{1,\varepsilon}$) for all ε since we find for all $\theta \in [2b(0),\infty)$ that

$$T^*\tilde{\lambda} = \exp\left(-\frac{\theta^2}{2t_0} + \frac{\theta b(t_0)}{t_0}\right) > \exp\left(-\frac{\theta^2}{2t_0} + \frac{\theta x_0}{t_0}\right) = g(\theta)$$

as $b(t_0) > x_0$ by assumption. Thus, $\tilde{\lambda}$ is admissible and $\|\tilde{\lambda}\| = 1$. Therefore, it suffices to minimise the $\sigma(\mathcal{M}, C)$-continuous mapping $\lambda \mapsto \|\lambda\|$ over the set

$$C_p^\varepsilon := (T^*)^{-1}\left(1 - C^+[2b(0),\infty)\right) \cap \mathcal{M}^+(\Omega_\varepsilon) \cap B_{\mathcal{M}^+(\Omega_\varepsilon)}(1).$$

The $\sigma(\mathcal{M}, C)$-compactness of C_p^ε now follows along the same lines as the compactness of C_d^ε. Thus, we can again conlcude with Lemma A.5 that $\lambda \mapsto \|\lambda\|$ attains its minimum

$p_{1,\varepsilon}$ at some measure $\lambda_{1,\varepsilon} \in C_p^\varepsilon$. □

We can now prove strong duality. The Lagrange function $L_1 : \mathcal{M}(\Omega_\varepsilon)\times\mathcal{M}[2b(0),\infty) \to [-\infty,\infty]$ associated with the ($P_{1,\varepsilon}$)-($D_{1,\varepsilon}$)-duality is defined as

$$L_1(\lambda,\mu) := \|\lambda\| + \langle g,\mu\rangle - \langle T^*\lambda,\mu\rangle + \mathcal{I}_{\mathcal{M}^+(\Omega_\varepsilon)}(\lambda) - \mathcal{I}_{\mathcal{M}^+[2b(0),\infty)}(\mu),$$

where

$$\mathcal{I}_M(x) := \begin{cases} 0, & \text{if } x \in M, \\ \infty, & \text{if } x \notin M, \end{cases}$$

for any set M. We will use the following simplifications later on:

$$\begin{aligned} \sup_{\mu\in\mathcal{M}[2b(0),\infty)} \inf_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} L_1(\lambda,\mu) &= \sup_{\mu\in\mathcal{M}^+[2b(0),\infty)} \inf_{\lambda\in\mathcal{M}^+(\Omega_\varepsilon)} (\|\lambda\| + \langle g - T^*\lambda,\mu\rangle) \\ &= \sup_{\mu\in\mathcal{M}^+[2b(0),\infty)} \left(\langle g,\mu\rangle + \inf_{\lambda\in\mathcal{M}^+(\Omega_\varepsilon)} \langle \lambda, 1 - T\mu\rangle\right) \qquad (2.1) \\ &= \sup_{\substack{\mu\in\mathcal{M}^+[2b(0),\infty) \\ T\mu\le 1}} \langle g,\mu\rangle = d_{1,\varepsilon}, \end{aligned}$$

$$\begin{aligned} \inf_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} \sup_{\mu\in\mathcal{M}[2b(0),\infty)} L_1(\lambda,\mu) &= \inf_{\lambda\in\mathcal{M}^+(\Omega_\varepsilon)} \sup_{\mu\in\mathcal{M}^+[2b(0),\infty)} (\|\lambda\| + \langle g - T^*\lambda,\mu\rangle) \\ &= \inf_{\lambda\in\mathcal{M}^+(\Omega_\varepsilon)} \left(\|\lambda\| + \sup_{\mu\in\mathcal{M}[2b(0),\infty)} \langle g - T^*\lambda,\mu\rangle\right) \qquad (2.2) \\ &= \inf_{\substack{\lambda\in\mathcal{M}^+(\Omega_\varepsilon) \\ T^*\lambda\ge g}} \|\lambda\| = p_{1,\varepsilon}. \end{aligned}$$

Moreover, define the *dual value function* $v_1 : C_0(\Omega_\varepsilon) \to (-\infty,\infty]$ by

$$v_1(f) := \inf_{\mu\in\mathcal{M}[2b(0),\infty)} L_{1,\mu}^*(f),$$

where $L_{1,\mu}^*$ denotes the conjugate of the mapping $L_{1,\mu}$.

Lemma 2.3. *The dual value function v_1 is convex and we have $-d_{1,\varepsilon} = v_1(0)$ and $v_1^{**}(0) = -p_{1,\varepsilon}$. In particular, we have $-d_{1,\varepsilon} \ge -p_{1,\varepsilon}$.*

Proof. We find with Lemma A.8(i) and our calculations in (2.1) that

$$\begin{aligned}
v_1^{**}(0) \le v_1(0) &= \inf_{\mu\in\mathcal{M}[2b(0),\infty)} L_{1,\mu}^*(0) \\
&= \inf_{\mu\in\mathcal{M}[2b(0),\infty)} \sup_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} (\langle 0,\lambda\rangle - L_{1,\mu}(\lambda)) \\
&= - \sup_{\mu\in\mathcal{M}[2b(0),\infty)} \inf_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} L_1(\lambda,\mu) = -d_{1,\varepsilon}.
\end{aligned}$$

Then, we can compute the conjugate $v_1^* : \mathcal{M}(\Omega_\varepsilon) \to [-\infty,\infty]$ of v_1 as

$$\begin{aligned}
v_1^*(\lambda) &= \sup_{f\in C_0(\Omega_\varepsilon)} (\langle f,\lambda\rangle - v_1(f)) \\
&= \sup_{\mu\in\mathcal{M}[2b(0),\infty)} \sup_{f\in C_0(\Omega_\varepsilon)} \Big(\langle f,\lambda\rangle - L_{1,\mu}^*(f)\Big) \\
&= \sup_{\mu\in\mathcal{M}[2b(0),\infty)} L_{1,\mu}^{**}(\lambda) \\
&= \sup_{\mu\in\mathcal{M}[2b(0),\infty)} L_{1,\mu}(\lambda),
\end{aligned}$$

where for the last equality we used Theorem A.9 which is applicable since the mapping $\mathcal{M}(\Omega_\varepsilon) \ni \lambda \mapsto L_{1,\mu}(\lambda) := L_1(\lambda,\mu)$ is closed in the sense of Section A.3 and convex for all $\mu \in \mathcal{M}[2b(0),\infty)$. Then, the biconjugate $of v_1$ can be computed as

$$\begin{aligned}
v_1^{**}(f) &= \sup_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} (\langle f,\lambda\rangle - v_1^*(\lambda)) \\
&= \sup_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} \inf_{\mu\in\mathcal{M}[2b(0),\infty)} (\langle f,\lambda\rangle - L_1(\lambda,\mu))\,. \qquad (2.3)
\end{aligned}$$

With the help of our calculations in (2.2), this yields

$$\begin{aligned}
v_1^{**}(0) &= \sup_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} \inf_{\mu\in\mathcal{M}[2b(0),\infty)} (\langle 0,\lambda\rangle - L_1(\lambda,\mu)) \\
&= -\inf_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} \sup_{\mu\in\mathcal{M}[2b(0),\infty)} L_1(\lambda,\mu) = -p_{1,\varepsilon}.
\end{aligned}$$

Moreover, note that the mapping v_1 never takes the value $-\infty$: Assume by contradiction that there exists $f \in C_0(\Omega_\varepsilon)$ such that $v_1(f) = -\infty$. We know that $v_1^{**} \le v_1$ and therefore we have $v_1^{**}(f) = -\infty$, as well. But then because of (2.3) we find that $\sup_\mu L(\lambda,\mu) = \infty$ for any $\lambda \in \mathcal{M}(\Omega_\varepsilon)$ and thus $p_{1,\varepsilon} = \infty$. But, we do know that a $(P_{1,\varepsilon})$-admissible solution exists and therefore this is impossible.

Finally, we show that v_1 is convex. To this end, let $\alpha \in (0,1)$ and $f_1, f_2 \in C_0(\Omega_\varepsilon)$. Note that L_1 is concave in its second component which is clear from its definition. Then, we find for any measures $\tilde{\mu}, \hat{\mu} \in \mathcal{M}^+[2b(0), \infty)$

$$\begin{aligned} v_1(\alpha f_1 + (1-\alpha) f_2) &= \inf_{\mu \in \mathcal{M}[2b(0),\infty)} L^*_{1,\mu}(\alpha f_1 + (1-\alpha) f_2) \\ &= \inf_{\mu \in \mathcal{M}[2b(0),\infty)} \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle \alpha f_1 + (1-\alpha) f_2, \lambda \rangle - L_1(\lambda, \mu)) \\ &\leq \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle \alpha f_1 + (1-\alpha) f_2, \lambda \rangle - L_1(\lambda, \alpha \tilde{\mu} + (1-\alpha)\hat{\mu})) \\ &\leq \alpha \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle f_1, \lambda \rangle - L_1(\lambda, \tilde{\mu})) \\ &\qquad + (1-\alpha) \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle f_2, \lambda \rangle - L_1(\lambda, \hat{\mu}))\,. \end{aligned}$$

Taking the infimum over $\tilde{\mu}, \hat{\mu}$ we obtain

$$\begin{aligned} v_1(\alpha f_1 + (1-\alpha) f_2) &\leq \alpha \inf_{\mu_1 \in \mathcal{M}[2b(0),\infty)} \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle f_1, \lambda \rangle - L_1(\lambda, \tilde{\mu})) \\ &\qquad + (1-\alpha) \inf_{\mu_2 \in \mathcal{M}[2b(0),\infty)} \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle f_2, \lambda \rangle - L_1(\lambda, \hat{\mu})) \\ &= \alpha v_1(f_1) + (1-\alpha) v_1(f_2). \end{aligned}$$

□

Note that we can conclude from Lemma 2.3 that $-d_{1,\varepsilon} \geq -p_{1,\varepsilon}$ which is equivalent to weak duality holding. Of course, we already showed weak duality above without using the additional structure of the dual value function v_1.

We will also need the following version of Theorem 5.42 from [AB06].

Lemma 2.4. *Let V be a locally convex space, $f : V \to (-\infty, \infty]$ a convex function and $x_0 \in V$. If there exists an open neighbourhood O of x_0 such that $\sup_{x \in O} f(x) < \infty$, then f is continuous in x_0.*

Now, we can prove the following lemma about strong duality.

Lemma 2.5. *Strong duality holds between the programs ($D_{1,\varepsilon}$) and ($P_{1,\varepsilon}$) and for the optimal solutions $\mu_{1,\varepsilon}$ and $\lambda_{1,\varepsilon}$ the following complementary slackness conditions hold*

$$T^*\lambda_{1,\varepsilon} = g \text{ holds } \mu_{1,\varepsilon}\text{-almost surely on } [2b(0), \infty) \text{ and}$$

$$T\mu_{1,\varepsilon} = 1 \text{ holds } \lambda_{1,\varepsilon}\text{-almost surely on } \Omega_\varepsilon.$$

Proof. Lemma 2.3 tells us that strong duality holds if $v_1^{**}(0) = v_1(0)$. By Theorem A.9 and the fact that we just showed v_1 to be convex, we can deduce that

$$v_1^{**}(0) = \mathrm{cl}(\mathrm{co}(v_1))(0) = \mathrm{lsc}(v_1)(0),$$

where cl, co and lsc denote the closure, the convex hull and the lower semi-continuous hull of a function as defined below in Appendix A.3. Now, use Lemma A.8 to obtain

$$v_1^{**}(0) = \sup_{O \in \mathcal{U}(0)} \inf_{f \in O \setminus \{0\}} v_1(f)$$

where $\mathcal{U}(0)$ is the set of all $\sigma(C, \mathcal{M})$-open neighbourhoods of 0. This means, we want

$$v_1(0) = \sup_{O \in \mathcal{U}(0)} \inf_{f \in O \setminus \{0\}} v_1(f),$$

i.e., continuity of v_1 at 0 with respect to the topology $\sigma(C, \mathcal{M})$. In order to show continuity we make use of Lemma 2.4. As a $\sigma(C, \mathcal{M})$-open neighbourhood of 0 we choose $O := \{\|f\|_\infty < 1\}$. Then we find for any $f \in O$ that

$$\begin{aligned} v_1(f) &= \inf_{\mu \in \mathcal{M}[2b(0),\infty)} L_{1,\mu}^*(f) = \inf_{\mu \in \mathcal{M}[2b(0),\infty)} \sup_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} (\langle f, \lambda \rangle - L_{1,\mu}(\lambda)) \\ &= \inf_{\mu \in \mathcal{M}^+[2b(0),\infty)} \sup_{\lambda \in \mathcal{M}^+(\Omega_\varepsilon)} (\langle f, \lambda \rangle - \|\lambda\| - \langle g, \mu \rangle + \langle T^* \lambda, \mu \rangle) \\ &\le \sup_{\lambda \in \mathcal{M}^+(\Omega_\varepsilon)} (\|f\|_\infty \|\lambda\| - \|\lambda\|) = 0 \end{aligned}$$

Now, Lemma 2.4 yields that v_1 is continuous at 0 and therefore $p_{1,\varepsilon} = -v_1^{**}(0) = -v_1(0) = d_{1,\varepsilon}$.

Moreover, the optimizers $\lambda_{1,\varepsilon}$ and $\mu_{1,\varepsilon}$ fulfil the complementary slackness conditions. By using that T and T^* are adjoint operators and that strong duality holds, i.e., that $d_{1,\varepsilon} = p_{1,\varepsilon}$, we find that

$$\begin{aligned} 0 \le \langle T^* \lambda_{1,\varepsilon} - g, \mu_{1,\varepsilon} \rangle &= \langle T^* \lambda_{1,\varepsilon}, \mu_{1,\varepsilon} \rangle - d_{1,\varepsilon} \\ &= \langle \lambda_{1,\varepsilon}, T\mu_{1,\varepsilon} \rangle - p_{1,\varepsilon} = \langle \lambda_{1,\varepsilon}, T\mu_{1,\varepsilon} - 1 \rangle \le 0. \end{aligned}$$

This means that $T^* \lambda_{1,\varepsilon} = g$ holds $\mu_{1,\varepsilon}$-almost surely on $[2b(0), \infty)$ and $T\mu_{1,\varepsilon} = 1$ holds $\lambda_{1,\varepsilon}$-almost surely on Ω_ε. □

We can now summarise our findings from Lemma 2.2 and Lemma 2.5.

Proposition 2.6. *The optimal values $p_{1,\varepsilon}$ and $d_{1,\varepsilon}$ of the linear programs ($P_{1,\varepsilon}$) and ($D_{1,\varepsilon}$) are attained by the solutions $\lambda_{1,\varepsilon}$ and $\mu_{1,\varepsilon}$, respectively. Strong duality holds and we have the following complementary slackness conditions*

$$T^*\lambda_{1,\varepsilon} = g, \qquad \text{holds } \mu_{1,\varepsilon}\text{-almost surely on } [2b(0),\infty) \qquad \text{and}$$
$$T\mu_{1,\varepsilon} = 1, \qquad \text{holds } \lambda_{1,\varepsilon}\text{-almost surely on } \Omega_\varepsilon.$$

2.2.2 Strong duality of the unrestricted programs

We now return to the original programs (D_1) and (P_1). We lift the results from the restricted programs ($D_{1,\varepsilon}$) and ($P_{1,\varepsilon}$) by considering suitable sequences of solutions where we let $\varepsilon \searrow 0$. We now consider measures $\lambda_{1,\varepsilon}$ as measures on all of $\Omega = (0, t_0]$ by continuing them as the null measure outside of Ω_ε.

If we denote by d_1 and p_1 the optimal values of (D_1) and (P_1), we obtain weak duality with analogous calculations as above (or simply by construction of the linear programs). We showed above that for any ($P_{1,\varepsilon}$)-admissible λ we have $\|\lambda\| \le 1$ and for the modified program ($D_{1,\varepsilon,mod}$) any admissible solution $\tilde{\mu}$ satisfies $\|\tilde{\mu}\| \le 1$ (recall that the solution μ of the unmodified program ($D_{1,\varepsilon}$) can be recovered from the solution $\tilde{\mu}$ of the modified program by $d\mu/d\tilde{\mu} = g^{-1}$ where $g(\theta) = r_\theta(t_0, x_0)$, cf. proof of Lemma 2.2 above). Note that the bound is independent of ε in both cases. The metrisation of the vague topology is possible on the total variation unit balls in both spaces (for example, cf. [Bau01], §31). From Theorem A.7 we can deduce that these unit balls are vaguely compact. Thus, there exists a sequence $\varepsilon_n \searrow 0$ and measures $\tilde{\mu}_1$ and λ_1 with $\|\tilde{\mu}_1\| \vee \|\lambda_1\| \le 1$ such that $\tilde{\mu}_{1,\varepsilon_n} \to \tilde{\mu}_1$ and $\lambda_{1,\varepsilon_n} \to \lambda_1$ vaguely.

Moreover, note that for every function $f : [2b(0), \infty) \to \infty$ with compact support, the function f/g also has compact support. This allows us to conclude that $\mu_{1,\varepsilon_n} = 1/g \cdot \tilde{\mu}_{1,\varepsilon_n} \xrightarrow{n\to\infty} 1/g \cdot \tilde{\mu}_1 =: \mu_1$ vaguely.

So, we have vague convergence of optimal measures for the restricted measures to some measures μ_1 and λ_1. These measures are good candidates for being the optimal measures in (D_1) and (P_1). However, we do not yet know whether μ_1 and λ_1 are even admissible in the respective programs, much less whether these solutions are optimal or whether strong duality holds.

Let us start by looking at admissibility. We know that μ_{1,ε_n} converges vaguely to μ_1 and that the mapping $\theta \mapsto r_\theta(t, b(t))$ is continuous on $[2b(0), \infty)$ for any $t \in \Omega$ and vanishes at infinity. By Lemma A.1 we can conclude that

$$T\mu_1(t) = \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu_1(d\theta) = \lim_{n\to\infty} \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu_{1,\varepsilon_n}(d\theta) \leq 1$$

and so μ_1 is indeed (D_1)-admissible.

Now, for λ_1 to be (P_1)-admissible, we need the additional assumption that $\|\lambda_{1,\varepsilon_n}\| \to \|\lambda_1\|$. Since then, we find by the Portemanteau theorem (see e.g., [Kle14], Theorem 13.16]) that $\lambda_{1,\varepsilon_n} \to \lambda_1$ weakly and not only vaguely. This assumption can be numerically investigated in an application by computing $\lambda_{1,\varepsilon}$ for smaller and smaller ε. The only thing that could go wrong is that $\lambda_{1,\varepsilon_n}$ pushes mass into 0 for $n \to \infty$ since the total mass of λ_1 would then be smaller than the mass of the $\lambda_{1,\varepsilon_n}$.

Moreover, we need $t \mapsto r_\theta(t, b(t))$ to be bounded on $(0, t_0]$. In order for this to be true, we need the additional assumption that $b'(0) < \infty$. Then, we can check the constraint for $\theta \in [2b(0), \infty)$. First, we find for $\theta > 2b(0)$ that

$$r_\theta(t, b(t)) = \exp\left(-\frac{\theta}{2t}(\theta - 2b(t))\right) \to 0$$

for $t \to 0$ since $\theta - 2b(t) > 0$ for t sufficiently small. The latter is true since we assumed that b is continuous. In the case that $\theta = 2b(0)$ we find that

$$r_{2b(0)}(t, b(t)) = \exp\left(2b(0)\left(\frac{b(t) - b(0)}{t}\right)\right) \to \exp\left(2b(0)b'(0)\right) < \infty$$

for $t \to 0$ since we assumed $0 < b(0) < \infty$ and $b'(0) < \infty$. In both cases we find that $t \mapsto r_\theta(t, b(t))$ is bounded. Then, we find by Portmanteau (again, see e.g., [Kle14], Theorem 13.16]) that

$$\int_{(0,t_0]} r_\theta(t, b(t))\lambda_1(dt) = \lim_{n\to\infty} \int_{(0,t_0]} r_\theta(t, b(t))\lambda_{1,\varepsilon_n}(dt) \geq g(\theta)$$

and so λ_1 is (P_1)-admissible.

Finally, we find by Lemma A.1 that $\liminf_{n\to\infty}\langle g, \mu_{1,\varepsilon_n}\rangle = \langle g, \mu_1\rangle$. Moreover, we have $\|\lambda_1\| \leq \liminf_{n\to\infty} \|\lambda_{1,\varepsilon_n}\|$ (which is already true due to vague convergence but of course also since we assumed $\lim_{n\to\infty} \|\lambda_{1,\varepsilon_n}\| = \|\lambda_1\|$). Thus,

$$\begin{aligned} p_1 \leq \|\lambda_1\| &\leq \liminf_{n\to\infty} \|\lambda_{1,\varepsilon_n}\| = \liminf_{n\to\infty} p_{1,\varepsilon_n} \\ &= \liminf_{n\to\infty} d_{1,\varepsilon_n} = \liminf_{n\to\infty}\langle g, \mu_{1,\varepsilon_n}\rangle = \langle g, \mu_1\rangle \leq d_1 \leq p_1. \end{aligned}$$

Note that all inequalities must therefore be equalities which shows several important things: λ_1 and μ_1 are (P_1)- and (D_1)-optimal and we have that $d_1 = p_1$, i.e., strong duality holds. Finally, we can again use the adjointness of T and T' as well as the fact that strong duality holds, i.e., $d_1 = p_1$ and thus find that

$$0 \leq \langle T'\lambda_1 - g, \mu_1 \rangle = \langle T'\lambda_1, \mu_1 \rangle - d_1 = \langle \lambda_1, T\mu_1 \rangle - p_1 = \langle \lambda_1, T\mu_1 - 1 \rangle \leq 0$$

and therefore we have $T^*\lambda_1 = g$ holds μ_1-almost surely on $[2b(0), \infty)$ and $T\mu_1 = 1$ holds λ_1-almost surely on Ω. We can again summarise our findings.

Theorem 2.7. *(a) There exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and measures $\lambda_1 \in \mathcal{M}^+(\Omega)$ and $\mu_1 \in \mathcal{M}^+[2b(0), \infty)$ such that $\lambda_{1,\varepsilon_n} \to \lambda_1$ and $\mu_{1,\varepsilon_n} \to \mu_1$ vaguely. In addition, μ_1 is (D_1)-admissible.*

(b) If $\|\lambda_{1,\varepsilon_n}\| \to \|\lambda_1\|$ and $b'(0) < \infty$, then λ_1 is (P_1)-admissible. Moreover, the optimal values d_1 and p_1 in (D_1) and (P_1) are obtained by μ_1 and λ_1, respectively, the optimal values coincide (i.e., strong duality holds) and the following complementary slackness conditions are satisfied:

$$\int r_\theta(t, b(t))\mu_1(d\theta) = 1 \text{ for } \lambda_1\text{-a.a. } t \in (0, t_0],$$
$$\int r_\theta(t, b(t))\lambda_1(dt) = r_\theta(t_0, x_0) \text{ for } \mu_1\text{-a.a. } \theta \in [2b(0), \infty).$$

2.3 Strong duality in the second set-up

Let us begin by recalling the definitions of the alternative primal problem

$$\begin{aligned} &\text{minimise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in \mathcal{M}^+[2b(0), \infty), \\ &&& \int r_\theta(t, b(t))\mu(d\theta) \geq 1 \text{ for any } t \in (0, t_0] \end{aligned} \tag{P_2}$$

and its corresponding alternative Lagrange dual problem

$$\begin{aligned} &\text{maximise} && \|\lambda\| \\ &\text{subject to} && \lambda \in \mathcal{M}^+(0, t_0], \\ &&& \int r_\theta(t, b(t))\lambda(dt) \leq r_\theta(t_0, x_0) \text{ for any } \theta \in [2b(0), \infty). \end{aligned} \tag{D_2}$$

With the same definitions of g, T, T' and T^* as well as Ω and Ω_ε as before we can again formulate the programs in their abbreviated forms and their constraint forms.

2.3.1 Strong duality of the restricted linear programs

Let us now directly consider the constraint forms. The reader can easily write down the abbreviated, unconstrained forms for herself. We consider

$$\begin{aligned} &\text{minimise} && \langle g, \mu\rangle \\ &\text{subject to} && \mu \in \mathcal{M}^+[2b(0), \infty), \\ & && T\mu - 1 \in C^+(\Omega_\varepsilon) \end{aligned} \tag{$P_{2,\varepsilon}$}$$

and its formal dual

$$\begin{aligned} &\text{maximise} && \|\lambda\| \\ &\text{subject to} && \lambda \in \mathcal{M}^+(\Omega_\varepsilon), \\ & && g - T^*\lambda \in C^+[2b(0), \infty). \end{aligned} \tag{$D_{2,\varepsilon}$}$$

In the same way as before, we find that T and T^* are adjoint operators with respect to $\langle f, \lambda\rangle = \int f d\lambda$ (defined on the respective spaces) and continuous. Denote the optimal values of ($P_{2,\varepsilon}$) and ($D_{2,\varepsilon}$) by $p_{2,\varepsilon}$ and $d_{2,\varepsilon}$, respectively. Then, we again have weak duality by construction which we can confirm with the following calculation: for any primal admissible $\mu \in \mathcal{M}^+[2b(0), \infty)$ and any dual feasible $\lambda \in \mathcal{M}^+(\Omega_\varepsilon)$

$$0 \le \|\lambda\| = \langle \lambda, 1\rangle \le \langle \lambda, T\mu\rangle = \langle T^*\lambda, \mu\rangle \le \langle g, \mu\rangle.$$

Now, we can show primal and dual attainment with an analogous result to Lemma 2.2.

Lemma 2.8. *There exist a ($P_{2,\varepsilon}$)-admissible $\mu_{2,\varepsilon}$ such that $p_{2,\varepsilon} = \langle g, \mu_{2,\varepsilon}\rangle$ and a ($D_{2,\varepsilon}$)-admissible $\lambda_{2,\varepsilon}$ such that $d_{2,\varepsilon} = \|\lambda_{2,\varepsilon}\|$.*

Proof. The proof follows along the same lines as the proof of Lemma 2.2 and we note that it is enough to show that there exists a primal feasible μ with $\|\mu\| < \infty$ and for all dual feasible λ that they are bounded by some constant. Let us begin by considering $\bar{\mu} = c \cdot \delta_{2b(0)}$ where we choose

$$c > \exp\left(-2b(0)\frac{b(t_0) - b(0)}{t_0}\right).$$

Then, for any $t \in \Omega_\varepsilon$

$$\begin{aligned} T\bar{\mu}(t) &= c \cdot \exp\left(-\frac{4b(0)^2}{2t} + \frac{2b(0)b(t)}{t}\right) \\ &= c \cdot \exp\left(2b(0)\frac{b(t)-b(0)}{t}\right) \\ &\geq c \cdot \exp\left(2b(0)\frac{b(t_0)-b(0)}{t_0}\right) > 1, \end{aligned}$$

i.e., $T\bar{\mu} - 1 \in C^+(\Omega_\varepsilon)$. So, we have found a primal feasible μ with $\|\mu\| = c < \infty$.
For the dual problem, we find that for any dual feasible λ and for $\theta = 2b(0)$ the constraint yields

$$\begin{aligned} g(2b(0)) = \exp\left(-\frac{2b(0)^2}{t_0} + \frac{2b(0)x_0}{t_0}\right) &\geq \int_{\Omega_\varepsilon} \exp\left(-\frac{2b(0)^2}{t} + \frac{2b(0)b(t)}{t}\right)\lambda(dt) \\ &= \int_{\Omega_\varepsilon} \exp\left(2b(0)\left(\frac{b(t)-b(0)}{t}\right)\right)\lambda(dt) \\ &\geq \int_{\Omega_\varepsilon} \exp\left(2b(0)\left(\frac{b(t_0)-b(0)}{t_0}\right)\right)\lambda(dt) \\ &\geq \exp\left(2b(0)\left(\frac{b(t_0)-b(0)}{t_0}\right)\right)\|\lambda\| \end{aligned}$$

where we used in the third step that $(b(t) - b(0))/t$ is monotone decreasing in t as b is concave. So, in particular, we find that for any feasible λ we have that

$$\|\lambda\| \leq \exp\left(2b(0)\frac{x_0 - b(t_0)}{t_0}\right) =: C$$

and so the norm of all feasible λ is bounded by a constant C. □

Next, we will deduce that strong duality holds. In order to be able to prove this, we will need to consider modified programs to be able to show strong duality in the same way as in the first set-up. The reader may rightly ask herself why we need this extra detour. The reason will be become apparent after the proof of Lemma 2.10 below. For now, the modified programs are

$$\begin{aligned} &\text{minimise} && \|\mu\| \\ &\text{subject to} && \mu \in \mathcal{M}^+[2b(0), \infty), \qquad\qquad (P_{2,\varepsilon,mod}) \\ & && T_{mod}\mu - 1 \in C^+(\Omega_\varepsilon) \end{aligned}$$

where

$$T_{mod}\mu(t) = \int_{[2b(0),\infty)} \frac{r_\theta(t,b(t))}{r_\theta(t_0,x_0)}\mu(d\theta)$$

and the formal dual program

$$\begin{aligned} &\text{maximise} && \|\lambda\| \\ &\text{subject to} && \lambda \in \mathcal{M}^+(\Omega_\varepsilon), \\ &&& 1 - T^*_{mod}\lambda \in C^+[2b(0),\infty). \end{aligned} \qquad (D_{2,\varepsilon,mod})$$

where

$$T^*_{mod}\lambda(\theta) = \int_{\Omega_\varepsilon} \frac{r_\theta(t,b(t))}{r_\theta(t_0,x_0)}\lambda(dt).$$

Note that we considered a modified version in the first set-up before where we also absorbed the function g into the feasible measures μ. As before, we can recover the solution of ($P_{2,\varepsilon}$) from the solution ($P_{2,\varepsilon,mod}$) via the relationship

$$\frac{d\mu_{1,\varepsilon}}{d\mu_{1,\varepsilon,mod}} = g^{-1}.$$

where $\mu_{1,\varepsilon}$ and $\mu_{1,\varepsilon,mod}$ denote the optimal solutions of ($P_{2,\varepsilon}$) and ($P_{2,\varepsilon,mod}$), respectively. Note that ($D_{2,\varepsilon}$) and ($D_{2,\varepsilon,mod}$) are actually the same problem where we just slightly modified the constraints by dividing by g. In particular, all characteristics we show for the modified programs ($P_{2,\varepsilon,mod}$) and ($D_{2,\varepsilon,mod}$) automatically carry over to ($P_{2,\varepsilon}$) and ($D_{2,\varepsilon}$).

In much the same way as before we define the Lagrange dual function as

$$L_2(\mu,\lambda) := \|\mu\| + \|\lambda\| - \langle T\mu,\lambda\rangle + \mathcal{I}_{\mathcal{M}^+[2b(0),\infty)}(\mu) - \mathcal{I}_{\mathcal{M}^+(\Omega_\varepsilon)}(\lambda).$$

With the same steps as before we can again explicitly calculate that

$$\sup_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} \inf_{\mu\in\mathcal{M}[2b(0),\infty)} L_2(\mu,\lambda) = d_{2,\varepsilon} \qquad \text{and} \qquad \inf_{\mu\in\mathcal{M}[2b(0),\infty)} \sup_{\lambda\in\mathcal{M}(\Omega_\varepsilon)} L_2(\mu,\lambda) = p_{2,\varepsilon}.$$

Using these calculations we can again consider $L_{2,\lambda}(\mu) := L_2(\mu,\lambda)$ which is closed and convex for any μ and λ. Defining $v_2 : C_0[2b(0), \infty) \to (-\infty, \infty]$ as

$$v_2(f) := \inf_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} L^*_{2,\lambda}(f)$$

as before we can formulate the analogon of Lemma 2.3.

Lemma 2.9. *The dual value function v_2 is convex and we have $-d_{2,\varepsilon,mod} = v_2(0)$ and $v_2^{**}(0) = -p_{2,\varepsilon,mod}$. In particular, we have $-d_{2,\varepsilon,mod} \geq -p_{2,\varepsilon,mod}$.*

Proof. The proof follows along the same lines as the proof of Lemma 2.3. Note that we used that L_1 is concave in its second argument to show that v_1 is convex. To accept that this also holds here note that $\|\lambda\| = \langle 1, \lambda \rangle$ is even a linear function in λ. □

Now, we can conclude that strong duality holds in a similar way as before.

Lemma 2.10. *Strong duality holds between the programs ($P_{2,\varepsilon,mod}$) and ($D_{2,\varepsilon,mod}$) and for the optimal solutions $\mu_{2,\varepsilon,mod}$ and $\lambda_{2,\varepsilon,mod}$ the following complementary slackness conditions hold*

$$T^*_{mod}\lambda_{2,\varepsilon,mod} = 1 \textit{ holds } \mu_{2,\varepsilon,mod}\textit{-almost surely on } [2b(0), \infty) \textit{ and}$$
$$T_{mod}\mu_{2,\varepsilon,mod} = 1 \textit{ holds } \lambda_{2,\varepsilon,mod}\textit{-almost surely on } \Omega_\varepsilon.$$

Proof. By Lemma A.8, it is again enough to show continuity of v_2 in 0. As before, we want to apply Lemma 2.4 to obtain this result. We choose $O := \{\|f\|_\infty < 1\}$ as a $\sigma(C, \mathcal{M})$-open neighbourhood of 0. For any $f \in O$ we have

$$\begin{aligned} v_2(f) &= \inf_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} L^*_{2,\lambda}(f) = \inf_{\lambda \in \mathcal{M}(\Omega_\varepsilon)} \sup_{\mu \in \mathcal{M}[2b(0),\infty)} (\langle f, \mu \rangle - L_{2,\lambda}(\mu)) \\ &= \inf_{\lambda \in \mathcal{M}^+(\Omega_\varepsilon)} \sup_{\mu \in \mathcal{M}^+[2b(0),\infty)} (\langle f, \mu \rangle - \|\mu\| - \|\lambda\| + \langle T_{mod}\mu, \lambda \rangle) \\ &\leq \sup_{\mu \in \mathcal{M}^+[2b(0),\infty)} (\|f\|_\infty \|\mu\| - \|\mu\|) = 0. \end{aligned}$$

Then, the proof for strong duality concludes as before: We obtain $d_{2,\varepsilon} = -v_2(0) = -v_2^{**}(0) = p_{2,\varepsilon}$. For the complementary slackness conditions we observe that the optimisers $\lambda_{2,\varepsilon,mod}$ and $\mu_{2,\varepsilon,mod}$ fulfil

$$\begin{aligned} 0 &\leq \langle T_{mod}\mu_{2,\varepsilon,mod} - 1, \lambda_{2,\varepsilon,mod} \rangle = \langle T\mu_{2,\varepsilon,mod}, \lambda_{2,\varepsilon,mod} \rangle - d_{2,\varepsilon,mod} \\ &= \langle \mu_{2,\varepsilon,mod}, T^*_{mod}\lambda_{2,\varepsilon,mod} \rangle - p_{2,\varepsilon,mod} = \langle \mu_{2,\varepsilon,mod}, T^*\lambda_{2,\varepsilon,mod} - 1 \rangle \leq 0. \end{aligned}$$

This means we have that $T^*_{mod}\lambda_{2,\varepsilon,mod} = 1$ holds $\mu_{2,\varepsilon,mod}$-almost surely on $[2b(0), \infty)$ and $T_{mod}\mu_{2,\varepsilon,mod} = 1$ holds $\lambda_{2,\varepsilon,mod}$-almost surely on Ω_ε. $\square$

Note that we needed the modification of the programs in this proof: our dual value function for the unmodified programs would have been

$$\begin{aligned} v_2(f) &= \inf_{\lambda\in\mathcal{M}^+(\Omega_\varepsilon)} \sup_{\mu\in\mathcal{M}^+[2b(0),\infty)} \left(\langle f,\mu\rangle - \langle g,\mu\rangle - \|\lambda\| + \langle T_{mod}\mu, \lambda\rangle\right) \\ &\leq \sup_{\mu\in\mathcal{M}^+[2b(0),\infty)} \left(\|f\|_\infty\|\mu\| - \langle g,\mu\rangle\right). \end{aligned}$$

But since $g(\theta) \to 0$ for $\theta \to \infty$, we could not have defined a $\sigma(C, \mathcal{M})$-open neighbourhood O of 0 such that $v_2(f)$ is bounded for all f. This necessitated our detour.

As noted above, the results carry over to the unmodified programs ($P_{2,\varepsilon}$) and ($D_{2,\varepsilon}$). We immediately obtain the following lemma.

Lemma 2.11. *Strong duality holds between the programs ($P_{2,\varepsilon}$) and ($D_{2,\varepsilon}$) and for the optimal solutions $\mu_{2,\varepsilon}$ and $\lambda_{2,\varepsilon}$ the following complementary slackness conditions hold*

$$\begin{aligned} &T^*\lambda_{2,\varepsilon} = 1 \text{ holds } \mu_{2,\varepsilon}\text{-almost surely on } [2b(0),\infty) \text{ and} \\ &T\mu_{2,\varepsilon} = 1 \text{ holds } \lambda_{2,\varepsilon}\text{-almost surely on } \Omega_\varepsilon. \end{aligned}$$

We can now summarise our findings from Lemma 2.8 and Lemma 2.11 in the following proposition.

Proposition 2.12. *The optimal values $p_{2,\varepsilon}$ and $d_{2,\varepsilon}$ of the linear programs ($P_{2,\varepsilon}$) and ($D_{2,\varepsilon}$) are attained by the solutions $\lambda_{2,\varepsilon}$ and $\mu_{2,\varepsilon}$, respectively. Strong duality holds and we have the following complementary slackness conditions*

$$\begin{aligned} &T^*\lambda_{2,\varepsilon} = g, \qquad \text{holds } \mu_{2,\varepsilon}\text{-almost surely on } [2b(0),\infty) \qquad \text{and} \\ &T\mu_{2,\varepsilon} = 1, \qquad \text{holds } \lambda_{2,\varepsilon}\text{-almost surely on } \Omega_\varepsilon. \end{aligned}$$

2.3.2 Strong duality of the unrestricted programs

As before, we can lift these results about the restricted programs ($P_{2,\varepsilon}$) and ($D_{2,\varepsilon}$) to the unrestricted programs (P_2) and (D_2). Let p_2 and d_2 denote the optimal values of the unrestricted programs. The weak duality $0 \leq d_2 \leq p_2$ again follows from the same calculation as above. Recall that we found for any $\varepsilon > 0$ that $\|\mu_{2,\varepsilon}\| \leq 1$ and for $\lambda_{2,\varepsilon}$

we found that

$$\|\lambda_{2,\varepsilon}\| \leq \exp\left(2b(0)\frac{x_0 - b(t_0)}{t_0}\right) =: C.$$

So, for any $\varepsilon > 0$ the masses of the optimisers $\mu_{2,\varepsilon}$ and $\lambda_{2,\varepsilon}$ are bounded by constants independent of ε. With the same metrisation argument as in the first set-up we can find a sequence $(\varepsilon_n)_n$ and measures μ_2 with $\|\mu_2\| \leq 1$ and λ_2 with $\|\lambda_2\| \leq C$ such that μ_{2,ε_n} converges to μ_2 vaguely and $\lambda_{2,\varepsilon_n}$ converges to λ_2 vaguely.

As before, we find that for any $t \in \Omega$ the mapping $\theta \mapsto r_\theta(t, b(t))$ is continuous and vanishes at infinity. So, we find by Lemma A.1 that

$$T\mu_2(t) = \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu_2(d\theta) = \lim_{n\to\infty} \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu_{2,\varepsilon_n}(d\theta) \geq 1$$

and so μ_2 is (P_2)-admissible. To accept that λ_2 is (D_2)-admissible, we need to make use of Urysohn's lemma (e.g., cf. [Lan93], p.40, Theorem 4.2). First, we observe that for any $\delta \in (0, t_0/4)$ we have $\emptyset \neq \Omega_{2\delta} \subset \Omega_\delta \subset \Omega$ and so by Urysohn's lemma we can find a continuous function $\phi^\delta : \Omega \to [0,1]$ such that $\phi^\delta(t) = 1$ for all $t \in \Omega_{2\delta}$ and $\phi^\delta(t) = 0$ for all $t \in \mathrm{cl}(\Omega \setminus \Omega_\delta)$. Note that $t \mapsto r_\theta(t, b(t))\phi^\delta(t)$ is a continuous mapping for any $\theta \in [2b(0), \infty)$. Thus, by Lemma A.1 we find that for any $\theta \in [2b(0), \infty)$

$$\begin{aligned}
\int_{(0,t_0]} r_\theta(t, b(t))\lambda_2(dt) &= \lim_{\delta\searrow 0} \int_{(0,t_0]} r_\theta(t, b(t))\mathbb{1}_{\Omega_{2\delta}}(t)\lambda_2(dt) \\
&\leq \lim_{\delta\searrow 0} \int_{(0,t_0]} r_\theta(t, b(t))\phi^\delta(t)\lambda_2(dt) \\
&= \lim_{\delta\searrow 0} \lim_{n\to\infty} \int_{(0,t_0]} r_\theta(t, b(t))\phi^\delta(t)\lambda_{2,\varepsilon_n}(dt) \\
&\leq \limsup_{\delta\searrow 0} \limsup_{n\to\infty} \int_{(0,t_0]} r_\theta(t, b(t))\lambda_{2,\varepsilon_n}(dt) \leq g(\theta)
\end{aligned}$$

and so in particular, λ_2 is (D_2)-admissible.

Now, recall that g is continuous and vanishes at infinity, so by Lemma A.1, we have that $\langle g, \mu_2\rangle = \lim_{n\to\infty}\langle g, \mu_{2,\varepsilon_n}\rangle$ and therefore

$$d_2 \leq p_2 \leq \langle g, \mu_2\rangle = \lim_{n\to\infty}\langle g, \mu_{2,\varepsilon_n}\rangle = \lim_{n\to\infty} p_{2,\varepsilon_n} = \lim_{n\to\infty} d_{2,\varepsilon_n} \leq d_2.$$

The last inequality is true as any ($D_{2,\varepsilon}$)-feasible λ is also (D_2)-feasible. As all inequalities in the above chain must be equalities, we can conclude that μ_2 is (P_2)-optimal, that $d_2 = p_2$, i.e., that strong duality holds and that $\lim_{n\to\infty} d_{2,\varepsilon_n} = d_2$.

In order to establish that λ_2 is (D_2)-optimal (and not only admissible as shown above) we will need an analogous additional assumption to the assumption in the first set-up, i.e., we will assume that $\|\lambda_{2,\varepsilon_n}\| \to \|\lambda_2\|$. Then, we find

$$\|\lambda_2\| = \lim_{n\to\infty} \|\lambda_{2,\varepsilon_n}\| = \lim_{n\to\infty} d_{2,\varepsilon_n} = d_2$$

as we already established above that $\lim_{n\to\infty} d_{2,\varepsilon_n} = d_2$. But that means that λ_2 is (D_2)-optimal. Finally, we find that

$$0 \leq \langle T'\lambda_2 - g, \mu_2\rangle = \langle T'\lambda_2, \mu_2\rangle - d_2 = \langle \lambda_2, T\mu_2\rangle - p_2 = \langle \lambda_2, T\mu_2 - 1\rangle \leq 0$$

and therefore we have that $T'\lambda_2 = g$ holds μ_2-almost surely on $[2b(0), \infty)$ and $T\mu_2 = 1$ holds λ_2-almost surely on Ω. We can again summarise our findings.

Theorem 2.13. *(a) There exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and measures $\lambda_2 \in \mathcal{M}^+(\Omega)$ and $\mu_2 \in \mathcal{M}^+[2b(0), \infty)$ such that $\lambda_{2,\varepsilon_n} \to \lambda_2$ and $\mu_{2,\varepsilon_n} \to \mu_2$ vaguely. The measures μ_2 and λ_2 are (P_2)- and (D_2)-admissible, respectively. Strong duality holds, i.e., $d_2 = p_2$ and μ_2 is (P_2)-optimal.*

(b) If $\|\lambda_{2,\varepsilon_n}\| \to \|\lambda_2\|$ and $b'(0) < \infty$, then λ_2 is (D_2)-optimal and the following complementary slackness conditions are satisfied:

$$\int r_\theta(t, b(t))\mu_2(d\theta) = 1 \text{ for } \lambda_2\text{-a.a. } t \in (0, t_0],$$
$$\int r_\theta(t, b(t))\lambda_2(dt) = r_\theta(t_0, x_0) \text{ for } \mu_2\text{-a.a. } \theta \in [2b(0), \infty).$$

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Note that in this second set-up, we have strong duality and optimality of μ_2 without any additional assumptions, while in Theorem 2.7 we needed the additional assumption $\|\lambda_{2,\varepsilon_n}\| \to \|\lambda_2\|$ in order to obtain strong duality and optimality of μ_1. In both cases, the additional assumption is necessary for the optimality of λ_1 and λ_2, respectively, as well as for the complementary slackness conditions to hold. This is due to the fact that any ($D_{2,\varepsilon}$)-feasible λ is also (D_2)-admissible as we consider λ to be the null measure outside of Ω_ε. However, a ($D_{1,\varepsilon}$)-admissible μ does not have to be (D_1)-admissible, as μ may not fulfil $T\mu = 1$ outside of Ω_ε.

2.4 On the existence of representing measures

In this section, we investigate the existence of a representing measure, i.e., given an analytic, concave boundary b with $b(0) > 0$ we want to prove the existence of a measure μ such that for all $t \le t_0$

$$1 = r(t, b(t)) = \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu(d\theta).$$

We have already proven something similar. In Theorem 2.7, the complementary slackness conditions state that

$$r(t, b(t)) = \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu_1(d\theta) = 1 \text{ for } \lambda_1\text{-a.a. } t \in (0, t_0]$$

where μ_1 and λ_1 are the optimal measures for (D_1) and (P_1), respectively. The analogous result holds for μ_2 and λ_2 due to Theorem 2.13. It is now enough to "lift" this result from "almost every" to "every" in the sense that if λ_1 or λ_2 put mass everywhere on $\Omega = (0, t_0]$, then we have that b is representable. As we want to make use of the complementary slackness conditions as well as strong duality throughout this section, we will assume the assumptions of Theorem 2.7 and 2.13 to hold, in particular, that $\|\lambda_{1,\varepsilon_n}\| \to \|\lambda_1\|$, $\|\lambda_{2,\varepsilon_n}\| \to \|\lambda_2\|$ and $b'(0) < \infty$. In addition, we will of course also assume that our initial assumptions hold, i.e., b is analytic and concave with $b(0) > 0$.

We start by noting that there exists an interesting measure which we call $\bar\lambda$. This measure is admissible in (P_1) and (D_2) and already fulfils the constraints in both programs with equality everywhere.

Lemma 2.14. *Denote the last hitting time of W to b by σ_b and write*

$$\bar\lambda(dt) = P(\sigma_b \in dt | W_0 = 0,\, W_{t_0} = x_0).$$

Then, $\bar\lambda$ is attainable for (P_1) and (D_2) and

$$\int r_\theta(t, b(t))\bar\lambda(dt) = r_\theta(t_0, x_0) \text{ for any } \theta \in [2b(0), \infty).$$

Proof. First, note that $\bar\lambda \in \mathcal{M}^+(0, t_0]$. Now, let $\theta \ge 2b(0)$ and consider a straight line g_θ with $g_\theta(0) = \theta/2$ lying above the boundary b. Let τ_θ be the first hitting time of W

to g_θ and choose a parameter a_θ according to Remark 1.4 such that

$$P(\tau_\theta < t | W_0 = 0, W_t = x) = a_\theta r_\theta(t, x) \text{ for any } x \leq g(t).$$

Then, using the notation from the proof of Proposition 1.3 and the strong Markov property of the Brownian bridge,

$$\begin{aligned} r_\theta(t_0, x_0) &= \frac{1}{a_\theta} P(\tau_\theta < t_0 | W_0 = 0, W_{t_0} = x_0) \\ &= \frac{1}{a_\theta} \int_{(0,t_0]} P_{(0,0)}^{(t,b(t))}(\tau_\theta \leq t) P_{(0,0)}^{(t_0,x_0)}(\sigma_b \in dt) \\ &= \frac{1}{a_\theta} \int_{(0,t_0]} a_\theta r_\theta(t, b(t)) P_{(0,0)}^{(t_0,x_0)}(\sigma_b \in dt) \\ &= \int_{(0,t_0]} r_\theta(t, b(t)) \bar{\lambda}(dt). \end{aligned}$$

So, in particular, $\bar{\lambda}$ is attainable in both (P_1) and (D_2). □

Note that $\bar{\lambda}$ is a strong contender to be optimal in both (P_1) and (D_2) as it already fulfils the constraint not only with inequality but even with equality everywhere. If $\bar{\lambda}$ really is the optimal measure in both programs, then this yields a stochastic interpretation of (P_1) and (D_2) as the programs determining the (conditional) last hitting time distribution of a standard Brownian motion to b. This would be a nice symmetry with the programs (D_1) and (P_2) used to determine the (conditional) first hitting distribution.

Recall that p_1 and d_1 denote the optimal values of (P_1) and (D_1) and that p_2 and d_2 denote the optimal values of (P_2) and (D_2). If we assume strong duality in both set-ups and since $\bar{\lambda}$ is attainable in both (P_1) and (D_2) we obtain

$$d_1 = p_1 \leq \|\bar{\lambda}\| \leq d_2 = p_2. \tag{2.4}$$

With the help of this inequality, we are able to formulate the following theorem which provides sufficient conditions for a general concave, analytic b to be representable.

Theorem 2.15. *Assume the prerequisites of Theorems 2.7 and 2.13 are met. Assume one of the following conditions is fulfilled:*

(i) $d_1 = p_2$

(ii) $p_1 = d_2$

Then, b is representable.

Proof. The proof is immediate from (2.4). If one of the conditions is fulfilled, we immediately have that $\bar{\lambda}$ is the optimal measure in both (P_1) and (D_2). As $\bar{\lambda}$ puts mass everywhere in $(0, t_0]$, we can conclude from either Theorem 2.7 or Theorem 2.13 that $r(t, b(t)) = 1$ for all $t \in (0, t_0]$ and so b is representable. □

Note that the conditions given in Theorem 2.15 are of course equivalent due to Equation (2.4). The theorem is stated in this way to stress that if the optimal values of the "μ-problems" (D_1) and (P_2) or the optimal values of the "λ-problems" (P_1) and (D_2) agree, then we have representability.

The conditions from Theorem 2.15 are a substantive improvement over the conditions that we usually impose to guarantee that b is representable. Where one usually would have to prove $r(t, b(t)) = 1$ for all $t \in (0, t_0]$, now one only has to check whether $d_1 = p_2$ or $p_1 = d_2$. In particular, these conditions can easily be checked in implementations (see Section 3.3 below).

Recall that linear boundaries b are representable by point measures. In particular, we have that $d_1 = p_2$ and therefore $\bar{\lambda}$ is the dual optimal measure. The following proof is more of a didactical nature to showcase how a representability proof for more general b might be treated.

Corollary 2.16. *If b is linear, then $p_1 = d_2$. In particular, $\bar{\lambda}$ is the optimal measure in both (P_1) and (D_2).*

Proof. We will show that $p_1 = d_2$ and therefore the optimal values of the original and the alternative set-up coincide. In particular, $\bar{\lambda}$ then is an optimal measure in both (P_1) and (D_2). Recall that the optimal measure λ_1 in (P_1) has to fulfil

$$T'\lambda_1(\theta) \geq g(\theta) \qquad \text{for } \mu_1 - \text{a.e. } \theta \in [2b(0), \infty).$$

So, in particular, we find for $\theta = 2b(0)$ that

$$\begin{aligned}
\exp\left(-\frac{2b(0)^2}{t_0} + \frac{2b(0)x_0}{t_0}\right) &\leq \int_{(0,t_0]} \exp\left(\frac{-2b(0)^2}{t} + \frac{2b(0)b(t)}{t}\right)\lambda_1(dt) \\
&= \int_{(0,t_0]} \exp\left(2b(0)\left(\frac{b(t)-b(0)}{t}\right)\right)\lambda_1(dt) \\
&\leq \exp\left(2b(0)b'(0)\right)\|\lambda_1\|
\end{aligned}$$

and so equivalently we have

$$\|\lambda_1\| \geq \exp\left(\frac{2b(0)}{t_0}\left(x_0 - b(0) - t_0 b'(0)\right)\right) =: A.$$

Similarly, we have that the optimal measure λ_2 in (D_2) has to satisfy

$$T'\lambda_2(\theta) \leq g(\theta) \qquad \text{for } \mu_2 - \text{a.e. } \theta \in [2b(0), \infty).$$

So, again we find for $\theta = 2b(0)$ that

$$\begin{aligned}
\exp\left(-\frac{2b(0)^2}{t_0} + \frac{2b(0)x_0}{t_0}\right) &\geq \int_{(0,t_0]} \exp\left(\frac{-2b(0)^2}{t} + \frac{2b(0)b(t)}{t}\right)\lambda_2(dt) \\
&= \int_{(0,t_0]} \exp\left(2b(0)\left(\frac{b(t) - b(0)}{t}\right)\right)\lambda_2(dt) \\
&\geq \int_{(0,t_0]} \exp\left(2b(0)\left(\frac{b(t_0) - b(0)}{t_0}\right)\right)\lambda_2(dt) \\
&= \exp\left(-\frac{2b(0)^2}{t_0} + \frac{2b(0)b(t_0)}{t_0}\right)\|\lambda_2\|
\end{aligned}$$

and so equivalently we have

$$\|\lambda_2\| \leq \exp\left(\frac{2b(0)}{t_0}\left(x_0 - b(t_0)\right)\right) =: B.$$

In particular, we already know that $A \leq \|\lambda_1\| = p_1 \leq d_2 = \|\lambda_2\| \leq B$. So, we need $A \geq B$ and this is the case if

$$x_0 - b(0) - t_0 b'(0) \geq x_0 - b(t_0) \qquad \Leftrightarrow \qquad b'(0) \leq \frac{b(t_0) - b(0)}{t_0}.$$

But this last condition is fulfilled since b is linear. So, we find that $\bar{\lambda}$ is now indeed the optimal measure in both (P_1) and (D_2). $\square$

As mentioned before Corollary 2.16, the course of the argument in the proof may be a starting point for a proof that more general b are representable and that does not rely on sufficient conditions like Theorem 2.15 does above. In particular, the ansatz used in the proof of Corollary 2.16 was to utilise the constraints of both programs which tell us that

$$T'\lambda_1(\theta) \geq g(\theta) \qquad \text{for } \mu_1 - \text{a.e. } \theta \in [2b(0), \infty)$$

as well as

$$T'\lambda_2(\theta) \leq g(\theta) \qquad \text{for } \mu_2 - \text{a.e. } \theta \in [2b(0), \infty)$$

to show $\|\lambda_2\| = d_2 \leq p_1 = \|\lambda_1\|$. Then, combine this with (2.4) where we have $\|\lambda_1\| = p_1 \leq d_2 = \|\lambda_2\|$. In the case of linear b we only used the constraints at $\theta = 2b(0)$. Incorporating the constraints for $\theta > 2b(0)$ might be a promising ansatz. For linear b the mass of the representing measure μ is concentrated in $2b(0)$ whereas for more general concave b there is mass in $(2b(0), \infty)$.

Moving away from the measure $\bar{\lambda}$, we can also give more sufficient conditions such that b is representable. To this end, we first make two observations about properties of attainable measures in (P_1) and (D_2), respectively.

Lemma 2.17. *Let λ be attainable for (P_1). Then, for every $\varepsilon \in (0, t_0)$ it holds that $\lambda([t_0 - \varepsilon, t_0]) > 0$.*

Proof. Let λ be such that

$$\int_\Omega r_\theta(t, b(t))\lambda(dt) \geq r_\theta(t_0, x_0) \qquad \text{for any } \theta \in [2b(0), \infty),$$

i.e.,

$$\int \exp\left(-\frac{1}{2}\theta^2\left(\frac{1}{t} - \frac{1}{t_0}\right) + \theta\left(\frac{b(t)}{t} - \frac{x_0}{t_0}\right)\right)\lambda(dt) \geq 1 \qquad \text{for any } \theta \in [2b(0), \infty).$$

We may write

$$\int \exp\left(-\frac{1}{2}\theta^2\left(\frac{1}{t} - \frac{1}{t_0}\right) + \theta\left(\frac{b(t)}{t} - \frac{x_0}{t_0}\right)\right)\lambda(dt) = \int \alpha(t)\exp(-\beta(t)(\theta - \gamma(t))^2)\lambda(dt)$$

for certain positive functions α, β and γ being bounded on $[0, t_0 - \varepsilon]$ for each $\varepsilon \in (0, t_0)$. Now, if $\varepsilon \in (0, t_0)$ is such that $\lambda([t_0 - \varepsilon, t_0]) = 0$, then dominated convergence yields the contradiction

$$\int \exp\left(-\frac{1}{2}\theta^2\left(\frac{1}{t} - \frac{1}{t_0}\right) + \theta\left(\frac{b(t)}{t} - \frac{x_0}{t_0}\right)\right)\lambda(dt) \to 0 \text{ as } \theta \to \infty.$$

□

Lemma 2.18. *Let λ be attainable for (D_2). Then, $\lambda(\{t_0\}) = 0$.*

Proof. As before, let λ be attainable but this time for (D_2), i.e., we have that for all $\theta \in [2b(0), \infty)$

$$\int_{(0,t_0]} r_\theta(t, b(t))\lambda(dt) \leq r_\theta(t_0, x_0).$$

So, equivalently, we obtain

$$\int_{(0,t_0]} \underbrace{\exp\left(-\frac{\theta^2}{2}\left(\frac{1}{t} - \frac{1}{t_0}\right) + \theta\left(\frac{b(t)}{t} - \frac{x_0}{t_0}\right)\right)}_{=:h(t)} \lambda(dt) \leq 1.$$

Assume that $\lambda(\{t_0\}) > 0$, then we find

$$1 \geq \int_\Omega h(t)\lambda(dt) \geq h(t_0) \cdot \lambda(\{t_0\}) = \exp\left(\theta\left(\frac{b(t_0) - x_0}{t_0}\right)\right) \cdot \lambda(\{t_0\}) \to \infty, \qquad \theta \to \infty$$

as $b(t_0) > x_0$ by assumption. This is a contradiction. So in particular, $\lambda(\{t_0\}) = 0$. □

With these two lemmata at hand, we can now formulate the following sufficient conditions such that b is representable.

Theorem 2.19. *Assume the prerequisites of Theorems 2.7 and 2.13 are met and assume the integrability condition from Theorem 1.6 (ii) is met, i.e., assume that there exist some t^* such that*

$$\int_{[0,\infty)} \exp\left(-\frac{\theta^2}{2t^*}\right) \mu_i(d\theta) < \infty$$

where $i = 1, 2$. Moreover, assume, one of the following conditions is met:

(i) $\lambda_1(\{t_0\}) = 0$

(ii) For every $\varepsilon \in (0, t_0)$ it holds $\lambda_2([t_0 - \varepsilon, t_0]) > 0$

Then, b is representable.

Proof. The proof uses Theorem 1.6 (ii) and the previous two lemmata (2.17) and (2.18). First, assume condition (i) holds. We know from Lemma 2.17 that λ_1 puts mass in every interval of the form $[t_0 - \varepsilon, t_0]$ for every $\varepsilon \in (0, t_0)$ but by assumption $\lambda_1(\{t_0\}) = 0$.

Due to the complementary slackness conditions from Theorem 2.7, we can conclude that there exists a strictly increasing sequence $t_1, t_2, \ldots \nearrow t_0$ such that

$$\int r_\theta(t_n, b(t_n))\mu_1(d\theta) = 1 \text{ for all } n \in \mathbb{N}.$$

Then, we can conclude from Theorem 1.6 (ii) that b is representable. The proof in the case that condition (ii) holds follows along the same lines with the help of Lemma 2.18 and Theorem 2.13. □

Note that Theorem 2.19 also offers sufficient conditions for b to be representable that are easier to check than the usual condition $r(t, b(t)) = 1$. In Chapter 3, we will investigate a new method to obtain numerical candidates for μ_1 and λ_1 or μ_2 and λ_2, respectively. Then, it is rather straightforward to check whether these measures fulfil the conditions from Theorem 2.19 at least numerically.

Chapter 3

Computational Method for the Linear Programming Approach

After a short introduction to existing numerical approaches for the inverse method of images, we give a convergence result for discretised versions of our programs and based on that a new algorithm. We also provide error bounds for the numerical distribution function of the first hitting time to a boundary b. This is followed by a numerical study of representability, i.e., a numerical study of the assumptions in Theorem 2.15. Finally, the chapter concludes with an investigation of two-sided boundaries.

3.1 Existing computational approaches for the inverse method of images

In [LRD02], the authors propose an approximation method for the inverse method of images. We adapt the notation from [LRD02] to our notation used above. Given a boundary b, the authors consider the equation

$$1 = \int_{[2b(0),\infty)} \exp\left(\frac{\theta b(t)}{t} - \frac{\theta^2}{2t}\right) \mu(d\theta)$$

for $t > 0$. The idea is now to approximate the actual but unknown representing measure μ with a measure $\tilde{\mu}$ such that the boundary $\tilde{b}$ generated by $\tilde{\mu}$ is close to b. The authors assume μ to be a weighted sum of point measures δ_{θ_r} with (positive or negative) weights w_r, $r = 1, \ldots, N$ for some $N \in \mathbb{N}$. Moreover, they choose a set of increasing time points

t_s, $s = 1, \dots, 2N$ to arrive at the simplified equation

$$1 = \sum_{r=1}^{N} w_r \exp\left(\theta_r \frac{b(t_s)}{t_s} - \frac{\theta_r^2}{2t_s}\right), \quad s = 1, \dots, 2N. \tag{3.1}$$

The values of θ_r can be pre-assigned in which case the given system of equations is linear in the weights w_r. Only N time points are then required to solve (3.1). The idea behind this discretisation is that the given boundary b and the boundary $\tilde{b}$ generated by the (signed) measure $\tilde{\mu}$ through the method of images are equal at the time points t_s. If the values of $\theta_r, r = 1, \dots, N$ are pre-assigned, N points are sufficient to solve the system of equations (3.1). The authors comment that a higher number of time points may be desirable to increase the accuracy of the approximation of b with $\tilde{b}$ but this may of course turn the system (3.1) singular.

The authors proceed to give approximate formulas for the distribution function and the density of the first hitting time where they take the formulas from the method of images and substitute the unknown representing measure μ with $\tilde{\mu}$ but keep the boundary b, which yields the formulas

$$\frac{b(t)}{2t^{3/2}} \phi\left(\frac{b(t)}{\sqrt{t}}\right) - \sum_{r=1}^{N} w_r \left(\frac{b(t) - \theta_r}{2t^{3/2}}\right) \phi\left(\frac{b(t) - \theta_r}{\sqrt{t}}\right)$$

for the density and for the distribution function

$$1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \sum_{r=1}^{N} w_r \Phi\left(\frac{b(t) - \theta_r}{\sqrt{t}}\right).$$

The authors also extend their method to two-sided boundaries (cf. Section 3.5, [LRD02]). Moreover, they investigate the approximation error by showing that for the first hitting times τ_b and $\tau_{\tilde{b}}$ to the boundaries b and $\tilde{b}$, respectively, it holds for all $t > 0$ that

$$|P(\tau_b < t) - P(\tau_{\tilde{b}} < t)| \to 0, \qquad \text{as } \bar{\varepsilon}_t := \sup_{0<s<t} |\tilde{b}(s) - b(s)| \to 0.$$

In other words: the distribution functions of τ_b and $\tau_{\tilde{b}}$ are close if b and $\tilde{b}$ are close. The authors do not, however, give any convergence result for the algorithm itself. The investigated examples of square-root and parabolic boundaries show fairly good approximations with deviations of $\tilde{b}$ from b especially for t small.

In [Zip13], Zipkin refines the numerical methods developed by among others [LRD02]. Again, we will slightly adapt the notation to the notation of this paper.

Given a boundary b, [Zip13] uses the same discretisations as [LRD02] but does not set the number of discretisations to N. Rather, denote by J the number of points with positive mass and by I the number of time discretisations. Then, set $w = (w_j)_{j=1,\dots,J}$ and let $\mathbb{1}$ be the vector of ones of length I, which enables us to restate Equation (3.1) as $Mw = \mathbb{1}$ for a suitable M. [Zip13] then relaxes this setting slightly by choosing slack variables s_i, $i = 1, \dots, I$ as well as a vector p of length I consisting of positive constants (e.g., $p = \mathbb{1}$). Then, the following linear program which has to be solved

$$\begin{aligned} \text{minimise} \quad & p^T s \\ \text{subject to} \quad & Mw + s = 1, \\ & w \geq 0, \\ & s \geq 0. \end{aligned}$$

The linear program's objective function minimises the deviations s. Solving this linear program will then give the weights $w_j, j = 1, \dots, J$ such that $\tilde{b}$ generated by $\tilde{\mu} = \sum_j w_j \delta_{\theta_j}$ most closely resembles b. [Zip13] does not give any convergence results for his algorithm. However, he finds that in examples the approximation $\tilde{b}$ of b generally works well on the discretised time interval $[t_1, t_I]$ but deviates from b outside that interval. The algorithm can nevertheless be seen as a substantial improvement on the algorithm from [LRD02]. Then, [Zip13] gives a generalisation of how to include not only point measures but also more general positive, σ-finite measures with densities.

Although the algorithms set out by [LRD02] and [Zip13] provide good approximations, they suffer from some drawbacks: in both cases, the measure has to be discretised as a weighted sum of point measures as well as the time axis has to be discretised. Moreover, for neither algorithm a convergence result is provided. Both problems are addressed in the following sections.

3.2 Convergence results and a new algorithm

In this section, we look at a possible implementation of the inverse method of images and to this end put forward an algorithm as well as convergence results. Similar to Section 2.2 above, this section relies on methods set out in [CKL22]. As before, we use analogous notations revealing parallel structures but also divergences where the problem at hand and the one presented in [CKL22] do not align.

Let us start by discretising the linear problem. To this end, let $\mu_i \in \mathcal{M}^+[2b(0), \infty)$, $i \in \mathbb{N}$, and let $U_n := \{\sum_{i=1}^n a_i\mu_i \mid a \in \mathbb{R}^n_+\}$ denote the positive cone generated by the measures $\mu_i, i = 1, \dots, n$. Restricting the linear program (P_2) to measures in U_n gives

$$\begin{aligned} &\text{minimise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in U_n, \\ & && \int r_\theta(t, b(t))\mu(d\theta) \geq 1 \text{ for any } t \in (0, t_0]. \end{aligned} \tag{$P_{2,n}$}$$

Then, we can prove the following existence and consistency result for our simplified program.

Proposition 3.1. *Assume $\mu_i \in \mathcal{M}^+(\mathbb{R}), i \in \mathbb{N}$ to be positive measures. Set $U_n := \{\sum_{i=1}^n a_i\mu_i \mid a \in \mathbb{R}^n_{\geq 0}\}$ and denote by U_∞ the closure of $\bigcup_{n\in\mathbb{N}} U_n$ with respect to the vague topology. For $n \in \mathbb{N} \cup \{\infty\}$ consider the linear program ($P_{2,n}$)*

$$\begin{aligned} &\text{minimise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in U_n, \\ & && \int r_\theta(t, b(t))\mu(d\theta) \geq 1 \textit{ for any } t \in (0, t_0] \end{aligned}$$

and assume that there exists $C \in \mathbb{R}_{>0}$ such that $C \cdot \mu_1$ is admissible in ($P_{2,1}$) and therefore in any ($P_{2,n}$) for $n \in \mathbb{N} \cup \{\infty\}$ and that $k \cdot \|\mu_1\| < \infty$. Then, the following assertions hold:

(a) Let $n \in \mathbb{N} \cup \{\infty\}$, then the linear program ($P_{2,n}$) attains its optimal value $p_{2,n}$ at some admissible measure $\mu^*_{2,n}$. The optimal value satisfies $p_{2,n} \leq k \cdot \|\mu_1\|$. Moreover, for $m \leq n$ the measure $\mu^*_{2,m}$ is ($P_{2,n}$)-admissible and it holds $p_{2,m} \geq p_{2,n}$.

(b) There exists a subsequence of optimisers $(\mu^*_{2,n_k})_k$ and a ($P_{2,\infty}$)-admissible measure ν_∞ such that $(\mu^*_{2,n_k})_k \to \nu_\infty$ vaguely. Moreover,

$$p_{2,\infty} \leq \int r_\theta(t_0, x_0)\nu_\infty(d\theta) \leq \inf_{n\in\mathbb{N}} p_{2,n} = \lim_{n\to\infty} p_{2,n}.$$

(c) If there exists a sequence $(\xi_n)_n$ with $\xi_n \in U_n$ converging weakly to some ($P_{2,\infty}$)-

optimal measure μ^*_∞ *as* $n \to \infty$ *and if*

$$\lim_{n\to\infty} \sup_{t\in(0,t_0]} \frac{|\int r_\theta(t,b(t))\mu_\infty(d\theta) - \int r_\theta(t,b(t))\xi_n(d\theta)|}{\int r_\theta(t,b(t))\mu_1(d\theta)} = 0,$$

then ν_∞ *from Assertion (b) is* $(P_{2,\infty})$*-optimal and* $(\mu^*_{2,n_k})_k$ *converges weakly to* ν_∞*. Moreover, we find*

$$p_{2,\infty} = \int r_\theta(t_0,x_0)\nu_\infty(d\theta) = \inf_{n\in\mathbb{N}} p_{2,n} = \lim_{n\to\infty} p_{2,n}.$$

Proof. (a) For any $n \in \mathbb{N}$ we again use the trick to slightly reformulate the program $(P_{2,n})$ by absorbing $r_\theta(t_0,x_0)$ into μ. The linear program now reads

$$\begin{aligned} &\text{minimise} && \|\mu\| \\ &\text{subject to} && \mu \in U_n, \\ &&& \int \frac{r_\theta(t,b(t))}{r_\theta(t_0,x_0)}\mu(d\theta) \geq 1 \text{ for any } t \in (0,t_0]. \end{aligned} \tag{$P'_{2,n}$}$$

We call $Z_{P'_{2,n}}$ the set of all admissible measures. Let $n \in \mathbb{N} \cup \{\infty\}$ and define $\mu_a := C \cdot \mu_1 \in U_n$. Recall that μ_a is $(P_{2,n})$-admissible by assumption. Now, set ${\mu'}_a := r_.(t_0,x_0) \cdot \mu_a$ and note that μ'_a is $(P'_{2,n})$-admissible and that every potential minimiser $\mu^* \in Z_{P'_{2,n}}$ satisfies

$$\|\mu^*\| \leq \|\mu'_a\| =: \rho$$

where $\rho < \infty$ by assumption. In particular, any potential solution $\mu^* \in Z_{P'_{2,n}}$ is contained in the vaguely compact ball $B_\mathcal{M}(\rho) = \{\mu \in \mathcal{M}[2b(0),\infty) \mid \|\mu\| \leq \rho\}$. Therefore, in the linear program $(P'_{2,n})$ it is sufficient to consider solutions from the set

$$Z_{P'_{2,n}} \cap B_\mathcal{M}(\rho) = \bigcap_{t\in[0,t_0]} H(t) \cap U_n \cap B_\mathcal{M}(\rho)$$

where $H(t) := \{\mu \in \mathcal{M}(\mathbb{R}) \mid \int \frac{r_\theta(t,b(t))}{r_\theta(t_0,x_0)}\mu(d\theta) \geq 1\}$. Note that $H(t)$ and U_n are closed with respect to the vague topology. So, we can conclude that $Z_{P'_{2,n}} \cap B_\mathcal{M}(\rho) \subset \mathcal{M}(\mathbb{R})$ is vaguely compact, as well. Then, the optimal value $p'_{2,n}$ in the linear program $(P'_{2,n})$ is attained by some measure $\mu'_{2,n} \in Z_{P'_{2,n}} \cap B_\mathcal{M}(\rho)$, cf.

Lemma A.5, as the functional $\mu \mapsto \|\mu\|$ is lower semi-continuous with respect to the vague topology due to Theorem A.7. Then, the optimal value $p_{2,n}$ in ($P_{2,n}$) is attained by $\mu^*_{2,n} := r_\cdot(t_0, x_0)^{-1} \cdot \mu'_{2,n}$.

(b) Recall that $\|\mu'_{2,n}\| \le \rho$ for any $n \in \mathbb{N}$. As before, Theorem A.7 tells us that the total variation unit ball is vaguely compact. So, we can find a subsequence $(\mu'_{n_k})_k$ and some measure $\nu'_\infty \in Z_{P'_{2,n}} \cap B_{\mathcal{M}}(\rho)$ such that $(\mu'_{n_k})_k \to \nu'_\infty$ vaguely. Then, we obtain that the subsequence $\mu^*_{2,n_k} = r_\cdot(t_0, x_0)^{-1} \cdot \mu'_{2,n_k}$ converges vaguely to $\nu_\infty = r_\cdot(t_0, x_0)^{-1} \cdot \nu'_\infty$ (cf. Lemma A.1). Let now $t \in (0, t_0]$. Then, the mapping $[2b(0), \infty) \to \mathbb{R}, \theta \mapsto \frac{r_\theta(t,b(t))}{r_\theta(t_0,x_0)}$ vanishes at infinity so we obtain

$$\int \frac{r_\theta(t, b(t))}{r_\theta(t_0, x_0)} \nu'_\infty(d\theta) = \lim_{k\to\infty} \int \frac{r_\theta(t, b(t))}{r_\theta(t_0, x_0)} \mu'_{n_k}(d\theta) \ge 1,$$

i.e., ν'_∞ is ($P'_{2,\infty}$)-admissible and therefore ν_∞ is ($P_{2,\infty}$)-admissible. Then, we have $p_{2,\infty} \le \int r_\theta(t_0, x_0)\nu_\infty(d\theta)$. Moreover, using vague convergence we find that $\|\nu'_\infty\| \le \liminf_{k\to\infty} \|\mu'_{1,n_k}\|$, cf. [Kle14], Lemma 13.15. As the sequence $(p_{2,n})_n$ is monotone, we can conclude that

$$p_{2,\infty} \le \int r_\theta(t_0, x_0)\nu_\infty(d\theta) = \|\nu'_\infty\| \le \liminf_{k\to\infty} \|\mu'_{1,n_k}\| = \inf_{n\in\mathbb{N}} p_{2,n} = \lim_{n\to\infty} p_{2,n}.$$

(c) Let $n \in \mathbb{N}$ and define $\eta_n := \xi_n + \varepsilon_n \mu_1$ where

$$\varepsilon_n := \sup_{t\in(0,t_0]} \frac{|\int r_\theta(t, b(t))\mu^*_\infty(d\theta) - \int r_\theta(t, b(t))\xi_n(d\theta)|}{\int r_\theta(t, b(t))\mu_1(d\theta)}.$$

By assumption, μ^*_∞ is ($P_{2,\infty}$)-optimal and therefore ($P_{2,\infty}$)-admissible. Then, we find for all $t \in (0, t_0]$

$$\begin{aligned}
&\int r_\theta(t, b(t))\eta_n(d\theta) - 1 \\
&\ge \int r_\theta(t, b(t))\eta_n(d\theta) - \int r_\theta(t, b(t))\mu^*_\infty(d\theta) \\
&= \int r_\theta(t, b(t))\mu_1(d\theta) \left(\varepsilon_n - \frac{\int r_\theta(t, b(t))\mu^*_\infty(d\theta) - \int r_\theta(t, b(t))\xi_n(d\theta)}{\int r_\theta(t, b(t))\mu_1(d\theta)} \right) \\
&\ge 0
\end{aligned}$$

which means that η_n is $(P_{2,n})$-admissible. With the help of Assertion (b) we obtain

$$\begin{aligned} p_{2,\infty} \leq \int r_\theta(t_0, x_0)\nu_\infty(d\theta) \leq p_{2,n} &\leq \int r_\theta(t_0, x_0)\eta_n(d\theta) \\ &= \int r_\theta(t_0, x_0)\xi_n(d\theta) + \varepsilon_n \int r_\theta(t_0, x_0)\mu_1(d\theta). \end{aligned}$$

Recall that the sequence $(\xi_n)_n$ was assumed to be weakly convergent to μ^*_∞ and therefore we obtain $\lim_{n\to\infty} \int r_\theta(t_0, x_0)\xi_n(d\theta) = \int r_\theta(t_0, x_0)\mu^*_\infty(d\theta) = p_{2,\infty}$. Putting this together, we finally find that

$$p_{2,\infty} \leq \int r_\theta(t_0, x_0)\nu_\infty(d\theta) \leq \lim_{n\to\infty} p_{2,n} \leq \lim_{n\to\infty} \int r_\theta(t_0, x_0)\eta_n(d\theta) = p_{2,\infty}$$

as $\lim_{n\to\infty} \varepsilon_n = 0$ by assumption. In particular, ν_∞ is $(P_{2,\infty})$-optimal. Also, we find that $\lim_{n\to\infty} \|\mu'_{2,n}\| = \lim_{n\to\infty} p_{2,n} = \int r_\theta(t_0, x_0)\nu_\infty(d\theta) = \|\nu'_\infty\|$. In Assertion (b), it was shown that a subsequence $(\mu'_{n_k})_k$ converges vaguely to ν'_∞. Recall that $\mathbb{R}$ is locally compact and Polish with respect to the Euclidean topology. Then, the Portmanteau theorem guarantees that $(\mu'_{2,n_k})_k$ converges weakly to the measure ν'_∞ (for example, cf. [Kle14], Theorem 13.16). Finally, we find that $\mu^*_{2,n_k} = r_\cdot(t_0, x_0)^{-1} \cdot \mu'_{2,n_k}$ converges weakly to $\nu_\infty = r_\cdot(t_0, x_0) \cdot \nu'_\infty$.

□

We can formulate an analogous statement for (D_1).

Proposition 3.2. *Assume $\mu_i \in \mathcal{M}^+(\mathbb{R}), i \in \mathbb{N}$, to be positive measures. Set $U_n := \{\sum_{i=1}^n a_i\mu_i \mid a \in \mathbb{R}^n_{\geq 0}\}$ and denote by U_∞ the closure of $\bigcup_{n\in\mathbb{N}} U_n$ with respect to the vague topology. For $n \in \mathbb{N} \cup \{\infty\}$ consider the linear program*

$$\begin{aligned} \text{maximise} \quad & \int r_\theta(t_0, x_0)\mu(d\theta) \\ \text{subject to} \quad & \mu \in U_n, \\ & \int r_\theta(t, b(t))\mu(d\theta) \leq 1 \text{ for any } t \in (0, t_0] \end{aligned} \tag{$D_{1,n}$}$$

and assume that there exists $C \in \mathbb{R}_{>0}$ such that $C \cdot \mu_1$ is admissible in $(D_{1,1})$ and therefore in any $(P_{2,n})$ for $n \in \mathbb{N} \cup \{\infty\}$ and that $C \cdot \|\mu_1\| < \infty$. Then,

(a) Let $n \in \mathbb{N} \cup \{\infty\}$, then the linear program $(D_{1,n})$ attains its optimal value $d_{1,n}$ at some admissible measure $\mu^*_{1,n}$. Moreover, for $m \leq n$ the measure $\mu^*_{1,m}$ is $(D_{1,n})$-admissible and it holds $d_{1,m} \leq d_{1,n}$.

(b) There exists a subsequence of optimisers $(\mu^*_{1,n_k})_k$ *and a* $(D_{1,\infty})$*-admissible measure* ν_∞ *such that* $(\mu^*_{1,n_k})_k \to \nu_\infty$ *vaguely. Moreover,*

$$d_{1,\infty} \geq \int r_\theta(t_0, x_0)\nu_\infty(d\theta) \geq \sup_{n\in\mathbb{N}} d_{1,n} = \lim_{n\to\infty} d_{1,n}.$$

(c) If there exists a sequence $(\xi_n)_n$ *with* $\xi_n \in U_n$ *converging weakly to some* $(D_{1,\infty})$*-optimal measure* μ_∞ *as* $n \to \infty$ *and if*

$$\lim_{n\to\infty} \inf_{t\in(0,t_0]} \frac{|\int r_\theta(t,b(t))\mu_\infty(d\theta) - \int r_\theta(t,b(t))\xi_n(d\theta)|}{\int r_\theta(t,b(t))\mu_1(d\theta)} = 0,$$

then ν_∞ *from Assertion (b) is* $(D_{1,\infty})$*-optimal and* $(\mu^*_{1,n_k})_k$ *converges weakly to* ν_∞*. Moreover, we find*

$$d_{1,\infty} = \int r_\theta(t_0, x_0)\nu_\infty(d\theta) = \sup_{n\in\mathbb{N}} d_{1,n} = \lim_{n\to\infty} d_{1,n}.$$

Proof. As before, consider the modified program

$$\begin{aligned} &\text{maximise} \quad \|\mu\| \\ &\text{subject to} \quad \mu \in U_n, \\ &\qquad\qquad\quad \int \frac{r_\theta(t,b(t))}{r_\theta(t_0,x_0)}\mu(d\theta) \leq 1 \text{ for any } t \in (0,t_0]. \end{aligned} \tag{$D'_{1,n}$}$$

Note that for all $(D'_{1,n})$-admissible μ we find by evaluating the constraint at $t = t_0$ that

$$\begin{aligned} 1 &\geq \int_{[2b(0),\infty)} \exp\left(-\frac{\theta^2}{2t_0} + \frac{\theta b(t_0)}{t_0} + \frac{\theta^2}{2t_0} - \frac{\theta x_0}{t_0}\right)\mu(d\theta) \\ &= \int_{[2b(0),\infty)} \exp\left(\theta\frac{b(t_0)-x_0}{t_0}\right)\mu(d\theta) \geq \int_{[2b(0),\infty)} \mu(d\theta) = \|\mu\|. \end{aligned}$$

Thus, all admissible solutions of $(D'_{1,n})$ are contained in $B_{\mathcal{M}}(1) = \{\mu \in \mathcal{M}[2b(0),\infty) \mid \|\mu\| \leq 1\}$. The rest of the proof follows along the same lines as the proof of Proposition 3.1. □

Similar results can be formulated and proven for the programs (P_1) and (D_2), see Appendix A.2.

Note that we can limit ourselves to point measures on a dense subset of $[2b(0),\infty)$ for our choices of the μ_i in the previous Propositions 3.1 and 3.2 and still be able to approximate every possible representing measure μ arbitrarily close. This is due to the fact that for every locally compact space E (in our case $E = [2b(0), \infty)$) the set of discrete Radon measures (i.e., the set U_∞ from the above propositions) is vaguely dense in $\mathcal{M}^+(E)$ (i.e., $\mathcal{M}^+[2b(0), \infty)$ in our case). This result can be found as Theorem 30.4 in [Bau01]. The inclusion of measures with densities, as done in [Zip13], is therefore not necessary for the convergence of the algorithm but may be useful for numerical reasons.

We can now outline a possible algorithm by considering the following simplification of our discretised linear problem ($P_{2,n}$). We consider the interval $[2b(0), 2b(0)+l]$, where $l > 0$ is some constant determining the interval length. Then, we choose a discretisation into n equidistant points θ_i, $i = 1, \ldots, n$, where $\theta_1 = 2b(0)$ and $\theta_n = 2b(0) + l$. Let $\mu_i, i = 1, \ldots, n$ be point measures at points θ_i and let again $U_n := \{\sum_{i=1}^n a_i \mu_i \mid a \in \mathbb{R}_+^n\}$ denote the positive cone generated by the measures $\mu_i, i = 1, \ldots, n$. Recall the restriction of the linear program ($P_{2,n}$) to measures in U_n:

$$\begin{aligned} &\text{minimise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in U_n, \\ & && \int r_\theta(t, b(t))\mu(d\theta) \geq 1 \text{ for any } t \in (0, t_0]. \end{aligned} \tag{$P'_{2,n}$}$$

This can then be simplified as

$$\begin{aligned} &\text{minimise} && \sum_{i=1}^n r_{\theta_i}(t_0, x_0) a_i \\ &\text{subject to} && a \in \mathbb{R}_+^n, \\ & && \sum_{i=1}^n r_{\theta_i}(t, b(t)) a_i \geq 1 \text{ for any } t \in (0, t_0]. \end{aligned} \tag{3.2}$$

In order to implement this, we make use of the cutting plane algorithm described in [LW92] and [Ito+10]. These two papers also provide good convergence results. We can now outline the following algorithm:

Step 1: Let the set of initial constraints $\Gamma_1 \subset (0, t_0]$ be the set $\{t_0\}$. Choose a maximum number of iterations k_{max} and set $k = 1$.

Step 2: Calculate the solution $a^{(k)} \in \mathbb{R}_+^n$ of the following finite dimensional linear program

$$\begin{aligned} \text{minimise} \quad & \sum_{i=1}^{n} r_{\theta_i}(t_0, x_0) a_i^{(k)} \mu(\theta_i) \\ \text{subject to} \quad & a^{(k)} \in \mathbb{R}_+^n, \\ & \sum_{i=1}^{n} r_{\theta_i}(t, b(t)) a_i^{(k)} \mu(\theta_i) \geq 1 \text{ for all } t \in \Gamma_k. \end{aligned} \tag{3.3}$$

Step 3: Calculate the point $t^{(k)} \in (0, t_0]$ where the constraint is most severely violated, i.e.,

$$t^{(k)} := \underset{t \in (0,t_0]}{\arg\min} \left\{ \sum_{i=1}^{n} r_{\theta_i}(t, b(t)) a_i^{(k)} \mu(\theta_i) \right\}.$$

Step 4: Add the point $t^{(k)}$ of the most severe violation of the constraints to the set of constraints, i.e., set $\Gamma_{k+1} := \Gamma_k \cup \{t^{(k)}\}$.

Step 5: If the maximum number of iterations is reached, i.e., $k = k_{max}$, terminate the algorithm and output the approximate solution

$$\tilde{a} := a^{(k)}.$$

Otherwise, increase the iteration counter $k \rightsquigarrow k+1$ and return to the second step.

Note that in Proposition 3.1 we do not assume the measures $\mu_i, i \in \mathbb{N}$, to be point measures as we assume in the algorithm. This indicates that the algorithm also works for a much larger class of "auxiliary measures" μ_i. Moreover, the proposed algorithm uses fewer assumptions than the algorithm proposed in [Zip13] as we only choose points in $[2b(0), \infty)$ where the algorithm can put point masses and the initial constraint at t_0. Then, the algorithm "chooses" the next points where the constraint is evaluated. In [Zip13], both the division of $[2b(0), \infty)$ as well as the points $t_1, \dots, t_m$ where the constraint is evaluated have to be chosen.

For the linear program (D_1), we use analogous simplifications and an analogous algorithm. For the linear programs (D_2) and (P_1), i.e., for the "λ"-problems, we divide the interval $(0, t_0]$ into n_λ equidistant points $0 < t_1 < \ldots < t_{n_\lambda} = t_0$. Again, we choose the measures λ_i for $i = 1, \dots, n_\lambda$ to be the point measures in t_i. We can then use analogous algorithms which are initialised with $\Gamma_1 = \{2b(0)\}$.

If we have obtained a candidate representing measure from one of the above outlined algorithms, it is interesting to know how far apart the cumulative distribution function generated by this measure is from the actual cumulative distribution function. Recall that if a measure μ represents b in the sense that $r(t, b(t)) = 1$ for all $t \in (0, t_0]$, then $r(t, x) = P(\tau \le t | W_t = x)$ and therefore the cumulative distribution function F of τ is given by

$$\begin{aligned} F(t) &= P(W_t \ge b(t)) + \int_{-\infty}^{b(t)} P(\tau \le t | W_t = x) p_t(0, x) dx \\ &= 1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \int_{-\infty}^{b(t)} r(t, x) p_t(0, x) dx. \end{aligned}$$

If we now have obtained a measure $\tilde{\mu}$ from our algorithms, we define $\tilde{r}(t, x) = \int_{[2b(0), \infty)} r_\theta(t, x) \tilde{\mu}(d\theta)$ and approximate the true cumulative distribution function F with

$$\tilde{F}(t) = 1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \int_{-\infty}^{b(t)} \tilde{r}(t, x) p_t(0, x) dx.$$

Now, we can give the following result on the approximation of F with $\tilde{F}$.

Proposition 3.3. *Let F be the true first hitting time distribution to a boundary b and $\tilde{\mu}$ a measure such that for all $t \in (0, t_0]$*

$$1 - \delta_1 \le \tilde{r}(t, b(t))^{-1} \le 1 + \delta_2$$

for some $\delta_1 \in [0, 1)$ and $\delta_2 > 0$. Then,

$$\sup_{t \in (0, t_0]} |F(t) - \tilde{F}(t)| \le \max(\delta_1, \delta_2).$$

In particular, we have

$$\lim_{\substack{\delta_1 \searrow 0 \\ \delta_2 \searrow 0}} \sup_{t \in (0, t_0]} |F(t) - \tilde{F}(t)| = 0.$$

Proof. Let $t \in (0, t_0]$. By assumption, we have for all $t \in (0, t_0]$

$$1 - \delta_1 \le \tilde{r}(t, b(t))^{-1} \le 1 + \delta_2$$

for some $\delta_1 \in [0, 1)$ and $\delta_2 > 0$. Then, Proposition 1.3 tells us that

$$(1 - \delta_1)\tilde{r}(t, x) \leq P_{(0,0)}(\tau \leq t | W_t = x) \leq (1 + \delta_2)\tilde{r}(t, x)$$

for all $x \leq b(t)$. Then, we find

$$\begin{aligned} F(t) &= 1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \int_{-\infty}^{b(t)} P(\tau \leq t | W_t = x) p_t(0, x) dx \\ &\leq 1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \int_{-\infty}^{b(t)} (1 + \delta_2)\tilde{r}(t, x) p_t(0, x) dx \\ &= (1 + \delta_2)\tilde{F}(t) - \delta_2 \left(1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right)\right) \\ &\leq (1 + \delta_2)\tilde{F}(t) \end{aligned}$$

where $p_t(0, x)$ is the transition kernel of a Brownian motion going from 0 to x in a time span t. In particular, we can conclude

$$F(t) - \tilde{F}(t) \leq \delta_2.$$

Analogously, we find

$$F(t) - \tilde{F}(t) \geq -\delta_1.$$

Together, we obtain

$$\sup_{t \in (0, t_0]} |F(t) - \tilde{F}(t)| \leq \max(\delta_1, \delta_2).$$

□

3.3 Numerical study of representability

In this section, we consider a number of concave, analytic boundaries b and investigate their representability. To this end, we have implemented the four programs as laid out at the end of Section 3.2. For all boundaries, we choose $t_0 = 1$, $x_0 = b(t_0) - 1$ and the maximum number of iterations $k_{max} = 20$. Moreover, for the "μ-problems" (D_1) and (P_2) we select a length of $l = 5$ and discretise the intervall $[2b(0), \infty)$ by setting $n = 100$ equidistant points in $[2b(0), 2b(0) + l]$ where the algorithm can put point mass. For the "λ-problems" (P_1) and (D_2) we discretise the interval $(0, t_0]$ by choosing $n_\lambda = 100$ equidistant points where the algorithm can put mass. Table 3.1 gives the optimal values of all these four programs as well as the computation time for the implementation of (D_1). The other programs have similar computation times. All programs were implemented in `R`. In Step 3 of the algorithm above, we have to determine the point where the constraint is most severely violated. In our implementation we used the package "RcppDE" and the function `DEoptim` for finding this extremal point. It is important to note that `DEoptim` is an implementation of the differential evolution algorithm and its result is a random variable. Therefore, different runs of our implementation may yield slightly different results. For more details on this see the documentation of the package "RcppDE" which can be found in [Mul+11]. The results from our runs can be found in Table 3.1. Moreover, we can consider the functions

$b = b(t)$	$d_{1,n}$	p_{1,n_λ}	d_{2,n_λ}	$p_{2,n}$	Time (in sec)
$1+t$	0.1353353	0.1353353	0.1353353	0.1353353	0.43
$\sqrt{1+t}$	0.1274203	0.1274203	0.1274203	0.1274203	0.45
$\log(2+t)$	0.2364878	0.2364878	0.2364878	0.2364878	0.61
$1+t^2$	0.1353353	0.1353353	0.9801987	0.9980020	0.52

Table 3.1: Numerical results for the optimal values and computations times for (D_1) with discretisations as defined above

$r_{\mu_{1,n}}(t, b(t)) := \int_{[2b(0),\infty)} r_\theta(t, b(t))\mu_{1,n}(d\theta)$ and $r_{\mu_{2,n}}(t, b(t))$ between 0 and $t_0 = 1$ and can consider the minima and maxima of the inverse of these functions, i.e., $r^{-1}_{\mu_{1,n}}$ and $r^{-1}_{\mu_{2,n}}$. According to Proposition 3.3, this will give us an idea of the quality of approximation of the true c.d.f. F with the numerical c.d.f. $\tilde{F}$. The results can be found in Table 3.2

$b = b(t)$	$\min r^{-1}_{\mu_{1,n}}$	$\max r^{-1}_{\mu_{1,n}}$	$\min r^{-1}_{\mu_{2,n}}$	$\max r^{-1}_{\mu_{2,n}}$
$1+t$	0.999999999953	1.000000000016	0.999999999996	1.000000000037
$\sqrt{1+t}$	0.999999986788	1.002635679587	0.992114401333	1.000003242113
$\log(2+t)$	0.999999954916	1.001276226376	0.998198226881	1.000000116299
$1+t^2$	1.000000005738	7.389040705848	0.135606226531	1.001999983886

Table 3.2: Numerical results for the minima and maxima of $r^{-1}_{\mu_{1,n}}$ and $r^{-1}_{\mu_{2,n}}$ on the interval $(0, t_0,]$

We can immediately see in Table 3.1 that in the case of the linear boundary $b(t) = 1+t$ the values of $d_{1,n}, p_{1,n_\lambda}, d_{2,n_\lambda}$ and $p_{2,n}$ agree in the first 8 digits. So, we can heuristically confirm both strong duality in both set-ups and that the conditions for representability from Theorem 2.15 are met, i.e., $d_1 = p_2$ and $p_2 = d_1$. This is, of course, not surprising as we already know that linear boundaries b are representable. So, this case serves as a sanity check on our algorithm.

Moreover, we can consider the boundaries $b_{\mu_{1,n}}$ and $b_{\mu_{2,n}}$ generated by the numerical solutions $\mu_{1,n}$ and $\mu_{2,n}$ and compare these boundaries to the boundary b that was the input for the optimisation problems. Figure 3.1 depicts the graphs of these three boundaries in the first picture, show the value of $r_{\mu_{1,n}}(t, b(t))$ and $r_{\mu_{2,n}}(t, b(t))$ between 0 and 1 in the second subfigure and the distribution function obtained by substituting the representing measure μ with the numerical solutions $\mu_{1,n}$ and $\mu_{2,n}$, respectively, i.e.,

$$F_{\mu_i}(t) := 1 - \Phi\left(\frac{b(t)}{\sqrt{t}}\right) + \int_0^\infty 1 - \Phi\left(\frac{\theta - b(t)}{\sqrt{t}}\right) \mu_i(d\theta)$$

for $i = 1, 2$. Note that F_{μ_i} yields an approximation of the true distribution function F in analytical form. We can see in Figure 3.1 that both numerical boundaries perfectly replicate the original boundary as the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue) are completely overlapped by the original boundary. In particular, the graphs of $r_{\mu_{i,n}}(t, b(t)), i = 1, 2$ which are equal to 1 support that b is represented by $\mu_{i,n}$. This is not very surprising as it is well known that linear boundaries are representable by measures μ which only put mass into $2b(0)$ which both $\mu_{1,n}$ and $\mu_{2,n}$ do. Moreover, we can see in Table 3.2 that $r^{-1}_{\mu_{1,n}}$ and $r^{-1}_{\mu_{2,n}}$ deviate from 1 by less than 10^{-10}. The deviation should be 0 and can probably be attributed to small numerical rounding errors. In particular, we know due to Proposition 3.3 that we get a very good approximation of the distribution function F.

Let us now consider the boundaries $b(t) = \sqrt{1+t}$ and $b(t) = \log(2+t)$, i.e., boundaries which are concave and monotone increasing. For both boundaries, we can observe (compare Table 3.1) that the values of $d_{1,n}, p_{1,n_\lambda}, d_{2,n_\lambda}$ and $p_{2,n}$ agree in the first 8 digits, i.e., we can again numerically confirm both strong duality in both set-ups as well as the conditions for representability from Theorem 2.15. Even though we choose $t_0 = 1$ we observe in Figure 3.2 that the boundary is very well replicated by the boundaries generated by $\mu_{2,n}$ and $\mu_{1,n}$ up to $t = 3$ with just slight deviations between $t = 2$ and $t = 3$. The graphs for $r_{\mu_{1,n}}(t, b(t))$ and $r_{\mu_{2,n}}(t, b(t))$ are virtually indistinguishable from 1 and we can see in Table 3.2 that $r^{-1}_{\mu_{1,n}}$ and $r^{-1}_{\mu_{2,n}}$ deviate from 1 by less than 10^{-2}. Thus, the graphs of the numerical distribution functions $F_{\mu_i}, i = 1, 2$ are very exact approximations for F due to Proposition 3.3. Moreover, note that Proposition 3.3 gives

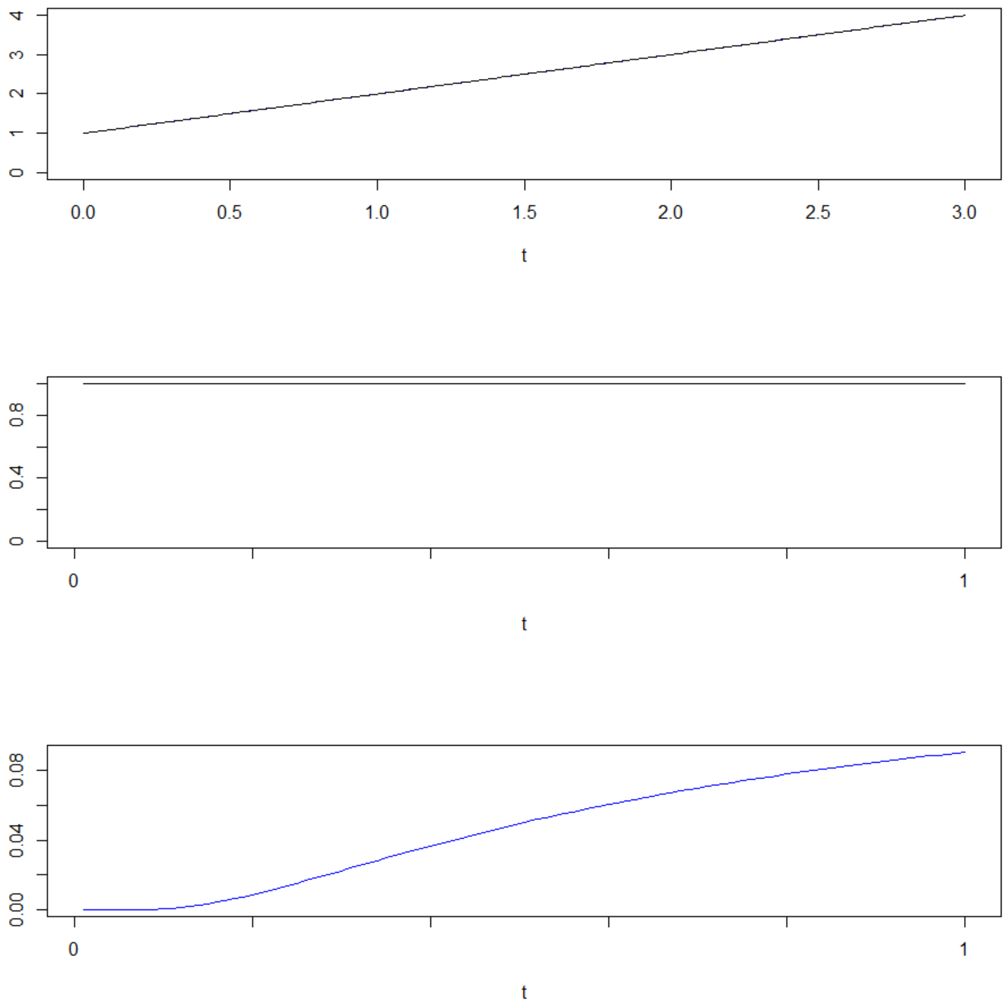

Figure 3.1: Numerical results for $b(t) = 1 + t$.
Top: graphs of the original boundary (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b(t))$ (red) and $r_{\mu_{2,n}}(t, b(t))$ (blue) between 0 and 1 overlapped by the function constant 1 (black).
Bottom: numerical approximations of the distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

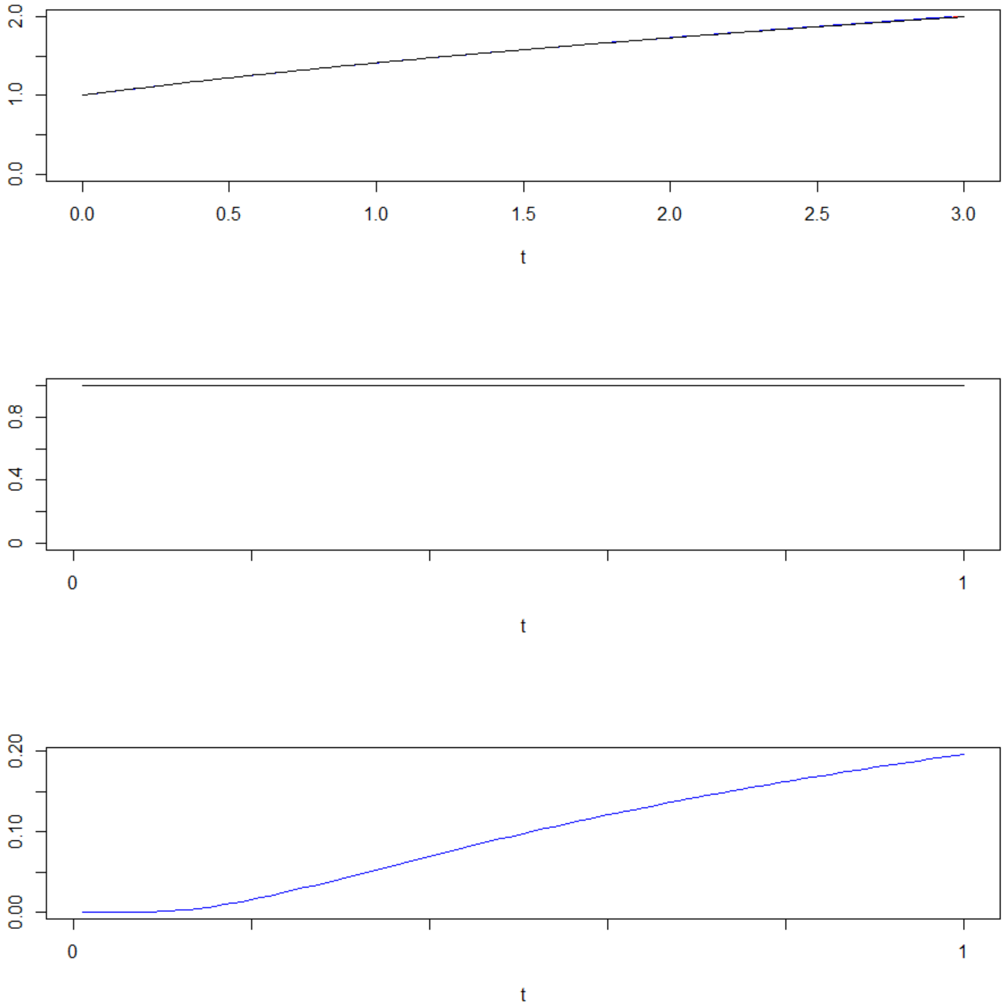

Figure 3.2: Numerical results for $b(t) = \sqrt{1+t}$
Top: graphs of the original boundary (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b(t))$ (red) and $r_{\mu_{2,n}}(t, b(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of the distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

a rather conservative estimate of the approximation error, so the true approximation error is probably much less.

For $b(t) = \log(2+t)$ we observe very similar results in Figure 3.3. The boundaries generated by $\mu_{i,n}, i = 1,2$ start to deviate slightly from the original b around $t = 2$. In much the same way as before, $r_{\mu_{1,n}}(t, b(t))$ and $r_{\mu_{2,n}}(t, b(t))$ are virtually indistinguishable from 1 and we can see in Table 3.2 that $r^{-1}_{\mu_{1,n}}$ and $r^{-1}_{\mu_{2,n}}$ deviate from 1 by less than 10^{-2}. So, the graphs of the numerical distribution functions F_{μ_i} are very exact approximations for F due to Proposition 3.3. Finally, we turn our attention to $b(t) = 1 + t^2$. We immediately see from Table 3.1 that $d_{1,n}$ and p_{1,n_λ} agree in the first 8 digits and d_{2,n_λ} and $p_{2,n}$ agree in the first 2 digits, so we can safely assume that strong duality holds in both set-ups. However, the gaps between the two different set-ups are very large. We immediately see in Figure 3.4 why the algorithms does not produce sensible results: both programs only allow for concave boundaries and the "most convex" the program can do is a linear boundary. Unsurprisingly, $r_{\mu_{1,n}}(t, b(t))$ is for the most part very far away from 1. The same holds true for $r_{\mu_{2,n}}(t, b(t))$ which is of course above 1 and not below as is $r_{\mu_{1,n}}(t, b(t))$. Expectedly, the numerical versions of the distribution functions are vastly different from one another.

Finally, we will investigate whether the conditions of Theorem 2.19 are numerically met. Whether λ_{1,n_λ} does not put mass into t_0 is hard to verify numerically, since the mass could be so small that it numerically vanishes. But we can investigate the mass which λ_{2,n_λ} puts close to t_0 for increasing n_λ. Due to the discretisation of the t axis, we look for mass in the interval $((n-1)t_0/n, t_0]$. In order to make sure that the mass near t_0 does not vanish, we need to ensure that x_0 is not too far below $b(t_0)$ as the hitting probability might become 0 numerically. So, we choose $x_0 = b(t_0) - 0.1$. All other parameters are chosen in the same way as before. Linear b are omitted in this test, as we already know that $\bar{\lambda}$, the conditional last hitting time distribution of W to b, is the optimal measure in this case (see Corollary 2.16 above). We obtain the values in Table 3.3 We can immediately see in Table 3.3 that for the concave boundaries

$b = b(t)$	$n_\lambda = 100$	$n_\lambda = 200$	$n_\lambda = 500$	$n_\lambda = 1000$
$\sqrt{1+t}$	0.233	0.282	0.276	0.283
$\log(2+t)$	0.293	0.290	0.296	0.055
$1+t^2$	0.000	0.000	0.000	0.000

Table 3.3: Numerical results for $\lambda_{2,n_\lambda}((n_\lambda - 1)t_0/n_\lambda, t_0]$. Values rounded to 3 decimal points.

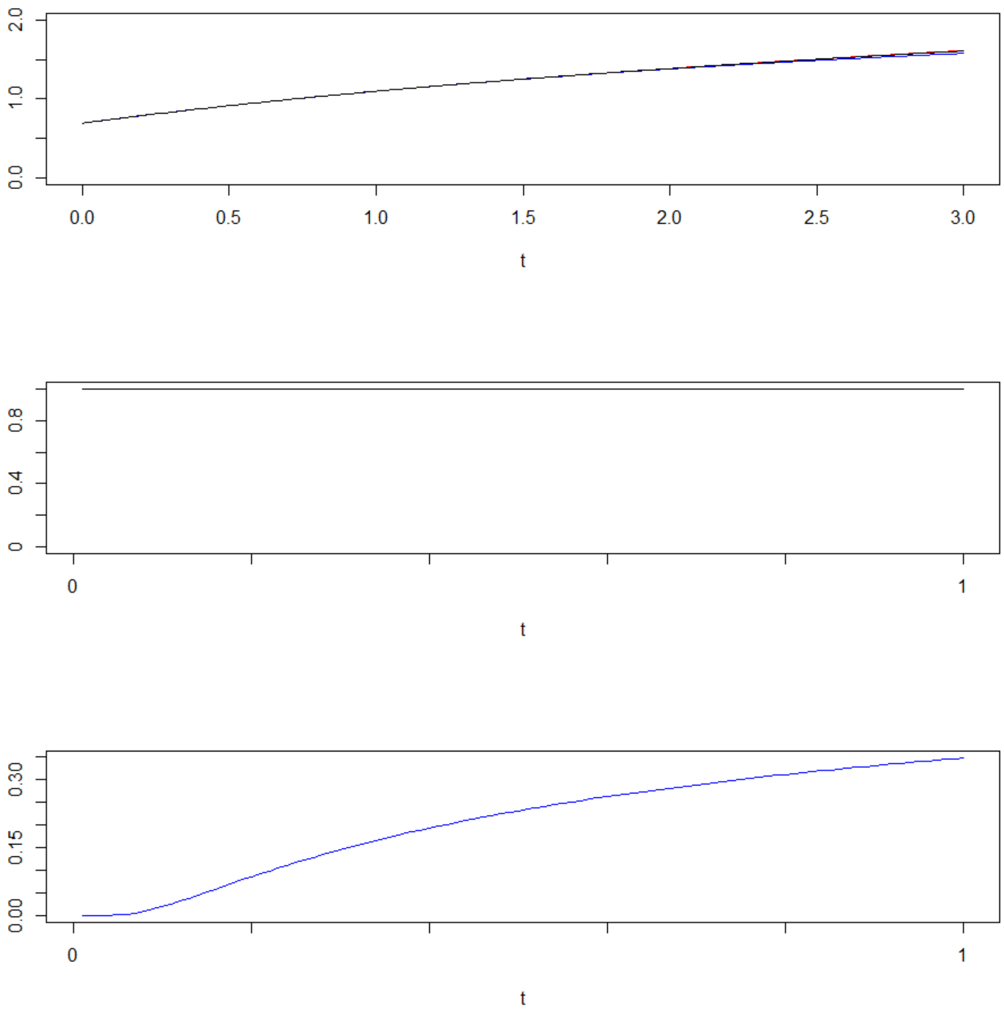

Figure 3.3: Numerical results for $b(t) = \log(2 + t)$
Top: graphs of the original boundary (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b(t))$ (red) and $r_{\mu_{2,n}}(t, b(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of the distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

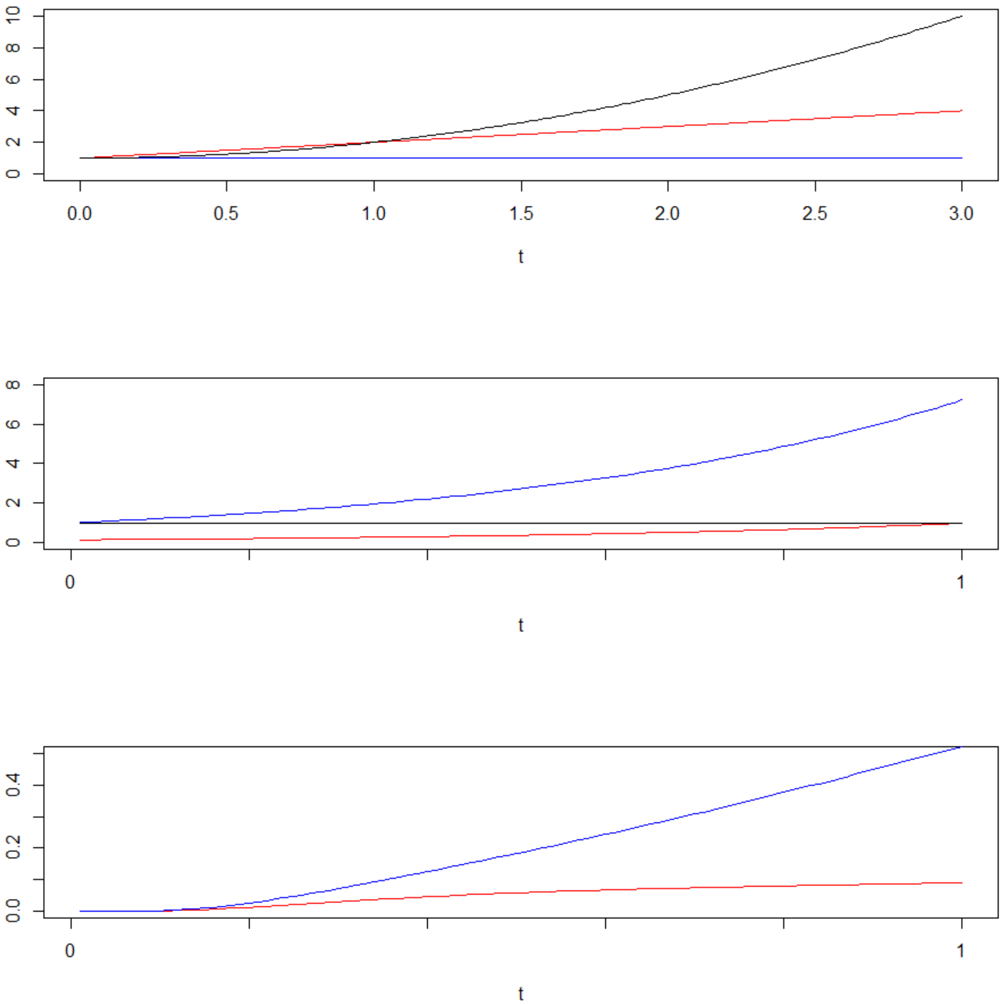

Figure 3.4: Numerical results for $b(t) = 1 + t^2$
Top: graphs of the original boundary (black) and the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b(t))$ (red) and $r_{\mu_{2,n}}(t, b(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of the distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

the measure $\lambda_{2,n}$ always puts mass near t_0 while we increase n_λ. This is yet another very good indicator that these boundaries are representable according to Theorem 2.19. Considering the fact that for the convex boundary there is no mass near t_0 for any choice of n_λ, this is an indicator that the condition from Theorem 2.19 is not only sufficient but might also be necessary.

Overall, the algorithm shows perfect replication of linear boundaries and very good replication of concave boundaries even beyond the controlled interval $(0, t_0]$. Unsurprisingly, for convex boundaries the programs do not find reasonable measures. The convergence times for the algorithms were in all cases very fast. In total, we found strong numerical evidence that concave boundaries are indeed representable in the sense of the method of images. Moreover, we obtain an approximation of the distribution function in analytical form which makes it easy to use in applications.

The algorithm also compares favourably with the algorithms from [LRD02] and [Zip13] as we only require a discretisation of the approximating measures but not of the time axis. The replication of the boundaries is very accurate even beyond the controlled interval $(0, t_0]$, something neither [LRD02] nor [Zip13] achieved. Finally, we also provide convergence results for the algorithms (cf. Prop. 3.1 and 3.2 above) which were also missing in both [LRD02] and [Zip13].

3.4 Numerical investigation of two-sided boundaries

At the end of Section 1.3, we have seen that the method of images extends to two-sided boundaries. We can therefore formulate linear programs for this problem which are stated analogously to the problems for the classic method of images with one boundary. Again, we can give two linear programs:

$$\begin{aligned} \text{maximise} \quad & \int r_\theta(t_0, x_0)\mu(d\theta) \\ \text{subject to} \quad & \mu \in \mathcal{M}^+((-\infty, 2b_1(0)] \cup [2b_2(0), \infty)), \\ & \int r_\theta(t, b_1(t))\mu(d\theta) \leq 1 \text{ for any } t \in (0, t_0] \\ & \int r_\theta(t, b_2(t))\mu(d\theta) \leq 1 \text{ for any } t \in (0, t_0] \end{aligned} \tag{$\tilde{D}_1$}$$

where we approximate the measure μ representing b_1 and b_2 “from below” as well as

$$\begin{aligned} &\text{minimise} && \int r_\theta(t_0, x_0)\mu(d\theta) \\ &\text{subject to} && \mu \in \mathcal{M}^+((-\infty, 2b_1(0)] \cup [2b_2(0), \infty)), \\ &&& \int r_\theta(t, b_1(t))\mu(d\theta) \geq 1 \text{ for any } t \in (0, t_0] \\ &&& \int r_\theta(t, b_2(t))\mu(d\theta) \geq 1 \text{ for any } t \in (0, t_0] \end{aligned} \tag{$\tilde{P}_2$}$$

where we approximate the measure μ representing b_1 and b_2 “from above”. The following proposition, which is an extension of Theorem 2.1 to the two-sided case, shows that the programs ($\tilde{D}_1$) and ($\tilde{P}_2$) indeed approximate the boundaries b_1 and b_2 from below and from above in a certain sense.

Proposition 3.4. *For each ($\tilde{D}_1$)-admissible $\tilde{\mu}_1$, we find that*

$$r_{\mu_1}(t_0, x_0) := \int r_\theta(t_0, x_0)\tilde{\mu}_1(d\theta) \leq P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0)$$

and for each ($\tilde{P}_2$)-admissible solution $\tilde{\mu}_2$ it holds

$$r_{\mu_2}(t_0, x_0) := \int r_\theta(t_0, x_0)\tilde{\mu}_2(d\theta) \geq P_{(0,0)}(\tau \leq t_0 | W_{t_0} = x_0)$$

Moreover, if there exists a representing measure $\tilde{\mu}$, i.e.,

$$\int r_\theta(t, b(t))\tilde{\mu}(d\theta) = 1 \qquad \textit{for any } t \in (0, t_0],$$

then this is a maximiser in ($\tilde{D}_1$) and a minimiser in ($\tilde{P}_2$).

Proof. The proof follows along the same lines as the proof of Theorem 2.1 but now of course using Proposition 1.7 instead of Proposition 1.3. □

To further analyse the problems ($\tilde{D}_1$) and ($\tilde{P}_2$), we can again consider the associated formal dual problems. For ($\tilde{D}_1$) the dual program is

$$\begin{aligned} &\text{minimise} && \|\lambda_1\| + \|\lambda_2\| \\ &\text{subject to} && \lambda \in \mathcal{M}^+(0, t_0], \\ &&& \int r_\theta(t, b_1(t))\lambda_1(dt) + \int r_\theta(t, b_2(t))\lambda_2(dt) \geq r_\theta(t_0, x_0) \\ &&& \text{for any } \theta \in (-\infty, 2b_1(0)] \cup [2b_2(0), \infty) \end{aligned} \tag{$\tilde{P}_1$}$$

and for ($\tilde{P}_2$) the dual program is

$$\begin{aligned} &\text{maximise} && \|\lambda_1\| + \|\lambda_2\| \\ &\text{subject to} && \lambda \in \mathcal{M}^+(0, t_0], \\ &&& \int r_\theta(t, b(t))\lambda(dt) + \int r_\theta(t, b_2(t))\lambda_2(dt) \leq r_\theta(t_0, x_0) \\ &&& \text{for any } \theta \in (-\infty, 2b_1(0)] \cup [2b_2(0), \infty). \end{aligned} \tag{$\tilde{D}_2$}$$

Finally, we can implement these problems in the same way as in Section 3.3. We choose again $t_0 = 1$ but now $x_0 = 0$ to land "in the middle" between $b_1(t_0)$ and $b_2(t_0)$. Moreover, we choose the maximum number of iterations as $k_{max} = 20$, the length of the intervals $[2b_1(0) - l, 2b_1(0)]$ and $[2b_2(0), 2b_2(0) + l]$ as $l = 5$ with $n = 100$ discretisations for the point measures. For the corresponding "λ-problems" we again let $n_\lambda = 100$. Denote the optimal values as $\tilde{d}_{1,n}$ for the numerical discretisation of ($\tilde{D}_1$) and analogously for the other programs. Then, the numerical results can be found in Table 3.4. First, we note

$b_1 = b_1(t)$	$b_2 = b_2(t)$	$\tilde{d}_{1,n}$	$\tilde{p}_{1,n_\lambda}$	$\tilde{d}_{2,n_\lambda}$	$\tilde{p}_{2,n}$	Time (in sec)
$-1 - t$	$1 + t$	0.0366190	0.0366190	0.0366313	0.0366313	0.80
$-\sqrt{1+t}$	$\sqrt{1+t}$	0.1096176	0.1096176	0.1096176	0.1096176	0.95
$-\log(2+t)$	$\log(2+t)$	0.4067019	0.4067085	0.4077246	0.4077253	0.89
$-\sqrt{1+t}$	$1 + t$	0.0731130	0.0731130	0.0731307	0.0731307	0.80
$-\log(2+t)$	$\sqrt{1+t}$	0.2600515	0.2600521	0.2600767	0.2600768	0.89

Table 3.4: Numerical results for the optimal values in the case with two boundaries b_1 and b_2 and computation times for ($\tilde{D}_1$) with discretisations as defined above

that the computation times are again very fast. We do not consider convex boundaries as we already saw very poor results in the set-up with one boundary in Section 3.3. Moreover, we only consider decreasing b_1 and increasing b_2 in order to avoid the case that b_1 and b_2 intersect in which case the Brownian motion inevitably hits one of the boundaries.

Now, considering the first case of symmetric linear boundaries, we can see that $\tilde{d}_{1,n}$ and $\tilde{p}_{1,n_\lambda}$ as well as $\tilde{d}_{2,n_\lambda}$ and $\tilde{p}_{2,n}$ agree in the first 7 digits, so we can safely assume that we have strong duality in both set-ups. The gap between the two set-ups is not 0 but smaller than $2 \cdot 10^{-5}$ which could be due to our implementation of the code. There is no good reason why the algorithm should assign different masses in the two set-ups. It can be noted however that the implementation of ($\tilde{D}_1$) assigns the correct mass to the point measures in (analogous to the one-sided case as noted in Remark 1.4) while the implementation of ($\tilde{P}_2$) assigns a little too less mass. We now consider as in Section 3.3 the plots of the boundaries b_1 and b_2 compared with the boundaries $b_{\mu_{1,n}}$ and $b_{\mu_{2,n}}$, i.e., the boundaries generated by the numerical solutions to ($\tilde{D}_1$) and ($\tilde{P}_2$), the plots of $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ between 0 and 1 and the plots of the numerical distribution functions F_{μ_i}. We see in Figure 3.5 that both $\mu_{1,n}$ as well as $\mu_{2,n}$ replicate the symmetric linear bounds perfectly. The graphs for $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ both are constant at 1 which supports that the boundaries b_1 and b_2 are indeed represented by $\mu_{1,n}$. Again, this is not very surprising since we know that linear boundaries are representable by measures μ that put the right amount of mass into both $2b_1(0)$ and $2b_2(0)$. We only show the values of $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ and not of $r_{\mu_{2,n}}(t, b_1(t))$ and $r_{\mu_{2,n}}(t, b_2(t))$ as the latter behave the same as the former (except the latter possibly deviate from 1 upwards and not downwards like the former).

Let us now turn our attention to the case of symmetric square root boundaries. Looking at Table 3.4 we see that all four values $\tilde{d}_{1,n}, \tilde{p}_{1,n_\lambda}, \tilde{d}_{2,n_\lambda}$, and $\tilde{p}_{2,n}$ agree. So, we can numerically confirm that we have strong duality in both set-ups as well as $\tilde{d}_1 = \tilde{p}_2$ and $\tilde{p}_1 = \tilde{d}_2$. The latter fact is a good indicator that the boundaries are indeed representable (as an analogue to the conditions from Theorem 2.15). We see in Figure 3.6 that both boundaries are incredibly well replicated by the boundaries generated by $\mu_{1,n}$ and $\mu_{2,n}$, respectively, even beyond the controlled interval up to $t_0 = 1$. Moreover, $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ are constant 1 which is again a strong indicator that these boundaries are indeed representable.

For the symmetric, logarithmic boundaries, we can see in Table 3.4 that the optimal values in both set-ups agree, i.e., we can numerically confirm strong duality in both set-ups but there is a slight gap of $\approx 10^{-3}$ between the set-ups. We observe in Figure 3.7 that the boundaries b_1 and b_2 are very well replicated by the bounds generated by $\mu_{1,n}$ and $\mu_{2,n}$ within the controlled interval up to $t_0 = 1$. Beyond that point we do see growing deviations. But again, we see that $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ are very close to 1 with just slight deviations which is a strong indicator that these boundaries are representable by the method of images.

We now consider the last two cases in Table 3.4, where we considered asymmetric boundaries. In both cases we see that we can numerically confirm strong duality but there is a slight gap of $\approx 2 \cdot 10^{-5}$ in both set-ups. Nevertheless, this is again a good indicator that these bounds are representable as well. We observe in Figure 3.8 that the asymmetric boundaries $b_1(t) = -\sqrt{1+t}$ and $b_2(t) = 1+t$ are very well replicated by the

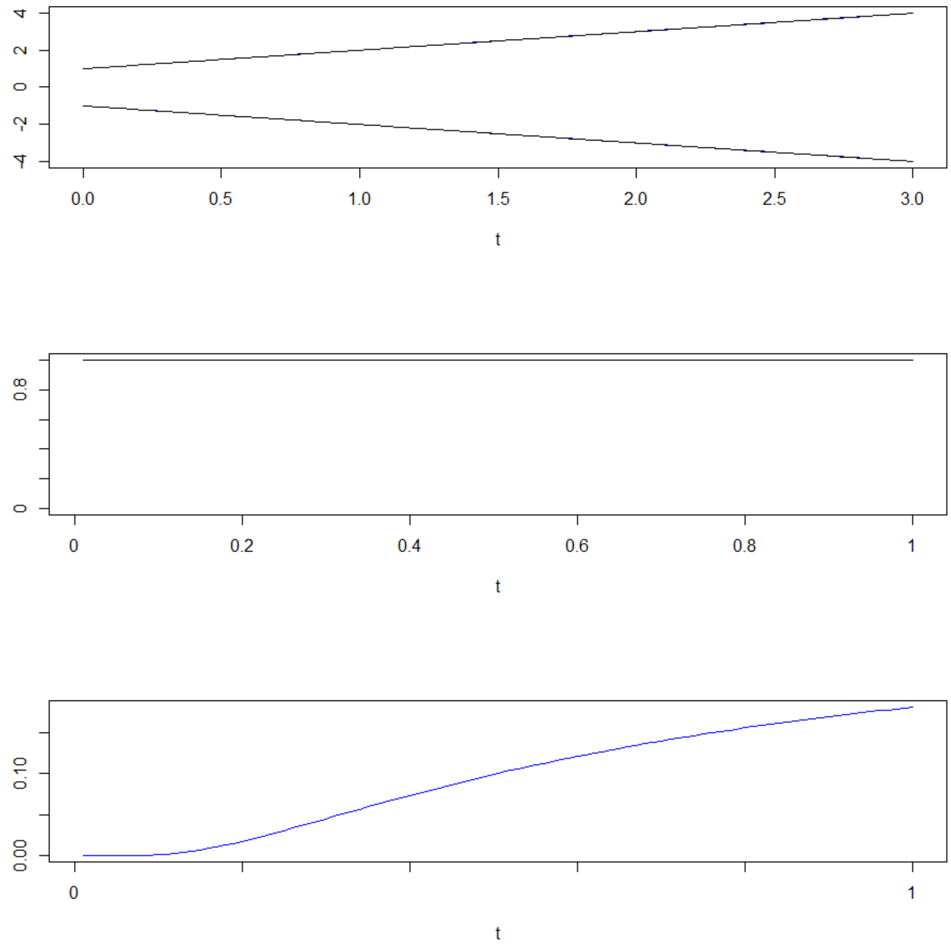

Figure 3.5: Numerical results for $b_1(t) = -1 - t$ and $b_2(t) = 1 + t$
Top: graphs of the original boundaries (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b_1(t))$ (red) and $r_{\mu_{1,n}}(t, b_2(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of the distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

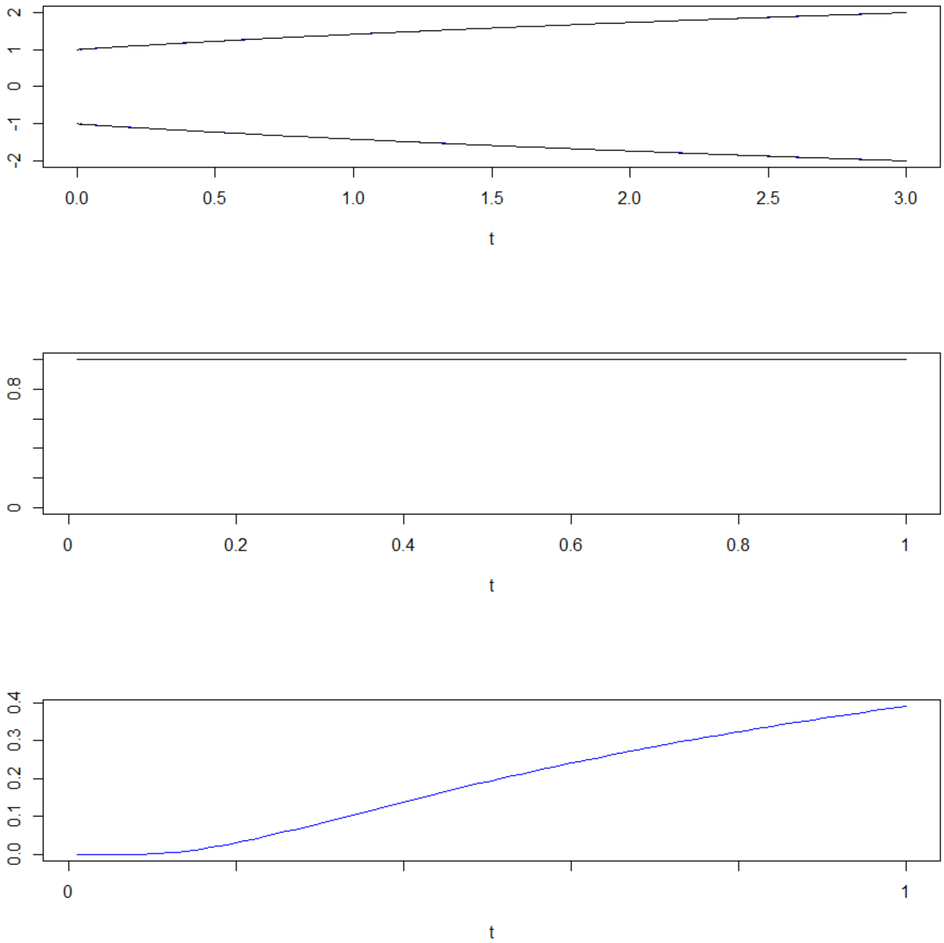

Figure 3.6: Numerical results for $b_1(t) = -\sqrt{1+t}$ and $b_2(t) = \sqrt{1+t}$
Top: graphs of the original boundaries (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b_1(t))$ (red) and $r_{\mu_{1,n}}(t, b_2(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

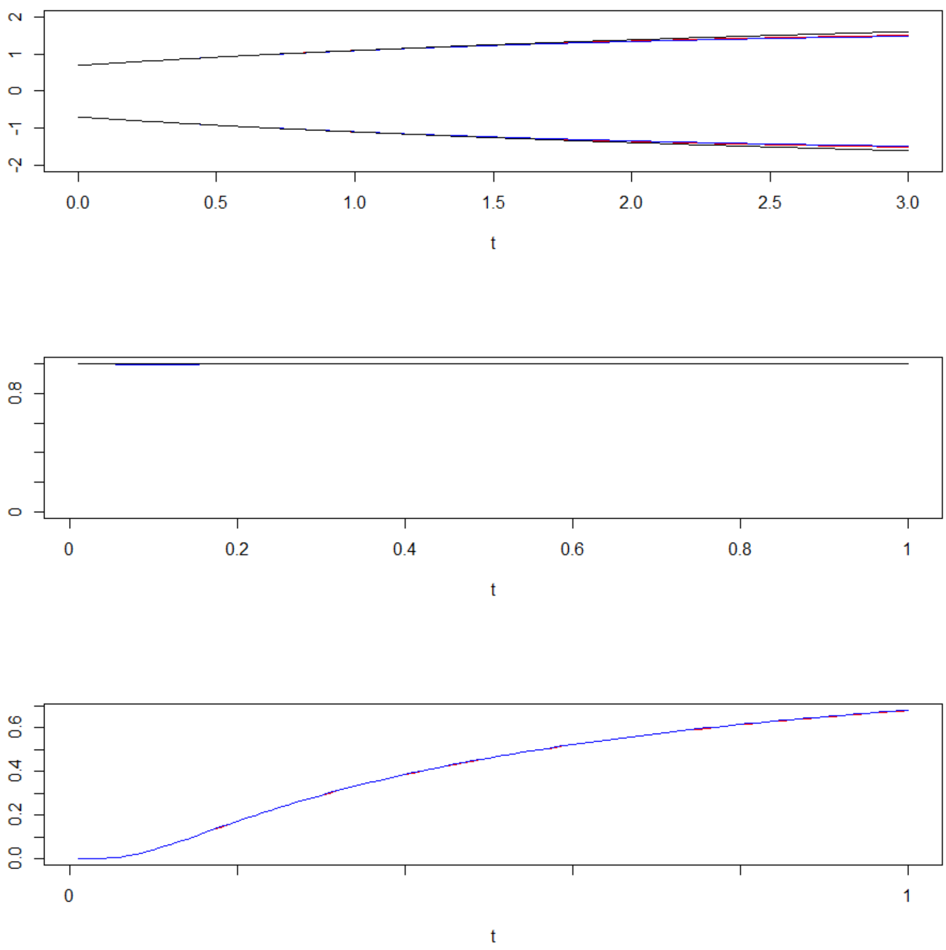

Figure 3.7: Numerical results for $b_1(t) = -\log(2+t)$ and $b_2(t) = \log(2+t)$
Top: graphs of the original boundaries (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b_1(t))$ (red) and $r_{\mu_{1,n}}(t, b_2(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

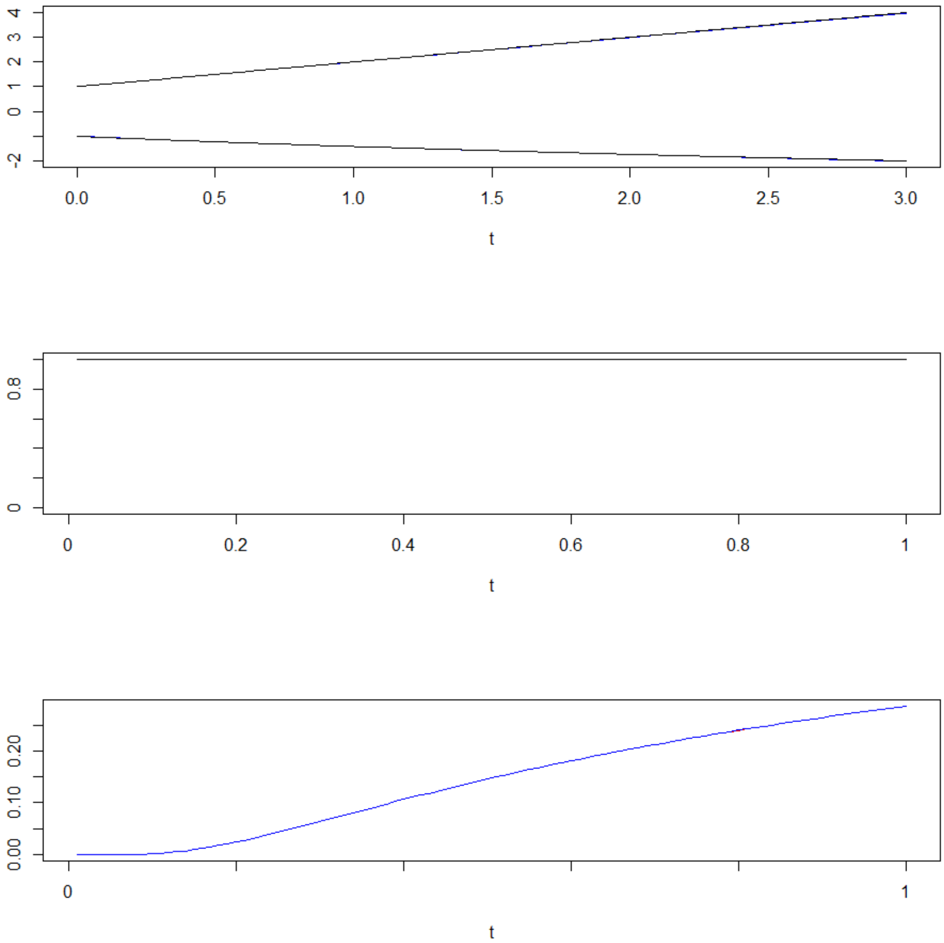

Figure 3.8: Numerical results for $b_1(t) = -\sqrt{1+t}$ and $b_2(t) = 1 + t$
Top: graphs of the original boundaries (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b_1(t))$ (red) and $r_{\mu_{1,n}}(t, b_2(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

boundaries generated by $\mu_{1,n}$ well beyond the controlled interval up to $t_0 = 1$ and also very well by the boundaries generated by $\mu_{2,n}$ up to $t = 2$. The values of $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ are both constant at 1 which implies that these boundaries are indeed representable.

In Figure 3.9 we observe that the boundaries $b_1(t) = -\log(2 + t)$ and $b_2(t) = \sqrt{1+t}$ are also very well replicated even beyond the controlled interval. The values of $r_{\mu_{1,n}}(t, b_1(t))$ and $r_{\mu_{1,n}}(t, b_2(t))$ are constant at 1 for the most time with only slight deviations around $t = 0$. In several examples above we have observed that there are slight gaps between the optimal values of the different set-ups. While for the case of linear, symmetric boundaries this should just not be the case and is most likely due to our implementation (or maybe peculiarities of the `DEoptim` function), some of the gaps can probably be explained by the fact that we choose the same discretisations for all boundaries and limited ourselves to $n = 100$ points in each of the intervals $[2b_1(0) - l, 2b_1(0)]$ and $[2b_2(0), 2b_2(0) + l]$. Indeed, choosing different interval lengths l and different numbers of discretisations n (and n_λ) lead to larger or smaller gaps between the two set-ups. Different boundaries require different amounts of mass points in different places. So the length of the intervals $[2b_1(0)-l, 2b_1(0)]$ and $[2b_2(0), 2b_2(0)+l]$ and the number of discretisation points do matter for the algorithms. As we wanted to make comparable results available, we limited ourselves to the same l, n and n_λ for all problems.

Overall, it can be said that in the set-up with two boundaries there is a strong indication that analogous duality results to those presented in Chapter 2 for one-sided boundaries hold as well as that these boundaries are representable in the sense of the method of images.

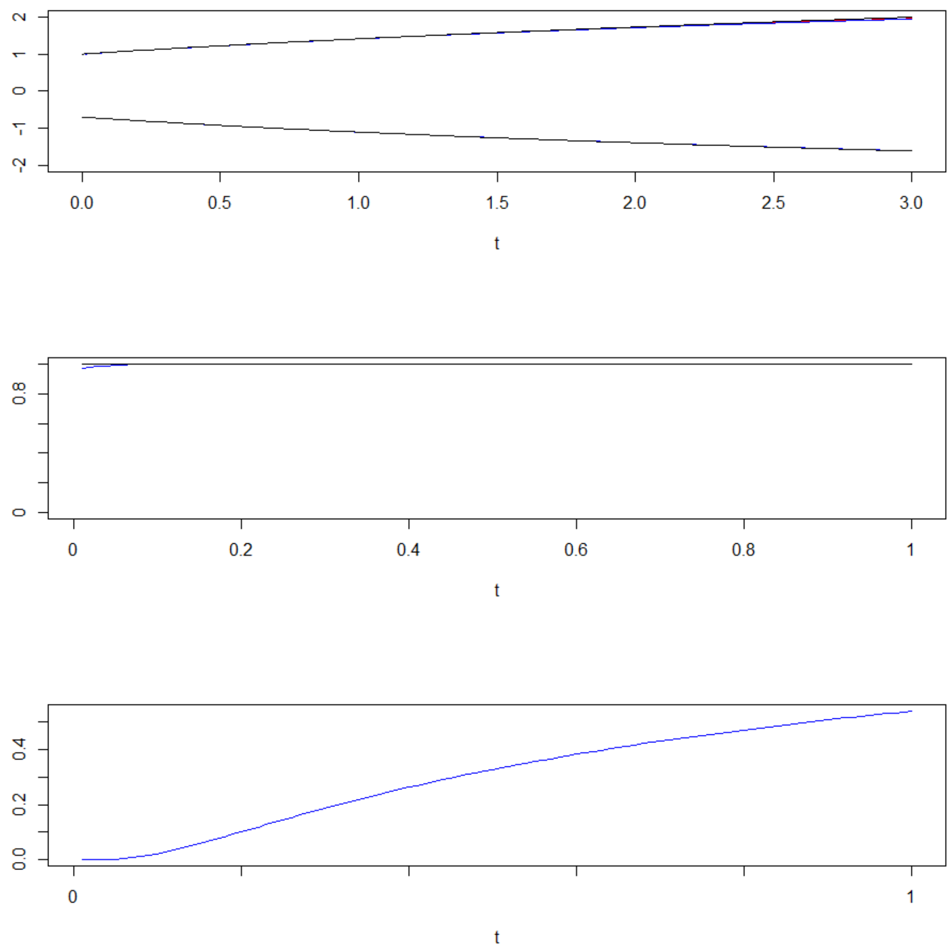

Figure 3.9: Numerical results for $b_1(t) = -\log(2+t)$ and $b_2(t) = \sqrt{1+t}$
Top: graphs of the original boundaries (black) which overlaps the boundaries generated by $\mu_{1,n}$ (red) and $\mu_{2,n}$ (blue).
Middle: corresponding values of $r_{\mu_{1,n}}(t, b_1(t))$ (red) and $r_{\mu_{1,n}}(t, b_2(t))$ (blue) between 0 and 1 compared with the function constant 1 (black).
Bottom: numerical approximations of distribution functions F_{μ_1} (red) and F_{μ_2} (blue).

Chapter 4

Boundary Hitting of Brownian Motion and Prices of American Options

In their articles [JM01] and [JM02], Jourdain and Martini explore how prices of American options can be replicated by prices of European options. The former are often hard or even impossible to obtain explicitly, while the latter can be derived more easily. While their first article [JM01] offers a theoretical framework to approximate prices of American options via European options, their second article [JM02] applies the framework to the American put option with pay-off $\psi : x \mapsto (K - x)^+$ for some $K > 0$ in the Black-Scholes framework. In this section we will explore the connection between the results in [JM02] and the method of images.

4.1 Representability of American options

In this section we introduce the concept of an European option *representing* an American option as set out in [JM01], [JM02], and [CKL22]. We quote the main results from these sources as necessary for our work. To this end, we look at the well-known Black-Scholes model (cf. [JM01])

$$\begin{aligned} dX_t &= rX_t dt + \sigma X_t dW_t \\ X_0 &= x, \end{aligned}$$

where X_t is the stock price at time t under the equivalent martingale measure, $x > 0$ is the initial stock price, r is the interest rate, $\sigma > 0$ is the volatility and $(W_t)_{t\geq 0}$ is a standard Brownian motion. Then,

$$\mathcal{A}_X f(x) = \frac{\sigma^2 x^2}{2} f''(x) + rxf'(x) - rf(x)$$

is the infinitesimal generator of the stochastic process $(X_t)_{t\geq 0}$ in the Black-Scholes model. For a pay-off function $g : \mathbb{R} \to \mathbb{R}_+$ the price of the corresponding American option with maturity t is given by

$$v^g_{am}(t, x) = \sup_{\tau \leq t} \mathbb{E}_x \left(e^{-r\tau} g\left(X_\tau\right)\right),$$

where the supremum is taken over all stopping times such that $\tau \leq t$ almost surely. The continuation set C is the set of all (t, x) such that $v^g_{am}(t, x) > g(x)$ (i.e., the set of all points (t, x) for which the holder of the American option with pay-off g does not yet want to exercise) and the stopping region C^c is the set of all (t, x) such that $v^g_{am}(t, x) = g(x)$ (i.e., the set of all (t, x) for which the holder of the American option with pay-off g wants to exercise immediately). Note that C and C^c are indeed complements of each other.

On the other hand, the price of an European option with pay-off function $f : \mathbb{R} \to \mathbb{R}_+$ and maturity t is given by

$$v^f_{eu}(t, x) = \mathbb{E}_x \left(e^{-rt} f(X_t)\right).$$

Note that we adapted the notation of [CKL22]. For a given European pay-off function f, Jourdain and Martini found under certain conditions an American pay-off function g *embedded* in the European pay-off f such that f *represents* g in the following sense (cf. [JM01], Theorem 5)

1. $v^f_{eu}(t, x) \geq v^g_{am}(t, x)$ for all t, x,
2. $v^f_{eu}(t, x) = v^g_{am}(t, x)$ for all $(t, x) \in \partial C$.

The second statement is phrased a bit differently in [JM01]: If there exists a continuous $\hat{t}(x) : \mathbb{R}_+ \to [0, T]$ such that for some $T > 0$

$$\forall x > 0 : \inf_{0\leq t\leq T} v^f_{eu}(t, x) = v^f_{eu}(\hat{t}(x), x),$$

then for all $(t,x) \in [0,T] \times \bar{\mathbb{R}}_+$ we have $v_{am}^g(t,x) = v_{eu}^f(t \vee \hat{t}(x), x)$ and $\hat{t}$ is the stopping boundary associated with the stopping problem (cf. [JM01], Theorem 5). In particular, the stopping boundary divides the space into the continuation set C and its complementary stopping set. A sufficient condition for the existence of the stopping boundary $\hat{t}$ is given in Proposition 7 in [JM01].

The conditions can be relaxed a bit further, in particular, it can be shown that if

1. $v_{eu}^f(t,x) \geq g(x)$ for all t, x,
2. $v_{eu}^f(t,x) = g(x)$ for all $(t,x) \in \partial C$,

then, we obtain representability in the above sense (cf. [CKL22], Prop. 2.2, Assertion 2, as well as [JM01], Theorem 5).

In [CKL22] the concept of an *embedded American option* is introduced: for a given European pay-off f a corresponding American pay-off g is defined such that f represents g in the above sense. As prices of American options are often hard to obtain, the reverse question seems to be more interesting, i.e., given an American pay-off g can we find a European pay-off f such that f represents g. In order to obtain this representing pay-off f, [CKL22] uses the notion of a European option representing an American option to formulate the following optimisation problem (cf. [CKL22], 3.8) for time to maturity $T \geq 0$ and initial stock price $X_0 = x_0$

$$\begin{aligned} &\text{minimise} && v_{eu,\mu}(T, x_0) \\ &\text{subject to} && \mu \in \mathcal{M}^+(\mathbb{R}) \\ & && v_{eu,\mu}(\theta, x) \geq g(x) \text{ for all } (\theta, x) \in [0,T] \times \mathbb{R}. \end{aligned} \tag{4.1}$$

Note that in this problem the notion of a pay-off function f was extended to include measures $\mu \in \mathcal{M}^+(\mathbb{R})$, as well. The value of the European option is therefore denoted by $v_{eu,\mu}$ and not by v_{eu}^f. If such a minimiser μ of this optimisation problem exists, it is called the *cheapest dominating European option (CDEO)* of g relative to (T, x_0). The optimisation problem was originally formulated in [Chr14]. Then, [CKL22] sets out a dual problem to the above linear program as well as a duality result (cf. Lemma 3.4 and Lemma 3.8, [CKL22]). With that in mind, it can be shown that a CDEO of g relative to (T, x_0) exists (cf. Theorem 3.1, [CKL22]) as well as that under certain sufficient conditions the CDEO actually represents the American option g (cf. Theorem 3.2, [CKL22]).

Note that the linear program (4.1) is similar to the programs (D_1) and (P_2) considered in Chapter 2. But while the proof of duality in Theorems 2.7 and 2.13 is similar to the proof of duality in [CKL22], there are some differences in the approach. Apart from minor differences (note for example the different roles played by g in [CKL22] compared to the role of g in the set-up of Chapter 2 above), there are two main differences. While in [CKL22] the problem is only approached "from above", the linear problems (D_1) and (P_2) approached the solution "from above" and "from below". Moreover, the stopping boundary in [CKL22] is just implicitly given and part of the solution of the optimisation problem, the boundary in our case is of course explicitly given. So while the basic structure of the proof could be retained, this necessitated changes and adaptations throughout the proofs.

In their second paper [JM02], Jourdain and Martini approach the question which (if any) European pay-off f represents the "classical" American put, i.e., the American option with pay-off function $g(x) = (K - x)^+$, where $K > 0$ is the so-called strike. They argue that if there was any hope to represent g, then the representing European pay-off $f = f_m$ should fulfil

$$\mathcal{A}_X f = m$$

for some measure m and find ([JM02], Lemma 1) that m has to be of the form

$$m(dr) = \frac{1}{2}\sigma^2 K^2 \delta_K(dr) - \frac{1}{2}\sigma^2 \alpha K h(dr)$$

where h is a positive measure on $(0, K)$ such that certain regularity conditions are fulfilled.

4.2 Connecting American options and the method of images

In this section, we connect the above concept of European options representing American options to the concept of representable boundaries in the sense of the method of images. In particular, we show that the candidate stopping boundary for the American option is representable in the sense of the method of images. To this end, we will transform the value function of an American option which is represented by some European pay-off

$f = f_m$ fulfilling $\mathcal{A}_X f = m$. We start by considering the simplified stopping problem

$$v(t,x) = \sup_{\tau \le t} \mathbb{E}_x \left(e^{-r\tau} g(W_\tau) \right),$$

where $t \in [0, \infty)$, $x \in \mathbb{R}$, $r > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a function with $g(x) = 0$ for $x \le 0$, $g'(0+) > 0$ and $\mathcal{A}g(x) < 0$ for $x > 0$. Here, the operator $\mathcal{A}$ is given by $\mathcal{A}h = \frac{1}{2}h'' - rh$ (cf. [PS06], 7.4.14). Denote by C the continuation region of the stopping problem. Note that this is the same problem as considered by Jourdain and Martini after a change of measure and a change of space. We consider this problem as it inhibits a connection to the method of image as we prove later on.

In order to see that we can rewrite the value function of an American option with pay-off g in this way, note that a geometric Brownian motion is of the form

$$X_t = X_0 e^{at + \sigma W_t} = X_0 \left(e^{bt + W_t} \right)^\sigma$$

where $a = r - \frac{\sigma^2}{2}$ and $b = \frac{r - \sigma^2/2}{\sigma}$. In particular, we can rewrite the original problem as

$$v(t,x) = \sup_{\tau \le t} \mathbb{E}_x \left(e^{-r\tau} g\left(X_\tau\right) \right) = \sup_{\tau \le t} \mathbb{E}_x \left(e^{-r\tau} \tilde{g}\left(W_\tau^b\right) \right),$$

where $(W_t^b)_t$ is a Brownian motion with drift b and for some function $\tilde{g}$. Moreover, we use a Girsanov transformation and further rewrite this problem as

$$\sup_{\tau \le t} \mathbb{E}_x \left(e^{-r\tau} \tilde{g}\left(W_\tau^b\right) \right) = \frac{1}{h(x)} \sup_{\tau \le t} \mathbb{E}_x \left(e^{-\tilde{r}\tau} \tilde{\tilde{g}}\left(W_\tau\right) \right)$$

for some $\tilde{r} > 0$ and some functions h and $\tilde{\tilde{g}}$. So, it is enough to consider the simplified stopping problem

$$v(t,x) = \sup_{\tau \le t} \mathbb{E}_x \left(e^{-r\tau} g(W_\tau) \right).$$

Now, we will consider an analogue to the value function of a European option: for functions f supported on $[0, \infty)$ consider

$$h_f(t,x) = \mathbb{E}_x \left(e^{-rt} f(W_t) \right).$$

Analogous to the approach in [JM01], we are now looking for a representing function $h = h_f$ for v, i.e., for a function h such that

1. for all $t \in [0, \infty)$, $x \in \mathbb{R}$: $h(t, x) \geq g(x)$,
2. for all $x \in \mathbb{R}$ there exists a unique $\hat{t} = \hat{t}(x)$: $h(\hat{t}(x), x) = g(x)$.

We also assume that the function $\hat{t}$ is continuous and increasing in x (which can be shown to hold under certain regularity conditions, cf. [JM01], Prop. 7). Indeed, if we have found such a representing function h then we already obtain representability in the sense of the previous section, i.e., we obtain

1. $h(t, x) \geq v(t, x)$ for all $t \in [0, \infty)$, $x \in \mathbb{R}$ (cf. [CKL22], Prop. 2.2, Assertion 2),
2. $h(t, x) = v(t, x)$ for all $(t, x) \in C$ (cf. [JM01], Theorem 5).

We now only take those f in account that [JM01] uses. In particular, $f = f_m$ is the solution of the (ordinary) differential equation $\mathcal{A}f = m$, where m is a measure on $[0, \infty)$ which can be decomposed in a (generalised) non-positive function on $(0, \infty)$ plus mass in 0. Recall that $\mathcal{A}f = \frac{1}{2}f'' - rf$. Then, the fundamental solutions of $\mathcal{A}f = 0$ are $e^{\lambda x}, e^{-\lambda x}$ where $\lambda := \sqrt{2r}$. Thus, we obtain the general solution of $\mathcal{A}f = m$ as (cf. [Wal00], 19.VII)

$$f_m(x) = ae^{\lambda x} + be^{-\lambda x} + \int_0^x \frac{e^{\lambda(x-y)} - e^{-\lambda(x-y)}}{\lambda} m(dy).$$

Now, we make use of the fact that $f_m(x) = 0$ for $x = 0$ and therefore the constants a and b have to be zero. Then, we obtain f_m for $x > 0$ as

$$f_m(x) = \int_0^x \frac{e^{\lambda(x-y)} - e^{-\lambda(x-y)}}{\lambda} m(dy).$$

Moreover, note that m puts mass $\frac{g'(0+)}{2}$ into 0.

The candidate $\hat{t}(x)$ for the stopping boundary is given as the critical points of $t \mapsto h_{f_m}(t, x)$, i.e. as the solution of

$$0 = \frac{\partial}{\partial t} h_{f_m}.$$

We will call these critical points the *candidate stopping boundary*. Now, let p be the transition kernel of a Brownian motion, i.e.,

$$p((0, x), (t, y)) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}.$$

Using the Kolmogorov forward equation (cf. for example [Øks00], Equation 8.6.35) which states that $\frac{\partial}{\partial t}p((0,x),(t,y)) = \frac{1}{2}\frac{\partial^2}{\partial y^2}p((0,x),(t,y))$, we find that

$$\begin{aligned}
\frac{\partial}{\partial t}h_{f_m} &= \frac{\partial}{\partial t}\left(e^{-rt}\int_{-\infty}^{\infty}p((0,x),(t,y))f_m(y)dy\right) \\
&= -re^{-rt}\int_{-\infty}^{\infty}p((0,x),(t,y))f_m(y)dy + e^{-rt}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}p((0,x),(t,y)f_m(y)dy \\
&= e^{-rt}\int_{-\infty}^{\infty}\mathcal{A}p((0,x),(t,y))f_m(y)dy \\
&= e^{-rt}\int_{-\infty}^{\infty}p((0,x),(t,y))m(dy),
\end{aligned}$$

where we can interchange integration and differentiation in the second step due to Lebesgue's theorem and where the last step follows since we assumed $\mathcal{A}f_m = m$. Note that f_m does not have to be in C^2 but that $\mathcal{A}f_m = m$ is defined in a "reasonable sense", i.e., for example in the sense of embedding functions and measures into the space of distributions (for more on this, cf. [Rud91], Section 6.11 onwards). The last step then essentially boils down to taking the derivative of a distribution (cf. [Rud91], Section 6.13). Note that p does not have compact support but can of course be approximated by test functions with compact support as $p((0,x),(t,y)) \to 0$ for $y \to \pm\infty$ exponentially fast.

Thus, in order to find the candidate stopping boundary, we have to solve

$$\begin{aligned}
0 &= \int p((0,x),(t,y))m(dy) \\
&= \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{x^2}{2t}}\frac{g'(0)}{2} - \int_{(0,\infty)}e^{-\frac{(x-y)^2}{2t}}(-\tilde{m})(dy)\right)
\end{aligned}$$

where $\tilde{m}$ denotes the measure m minus the mass in 0. Recall that m is a non-positive measure on $(0,\infty)$ and therefore $-\tilde{m}$ is a non-negative measure on $(0,\infty)$. So, we obtain

$$a = \int_{(0,\infty)}e^{-\frac{y^2-2yx}{2t}}(-\tilde{m})(dy)$$

where $a = \frac{g'(0)}{2}$. But this is exactly the integral equation from the method of images. This means that if $f = f_m$ represents g in the above sense we find that $b_g = \hat{t}^{-1}$ fulfils the integral equation from the method of images (recall that we can assume $\hat{t}$ to be continuous and increasing, so indeed invertible). Let us summarise our findings in the following theorem.

Theorem 4.1. *For European reward functions $f = f_m$ satisfying $\mathcal{A}f = m$, the candidate stopping boundary $\hat{t}$ of the American option g represented by f is always representable in the sense of the method of images.*

In particular, we can now immediately transfer the properties which boundaries from the method of images show to the stopping boundaries in the Jourdain and Martini context. In particular, these are concave and analytic (as functions in t).

Corollary 4.2. *Let now t_m be the stopping boundary (parametrised in x) from the method of images applied to $\tilde{m}$. Then, we have that g is represented by $h = h_f = h_{f_m}$ if and only if*

$$g(x) = h_{f_m}\left(t_m(x), x\right) = \mathbb{E}_x\left(e^{-rt_m(x)} f_m\left(W_{t_m(x)}\right)\right).$$

So, there exists a close connection of the stopping boundaries from the context of Jourdain and Martini and the stopping boundaries from the method of images. Indeed, given an American pay-off g, one would now like to apply the inverse method of images to obtain the measure m to then find the European pay-off $f = f_m$ that represents g in the sense of Jourdain and Martini. But alas, we have only shown that the following holds: given $f = f_m$ representing g in the sense of Jourdain and Martini, the resulting stopping boundary b_g also is the stopping boundary of the method of images applied to the measure $-\tilde{m}$.

In order to show the existence of representing European options, it now seems natural to show representability (in the sense of the method of images) of the stopping boundary associated to an American pay-off. This then immediately gives the candidate for the representing European option. Therefore, relevant open questions remaining are: Which curves are representable in the sense of the method of images (Chapter 2 gave sufficient conditions)? What is then a sufficient condition for a representing European to exist?

Chapter 5

Boundary Hitting of Brownian Motion via Integral Equations

This chapter is based on the article "Uniqueness of First Passage Time Distributions via Fredholm Integral Equations" which is available as a pre-print here: [CFH23]. The paper has been slightly adapted to fit the other chapters.

In this chapter, we return to the FPT problem and investigate Fredholm integral equations connected to the first hitting distribution. We will first derive known Fredholm equations and then show that these determine the first hitting distribution uniquely. Finally, we will discuss possible extensions.

5.1 The first passage time problem and Fredholm integral equations

Let W be a standard Brownian motion with $W_0 = 0$ and let $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with $b(0) > 0$. Recall that the first passage time (from below) is defined as

$$\tau := \inf\{t \geq 0 | W_t \geq b(t)\}$$

and that we seek to determine the distribution F of τ, i.e., $F(t) = P(\tau \leq t)$.

We denote the transition kernel of W by p and assume $b : \mathbb{R}^+ \to \mathbb{R}$ to be Lipschitz continuous with constant L and b to have linear growth, i.e., $\limsup_{t\to\infty} \frac{b(t)}{t} \leq d$ for some $d \in \mathbb{R}$. This implies that there exist $M, m > 0$ such that $b(t) \leq M + mt$. There are different approaches to establishing a Fredholm-type integral equation for the distribution F of τ. A direct way is as follows: Using the strong Markov property, we obtain

$$\begin{aligned} p((0,0),(t,x)) &= P(W_t \in dx) \\ &= \int_0^t P_{(s,b(s))}(W_t \in dx)P(\tau \in ds) + P(W_t \in \underset{\cdot}{x}, \tau > t). \end{aligned} \tag{5.1}$$

Since b is continuously differentiable, $P(\tau \in ds)$ has a continuous density function f (see e.g., [Fer82b]). Then,

$$1 = \frac{p((0,0),(t,x))}{p((0,0),(t,x))} = \frac{\int_0^t P_{(s,b(s))}(W_t \in dx)f(s)ds}{p((0,0),(t,x))} + \frac{P(W_t \in dx, \tau > t)}{p((0,0),(t,x))}. \tag{5.2}$$

For $c > d$ we set $x = ct$ and let $t \to \infty$. The last term can be reformulated as

$$\frac{P(W_t \in d(ct), \tau > t)}{p((0,0),(t,ct))} = P(\tau > t \mid W_t = ct) \xrightarrow{t\to\infty} 0$$

since $\limsup_{t\to\infty} \frac{b(t)}{t} \leq d$ and $c > d$. On the other hand, using linear growth of b and the dominated convergence theorem, the first term can be rewritten as

$$\begin{aligned} \lim_{t\to\infty} \frac{\int_0^t P_{(s,b(s))}(W_t \in d(ct))f(s)ds}{p((0,0),(t,ct))} &= \lim_{t\to\infty} \int_0^t \frac{p((s,b(s)),(t,ct))f(s)}{p((0,0),(t,ct))} ds \\ &= \int_0^\infty e^{-\frac{c^2}{2}s + cb(s)} f(s)ds. \end{aligned}$$

Combining this, we obtain

$$1 = \int_0^\infty e^{-\frac{c^2}{2}s + cb(s)} f(s)ds \tag{5.3}$$

for all $c > d$ such that the integral exists. A visualization of the setting can be seen in Figure 5.1. Now, the task is: For a given boundary b we would like to find f with $\int_0^\infty f(s)ds = P(\tau < \infty)$, such that (5.3) holds for all c.

5.2 Uniqueness

A central question is if solutions to (5.3) are unique, i.e., if (5.3) fully characterizes the distribution of τ. Our ansatz is inspired by recent results for the Fredholm representation for discounted stopping problems with finite time horizon in [CF21]. We use the following

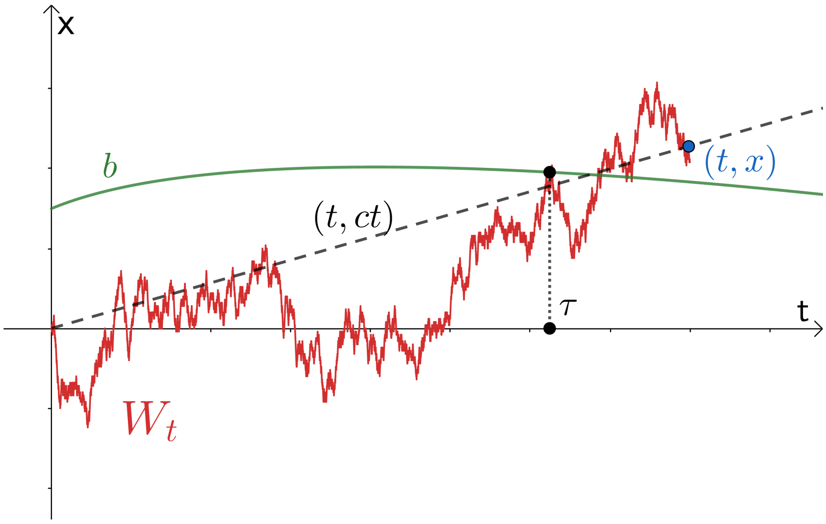

Figure 5.1: Visualization of the setting

notation for $J \subset (-\infty, 0]$ measurable and $n \in \mathbb{N}_0$:

$$I_J^f(n)(c) := \int_J e^{-\frac{c^2}{2}s+cb(s)} s^n f(s) ds$$
$$I^f(n)(c) := I^f_{[0,\infty)}(n)(c)$$
$$I_J^f(c) := I_J^f(0)(c),$$

and for two different continuous functions f and $\tilde{f}$

$$D_J(n)(c) := I_J^f(n)(c) - I_J^{\tilde{f}}(n)(c)$$
$$D(n)(c) := D_{[0,\infty)}(n)(c).$$

Lemma 5.1. *Let $t \geq 0$ be fixed and $\varepsilon > 0$, then*

$$\lim_{c\to\infty} \frac{I^f_{[t,\infty)}(n)(c)}{I^f_{[t,t+\varepsilon]}(n)(c)} = 1.$$

If additionally $f(t) \neq \tilde{f}(t)$, then

$$\lim_{c\to\infty} \frac{D_{[t,\infty)}(n)(c)}{D_{[t,t+\varepsilon]}(n)(c)} = 1.$$

Proof. Let (c_i) be a sequence such that $c_i \to \infty$ for $i \to \infty$. We first show that for all

$\varepsilon > 0$ and $t \geq 0$ it holds

$$\lim_{i\to\infty} \frac{I^f_{[t,\infty)}(n)(c_i)}{I^f_{[t,t+\varepsilon]}(n)(c_i)} = 1. \tag{5.4}$$

Equation (5.4) is equivalent to

$$\lim_{i\to\infty} \frac{I^f_{[t+\varepsilon,\infty)}(n)(c_i)}{I^f_{[t,t+\varepsilon]}(n)(c_i)} = 0.$$

The numerator and denominator are positive. Let us derive an upper bound for $I^f_{[t+\varepsilon,\infty)}(n)(c)$. Since f is continuous and integrable we can set $f_1 = \max\{f(s) \mid s \in [t+\varepsilon,\infty)\}$. For positive s and n, the function $s^n e^{-s}$ has its maximum value at $s = n$, hence we can use that $s^n e^{-s} \leq n^n e^{-n}$. For c large enough we have

$$\begin{aligned}
I^f_{[t+\varepsilon,\infty)}(n)(c) &= \int_{t+\varepsilon}^{\infty} e^{-\frac{1}{2}c^2 s + cb(s)} s^n f(s) \underset{\cdot}{s} \\
&\leq n^n e^{-n} f_1 \int_{t+\varepsilon}^{\infty} e^{-\frac{1}{2}c^2 s + c(M+ms)+s} \underset{\cdot}{s} \\
&= \frac{n^n e^{-n} f_1}{\frac{c^2}{2} - 1 - cm} e^{(t+\varepsilon)(cm - \frac{c^2}{2} - 1) + cM}.
\end{aligned}$$

We now derive a lower bound for $I^f_{[t,t+\varepsilon]}(n)(c_i)$. The function b is continuous, so we can set $b_1 := \min\{b(s) \mid s \in [t+\frac{\varepsilon}{2}, t+\varepsilon]\}$. Furthermore, let $f_2 := \min\{f(s) \mid s \in [t+\frac{\varepsilon}{2}, t+\varepsilon]\}$. Note that $f_2 > 0$, since b is locally Lipschitz continuous. We have

$$\begin{aligned}
I^f_{[t,t+\varepsilon]}(n)(c) &= \int_{t}^{t+\varepsilon} e^{-\frac{1}{2}c^2 s + cb(s)} s^n f(s) \underset{\cdot}{s} \\
&\geq \left(t + \frac{\varepsilon}{2}\right)^n f_2 \int_{t+\frac{\varepsilon}{2}}^{t+\varepsilon} e^{-\frac{1}{2}c^2 s + cb_1} \underset{\cdot}{s} \\
&= \left(t + \frac{\varepsilon}{2}\right)^n f_2 \frac{2}{c^2} \left(e^{-\frac{c^2}{2}\left(t+\frac{\varepsilon}{2}\right) + cb_1} - e^{-\frac{c^2}{2}(t+\varepsilon) + cb_1} \right).
\end{aligned}$$

Putting these results together we find

$$\begin{aligned}
0 &\leq \lim_{i\to\infty} \frac{I^f_{[t+\varepsilon,\infty)}(n)(c_i)}{I^f_{[t,t+\varepsilon]}(n)(c_i)} \\
&\leq \lim_{i\to\infty} \frac{\frac{n^n e^{-n} f_1}{\frac{c_i^2}{2} + 1 - cm} e^{(t+\varepsilon)(c_i m - \frac{c_i^2}{2} - 1) + c_i M}}{\left(t + \frac{\varepsilon}{2}\right)^n f_2 \frac{2}{c_i^2} \left(e^{-\frac{c_i^2}{2}\left(t+\frac{\varepsilon}{2}\right) + c_i b_1} - e^{-\frac{c_i^2}{2}(t+\varepsilon) + c_i b_1} \right)} = 0
\end{aligned}$$

which shows the first claim.
Let now $f(t) \neq \tilde{f}(t)$. We assume w.l.o.g. that $f(t) > \tilde{f}(t)$. Since f and $\tilde{f}$ are continuous, there exists $\delta > 0$ such that $f(s) > \tilde{f}(s)$ for all $s \in [t, t+\delta]$. Then, the term

$$D_{[t,t+\delta]}(n)(c_i) = \int_t^{t+\delta} e^{-\frac{1}{2}c^2 s + cb(s)} s^n (f(s) - \tilde{f}(s))\underset{.}{s}$$

is positive and analogously to the calculations above we obtain

$$\lim_{i\to\infty} \frac{|D_{[t+\delta,\infty)}(n)(c_i)|}{D_{[t,t+\delta]}(n)(c_i)} \leq \lim_{i\to\infty} \frac{I^{\max(f,\tilde{f})}_{[t+\delta,\infty)}(n)(c_i)}{D_{[t,t+\delta]}(n)(c_i)} = 0.$$

If $\varepsilon \leq \delta$ we can set $\delta = \varepsilon$ and we are done. If $\varepsilon > \delta$, we have

$$\begin{aligned} D_{[t,t+\varepsilon]}(n)(c_i) &= \int_t^{t+\delta} e^{-\frac{1}{2}c_i^2 s + c_i b(s)} s^n (f(s) - \tilde{f}(s))\underset{.}{s} \\ &\quad + \int_{t+\delta}^{t+\varepsilon} e^{-\frac{1}{2}c_i^2 s + c_i b(s)} s^n (f(s) - \tilde{f}(s))\underset{.}{s} \\ &\geq \int_t^{t+\delta} e^{-\frac{1}{2}c_i^2 s + c_i b(s)} s^n (f(s) - \tilde{f}(s))\underset{.}{s} \\ &\quad - \int_{t+\delta}^{t+\varepsilon} e^{-\frac{1}{2}c_i^2 s + c_i b(s)} s^n \max(f(s), \tilde{f}(s))\underset{.}{s} \end{aligned}$$

where the integrand of the first term is non-negative and for c_i large enough the whole rhs term is non-negative. Using this inequality, we obtain

$$\lim_{i\to\infty} \frac{|D_{[t+\varepsilon,\infty)}(n)(c_i)|}{D_{[t,t+\varepsilon]}(n)(c_i)} \leq \lim_{i\to\infty} \frac{|D_{[t+\varepsilon,\infty)}(n)(c_i)|}{D_{[t,t+\delta]}(n)(c_i) - I^{\max(f,\tilde{f})}_{[t+\delta,\infty)}(n)(c_i)} = 0.$$

For c large enough the denominator is positive, hence, it follows

$$\lim_{i\to\infty} \frac{D_{[t,\infty)}(n)(c_i)}{D_{[t,t+\varepsilon]}(n)(c_i)} = 1.$$

□

Lemma 5.2. *If $I^f(c) = I^{\tilde{f}}(c)$ for all $c > C$ for some constant $C > 0$, then*

$$I^f(n)(c) = I^{\tilde{f}}(n)(c),$$

for all $n \in \mathbb{N}_0$ and $c > C$ s.t. the integral exists.

Proof. The proof is by induction, i.e., we assume that

$$\int_0^\infty e^{-\frac{c^2}{2}s+cb(s)}s^n f(s)ds = \int_0^\infty e^{-\frac{c^2}{2}s+cb(s)}s^n \tilde{f}(s)ds$$

for all $c > C$. Multiplying both sides with $e^{-cb(0)}$ we obtain

$$\int_0^\infty e^{-\frac{c^2}{2}s+c(b(s)-b(0))}s^n f(s)ds = \int_0^\infty e^{-\frac{c^2}{2}s+c(b(s)-b(0))}s^n \tilde{f}(s)ds.$$

Taking derivatives w.r.t. c on both sides, it follows

$$\begin{aligned}&\int_0^\infty (-cs+b(s)-b(0))e^{-\frac{c^2}{2}s+c(b(s)-b(0))}s^n f(s)ds\\ =&\int_0^\infty (-cs+b(s)-b(0))e^{-\frac{c^2}{2}s+c(b(s)-b(0))}s^n \tilde{f}(s)ds.\end{aligned} \tag{5.5}$$

We assume that $I^f(n+1) \neq I^{\tilde{f}}(n+1)$. Since $I^f(n+1)(c)$ is analytic, we find a sequence $c_i \to \infty$ such that $I^f(n+1)(c_i) \neq I^{\tilde{f}}(n+1)(c_i)$. Rearranging (5.5) this leads to

$$c_i = \frac{\int_0^\infty e^{-\frac{c_i^2}{2}s+c_i(b(s)-b(0))}(b(s)-b(0))s^n(\tilde{f}(s)-f(s))ds}{\int_0^\infty e^{-\frac{c_i^2}{2}s+c_i(b(s)-b(0))}s^{n+1}(f(s)-\tilde{f}(s))ds}. \tag{5.6}$$

Since b is Lipschitz continuous, we find $L > 0$ such that $|b(t) - b(0)| \leq Lt$. Let us assume that $f(0) \neq \tilde{f}(0)$. Using Lemma 5.1, we obtain

$$\begin{aligned}\infty = \lim_{i\to\infty} c_i &= \lim_{i\to\infty} \frac{\int_0^\infty e^{-\frac{c_i^2}{2}s+c_ib(s)}(b(s)-b(0))s^n(\tilde{f}(s)-f(s))\underset{.}{s}}{\int_0^\infty e^{-\frac{c_i^2}{2}s+c_ib(s)}s^{n+1}(f(s)-\tilde{f}(s))\underset{.}{s}}\\
&\leq \lim_{i\to\infty} \frac{\int_0^\varepsilon e^{-\frac{c_i^2}{2}s+c_ib(s))}Ls^{n+1}(\tilde{f}(s)-f(s))\underset{.}{s}}{D(n+1)}\\
&\quad + \lim_{i\to\infty} \frac{\int_\varepsilon^\infty e^{-\frac{c_i^2}{2}s+c_ib(s))}\Big((M+ms)s^n\Big)(\tilde{f}(s)-f(s))\underset{.}{s}}{D(n+1)}\\
&= L + 0 < \infty.\end{aligned}$$

This is a contradiction so the assumption $f(0) \neq \tilde{f}(0)$ is false. Let us now consider the second case that $f(0) = \tilde{f}(0)$. We set $t^* := \inf\{s \mid \tilde{f}(s) \neq f(s)\}$. Since f and $\tilde{f}$ are continuous, we find sequences t_j and ε_j such that $t_j \to t^*$ and $f(t) \neq \tilde{f}(t)$ for

$t \in [t_j, t_j + \varepsilon_j]$. In the same way as before it follows that

$$\lim_{i\to\infty} \frac{\int_{t_j}^{\infty} e^{-\frac{c_i^2}{2}s + c_i b(s)} (b(s) - b(t_j)) s^n (\tilde{f}(s) - f(s)) \d s}{\int_{t_j}^{\infty} e^{-\frac{c_i^2}{2}s + c_i b(s)} s^{n+1} (f(s) - \tilde{f}(s)) \d s} \leq L_j < \infty.$$

By the dominated convergence theorem we find that

$$\begin{aligned}
&\lim_{i\to\infty} \frac{\int_{t^*}^{\infty} e^{-\frac{c_i^2}{2}s + c_i b(s)} (b(s) - b(t^*)) s^n (\tilde{f}(s) - f(s)) \d s}{\int_{t^*}^{\infty} e^{-\frac{c_i^2}{2}s + c_i b(s)} s^{n+1} (f(s) - \tilde{f}(s)) \d s} \\
&= \lim_{i\to\infty} \lim_{j\to\infty} \frac{\int_{t_j}^{\infty} e^{-\frac{c_i^2}{2}s + c_i b(s)} (b(s) - b(t_j)) s^n (\tilde{f}(s) - f(s)) \d s}{\int_{t_j}^{\infty} e^{-\frac{c_i^2}{2}s + c_i b(s)} s^{n+1} (f(s) - \tilde{f}(s)) \d s} \\
&= \lim_{j\to\infty} L_j < \infty.
\end{aligned}$$

Again, this is a contradiction to (5.6) which concludes the proof. □

We can now state the uniqueness theorem for (5.3).

Theorem 5.3. *Let b be continuously differentiable with linear growth and*

$$-\infty < \liminf_{t\to 0} \frac{b(t) - b(0)}{t} \leq \limsup_{t\to 0} \frac{b(t) - b(0)}{t} < \infty.$$

If there exist $d \geq 0$ s.t. (5.3) holds for all $c \geq d$ then the representation defines f uniquely in the class of continuous functions.

Proof. Let $I^f(c) = I^{\tilde{f}}(c)$ for all $c > d$. By Lemma 5.2 we have $I^f(n)(c) = I^{\tilde{f}}(n)(c)$ for all $c > d$ and $n \in \mathbb{R}$. We rewrite $I^f(n)(c)$ as

$$I^f(n)(c) = \int_0^{\infty} e^{-s} s^n \mu(ds)$$

with a measure $\mu = g \circ \lambda$ where g denotes the density function

$$g(s) := e^s e^{-\frac{1}{2}c^2 s + cb(s)} f(s)$$

and λ denotes the Lebesgue measure on $\mathbb{R}$. A measure $\tilde{\mu}$ with density function $\tilde{g}$ is defined analogously via $\tilde{f}$. Recall that the Laguerre exponential polynomials $p(-s)e^{-s}$ lie dense in $L^2([0, \infty))$, see [AK72, Lemma 1. (ii)]. By Lemma 5.2 and linearity of the integral we have

$$\int e^{-s} q(s) g(s) ds = \int e^{-s} q(s) \tilde{g}(s) ds$$

for all polynomials q. Let now $h \in L^2([0,\infty))$ and (q_n) be a sequence of polynomials with $e^{-s} q_n \xrightarrow{L^2} h$. For c large enough, g and $\tilde{g}$ are positive, bounded, and in L^2. It follows that

$$\lim_{n\to\infty} \int e^{-s} q_n(s) g(s) ds = \int h(s) g(s) ds$$

and

$$\lim_{n\to\infty} \int e^{-s} q_n(s) \tilde{g}(s) ds = \int h(s) \tilde{g}(s) ds,$$

hence,

$$\int h(s) \mu(ds) = \int h(s) \tilde{\mu}(ds)$$

for all $g \in L^2(B)$.

It follows that $\mu = \tilde{\mu}$ a.e. Since f and $\tilde{f}$ are continuous we have $f = \tilde{f}$. $\square$

Example 5.4. Let $b(t) \equiv b_0 > 0$. Then

$$1 = \int_0^\infty e^{-\frac{c^2}{2}s + cb_0} f(s) ds$$

for all $c > 0$. We set

$$f(s) = \frac{b_0}{\sqrt{2\pi s^3}} e^{-\frac{b_0^2}{2s}},$$

then

$$\int_0^\infty \frac{b_0}{\sqrt{2\pi s^3}} e^{-\frac{c^2}{2}s + cb_0 - \frac{b_0^2}{2s}} ds = \int_0^\infty \frac{b_0}{\sqrt{2\pi s^3}} e^{-\frac{(b_0 - cs)^2}{2s}} ds = 1,$$

so f indeed solves the problem. By Theorem 5.3 we can conclude that f is the density function of the distribution of τ, which is of course a well known result.

5.3 Outlook and discussion

A main contribution of this paper is that the FPT problem has been reduced to the solution of an integral equation because of the uniqueness result and thus the well known methods for such equations can be applied. The integral equation we are faced with here is a so-called Fredholm integral equation of the first kind. For these, numerical approaches suffer from the fact that these integral equations are usually “ill-posed” in the sense of Hadamard. There are, however, quite a few approaches. The book [Waz11] provides a detailed overview. Among these methods are the method of regularization and the homotopy perturbation method. The more recent review article [YZ19] presents even more methods such as the wavelet method. The paper [CMW19] presents an algorithm to numerically solve Fredholm equations of the first kind and applies it to the FPT problem and the same integral equation as used in this paper (although the notation differs slightly). Due to the large number of results and methods, we also refer the interested reader to the literature cited in the above sources.

From a mathematical perspective, it seems natural to generalise this approach to the n-dimensional case, i.e., to the case of determining the distribution of the first passage time of a n-dimensional Brownian motion to some n-dimensional surface. As there is no canonical generalisation, we will present a possible approach in this section for future research. To this end, let $W_t = (W_t^1, \dots, W_t^n)$ be an n-dimensional standard Brownian motion. Let $d : \mathbb{R}^n \to [0, \infty]$ be a continuous function such that $\tau := \inf\{t \mid d(W_t) \geq t\}$ is an almost surely finite stopping time. A visualization of the setting can be seen in Figure 5.2. We denote by $h : \mathbb{R}^n \to \mathbb{R}$ the density of the distribution of W_τ. As above we decompose the transition kernel p via first passage of d. Note that in contrast to the function b above, d maps from *space* to *time*. We have

$$
\begin{aligned}
p((0,0),(t,\mathbf{x})) = &\int_{d^{-1}([0,t])} p\Big((d(\mathbf{y}),\mathbf{y}),(t,\mathbf{x})\Big) P(W_\tau \in d\mathbf{y}) \\
&+ P(W_t \in dx, \tau > t)
\end{aligned}
$$

and

$$
1 = \frac{\int_{d^{-1}([0,t])} p((d(\mathbf{y}),\mathbf{y}),(t,\mathbf{x})h(\mathbf{y})d\mathbf{y}}{p((0,0),(t,\mathbf{x}))} + \frac{P(W_t \in dx, \tau > t)}{p((0,0),(t,\mathbf{x}))}.
$$

We set $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{x} = \mathbf{c}t$ and let $t \to \infty$. As before the last term vanishes, if we choose c appropriately. We are left with

$$
\lim_{t\to\infty} \int_{d^{-1}([0,t])} \frac{p((d(\mathbf{y}),\mathbf{y}),(t,\mathbf{c}t)}{p((0,0),(t,\mathbf{c}t))} h(\mathbf{y})d\mathbf{y} = \int_{d^{-1}([0,\infty))} e^{-\frac{\|\mathbf{c}\|^2}{2}d(\mathbf{y})+\mathbf{c}\cdot\mathbf{y}} h(\mathbf{y})d\mathbf{y},
$$

where $\cdot$ denotes the standard scalar product and $\| \; \|$ the corresponding norm. If $d(\mathbf{y}) = \infty$ the integrand vanishes, so we can write the resulting equation as

$$
1 = \int_{\mathbb{R}^n} e^{-\frac{\|\mathbf{c}\|^2}{2}d(\mathbf{y})+\mathbf{c}\cdot\mathbf{y}} h(\mathbf{y})d\mathbf{y} \tag{5.7}
$$

for $\mathbf{c} \in \mathbb{R}^n$ such that the integral exists. The density f of the distribution of τ can be obtained via

$$f(t) = \int_{h^{-1}(t)} h(\mathbf{y}) d\sigma$$

where $d\sigma$ denotes integration over the (hyper) surface $h^{-1}(t)$. The proof of uniqueness seems to us to be possible in principle under suitable additional preconditions with the methods presented here, but it is beyond the scope of this chapter.

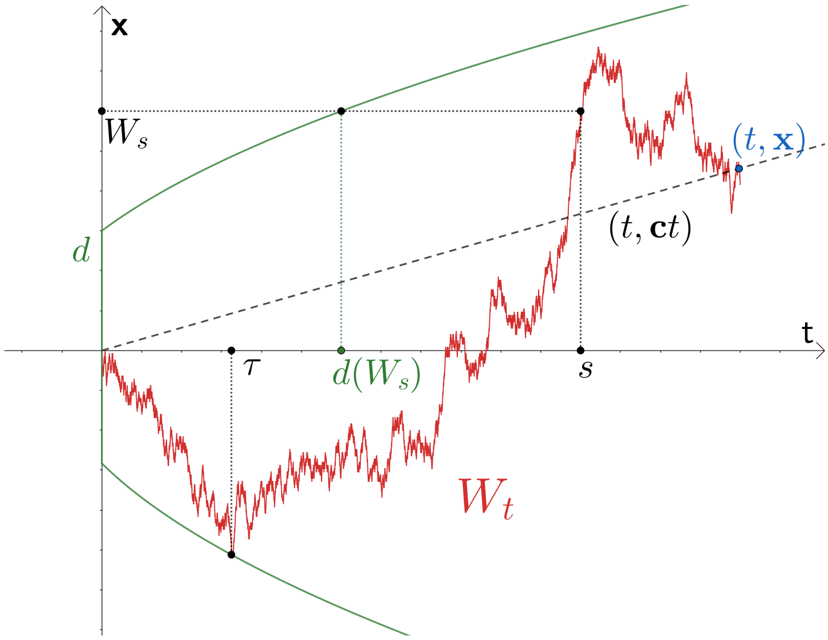

Figure 5.2: One-dimensional visualization of the parametrization via $\mathbf{x}$

Chapter 6

Conclusion

This thesis gives deeper insight into the FPT problem for Brownian motion and in particular into the method of images. Proposition 1.3 is an extension of a classical result from the method of images and establishes straightforward error bounds on the first hitting time distribution of a boundary b up to some point t_0 if we can bound the function $t \mapsto r_\mu(t, b(t))^{-1}$ up to t_0. This proposition is helpful to show that the linear programs (D_1) and (P_2) yield upper and lower bounds on the first hitting time distribution as seen in Theorem 2.1. Note we do not use that b is concave or analytic in the proof of Proposition 1.3 or in the proof of Theorem 2.1. In particular, the linear programs give upper and lower bounds for the first hitting time distribution to boundaries b that do not need to be concave or analytic.

For concave, analytic b we see a new duality structure that has not been given before. Strong duality results are established in Theorems 2.7 and 2.13 and help to give sufficient conditions for b to be representable by a positive, σ-finite measure μ. In particular, the duality structure may reveal a close connection of the first hitting time and the last hitting time as seen above with the measure $\bar{\lambda}$ of the conditional last hitting time distribution in Theorem 2.15. More sufficient conditions for b to be representable are based mainly on the analyticity of b in Theorem 2.19. The duality structures as well as the sufficient conditions merit future research. In particular, the measure $\bar{\lambda}$ seems to be very promising to hopefully dispose of sufficient conditions and show representability of all concave, analytic b. Moreover, the duality results and sufficient conditions from Chapter 2 should carry over to the case of two-sided boundaries with appropriate adaptations based on the programs for two-sided boundaries which we formulated above as ($\tilde{D}_1$), ($\tilde{P}_1$), ($\tilde{D}_2$), and ($\tilde{P}_2$).

In Chapter 3, we establish convergence results for our linear programs and an explicit error bound for the approximation of the c.d.f. F of the first hitting time τ. The error bound is given in Proposition 3.3 and again builds upon Proposition 1.3 which shows its usefulness throughout this thesis. A new algorithm is given which substantially improves upon existing algorithms by requiring less discretisations and offering considerably faster convergence speeds than existing algorithms. The qualitative study of concave, analytic boundaries b conducted in Section 3.3 confirm that the sufficient conditions for representability given in Theorems 2.15 and 2.19 are easily checked numerically.

The use of representability goes beyond the FPT problem, as we see in Chapter 4. We establish a connection to the representability of American options by European options where the candidate stopping boundaries are shown to be representable in the sense of the method of images in Theorem 4.1. This connection may be useful for future research trying to determine whether American options or, in particular, the American put are representable by European options.

Finally, Chapter 5 establishes the important result that the first hitting time distribution is actually uniquely determined by the set of Fredholm equations given in (5.3). This result, which is given in Theorem 5.3, validates the use of Fredholm integral equations to determine the first hitting time distribution. Moreover, an ansatz is given how this approach may carry over to the multi-dimensional case but this requires future research to establish sensible conditions on the multi-dimensional surface to be hit.

Chapter A

Appendix

A.1 Auxiliary results

Lemma A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function vanishing at infinity. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $\mathbb{R}$ converging vaguely to some measure μ. Assume there exists some $0 < C < \infty$ such that the total variations of μ_n and μ are bounded by C, i.e., $\|\mu_n\| < C$ for all $n \in \mathbb{N}$ and also $\|\mu\| < C$. Then,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\mu_n(dx) = \int_{\mathbb{R}} f(x)\mu(dx).$$

Proof. Let $\varepsilon > 0$. As f is vanishing at infinity, we find $0 < K < \infty$ such that $|f(x)| < \varepsilon/(4\cdot C)$ for all x with $|x| > K$. Moreover, we have by vague convergence

$$\lim_{n\to\infty} \int_{-K}^{K} f(x)\mu_n(dx) = \int_{-K}^{K} f(x)\mu(dx).$$

With this in mind, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \left| \int_{\mathbb{R}} f(x)\mu_n(dx) - \int_{\mathbb{R}} f(x)\mu(dx) \right| \\
\leq &\lim_{n\to\infty} \Bigg(\int_{-\infty}^{-K} |f(x)|\mu_n(dx) + \int_{-\infty}^{-K} |f(x)|\mu(dx) + \left| \int_{-K}^{K} f(x)\mu_n(dx) - \int_{-K}^{K} f(x)\mu(dx) \right| \\
&+ \int_{K}^{\infty} |f(x)|\mu_n(dx) + \int_{K}^{\infty} |f(x)|\mu(dx) \Bigg) \\
\leq &4\cdot C \cdot \frac{\varepsilon}{4\cdot C} = \varepsilon.
\end{aligned}$$

$\square$

Lemma A.2. *Let $b_1, b_2 : \mathbb{R}_+ \to \mathbb{R}$ be two boundaries with $b_1(t) \leq b_2(t)$ for all $t \in (0, t_0]$, $b_1(0) < W_0 < b_2(0)$ and define $\tau := \inf\{t \geq 0 : W_t \notin (b_1(t), b_2(t))\}$. If μ is a measure on $\mathbb{R}$ with $\mu(\{0\}) = 0$ and we have that both $r(t, b_1(t)) = 1$ and $r(t, b_2(t)) = 1$ for all $t \leq t_0$, where*

$$r(t,x) = \int_{(-\infty,\infty)} r_\theta(t,x)\mu(d\theta),$$

then, the distribution function of τ is given by

$$F(t) = 1 - \Phi\left(\frac{b_2(t)}{\sqrt{t}}\right) + \Phi\left(\frac{b_1(t)}{\sqrt{t}}\right) + \int_{-\infty}^{\infty} \left(\Phi\left(\frac{\theta - b_1(t)}{\sqrt{t}}\right) - \Phi\left(\frac{\theta - b_2(t)}{\sqrt{t}}\right)\right)\mu(d\theta).$$

Proof. We find with the help of Proposition 1.7 that

$$\begin{aligned}
F(t) &= P\left(\tau \leq t, W_t \in (-\infty, b_1(t)) \cup (b_2(t), \infty)\right) + P\left(\tau \leq t, W_t \in [b_1(t), b_2(t)]\right) \\
&= P\left(W_t \in (-\infty, b_1(t)) \cup W_t \in (b_2(t), \infty)\right) + \int_{b_1(t)}^{b_2(t)} P(\tau \leq t | W_t = y) f_{W_t}(y) dy \\
&= \Phi\left(\frac{b_1(t)}{\sqrt{t}}\right) + 1 - \Phi\left(\frac{b_2(t)}{\sqrt{t}}\right) + \int_{b_1(t)}^{b_2(t)} \int_{-\infty}^{\infty} r_\theta(t,y)\mu(d\theta) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{y^2}{t}\right) dy \\
&= 1 - \Phi\left(\frac{b_2(t)}{\sqrt{t}}\right) + \Phi\left(\frac{b_1(t)}{\sqrt{t}}\right) + \int_{-\infty}^{\infty} \int_{b_1(t)}^{b_2(t)} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\frac{(y-\theta)^2}{t}\right) dy \mu(d\theta) \\
&= 1 - \Phi\left(\frac{b_2(t)}{\sqrt{t}}\right) + \Phi\left(\frac{b_1(t)}{\sqrt{t}}\right) + \int_{-\infty}^{\infty} \Phi\left(\frac{\theta - b_1(t)}{\sqrt{t}}\right) - \Phi\left(\frac{\theta - b_2(t)}{\sqrt{t}}\right) \mu(d\theta).
\end{aligned}$$

□

A.2 Additional convergence results

In Section 3.2 we stated and proved convergence results for (P_2) and for (D_1), the "μ-problems". Analogous results can be given for (P_1) and (D_2), the "λ-problems".

Proposition A.3. *Assume $\lambda_i \in \mathcal{M}^+(\mathbb{R}), i \in \mathbb{N}$, to be positive measures. Set $V_n := \{\sum_{i=1}^n a_i\lambda_i \mid a \in \mathbb{R}^n_{\geq 0}\}$ and denote by V_∞ the closure of $\bigcup_{n\in\mathbb{N}} V_n$ with respect to the vague* *topology. For $n \in \mathbb{N} \cup \{\infty\}$ consider the linear program*

$$\begin{aligned} &\text{minimise} \quad \|\lambda\| \\ &\text{subject to} \quad \lambda \in V_n, \\ &\qquad\qquad \int r_\theta(t, b(t))\lambda(dt) \geq r_\theta(t_0, x_0) \text{ for any } \theta \in [2b(0), \infty) \end{aligned} \tag{$P_{1,n}$}$$

and assume that there exists $C \in \mathbb{R}_{>0}$ such that $C \cdot \lambda_1$ is admissible in $(P_{1,1})$ and therefore in any $(P_{1,n})$ for $n \in \mathbb{N} \cup \{\infty\}$ and that $C \cdot \|\lambda_1\| < \infty$. Then,

(a) Let $n \in \mathbb{N} \cup \{\infty\}$. Then the linear program $(P_{1,n})$ attains its optimal value $p_{1,n}$ at some admissible measure $\lambda^*_{1,n}$. The optimal value satisfies $p_{1,n} \leq k \cdot \|\lambda_1\|$. Moreover, for $m \leq n$ the measure $\lambda^*_{1,m}$ is $(P_{1,n})$-admissible and it holds $p_{1,m} \geq p_{1,n}$.

(b) There exists a subsequence of optimisers $(\lambda^*_{1,n_k})_k$ and a $(P_{1,\infty})$-admissible measure κ_∞ such that $(\lambda^*_{1,n_k})_k \to \kappa_\infty$ vaguely. Moreover,

$$p_{1,\infty} \leq \int r_\theta(t_0, x_0)\kappa_\infty(dt) \leq \inf_{n\in\mathbb{N}} p_{1,n} = \lim_{n\to\infty} p_{1,n}.$$

(c) If there exists a sequence $(\xi_n)_n$ with $\xi_n \in V_n$ converging weakly to some $(P_{1,\infty})$-admissible measure λ^*_∞ as $n \to \infty$ and if

$$\lim_{n\to\infty} \sup_{\theta\in[2b(0),\infty)} \frac{|\int r_\theta(t, b(t))\lambda_\infty(dt) - \int r_\theta(t, b(t))\xi_n(dt)|}{\int r_\theta(t, b(t))\lambda_1(dt)} = 0,$$

then κ_∞ from Assertion (b) is $(P_{1,\infty})$-optimal and $(\lambda^*_{1,n_k})_k$ converges weakly to κ_∞. Moreover, we find

$$p_{1,\infty} = \int r_\theta(t_0, x_0)\kappa_\infty(dt) = \inf_{n\in\mathbb{N}} p_{1,n} = \lim_{n\to\infty} p_{1,n}.$$

Proof. The proof is analogous to the proof of Proposition 3.1. □

Proposition A.4. *Assume $\lambda_i \in \mathcal{M}^+(\mathbb{R}), i \in \mathbb{N}$, to be positive measures. Set $V_n := \{\sum_{i=1}^n a_i\lambda_i \mid a \in \mathbb{R}^n_{\geq 0}\}$ and denote by V_∞ the closure of $\bigcup_{n\in\mathbb{N}} V_n$ with respect to the vague* *topology. For $n \in \mathbb{N} \cup \{\infty\}$ consider the linear program*

$$\begin{aligned} &\text{maximise} \quad \|\lambda\| \\ &\text{subject to} \quad \lambda \in V_n, \\ &\qquad \int r_\theta(t, b(t))\lambda(dt) \leq r_\theta(t_0, x_0) \text{ for any } \theta \in [2b(0), \infty) \end{aligned} \tag{$D_{2,n}$}$$

and assume that there exists $C \in \mathbb{R}_{>0}$ such that $C \cdot \lambda_1$ is admissible in $(D_{2,1})$ and therefore in any $(D_{2,n})$ for $n \in \mathbb{N} \cup \{\infty\}$ and that $C \cdot \|\lambda_1\| < \infty$. Then,

(a) Let $n \in \mathbb{N} \cup \{\infty\}$. Then, the linear program $(D_{2,n})$ attains its optimal value $d_{2,n}$ at some admissible measure $\lambda^*_{2,n}$. Moreover, for $m \leq n$ the measure $\lambda^*_{2,m}$ is $(D_{2,n})$-admissible and we have $d_{2,m} \leq d_{2,n}$.

(b) There exists a subsequence of optimisers $(\lambda^*_{2,n_k})_k$ and a $(D_{2,\infty})$-admissible measure κ_∞ such that $(\lambda^*_{2,n_k})_k \to \kappa_\infty$ vaguely. Moreover,

$$d_{2,\infty} \geq \int r_\theta(t_0, x_0)\kappa_\infty(dt) \geq \inf_{n\in\mathbb{N}} d_{2,n} = \lim_{n\to\infty} d_{2,n}.$$

(c) If there exists a sequence $(\xi_n)_n$ with $\xi_n \in V_n$ converging weakly to some $(D_{2,\infty})$-admissible measure λ^*_∞ as $n \to \infty$ and if

$$\lim_{n\to\infty} \inf_{\theta\in[2b(0),\infty)} \frac{\left|\int r_\theta(t, b(t))\lambda_\infty(dt) - \int r_\theta(t, b(t))\xi_n(dt)\right|}{\int r_\theta(t, b(t))\lambda_1(d\theta)} = 0,$$

then κ_∞ from Assertion (b) is $(D_{2,\infty})$-optimal and λ^*_{2,n_k} converges weakly to κ_∞. Moreover, we find

$$d_{2,\infty} = \int r_\theta(t_0, x_0)\kappa_\infty(dt) = \inf_{n\in\mathbb{N}} d_{2,n} = \lim_{n\to\infty} d_{2,n}.$$

Proof. Note that for all $(D_{2,n})$-admissible λ we find by evaluating the constraint at $\theta = 2b(0)$ that

$$\begin{aligned} \exp\left(-\frac{2b(0)^2}{t_0} + \frac{2b(0)x_0}{t_0}\right) &\geq \int_\Omega \exp\left(2b(0)\frac{b(t) - b(0)}{t}\right)\lambda(dt) \\ &\geq \exp\left(2b(0)\frac{b(t_0) - b(0)}{t_0}\right)\|\lambda\| \end{aligned}$$

where we use in the last step that $t \to \frac{b(t)-b(0)}{t}$ is a decreasing function. But then, we obtain

$$\|\lambda\| \leq \exp\left(2b(0)\frac{x_0 - b(t_0)}{t_0}\right) =: C.$$

Thus, all admissible solutions of ($D_{2,n}$) must be contained in $B_{\mathcal{M}}(C) = \{\lambda \in \mathcal{M}(0, t_0] \mid \|\lambda\| \leq C\}$. The rest of the proof follows along the same lines as the proof of Proposition 3.1. □

A.3 Some concepts from functional analysis

This section aims at introducing some results from functional analysis that are necessary for the proofs in the main thesis above or help to understand the concepts of infinite-dimensional linear programming that are introduced in Section A.4 below. These results are well-known and a reader well-versed in these subjects may skip this section or use it as a reference to refresh her knowledge on these results. This section uses definitions and results from [Roc74], [Con90], [MV92] and [Len17] and quotes these as necessary.

We start by considering lower semi-continuous functions. The definition and their properties can be found in [Roc74], Chapter 3. Let X be a topological space and let $f : X \to [-\infty, \infty]$ be an extended real-valued function on X. Then, f is called *lower semi-continuous* if the set $\{x | f(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$. A function is called *upper semi-continuous* if $-f$ is lower semi-continuous. If X is a metric space, then f is lower semi-continuous if and only if

$$f(x) \leq \liminf_{x' \to x} f(x')$$

for all $x \in X$. One can show that lower semi-continuous functions have the following properties.

Lemma A.5. *Let f, g and $(f_i)_{i\in I}$ be lower semi-continuous functions on X and I some non-empty index set. Then,*

(i) The functions $f + g$, $f \wedge g$ and $f \vee g$ are lower semi-continuous.

(ii) The function $x \mapsto \sup_{i\in I} f_i(x)$ is lower semi-continuous.

(iii) If f is bounded from below on some compact set $K \subset X$, then f attains its minimum on K.

(iv) The function f is lower semi-continuous if and only if its epigraph*, i.e.,*

$$\mathrm{epi}(f) := \{(x,s) \in X \times \mathbb{R} | f(x) \leq s\}$$

is closed with respect to the product topology on $X \times \mathbb{R}$.

Now, we turn our attention to locally convex spaces. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let V be a vector space over $\mathbb{K}$. Recall that a semi-norm p on V fulfils $p(\lambda x) = |\lambda| p(x)$ and $p(x+y) \leq p(x) + p(y)$ for all $x, y \in V$ and all $\lambda \in \mathbb{K}$. Let I be an arbitrary index set and let $P := (p_i)_{i \in I}$ be a family of semi-norms on V. Then, the initial topology induced by these semi-norms is given by

$$\mathcal{T}_P := \{O \subset V | \forall x_0 \in O \exists F \subset I \text{ finite } \exists \varepsilon \in (0,\infty)^F : B_\varepsilon^F(x_0) \subset O\},$$

where $B_\varepsilon^F(x_0) := \bigcap_{i \in F}\{x \in V | p_i(x - x_0) < \varepsilon_i\}$. The initial topology is the coarsest topology on V such that $x \mapsto p_i(x - x_0)$ is continuous for all $i \in I$, $x_0 \in X$. We call $\mathcal{T}_P$ the *locally convex topology* on V generated by P and $(V, \mathcal{T}_P)$ a *locally convex space*. Note that the definition given above differs from the definition given in [Con90], Chapter 4, where locally convex spaces are introduced to be always Hausdorff which they do not have to be in our definition given above.

As usual, the space of all $\mathcal{T}_P$-continuous linear functionals on V is called *dual space* and is denoted by $(V, \mathcal{T}_P)^*$ or shortly V^*. Moreover, for $\mathbb{K}$-vector spaces V and W let $\langle V, W\rangle$ be an *algebraic pairing*, i.e., a bilinear mapping $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{K}$. Then, $P := \{x \mapsto |\langle x, y\rangle| \,|\, y \in W\}$ is a family of semi-norms as can be easily verified. Now, we can consider the locally convex topology on V induced by P. This topology is called the *weak topology* on V induced by W and is denoted by $\sigma(V, W)$.

Now, we can define *dual operators*. Let $\langle V_1, W_1\rangle_1$ and $\langle V_2, W_2\rangle_2$ be algebraic pairings of vector space V_1, W_1, V_2, and W_2 and let $T : V_1 \to V_2$ be a $\sigma(V_1, W_1)$-$\sigma(V_2, W_2)$-continuous mapping. Then, it can be shown that there exists a unique $\sigma(W_2, V_2) - \sigma(W_1, V_1)$-continuous, linear operator T^* such that

$$\langle Tx, y\rangle_2 = \langle x, T^* y\rangle_1$$

for all $x \in V_1$, $y \in W_2$. The following lemma can for example be found as Lemma 5.17 in [Len17].

Lemma A.6. *Let $\langle V_1, W_1\rangle_1$ and $\langle V_2, W_2\rangle_2$ be algebraic pairings. Suppose that $T : V_1 \to$*

V_2 and $T^* : W_2 \to W_1$ are linear mappings satisfying

$$\langle Tx, y\rangle_2 = \langle x, T^* y\rangle_1$$

for all $x \in V_1$, $y \in W_2$. Then, T is $\sigma(V_1, W_1) - \sigma(V_2, W_2)$-continuous and T^* is $\sigma(W_2, V_2) - \sigma(W_1, V_1)$-continuous.

Moreover, we consider the following formulation of the Alaoglu-Bourbaki theorem which is due to Theorem 23.5 in [MV92].

Theorem A.7 (Alaoglu-Bourbaki)**.** *Let V be a locally convex Hausdorff space and let $U \subset V$ be a neighbourhood of 0. Then, the set*

$$U^\circ := \{x^* \in V^* | \mathrm{Re}(x^*(x)) \leq 1 \forall x \in U\}$$

is $\sigma(V^*, V)$-compact.

Now, we recall some results concerning conjugate functions. The following definitions can for example be found in [Roc74], Chapter 3. Let $(V, \mathcal{T}_P)$ be a locally convex space over $\mathbb{R}$ and $f : V \to [-\infty, \infty]$ a function. The *conjugate* $f^* : V^* \to [-\infty, \infty]$ of f is defined as

$$f^*(x^*) := \sup_{x \in V} (\langle x, x^*\rangle - f(x)).$$

The mapping $f \mapsto f^*$ is called the *Fenchel transform*. Going one step further, we can define the *biconjugate* $f^{**} : V \to [-\infty, \infty]$ of f as

$$f^{**}(x) := (f^*)^*(x) = \sup_{x^* \in V^*} (\langle x, x^*\rangle - f^*(x^*)).$$

Moreover, the *lower semi-continuous hull* $\mathrm{lsc}(f)$ of f is the largest semi-continuous function that is smaller than f, i.e.,

$$\mathrm{lsc}(f)(x) := \sup\{h(x) | h \text{ lower semi-continuous and } h \leq f\}.$$

The *convex hull* $\mathrm{co}(f)$ of f is the largest convex function smaller than f, i.e.,

$$\mathrm{co}(f)(x) := \sup\{h(x) | h \text{ convex and } h \leq f\}.$$

Finally, the *closure* $\mathrm{cl}(f)$ of f is given by

$$\mathrm{cl}(f) := \begin{cases} \mathrm{lsc}(f), & \text{if } \mathrm{lsc}(f)(x) > -\infty \text{ for all } x \in V, \\ -\infty, & \text{otherwise.} \end{cases}$$

A function f is said to be closed if $f = \mathrm{cl}(f)$. Then, one can show the following properties of these objects. For a proof, see for example Lemma 5.25 in [Len17].

Lemma A.8. *The following assertions hold:*

(i) For any $x \in V$ and $x^* \in V^*$, it holds $\langle x, x^* \rangle \leq f(x) + f^*(x^*)$ and $f^{**}(x) \leq f(x)$.

(ii) The conjugate and biconjugate of f are convex and lower semi-continuous.

(iii) The lower semi-continuous hull of f is the largest lower semi-continuous minorant of f. In other words, the function $\mathrm{lsc}(f)$ is lower semi-continuous, $\mathrm{lsc}(f) \leq f$ and $h \leq \mathrm{lsc}(f)$ for any other lower semi-continuous function h satisfying $h \leq f$. For any $x \in V$, we have

$$\mathrm{lsc}(f)(x) = \sup_{O \in \mathcal{U}(x)} \inf_{y \in O} f(y)$$

where $\mathcal{U}(x)$ denotes the collection of all open sets containing x. Moreover, the epigraph of $\mathrm{lsc}(f)$ coincides with the closure of $\mathrm{epi}(f)$ with respect to the product topology $\mathcal{T}_P \otimes \mathcal{T}_{\mathbb{R}}$ on $V \times \mathbb{R}$, i.e.,

$$\mathrm{epi}(\mathrm{lsc}(f)) = \mathrm{cl}(\mathrm{epi}(f)).$$

(iv) The convex envelope of f is the largest convex minorant of f. In other words, the function $\mathrm{co}(f)$ is convex, $\mathrm{co}(f) \leq f$ and $h \leq \mathrm{co}(f)$ for any other convex function h satisfying $h \leq f$. Moreover, the epigraph of $\mathrm{co}(f)$ coincides with the convex hull of $\mathrm{epi}(f)$, i.e.,

$$\mathrm{epi}(\mathrm{co}(f)) = \mathrm{co}(\mathrm{epi}(f)).$$

The following theorem can be found as Theorem 5, [Roc74] and gives the connection between f and f^{**}.

Theorem A.9 (Fenchel-Moreau)**.** *Let $f : V \to [-\infty, \infty]$ be an extended real-valued mapping. Then,*

(i) *The conjugate* f^* *constitutes a closed, convex function on the dual space* V^* *and we have* $f^{**} = \mathrm{cl}(\mathrm{co}(f))$.

(ii) *The Fenchel transform induces a one-to-one correspondence between the closed, convex functions on* V *and closed, convex functions on* V^*.

A.4 Introduction to infinite-dimensional optimisation

In this section, we will introduce some concepts from infinite-dimensional optimisation. It is relying on [Roc74] which Rockafellar wants to be understood as complementary reading to [Roc70]. Indeed, [Roc74] sets out the same concepts as [Roc70] but refrains from introducing most concepts of convex analysis and rather concentrates on setting out the theory in more general terms.

We start by defining an *abstract optimization problem*: Let X be a real linear space and $C \subset X$ a subset of X. Furthermore, let $f : C \to [-\infty, \infty]$. Then, the abstract optimization problem is to minimise $f(x)$ for $x \in C$. The problem is said to be *convex* if f is convex. Often, f is set to ∞ outside of C so we can consider minimising f over all of X.

We now embed this problem into a class of optimisation problems depending on a parameter $u \in U$ for some vector space U: minimise $F(x, u)$ over all $x \in X$. The representation F is chosen in such a way that $f(x) = F(x, 0)$ for all $x \in X$. We will consider an example to illustrate this embedding.

Example A.10. Let f_i be real-valued convex functions on a non-empty convex set C in the linear space X for all $i = 0, 1, \ldots, m$. The problem is to minimise f over X where

$$f(x) = \begin{cases} f_0(x), & \text{if } x \in C, f_i(x) \leq 0 \text{ for all } i = 1, \ldots, m \\ \infty, & \text{else.} \end{cases}$$

This represents our "usual" convex problems in the finite dimensional case. Now, a way to parametrise this problem for $u \in U = \mathbb{R}^m$ is to set

$$F(x, u) = \begin{cases} f_0(x), & \text{if } x \in C, f_i(x) \leq u_i \text{ for all } i = 1, \ldots, m \\ \infty, & \text{else.} \end{cases}$$

Then, $F(x,0) = f(x)$.

Now return to the problem of minimising a function f over X and suppose we have chosen a representation $F(x,0) = f(x)$ with $F : X \times U \to [-\infty, \infty]$. Furthermore, assume that X is paired with a space V and U is paired with a space Y in the sense of algebraic pairings considered in Section A.3. A topology on x is said to be *compatible* with the pairing if it turns X into a locally convex space and all linear functions of the form

$$\langle \cdot, v \rangle : x \to \langle x, v \rangle$$

for $v \in V$ are continuous. A topology on V is compatible if the analogous of the above holds. That means if X and V are called paired spaces, we will assume that not only a pairing has been selected but also that compatible topologies for X and V in accordance with the pairing have been chosen.

We can now define the *Lagrangian* function L on $X \times Y$ by

$$L(x,y) = \inf\{F(x,u) + \langle u, y \rangle | u \in U\}$$

This definition of the Lagrangian indeed breaks down into the "usual" definition of the Lagrangian. In order to illustrate this, we will take a look back at our example.

Example A.11. Consider f and F as in Example A.10, i.e., in particular

$$F(x,u) = \begin{cases} f_0(x), & \text{if } x \in C, f_i(x) \leq u_i \text{ for all } i = 1, \dots, m \\ \infty, & \text{else.} \end{cases}$$

Then, it is easy to see that

$$L(x,y) = \begin{cases} f_0(x) + y_1 f_1(x) + \ldots + y_m f_m(x), & \text{if } x \in C, y \in \mathbb{R}^m_+ \\ -\infty & \text{if } x \in C, y \notin \mathbb{R}^m_+ \\ \infty & \text{if } x \notin C. \end{cases}$$

It can now be shown that

$$f(x) = \sup_{y \in Y} L(x,y).$$

Indeed, [Roc74] establishes this in Theorem 6 (which in turn is based on a far more general statement in Theorem 5 in the same book). Now, the *dual problem* can be defined by

$$\text{maximise} \qquad g(y)$$

over all $y \in Y$ where

$$g(y) = \inf_{x \in X} L(x, y).$$

Now, we say that *strong duality* holds if

$$\inf_{x \in X} f(x) = \sup_{y \in Y} g(y)$$

and therefore to the problem of finding a *saddle-value* of L, i.e., a pair $(\tilde{x}, \tilde{y})$ such that

$$L(x, \tilde{y}) \geq L(\tilde{x}, \tilde{y}) \geq L(\tilde{x}, y) \qquad \text{for all } x \in X, y \in Y.$$

Note that by the definition of the *subgradient* ∂f of a function f we have that $\tilde{x}$ solves the primal problem, i.e., minimises f if and only if $0 \in \partial f(\tilde{x})$. This notation is necessary as we cannot assume f to be differentiable everywhere or even anywhere. This notion of course simplifies if f is differentiable.

Furthermore, note that $(0, 0) \in \partial L(\tilde{x}, \tilde{y})$ if and only if $(\tilde{x}, \tilde{y})$ is a saddle-point of L (recall the slightly different definition of a subgradient for a concave function). The relation $(0, 0) \in \partial L(\tilde{x}, \tilde{y})$ is called the *(abstract) Kuhn-Tucker conditions* for the primal problem. The abstract Kuhn-Tucker conditions can be shown to simplify to the "usual" KKT conditions (cf. chapter 10 in [Roc74]).

Moreover, these abstract Kuhn-Tucker conditions behave in the same way as the "usual" KKT conditions as the following theorem states (cf. Theorem 15, [Roc74]).

Theorem A.12. *Let F be closed convex in u, then the following are equivalent.*

1. *$\tilde{x}$ solves the primal problem and $\tilde{y}$ solves the dual problem and strong duality holds (i.e., $\inf f = \sup g$).*

2. *The pair $(\tilde{x}, \tilde{y})$ satisfies the abstract Kuhn-Tucker condition.*

If F is not closed convex, then the abstract Kuhn-Tucker conditions still remain a necessary condition for strong duality and the existence of solutions $\tilde{x}$ and $\tilde{y}$ for the primal and dual problem. So like in the finite-dimensional case, the convexity of the problem ensures equivalence of these statements. The following theorem is a generalised version of the famous Slater's condition (cf. Theorem 18, [Roc74]) which is a sufficient condition for strong duality to hold.

Theorem A.13. *Let F be convex. Assume there is an $x \in X$ such that the function $u \to F(u, x)$ is bounded above on a neighbourhood of 0. Then,* $\inf f = \sup g$ *(i.e., strong duality holds) and there exists at least one $\tilde{y}$ at which g attains its supremum.*

This condition is indeed equivalent to the "usual" Slater condition as shown in Chapter 8 in [Roc74].

Notations

$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$\{0, 1, 2, 3, \ldots\}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	$(0, \infty)$
ϕ	density of the standard normal distribution
Φ	cumulative distribution function of the standard normal distribution
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$
$P_{(0,x)}$	probability measure of a Brownian motion started in x at time 0
$P_{(0,x)}^{(t,y)}$	probability measure of a Brownian bridge running from x at time 0 to y at time t
$C(S)$	continuous functions from S to $\mathbb{R}$
$C_b(S)$	continuous, bounded functions from S to $\mathbb{R}$
$C_0(S)$	continuous functions from S to $\mathbb{R}$ vanishing at infinity
$C_c(S)$	continuous, compactly supported functions from S to $\mathbb{R}$
$C^+(S)$	cone of non-negative elements of $C(S)$, analogous notation for $C_b(S)$ etc.
$\mathcal{M}(S)$	regular, σ-finite Borel measures on S
$\mathcal{M}^+(S)$	cone of non-negative elements of $\mathcal{M}(s)$
$\sigma(v, W)$	weak topology on V induced by W
T^*	dual operator of the operator T
f^*	conjugate of the function f
f^{**}	biconjugate of the function f
$B_V(x, r)$	closed ball of radius r around x in vector space V
$\mathcal{U}(x)$	system of all open sets containing x in some topological space
$\mathrm{lsc}(f)$	lower-semi-continuous hull of a function f
$\mathrm{co}(f)$	convex hull of a function f
$\mathrm{cl}(f)$	closure of a function f

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertation - abgesehen von der Beratung durch meinen Betreuer Herrn Prof. Dr. Sören Christensen - nach Inhalt und Form eigenständig angefertigt habe. Dabei habe ich die Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft eingehalten. Die Arbeit hat weder ganz noch zum Teil einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen und ist bis auf den unten erwähnten Preprint weder ganz noch zum Teil veröffentlicht oder zur Veröffentlichung eingereicht worden. Des Weiteren habe ich noch keinen Promotionsversuch unternommen. Mir wurde kein akademischer Grad entzogen. Teile dieser Arbeit wurden in folgenden Artikeln vorab zur Veröffentlichung eingereicht:

S. Christensen, S. Fischer, and O. Hallmann. *Uniqueness of First Passage Time Distributions via Fredholm Integral Equations.* 2023. URL: https://arxiv.org/abs/2303.05450

Oskar Felix Hallmann
Hamburg, 26.7.2023