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Constrained quantization for probability distributions

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CONSTRAINED QUANTIZATION FOR PROBABILITY DISTRIBUTIONS

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Abstract. In this paper, for a Borel probability measure P on a Euclidean space $\mathbb{R}^k$, we extend the definitions of nth unconstrained quantization error, unconstrained quantization dimension, and unconstrained quantization coefficient, which traditionally in the literature known as nth quantization error, quantization dimension, and quantization coefficient, to the definitions of nth constrained quantization error, constrained quantization dimension, and constrained quantization coefficient. The work in this paper extends the theory of quantization and opens a new area of research. In unconstrained quantization, the elements in an optimal set are the conditional expectations in their own Voronoi regions, and it is not true in constrained quantization. In unconstrained quantization, if the support of P contains infinitely many elements, then an optimal set of n-means always contains exactly n elements, and it is not true in constrained quantization. It is known that the unconstrained quantization dimension for an absolutely continuous probability measure equals the Euclidean dimension of the underlying space. In this paper, we show that this fact is not true as well for the constrained quantization dimension. It is known that the unconstrained quantization coefficient for an absolutely continuous probability measure exists as a unique finite positive number. From work in this paper, it can be seen that the constrained quantization coefficient for an absolutely continuous probability measure can be any nonnegative number depending on the constraint that occurs in the definition of nth constrained quantization error.

1. Introduction

The most common form of quantization is rounding-off. Its purpose is to reduce the cardinality of the representation space, in particular, when the input data is real-valued. It has broad applications in communications, information theory, signal processing, and data compression (see [GG, GL, GL1, GN, P, Z1, Z2]).

For $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, let $\mathbb{R}^k$ be a Euclidean space equipped with a metric d. Let P be a Borel probability measure on $\mathbb{R}^k$ and $r \in (0, +\infty)$. Let $S \subseteq \mathbb{R}^k$ be such that S is closed. Then, the distortion error for P, of order r, with respect to a set $\alpha \subseteq S$, denoted by $V_r(P; \alpha)$, is defined as

$$V_r(P; \alpha) = \int \min_{a \in \alpha} d(x, a)^r dP(x).$$

Then, for $n \in \mathbb{N}$, the *nth constrained quantization error* for P, of order r, is defined as

$$V_{n,r} := V_{n,r}(P) = \inf \Big\{ V_r(P; \alpha) : \alpha \subseteq S,\ 1 \leq \text{card}(\alpha) \leq n \Big\}, \tag{1}$$

where card(A) represents the cardinality of a set A. If in the definition of nth constrained quantization error, the set S, known as *constraint*, is chosen as the set $\mathbb{R}^k$ itself, then the *nth constrained quantization error* is referred to as the *nth unconstrained quantization error*, which traditionally in the literature is referred to as the *nth quantization error*. For the details of the mathematical treatment of unconstrained quantization, one is referred to [GL]. For the probability measure P, we make the standard assumption that $\int d(x, 0)^r dP(x) < \infty$. This ensures us that there is a set $\alpha \subseteq S$ for which the infimum in (1) exists. For a finite set $\alpha \subset \mathbb{R}^k$, and $a \in \alpha$, by $M(a|\alpha)$ we denote the set of all elements in $\mathbb{R}^k$ which are nearest to a among all the elements in α, i.e., $M(a|\alpha) = \{x \in \mathbb{R}^k : d(x, a) = \min_{b \in \alpha} d(x, b)\}$. $M(a|\alpha)$ is called the *Voronoi region* in $\mathbb{R}^k$ generated by $a \in \alpha$. On the other hand, the set $\{M(a|\alpha) : a \in \alpha\}$ is called the *Voronoi diagram* or *Voronoi tessellation* of $\mathbb{R}^k$ with respect to the set α. In the case of unconstrained quantization as described in [GL], the elements in an optimal set are the conditional expectations in

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their own Voronoi regions. This fact is not true in the case of constrained quantization. Because of that, in the case of constrained quantization, a set α for which the infimum in (1) exists and contains no more than n elements is called an *optimal set of n-points* instead of calling it as an *optimal set of n-means*. Elements of an optimal set are called *optimal elements*. In unconstrained quantization, as described in [GL], if the support of P contains infinitely many elements, then an optimal set of n-means always contains exactly n elements. It is not true in constrained quantization. In the case of constrained quantization, an optimal set of one-point always exists; on the other hand, if the support of P contains infinitely many elements, then an optimal set of n-points for any $n \geq 2$ does not necessarily contain exactly n elements. Notice that unconstrained quantization, as described in [GL], is a special case of constrained quantization. There are some properties in unconstrained quantization that are not true in constrained quantization. This paper deals with $r = 2$ and the metric on $\mathbb{R}^k$ as the Euclidean metric denoted by $\|\cdot\|$. Thus, instead of writing $V_r(P;\alpha)$ and $V_{n,r} := V_{n,r}(P)$ we will write them as $V(P;\alpha)$ and $V_n := V_n(P)$.

1.1. **Delineation.** In this paper, we have determined the optimal sets of n-points and the nth constrained quantization errors for all $n \in \mathbb{N}$ for different uniform distributions: in Section 3, the uniform distribution has support a closed interval $[a, b]$ and the optimal elements lie on another line segment; in Section 4, the uniform distribution has support a circle, and the optimal elements lie on another circle; in Section 5, the uniform distribution has support a chord of a circle, and the optimal elements lie on that circle; and in Section 6 the uniform distribution has support a closed interval on a line outside of a circle and the optimal elements lie on that circle. Finally, in Section 7, we give the definitions of the constrained quantization dimension and the constrained quantization coefficient and, with different examples, show the differences between the constrained and the unconstrained quantization dimensions, and the constrained and the unconstrained quantization coefficients.

2. Preliminaries

In this section, we give some basic notations and definitions which we have used throughout this paper. For any two elements $\tilde a := (a_1, a_2)$ and $\tilde b := (b_1, b_2)$ in $\mathbb{R}^2$, we write $\rho(\tilde a, \tilde b) := \|(a_1, a_2) - (b_1, b_2)\|^2 = (a_1-b_1)^2 + (a_2-b_2)^2$, which gives the squared Euclidean distance between the two elements (a_1, a_2) and (b_1, b_2). Let p and q be two elements that belong to an optimal set of n-points for some positive integer n, and let e be an element on the boundary of the Voronoi regions of the elements p and q. Since the boundary of the Voronoi regions of any two elements is the perpendicular bisector of the line segment joining the elements, we have

$$\|p-e\| = \|q-e\| \text{ yielding } \|p-e\|^2 = \|q-e\|^2, \text{ i.e., } \rho(p,e) - \rho(q,e) = 0.$$

We call such an equation a *canonical equation*. Let P be a Borel probability measure on $\mathbb{R}^2$. Then, by $dP(x, y)$ it is meant that

$$dP(x,y) = \left\{\begin{array}{cl} P(dxdy) = f(x,y)\,dxdy & \text{if } x \text{ and } y \text{ are variables,} \\ dP(x) = P(dx) = f(x,y)\,dx & \text{if } x \text{ is a variable and } y \text{ is a constant,} \\ dP(y) = P(dy) = f(x,y)\,dy & \text{if } y \text{ is a variable and } x \text{ is a constant,} \end{array}\right.$$

where $f(x, y)$ is the probability density function for the probability measure P, i.e., $f(x, y)$ is a real-valued function on $\mathbb{R}^2$ with the following properties: $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$, and

$$\left\{\begin{array}{cl} \int_{\mathbb{R}^2} f(x,y)\,dxdy = 1 & \text{if } x \text{ and } y \text{ are variables,} \\ \int_{\mathbb{R}} f(x,y)\,dx = 1 & \text{if } x \text{ is a variable and } y \text{ is a constant,} \\ \int_{\mathbb{R}} f(x,y)\,dy = 1 & \text{if } y \text{ is a variable and } x \text{ is a constant.} \end{array}\right.$$

Let P be a Borel probability measure on $\mathbb{R}^2$ with probability density function $f(x, y)$. Suppose P has support a line segment L_x which is parallel to the x-axis, i.e., y is constant on L_x. Then, notice that $P(\mathbb{R}^2 \setminus L_x) = 0$, and $dP(x, y) = dP(x)$ on L_x. Hence, for any $(a, b) \in \mathbb{R}^2$,

$$\int_{\mathbb{R}^2} \rho((x,y),(a,b))\,dP(x,y) = \int_{L_x} \rho((x,y),(a,b))\,dP(x) = \int_{L_x} \rho((x,y),(a,b))f(x,y)\,dx.$$

Similarly, if P has support a line segment L_y which is parallel to the y-axis, i.e., x is constant on L_y, then

$$\int_{\mathbb{R}^2} \rho((x,y),(a,b))\,dP(x,y) = \int_{L_y} \rho((x,y),(a,b))\,dP(y) = \int_{L_y} \rho((x,y),(a,b)) f(x,y)\,dy.$$

3. Constrained quantization when the support lies on a line segment and the optimal elements lie on another line segment

Let $a, b \in \mathbb{R}$ with $a < b$, and $c, m \in \mathbb{R}$. Let L be a line given by $y = mx + c$, the parametric representation of which is

$$L := \{(x, mx + c) : x \in \mathbb{R}\}.$$

Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } y = 0\}$. Then, the probability density function f for P is given by

$$f(x,y) = \left\{\begin{array}{cc} \frac{1}{b-a} & \text{if } a \leq x \leq b \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{array}\right.$$

Notice that $dP(x,y) = dP(x) = P(dx) = f(x,0)dx$. In this section, we determine the optimal sets of n-points and the nth constrained quantization errors for the probability measure P for all positive integers n so that the elements in the optimal sets lie on the line L between the two elements $(d, md+c)$ and $(e, me+c)$, where $d, e \in \mathbb{R}$ with $d < e$.

Let us now give the following Theorem.

Theorem 3.1. *Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x,y) \in \mathbb{R}^2 : a \leq x \leq b$ and $y = 0\}$. For $n \in \mathbb{N}$ with $n \geq 2$, let $\alpha_n := \{(a_i, ma_i + c) : 1 \leq i \leq n\}$ be an optimal set of n-points for P so that the elements in the optimal sets lie on the line L between the two elements $(d, md + c)$ and $(e, me + c)$, where $d, e \in \mathbb{R}$ with $d < e$. Assume that*

$$\max\{a, (m^2+1)d + mc\} = a \text{ and } \min\{b, (m^2+1)e + mc\} = b.$$

Then, $a_i = \frac{2i-1}{2n(1+m^2)}(b-a) + \frac{a-cm}{1+m^2}$ for $1 \leq i \leq n$ with quantization error

$$V_n = \frac{1}{12\left(m^2+1\right)n^3}\Big(-48(a-b)^2m^2 + (a-b)(a-b+72cm+8(11a-2b)m^2)n \\ -12(a-b)m(5c+(4a+b)m)n^2 + 12(c+am)^2n^3\Big).$$

Proof. For $n \geq 2$, let $\alpha_n := \{(a_i, ma_i + c) : 1 \leq i \leq n\}$ be an optimal set of n-points on L such that $d \leq a_1 < a_2 < \cdots < a_{n-1} < a_n \leq e$. Notice that the boundary of the Voronoi region of the element (a_1, ma_1+c) intersects the support of P at the elements $(a, 0)$ and $((m^2+1)\frac{(a_1+a_2)}{2}+mc, 0)$, the boundary of the Voronoi region of (a_n, ma_n+c) intersects the support of P at the elements $((m^2+1)\frac{(a_{n-1}+a_n)}{2}+mc, 0)$ and $(b, 0)$. On the other hand, the boundaries of the Voronoi regions of $(a_i, ma_i + c)$ for $2 \leq i \leq n-1$ intersect the support of P at the elements $((m^2+1)\frac{(a_{i-1}+a_i)}{2} + mc, 0)$ and $((m^2+1)\frac{(a_i+a_{i+1})}{2} + mc, 0)$. Since the Voronoi regions of the elements in an optimal set must have positive probability, we have

$$\max\{a, (m^2+1)d + mc\} \leq a_1 < a_2 < \cdots < a_n \leq \min\{b, (m^2+1)e + mc\}.$$

Let us consider the following two cases:

Case 1: $n = 2$.

In this case, the distortion error due to the set α_2 is given by

$$\begin{aligned}V(P;\alpha_2)&=\int_{\mathbb{R}}\min_{a\in\alpha_2}\|(x,0)-a\|^2dP(x)\\&=\frac{1}{b-a}\Big(\int_a^{(m^2+1)\frac{(a_1+a_2)}{2}+mc}\rho((x,0),(a_1,ma_1+c))\,dx\\&\qquad+\int_{(m^2+1)\frac{(a_1+a_2)}{2}+mc}^b\rho((x,0),(a_2,ma_2+c))\,dx\Big).\end{aligned}$$

Notice that $V(P;\alpha_2)$ is not always differentiable with respect to a_1 and a_2. By the hypothesis, we have

$$\max\{a,(m^2+1)d+mc\}=a \text{ and } \min\{b,(m^2+1)e+mc\}=b.$$

This guarantees that $V(P;\alpha_2)$ is differentiable with respect to a_1 and a_2. Since $\frac{\partial}{\partial a_1}V(P;\alpha_2)=0$ and $\frac{\partial}{\partial a_n}V(P;\alpha_2)=0$, we deduce that

$$-3a_1m^2+a_2m^2+2a-3a_1+a_2-2cm=0 \text{ and } a_1m^2-3a_2m^2+a_1-3a_2+2b-2cm=0$$

implying

$$a_1=\frac{1}{4(1+m^2)}(b-a)+\frac{a-cm}{1+m^2} \text{ and } a_2=\frac{3}{4(1+m^2)}(b-a)+\frac{a-cm}{1+m^2}$$

with quantization error

$$V_2=\frac{a^2\left(16m^2+1\right)+2ab\left(8m^2-1\right)+48acm+b^2\left(16m^2+1\right)+48bcm+48c^2}{48\left(m^2+1\right)}.$$

Case 2: $n\geq 3$.

In this case, the distortion error due to the set α_n is given by

$$\begin{aligned}V(P;\alpha_n)&=\int_{\mathbb{R}}\min_{a\in\alpha_n}\|(x,0)-a\|^2dP(x)\\&=\frac{1}{b-a}\Big(\int_a^{(m^2+1)\frac{(a_1+a_2)}{2}+mc}\rho((x,0),(a_1,ma_1+c))\,dx\\&\quad+\sum_{i=2}^{n-1}\int_{(m^2+1)\frac{(a_{i-1}+a_i)}{2}+mc}^{(m^2+1)\frac{(a_i+a_{i+1})}{2}+mc}\rho((x,0),(a_i,ma_i+c))\,dx\\&\qquad+\int_{(m^2+1)\frac{(a_{n-1}+a_n)}{2}+mc}^b\rho((x,0),(a_n,ma_n+c))\,dx\Big).\end{aligned}$$

Since $V(P;\alpha_n)$ gives the optimal error and is always differentiable with respect to a_i for $2\leq i\leq n-1$, we have $\frac{\partial}{\partial a_i}V(P;\alpha_n)=0$ yielding

$$a_{i+1}-a_i=a_i-a_{i-1} \text{ for } 2\leq i\leq n-1$$

implying

$$a_2-a_1=a_3-a_2=\cdots=a_n-a_{n-1}=k \tag{2}$$

for some real k. Due to the same reasoning as given in *Case 1*, we have $\frac{\partial}{\partial a_1}V(P;\alpha_n)=0$ and $\frac{\partial}{\partial a_n}V(P;\alpha_n)=0$, i.e.,

$$2(a-cm)-3a_1\left(m^2+1\right)+a_2\left(m^2+1\right)=0 \text{ and } a_{n-1}\left(m^2+1\right)-3a_n\left(m^2+1\right)+2(b-cm)=0$$

implying

$$a_1=\frac{a-cm}{1+m^2}+\frac{k}{2} \text{ and } a_n=\frac{b-cm}{m^2+1}-\frac{k}{2}. \tag{3}$$

Now we have

$$b-a=(a_1-a)+\sum_{i=2}^{n-1}(a_i-a_{i-1})+(b-a_n)=\left(\frac{a-cm}{1+m^2}+\frac{k}{2}-a\right)+(n-1)k+\left(b-\frac{b-cm}{1+m^2}+\frac{k}{2}\right),$$

which implies $k=\frac{b-a}{n(1+m^2)}$. Putting $k=\frac{b-a}{n(1+m^2)}$, by the expressions given in (2) and (3), we deduce that

$$a_i=\frac{2i-1}{2n(1+m^2)}(b-a)+\frac{a-cm}{1+m^2} \text{ for } 1\leq i\leq n.$$

To obtain the quantization error V_n, we proceed as follows:

Since the probability distribution P is uniform on its support, Equation (2) helps us to deduce that the distortion errors contributed by $a_2, a_3, \cdots, a_{n-1}$ in their own Voronoi regions are equal, i.e., each term in the sum

$$\sum_{i=2}^{n-1}\int_{(m^2+1)\frac{(a_{i-1}+a_i)}{2}+mc}^{(m^2+1)\frac{(a_i+a_{i+1})}{2}+mc}\rho((x,0),(a_i, ma_i+c))\,dx$$

have the same value. Now, putting the values of a_i for $2\leq i\leq n$ in terms of a_1 and k, we have

$$\begin{aligned}V(P;\alpha_n)&=\int_{\mathbb{R}}\min_{a\in\alpha_n}\|(x,0)-a\|^2dP(x)=\frac{1}{b-a}\Big(\int_a^{(m^2+1)\frac{(2a_1+k)}{2}+mc}\rho\Big((x,0),(a_1,ma_1+c)\Big)\,dx\\&+(n-2)\int_{(m^2+1)\frac{(2a_1+k)}{2}+mc}^{(m^2+1)\frac{(2a_1+3k)}{2}+mc}\rho\Big((x,0),(a_1+k,m(a_1+k)+c)\Big)\,dx\\&+\int_{(m^2+1)\frac{(2a_1+k(2n-3))}{2}+mc}^{b}\rho\Big((x,0),(a_1+k(n-1),m(a_1+k(n-1))+c)\Big)\,dx\Big).\end{aligned}$$

Upon simplification, and putting $a_1=\frac{b-a}{2(m^2+1)n}+\frac{a-cm}{m^2+1}$ and $k=\frac{b-a}{(m^2+1)n}$ in the above expression, we have the quantization error as

$$\begin{aligned}V_n=&\frac{1}{12\,(m^2+1)\,n^3}\Big(-48(a-b)^2m^2+(a-b)(a-b+72cm+8(11a-2b)m^2)n\\&-12(a-b)m(5c+(4a+b)m)n^2+12(c+am)^2n^3\Big).\end{aligned}$$

Thus, the proof of the theorem is complete. □

Let us now give the following corollary.

Corollary 3.2. Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq 2 \text{ and } y=0\}$. For $n\in\mathbb{N}$ with $n\geq 2$, let α_n be an optimal set of n-points for P such that the elements in the optimal set lie on the line $y=\sqrt{3}x$ between the elements $(0,0)$ and $(2,2\sqrt{3})$. Then,

$$\alpha_n=\Big\{\Big(\frac{2i-1}{4n},\frac{2i-1}{4n}\sqrt{3}\Big) : 1\leq i\leq n\Big\} \text{ and } V_n=\frac{144n^2+196n-576}{48n^3}.$$

Proof. Putting $a=0$, $b=2$, $m=\sqrt{3}$, $c=0$, $d=0$, and $e=2$ in Theorem 3.1, we see that

$$\max\{a,(m^2+1)d+mc\}=0=a \text{ and } \min\{b,(m^2+1)e+mc\}=2=b.$$

Hence, by Theorem 3.1, we obtain the optimal sets α_n and the corresponding quantization errors V_n as follows:

$$\alpha_n=\Big\{\Big(\frac{2i-1}{4n},\frac{2i-1}{4n}\sqrt{3}\Big) : 1\leq i\leq n\Big\} \text{ and } V_n=\frac{144n^2+196n-576}{48n^3}.$$

Thus, the proof of the corollary is complete. □

Remark 3.3. If $m=0$, $c=0$, $d=a$ and $e=b$, then by Theorem 3.1, the optimal set n-points is given by $\alpha_n := \{a+\frac{2i-1}{2n}(b-a) : 1\leq i\leq n\}$, and the corresponding quantization error is $V_n := V_n(P)=\frac{(a-b)^2}{12n^2}$, which is Theorem 2.1.1 in [RR]. Thus, Theorem 3.1 generalizes Theorem 2.1.1 in [RR].

The following proposition plays an important role in finding the optimal sets of n-points.

Proposition 3.4. *Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b$ and $y = 0\}$. For $n \in \mathbb{N}$ with $n \geq 2$, let $\alpha_n := \{(a_i, ma_i + c) : 1 \leq i \leq n\}$ be an optimal set of n-points for P so that the elements in the optimal sets lie on the line L between the two elements $(d, md + c)$ and $(e, me + c)$, where $d, e \in \mathbb{R}$ with $d < e$. Assume that one or both of the following two conditions are not true:*

$$\max\{a, (m^2 + 1)d + mc\} = a \text{ and } \min\{b, (m^2 + 1)e + mc\} = b,$$

If $(m^2 + 1)d + mc > a$, (or $(m^2 + 1)e + mc < b$), then there exists a positive integer N such that for all $n \geq N + 1$, the optimal sets α_n always contain the end element $(d, md + c)$, (or $(e, me + c)$). If $(m^2 + 1)d + mc > a$ and $(m^2 + 1)e + mc < b$, then there exists a positive integer N such that for all $n \geq N + 1$, the optimal sets α_n always contain the end elements $(d, md + c)$ and $(e, me + c)$.

Proof. Let $\alpha_n := \{(a_i, ma_i + c) : 1 \leq i \leq n\}$ be an optimal set of n-points for P so that the elements in the optimal sets lie on the line L between the two elements $(d, md + c)$ and $(e, me + c)$, where $d, e \in \mathbb{R}$ with $d < e$. By Theorem 3.1, we know that

$$a_i = \frac{2i - 1}{2n(1 + m^2)}(b - a) + \frac{a - cm}{1 + m^2} \text{ for } 1 \leq i \leq n.$$

Suppose that $(m^2 + 1)d + mc > a$. Let $n = N$ be the largest positive integer such that

$$(m^2 + 1)d + mc < a_1, \text{ i.e., } (m^2 + 1)d + mc < \frac{1}{2N(1 + m^2)}(b - a) + \frac{a - cm}{1 + m^2}. \tag{4}$$

Notice that the sequence $\{\frac{1}{2n(1+m^2)}(b - a) + \frac{a-cm}{1+m^2}\}$ is strictly decreasing, and hence for all $n \geq N + 1$, the optimal sets α_n always contain the end element $(d, md + c)$. Suppose that $(m^2 + 1)e + mc < b$. Let $n = N$ be the largest positive integer such that

$$a_N < (m^2 + 1)e + mc, \text{ i.e., } \frac{2N - 1}{2N(1 + m^2)}(b - a) + \frac{a - cm}{1 + m^2} < (m^2 + 1)e + mc. \tag{5}$$

Notice that the sequence $\{\frac{2n-1}{2n(1+m^2)}(b - a) + \frac{a-cm}{1+m^2}\}$ is strictly increasing, and hence for all $n \geq N + 1$, the optimal sets α_n always contain the end element $(e, me + c)$. Next, suppose that $(m^2 + 1)d + mc > a$ and $(m^2 + 1)e + mc < b$. Choose N_1 and N_2 same as N described in (4) and (5), respectively. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N + 1$, the optimal sets α_n always contain the end elements $(d, md + c)$ and $(e, me + c)$. □

Note 3.5. In the following, we state and prove two theorems, Theorem 3.6 and Theorem 3.8. To facilitate the proofs in both the theorems, Proposition 3.4 can be used. However, in the proof of Theorem 3.6, we have not used Proposition 3.4; on the other hand, in the proof of Theorem 3.8, we have used Proposition 3.4.

Theorem 3.6. *Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2$ and $y = 0\}$. For $n \in \mathbb{N}$, let $\alpha_n := \{(a_i, 1) : 1 \leq i \leq n\}$ be an optimal set of n-points for P so that the elements in the optimal sets lie on the line $y = 1$ between the two elements $(\frac{1}{2}, 1)$ and $(\frac{3}{2}, 1)$. Then, $\alpha_1 = \{(1, 1)\}$, $\alpha_2 = \{(\frac{1}{2}, 1), (\frac{3}{2}, 1)\}$, and for $n \geq 3$, we have*

$$a_i = \begin{cases} \frac{1}{2} & \text{if } i = 1, \\ \frac{1}{2} + \frac{(i-1)}{(n-1)} & \text{if } 2 \leq i \leq n - 1, \\ \frac{3}{2} & \text{if } i = n, \end{cases}$$

and the quantization error for n-points is given by $V_n = \frac{25n^2-50n+26}{24(n-1)^2}$.

Proof. The proofs of $\alpha_1 = \{(1, 1)\}$, $\alpha_2 = \{(\frac{1}{2}, 1), (\frac{3}{2}, 1)\}$ are routine. We just give the proof for $n \geq 3$. Let $\alpha_n := \{(t, 1) : t = a_i \text{ for } 1 \leq i \leq n\}$ be an optimal set of n-points such that $\frac{1}{2} \leq a_1 < a_2 < \cdots < a_{n-1} < a_n \leq \frac{3}{2}$. Notice that the boundary of the Voronoi region of the element $(a_1, 1)$ intersects the support of P at the elements $(0, 0)$ and $(\frac{1}{2}(a_1 + a_2), 0)$, the boundary of the Voronoi region of $(a_n, 1)$ intersects the support of P at the elements $(\frac{1}{2}(a_{n-1} + a_n), 0)$ and $(2, 0)$. On the other hand, the boundaries of the

Voronoi regions of $(a_i, 1)$ for $2 \leq i \leq n-1$ intersect the support of P at the elements $(\frac{1}{2}(a_{i-1}+a_i), 0)$ and $(\frac{1}{2}(a_i+a_{i+1}), 0)$. Thus, the distortion error due to the set α_n is given by

$$\begin{aligned} V(P;\alpha_n) &= \int_{\mathbb{R}} \min_{a\in\alpha_n} \|(x,0)-a\|^2 dP(x) \\ &= \int_0^{\frac{1}{2}(a_1+a_2)} \frac{1}{2}\left((x-a_1)^2+1\right) dx + \sum_{i=2}^{n-1} \int_{\frac{1}{2}(a_{i-1}+a_i)}^{\frac{1}{2}(a_{i+1}+a_i)} \frac{1}{2}\left((x-a_i)^2+1\right) dx \\ &\quad + \int_{\frac{1}{2}(a_{n-1}+a_n)}^{2} \frac{1}{2}\left((x-a_n)^2+1\right) dx. \end{aligned}$$

Since $V(P;\alpha_n)$ gives the optimal error and is differentiable with respect to a_i for $2 \leq i \leq n-1$, we have $\frac{\partial}{\partial a_i} V(P;\alpha_n) = 0$ implying

$$a_{i+1} - a_i = a_i - a_{i-1} \text{ for } 2 \leq i \leq n-1.$$

This yields the fact that

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = k \tag{6}$$

for some real number $0 < k < 1$. By the equations in (6), we see that each term in the sum $\sum_{i=2}^{n-1} \int_{\frac{1}{2}(a_{i-1}+a_i)}^{\frac{1}{2}(a_{i+1}+a_i)} \frac{1}{2}\left((x-a_i)^2+1\right) dx$ have the same value. Again, by the equations in (6) we have

$$a_2 = k + a_1,\ a_3 = 2k + a_1, \cdots, a_n = (n-1)k + a_1.$$

Hence,

$$\begin{aligned} V(P;\alpha_n) = &\int_0^{\frac{1}{2}(2a_1+k)} \frac{1}{2}\left((x-a_1)^2+1\right) dx + (n-2)\int_{\frac{1}{2}(2a_1+k)}^{\frac{1}{2}(2a_1+3k)} \frac{1}{2}\left((x-(a_1+k))^2+1\right) dx \\ &+ \int_{\frac{1}{2}(2a_1+k(2n-3))}^{2} \frac{1}{2}\left((x-(a_1+k(n-1)))^2+1\right) dx, \end{aligned}$$

which upon simplification yields

$$\begin{aligned} V(P;\alpha_n) = \frac{1}{24}\Big(&-12a_1(k(n-1)-2)\left(a_1+k(n-1)-2\right) \\ &- k(n-1)\left(4k^2n^2 - 8k(k+3)n + 3(k+4)^2\right) + 56\Big), \end{aligned}$$

which is minimum if $a_1 = \frac{1}{2}$ and $k = \frac{1}{n-1}$, and the minimum value is $\frac{25n^2-50n+26}{24(n-1)^2}$. As $k = \frac{1}{n-1}$ and $a_1 = \frac{1}{2}$, using the expression (6), we obtain

$$a_i = \left\{\begin{array}{cc} \frac{1}{2} & \text{if } i = 1, \\ \frac{1}{2} + \frac{(i-1)}{(n-1)} & \text{if } 2 \leq i \leq n-1, \\ \frac{3}{2} & \text{if } i = n, \end{array}\right.$$

with quantization error $V_n = \frac{25n^2-50n+26}{24(n-1)^2}$. Thus, the proof of the theorem is complete. □

Remark 3.7. Comparing Theorem 3.6 with Proposition 3.4, we have $a = 0$, $b = 2$, $m = 0$, $c = 1, d = \frac{1}{2}$, and $e = \frac{3}{2}$, and so

$$(m^2+1)d + mc = \frac{1}{2} > a \text{ and } (m^2+1)e + mc = \frac{3}{2} < b.$$

$n = N_1$ be the largest positive integer such that

$$(m^2+1)d + mc < \frac{1}{2N_1(1+m^2)}(b-a) + \frac{a-cm}{1+m^2},$$

which is true if $N_1 < 2$, i.e., $N_1 = 1$. Let $n = N_2$ be the largest positive integer such that

$$\frac{2N-1}{2N(1+m^2)}(b-a) + \frac{a-cm}{1+m^2} < (m^2+1)e + mc,$$

which is true if $N_2 < 2$, i.e., $N_2 = 1$. Take $N = \max\{N_1, N_2\}$. Then, $N = 1$. By Proposition 3.4, we can conclude that for all $n \geq 2$, the optimal sets α_n will contain the end elements $(\frac{1}{2}, 1)$ and $(\frac{3}{2}, 1)$, which is clearly true by Theorem 3.6.

Theorem 3.8. *Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2$ and $y = 0\}$. For $n \in \mathbb{N}$, let $\alpha_n := \{(a_i, 1) : 1 \leq i \leq n\}$ be an optimal set of n-points for P so that the elements in the optimal sets lie on the line $y = 1$ between the two elements $(0, 1)$ and $(\frac{28}{15}, 1)$, i.e., $0 \leq a_1 < a_2 < \cdots < a_n \leq \frac{28}{15}$. Then, $\alpha_1 = \{(1, 1)\}$, and for $1 \leq n \leq 7$,*

$$\alpha_n = \Big\{\Big(\frac{2i-1}{n}, 1\Big) : 1 \leq i \leq n\Big\}.$$

On the other hand, for $n \geq 8$, we obtain

$$a_i = \left\{ \begin{array}{cc} \frac{28(2i-1)}{15(2n-1)} & \text{if } 1 \leq i \leq n-1, \\ \frac{28}{15} & \text{if } i = n, \end{array} \right.$$

and the quantization error for n-points is given by $V_n = \frac{7(5788(n-1)n+3015)}{10125(1-2n)^2}$.

Proof. Let $\alpha_n := \{(t, 1) : t = a_i$ for $1 \leq i \leq n\}$ be an optimal set of n-points such that $0 \leq a_1 < a_2 < \cdots < a_{n-1} < a_n \leq \frac{28}{15}$ for all $n \in \mathbb{N}$. Using Proposition 3.4, it can be proved that for all $n \geq 8$, the optimal sets always contain the end element $\frac{28}{15}$, i.e., $a_n = \frac{28}{15}$ for all $n \geq 8$. The proofs of $\alpha_1 = \{(1, 1)\}$, and for $1 \leq n \leq 7$,

$$\alpha_n = \Big\{\Big(\frac{2i-1}{n}, 1\Big) : 1 \leq i \leq n\Big\},$$

are routine. Here we prove the optimal sets of n-points for all $n \geq 8$. Proceeding in the similar lines as given in the proof of Theorem 3.6, we have

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} = k$$

for some real k, which implies

$$a_1 = a_n - (n-1)k,\ a_2 = a_n - (n-2)k, \cdots, a_{n-1} = a_n - k.$$

Also, by using $\frac{\partial}{\partial a_1} V(P; \alpha_n) = 0$, we get $3a_1 - a_2 = 0$, which implies that $a_1 = \frac{k}{2}$. Now we have

$$\frac{k}{2} = a_1 = a_n - (n-1)k = \frac{28}{15} - (n-1)k,$$

this yields $k = \frac{56}{15(2n-1)}$. Using $a_n = \frac{28}{15}$ and $k = \frac{56}{15(2n-1)}$, we get $a_i = \frac{28(2i-1)}{15(2n-1)}$ for $1 \leq i \leq n-1$ with quantization error

$$\begin{aligned} V(P; \alpha_n) =& \frac{1}{2}\Bigg(\int_0^{\frac{1}{2}(2a_n - k(2n-3))} \Big((x - (a_n - k(n-1)))^2 + 1\Big)\, dx \\ & + (n-2)\int_{\frac{1}{2}(2a_n-3k)}^{\frac{1}{2}(2a_n-k)} \Big((x - (a_n - k))^2 + 1\Big)\, dx + \int_{\frac{1}{2}(2a_n-k)}^{2} \Big((x - a_n)^2 + 1\Big)\, dx\Bigg) \\ =& \frac{1}{24}\Big(12k^2n^2a_n - 24k^2na_n + 12k^2a_n - 12kna_n^2 + 12ka_n^2 + 24a_n^2 - 48a_n - 4k^3n^3 \\ & + 12k^3n^2 - 11k^3n + 3k^3 + 56\Big) = \frac{7(5788(n-1)n + 3015)}{10125(1-2n)^2}. \end{aligned}$$

This completes the proof. $\square$

4. Constrained quantization when the support lies on a circle and the optimal elements lie on another circle

Let $O(0,0)$ be the center of the Cartesian plane. Let C be the unit circle given by the parametric equations:

$$C := \{(x,y) : x = \cos\theta,\ y = \sin\theta \text{ for } 0 \leq \theta \leq 2\pi\}.$$

Let the positive direction of the x-axis cut the circle at the element A_0, i.e., A_0 is represented by the parametric value $\theta = 0$. Let s be the distance of an element on C along the arc starting from the element A_0 in the counterclockwise direction. Then,

$$ds = \sqrt{\Big(\frac{dx}{d\theta}\Big)^2 + \Big(\frac{dy}{d\theta}\Big)^2}\, d\theta = d\theta.$$

Let P be a uniform distribution with support to the unit circle C. Then, the probability density function $f(x,y)$ for P is given by

$$f(x,y) = \begin{cases} \frac{1}{2\pi} & \text{if } (x,y) \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have $dP(s) = P(ds) = f(x,y)ds = \frac{1}{2\pi}d\theta$. Moreover, we know that if $\hat\theta$ radians is the central angle subtended by an arc of length S of the unit circle, then $S = \hat\theta$, and

$$P(S) = \int_S dP(s) = \frac{1}{2\pi}\int_S d\theta = \frac{\hat\theta}{2\pi}.$$

Let L be a concentric circle with C, and L has radius a, i.e., the parametric representation of the circle L is given by

$$L := \{(x,y) : x = a\cos\theta,\ y = a\sin\theta \text{ for } 0 \leq \theta \leq 2\pi\}.$$

In this section, we determine the optimal sets of n-points and the nth constrained quantization errors for the uniform distribution P on C under the condition that the elements in an optimal set lie on the circle L. Let the line OA_0 cut the circle L at the element B_0, i.e., B_0 is represented on the circle L by the parameter $\theta = 0$.

Proposition 4.1. *Any element on the circle L forms an optimal set of one-point with quantization error $V_1 = 1 + a^2$.*

Proof. Let $\alpha := \{(a\cos\theta, a\sin\theta)\}$, where $0 \leq \theta \leq 2\pi$, forms an optimal set of one-point. Then, the distortion error $V(P;\alpha)$ is given by

$$V(P;\alpha) = \int_C \frac{1}{2\pi}\rho((\cos\theta, \sin\theta), (a\cos\theta, a\sin\theta))\, d\theta = 1 + a^2,$$

which does not depend on θ for any $0 \leq \theta \leq 2\pi$. Hence, any element on the circle L forms an optimal set of one-point, and the quantization error for one-point is given by $V_1 = 1 + a^2$. □

Proposition 4.2. *A set of the form $\{(a\cos\theta, a\sin\theta), (-a\cos\theta, -a\sin\theta)\}$, where $0 \leq \theta \leq 2\pi$, forms an optimal set of two-points with quantization error $V_2 = 1 + a^2 - \frac{4a}{\pi}$.*

Proof. Let $\alpha := \{(a\cos\theta_1, a\sin\theta_1), (a\cos\theta_2, a\sin\theta_2)\}$, where $0 \leq \theta_1 < \theta_2 \leq 2\pi$, form an optimal set of two-points. Notice that the boundary of the Voronoi regions of the two elements in the optimal set is the line joining the two points given by the parameters $\theta = \frac{\theta_1+\theta_2}{2}$ and $\theta = \pi + \frac{\theta_1+\theta_2}{2}$. Then, the distortion error is given by

$$\begin{aligned} V(P;\alpha) &= \frac{1}{2\pi}\Big(\int_{-\pi+\frac{\theta_1+\theta_2}{2}}^{\frac{\theta_1+\theta_2}{2}} \rho\Big((\cos\theta,\sin\theta),(a\cos\theta_1,a\sin\theta_1)\Big)\,d\theta \\ &\qquad + \int_{\frac{\theta_1+\theta_2}{2}}^{\pi+\frac{\theta_1+\theta_2}{2}} \rho\Big((\cos\theta,\sin\theta),(a\cos\theta_2,a\sin\theta_2)\Big)\,d\theta\Big) \\ &= \frac{1}{2\pi}\Big(\int_{-\pi+\frac{\theta_1+\theta_2}{2}}^{\frac{\theta_1+\theta_2}{2}} \Big(1+a^2-2a\cos(\theta-\theta_1)\Big)\,d\theta \\ &\qquad + \int_{\frac{\theta_1+\theta_2}{2}}^{\pi+\frac{\theta_1+\theta_2}{2}} \Big(1+a^2-2a\cos(\theta-\theta_2)\Big)\,d\theta\Big), \end{aligned}$$

which upon simplification yields that

$$V(P;\alpha) = \frac{1}{2\pi}\Big((1+a^2)2\pi - 8a\sin\frac{\theta_2-\theta_1}{2}\Big).$$

Since $0 < \frac{\theta_2-\theta_1}{2} < \pi$, we can say that $V(P;\alpha)$ is minimum if $\theta_2-\theta_1=\pi$. Thus, an optimal set of two-points is given by $\{(a\cos\theta, a\sin\theta), (-a\cos\theta,-a\sin\theta)\}$ for $0\leq\theta\leq 2\pi$ with quantization error $V_2 = 1+a^2-\frac{4a}{\pi}$, which yields the proposition. □

Theorem 4.3. *Let α_n be an optimal set of n-points for the uniform distribution P on the unit circle C for $n\in\mathbb{N}$ with $n\geq 3$. Then,*

$$\alpha_n = \Big\{\Big(a\cos\frac{(2i-1)\pi}{n}, a\sin\frac{(2i-1)\pi}{n}\Big) : i=1,2,\cdots,n\Big\}$$

and the corresponding quantization error is given by $V_n = a^2+1-\frac{2an}{\pi}\sin\frac{\pi}{n}$.

Proof. Let $\alpha_n := \{a_1,a_2,\cdots,a_n\}$, where $a_i = (a\cos\theta_i, a\sin\theta_i)$, be an optimal set of n-points for P with $n\geq 3$ such that the elements in the optimal set lie on the circle L. Let the boundary of the Voronoi regions of a_i cut the circle L, in fact also the circle C, at the elements given by the parameters θ_{i-1} and θ_i, where $1\leq i\leq n$. Since the circles have rotational symmetry, without any loss of generality, we can assume that $\theta_0=0$, and $\theta_n=2\pi$. Then, each a_i on L has the parametric representation $\frac{1}{2}(\theta_{i-1}+\theta_i)$ for $1\leq i\leq n$. Then, the quantization error for n-points is given by

$$\begin{aligned} V(P;\alpha) &= \int_C \min_{u\in\alpha}\rho((\cos\theta,\sin\theta)-u)\,dP(s) \\ &= \sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} \frac{1}{2\pi}\rho\Big(((\cos\theta,\sin\theta),(\frac{1}{2}\cos\frac{\theta_{i-1}+\theta_i}{2},\frac{1}{2}\sin\frac{\theta_{i-1}+\theta_i}{2}))\Big)\,d\theta \\ &= \sum_{i=1}^n \int_{\theta_{i-1}}^{\theta_i} \frac{1}{2\pi}\Big(a^2-2a\cos(-\frac{\theta_{i-1}}{2}-\frac{\theta_i}{2}+\theta)+1\Big)\,d\theta \\ &= \sum_{i=1}^n \frac{1}{2\pi}\Big((a^2+1)(\theta_i-\theta_{i-1})-4a\sin\frac{\theta_i-\theta_{i-1}}{2}\Big), \end{aligned}$$

upon simplification, which yields

$$V(P;\alpha) = a^2+1-\frac{2a}{\pi}\sum_{i=1}^n \sin\frac{\theta_i-\theta_{i-1}}{2}. \tag{7}$$

Since $V(P;\alpha)$ gives the optimal error and is differentiable with respect to θ_i for all $1\leq i\leq n-1$, we have $\frac{\partial}{\partial\theta_i}V(P;\alpha)=0$. For $1\leq i\leq n-1$, the equations $\frac{\partial}{\partial\theta_i}V(P;\alpha)=0$ implies that

$$\cos\frac{\theta_i-\theta_{i-1}}{2} = \cos\frac{\theta_{i+1}-\theta_i}{2} \text{ yielding } \frac{\theta_i-\theta_{i-1}}{2} = \frac{\theta_{i+1}-\theta_i}{2}, \text{ or } \frac{\theta_i-\theta_{i-1}}{2} = 2\pi-\frac{\theta_{i+1}-\theta_i}{2}.$$

Without any loss of generality, for $1 \leq i \leq n-1$ we can take $\frac{\theta_i-\theta_{i-1}}{2} = \frac{\theta_{i+1}-\theta_i}{2}$. This yields the fact that

$$\theta_1 - \theta_0 = \theta_2 - \theta_1 = \theta_3 - \theta_2 = \cdots = \theta_n - \theta_{n-1} = \frac{2\pi}{n}.$$

Thus, we have $\theta_i = \frac{2\pi i}{n}$ for $i = 1, 2, \cdots, n$. Hence, if $\alpha_n := \{a_1, a_2, \cdots, a_n\}$ is an optimal set of n-points, then

$$a_i = \Big(a\cos\frac{(2i-1)\pi}{n}, a\sin\frac{(2i-1)\pi}{n}\Big) \text{ for } i = 1, 2, \cdots, n,$$

and the quantization error for n-points, by (7), is given by

$$V_n = V(P; \alpha) = a^2 + 1 - \frac{2an}{\pi}\sin\frac{\pi}{n}.$$

Thus, the proof of the theorem is complete. □

5. Constrained quantization when the support lies on a chord of a circle and the optimal elements lie on the circle

Let C be a circle with center $(0,0)$ and radius 1, i.e, the equation of the circle is $x^2 + y^2 = 1$, whose parametric representation is $x = \cos\theta$ and $y = \sin\theta$, where $0 \leq \theta \leq 2\pi$. Thus, if $(\cos\theta, \sin\theta)$ is an element on the circle, we will represent it by θ. Let P be a Borel probability measure on $\mathbb{R}^2$ such that P has support a chord of the circle, and P is uniform on its support. We now investigate the optimal sets of n-points and the nth constrained quantization errors for all $n \in \mathbb{N}$ so that the optimal elements lie on the circle. The two cases can happen as described in the following two subsections.

5.1. **Chord is a diameter of the circle.** Without any loss of generality, let us consider the horizontal diameter as the support of P, i.e., the support of P is the closed interval $[-1, 1]$. Then, the probability density function is given by

$$f(x,y) = \left\{\begin{array}{cc} \frac{1}{2} & \text{if } -1 \leq x \leq 1 \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{array}\right.$$

Notice that $d(P(x,y) = dP(x) = f(x,0)dx$. We know that an optimal set of one-point always exists. Let α_n be an optimal set of n-points for any $n \geq 2$. Since the boundary of the Voronoi regions of any two optimal elements, in this case, passes through the center of the circle, from the geometry, we see that among n Voronoi regions, only two Voronoi regions contain elements from the support of P. Hence, an optimal set of n-points for any $n \geq 2$ contains exactly two elements. We now calculate the optimal sets of one-point and two-points in the following propositions:

Proposition 5.1.1. *Any element on the circle forms an optimal set of one-point with quantization error* $V_1 = \frac{4}{3}$.

Proof. Let $(\cos\theta, \sin\theta)$ be an element on the circle. Then, the distortion error for P with respect to this element is given by

$$V(P; \{(\cos\theta, \sin\theta)\}) = \int_{-1}^{1} \rho((x,0), (\cos\theta, \sin\theta))\, dP(x) = \frac{1}{2}\int_{-1}^{1} \rho((x,0), (\cos\theta, \sin\theta))\, dx = \frac{4}{3},$$

which does not depend on θ. Hence, any element on the circle forms an optimal set of one-point with quantization error $V_1 = \frac{4}{3}$. □

Proposition 5.1.2. *The set* $\{(-1,0), (1,0)\}$ *forms an optimal set of two-points with quantization error* $V_2 = \frac{1}{3}$.

Proof. From the geometry, we see that the boundary of any two elements on the circle passes through the center of the circle. Thus, in an optimal set of two-points, one Voronoi region will contain the left half, and the other Voronoi region will contain the right half of the support of P. Hence, by the routine calculation, we can show that $\{(-1,0), (1,0)\}$ forms an optimal set of two-points with quantization error

$$V_2 = \frac{1}{2}\Big(\int_{-1}^{0} \rho((x,0), (-1,0))dx + \int_{0}^{1} \rho((x,0), (1,0))dx\Big) = \frac{1}{3}.$$

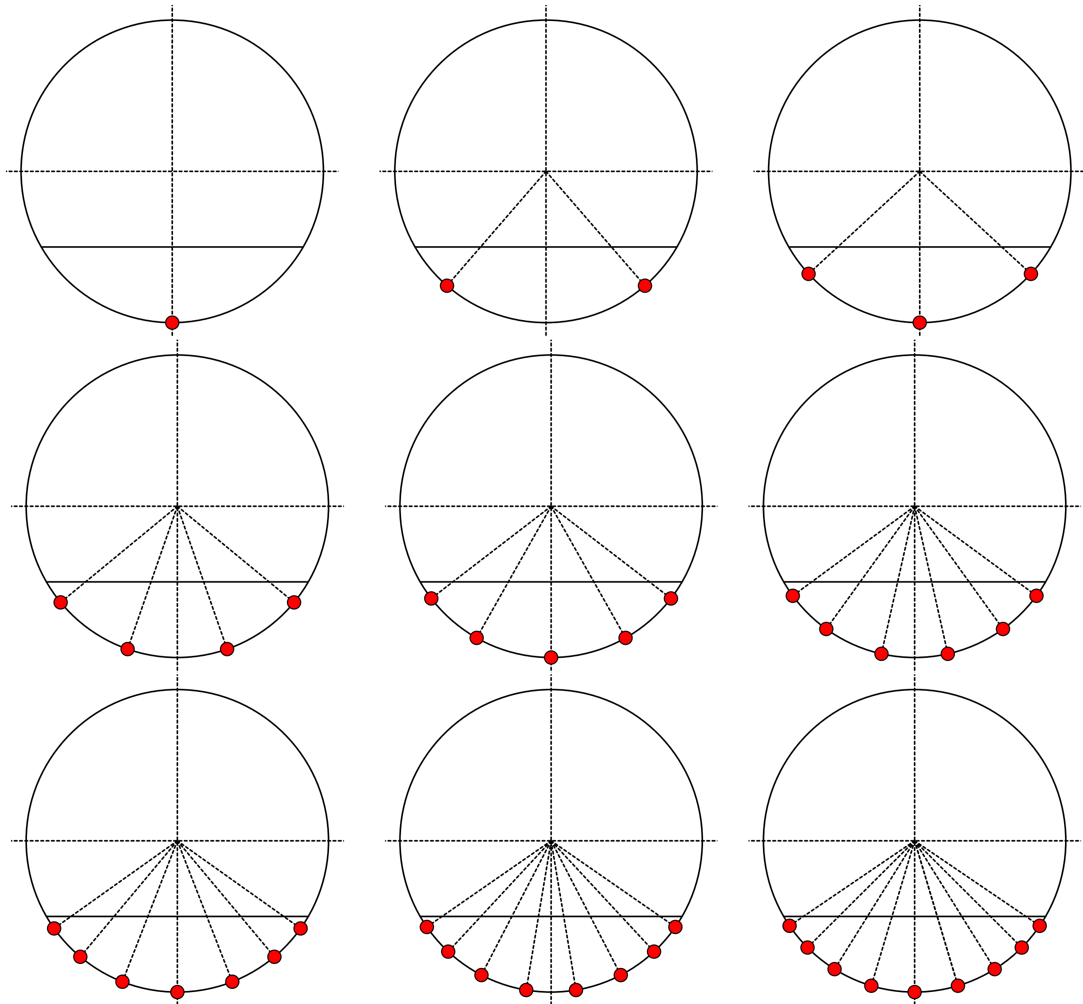

FIGURE 1. Optimal configuration of n elements for $1 \leq n \leq 9$.

Thus, the proof of the proposition is complete. □

5.2. **Chord is not a diameter of the circle.** In this case, for definiteness sake, we investigate the optimal sets of n-points and the nth constrained quantization errors for a Borel probability measure P on $\mathbb{R}^2$ such that P has support the chord $y = -\frac{1}{2}$ for $-\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2}$, and P is uniform there. Then, the probability density function for P is given by

$$f(x, y) = \left\{\begin{array}{cc} \frac{1}{\sqrt{3}} & \text{if } -\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2} \text{ and } y = -\frac{1}{2}, \\ 0 & \text{otherwise.} \end{array}\right.$$

Recall that the circle has rotational symmetry. Thus, for any other chord, the technique of finding the optimal sets of n-points and the nth constrained quantization errors will be similar. Notice that $dP(x, y) = dP(x) = P(dx) = f(x, -\frac{1}{2})dx$, where x varies over the line $y = -\frac{1}{2}$. The arc of the circle subtended by the chord is represented by θ for $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$. Moreover, the circle is geometrically symmetric with respect to the line $y = 0$, and also the probability measure is symmetric with respect to the line $y = 0$, i.e., if two intervals of the same length lie on the support of P and are equidistant from the line $y = 0$, then they have the same probability. In proving the results, we can use this symmetry of the circle.

Proposition 5.2.1. *The set* $\{(0, -1)\}$ *forms an optimal set of one-point with quantization error* $V_1 = \frac{1}{2}$.

Proof. Let us consider an element $(\cos\theta, \sin\theta)$ on the circle. The distortion error for P with respect to the set $\{(\cos\theta, \sin\theta)\}$ is given by

$$V(P; \{(\cos\theta, \sin\theta)\}) = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{3}}\rho((x, -\frac{1}{2}), (\cos\theta, \sin\theta))\, dx = \sin\theta + \frac{3}{2},$$

the minimum value of which is $\frac{1}{2}$ and it occurs when $\theta = \frac{3\pi}{2}$ (see Figure 1). Thus, the proof of the proposition is yielded. □

Proposition 5.2.2. *The optimal set of two-points is given by*

$$\Big\{2\pi - 2\tan^{-1}\Big(\frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{2}\Big), \pi + 2\tan^{-1}\Big(\frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{2}\Big)\Big\}$$

with quantization error $V_2 = \frac{1}{2}\left(3 - \sqrt{7}\right)$.

Proof. Since the probability measure is symmetric with respect to the line $y = 0$, we can assume that in an optimal set of two-points, the Voronoi region of one element will contain the left half of the chord, and the Voronoi region of the other element will contain the right half of the chord, i.e., the boundary of the two Voronoi regions is the y-axis. Let the left element is $(\cos\theta, \sin\theta)$. Then, due to symmetry, the distortion error for the two elements is given by

$$2\int_{-\frac{\sqrt{3}}{2}}^{0} \frac{1}{\sqrt{3}}\rho((x, -\frac{1}{2}), (\cos\theta, \sin\theta))\, dx = \sin\theta + \frac{1}{2}\sqrt{3}\cos\theta + \frac{3}{2},$$

which is minimum if $\theta = 2\pi - 2\tan^{-1}(\frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{2})$, and the minimum value is $\frac{1}{2}\left(3 - \sqrt{7}\right)$. Thus, the one element is represented by $\theta = 2\pi - 2\tan^{-1}(\frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{2})$, and due to symmetry the other element is represented by $\theta = \pi + 2\tan^{-1}(\frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{2})$ with quantization error for two-points $V_2 = \frac{1}{2}\left(3 - \sqrt{7}\right)$ (see Figure 1). Thus, the proof of the proposition is complete. □

Remark 5.2.3. Due to the symmetry of the probability measure P and the geometrical symmetry of the circle, we can assume that in an optimal set of n-points, where $n \geq 3$, if n is even, then there are $\frac{n}{2}$ elements to the left of the y-axis and $\frac{n}{2}$ elements to the right of the y-axis. if n is odd, then there are $\frac{n-1}{2}$ elements to the left of the y-axis and $\frac{n-1}{2}$ elements to the right of the y-axis, and the remaining one element will be the element $(-1, 0)$. Moreover, whether n is even or odd, the set of elements on the left side and the set of elements on the right side are reflections of each other with respect to the y-axis. Due to this fact, in the sequel of this section, we calculate the optimal sets of n-points for $n = 8$ and $n = 9$. Following a similar technique, whether n is even or odd, one can calculate the locations of elements for any positive integer $n \geq 3$.

Proposition 5.2.4. *The optimal set of eight-points is given by*

$$\begin{aligned}&\{(-0.821938, -0.569577), (-0.680768, -0.732499), (-0.4608, -0.887504),\\ &(-0.164598, -0.986361), (0.821938, -0.569577), (0.680768, -0.732499),\\ &(0.4608, -0.887504), (0.164598, -0.986361)\}\end{aligned}$$

with quantization error $V_8 = 0.12327$.

Proof. Let $\alpha_8 := \{\theta_1, \theta_2, \cdots \theta_8\}$ be an optimal set of eight-points. Without any loss of generality, we can assume that $\theta_1 < \theta_2 < \cdots < \theta_8$. Due to symmetry as mentioned in Remark 5.2.3, the boundary of the Voronoi regions of θ_4 and θ_5 is the y-axis, and the elements on the right side of y-axis are the reflections of the elements in the left side of y-axis with respect to the y-axis. Thus, it is enough to calculate the first four elements $\theta_1, \theta_2, \theta_3, \theta_4$. Let the boundaries of the Voronoi regions of θ_i and θ_{i+1} intersects the support of P at the element $(a_i, -\frac{1}{2})$, where $1 \leq i \leq 3$. Because of the symmetry, the distortion error is given by

$$V(P;\alpha_8) = 2\Big(\int_{-\frac{\sqrt{3}}{2}}^{a_1} \rho((x,0),(\cos\theta_1,\sin\theta_1))\,dP(x) + \sum_{i=1}^{2}\int_{a_i}^{a_{i+1}} \rho((x,0),(\cos\theta_{i+1},\sin\theta_{i+1}))\,dP(x) + \int_{a_3}^{0} \rho((x,0),(\cos\theta_4,\sin\theta_4))\,dP(x)\Big). \tag{8}$$

The canonical equations are

$$\rho((a_i,-\frac{1}{2}),(\cos\theta_i,\sin\theta_i)) - \rho((a_i,-\frac{1}{2}),(\cos\theta_{i+1},\sin\theta_{i+1})) = 0 \text{ for } i=1,2,3.$$

Solving the canonical equations, we have

$$a_1 = \frac{\sin\theta_1-\sin\theta_2}{2(\cos\theta_1-\cos\theta_2)}, a_2 = \frac{\sin\theta_2-\sin\theta_3}{2(\cos\theta_2-\cos\theta_3)}, a_3 = \frac{\sin\theta_3-\sin\theta_4}{2(\cos\theta_3-\cos\theta_4)}.$$

Putting the values of a_1, a_2, a_3 in (8), we see that $V(P;\alpha_8)$ is a function of θ_i for $i = 1, 2, 3, 4$. Since $V(P;\alpha_8)$ is optimal we have

$$\frac{\partial}{\partial\theta_i}V(P;\alpha_8) = 0 \text{ for } i=1,2,3,4.$$

Solving the above four equations, we obtain the values of θ_i for which $V(P;\alpha_8)$ is minimum as

$$\theta_1 = 3.74758, \theta_2 = 3.96358, \theta_3 = 4.23349, \theta_4 = 4.54704.$$

Due to symmetry $\theta_5, \theta_6, \theta_7, \theta_8$ can also be obtained. Recall that θ_i represents the element $(\cos\theta_i, \sin\theta_i)$. Thus, we obtain the optimal set of eight-points as mentioned in the proposition with quantization error $V_8 = 0.12327$ (see Figure 1). Thus, the proof of the proposition is complete. □

Proposition 5.2.5. *The optimal set of nine-points is given by*

$$\{(-0.827126,-0.562016),(-0.708531,-0.70568),(-0.529525,-0.848294),(-0.286494,-0.958082),$$
$$(0.,-1),(0.827126,-0.562016),(0.708531,-0.70568),(0.529525,-0.848294),(0.286494,-0.958082)\}$$

with quantization error $V_9 = 0.122546$.

Proof. Recall Remark 5.2.3. We can assume that the optimal set of nine-points is $\alpha_9 = \{\theta_i : 1 \leq i \leq 9\}$ such that $\theta_i < \theta_{i+1}$ for $1 \leq i \leq 8$, where $\theta_5 = \frac{3\pi}{2}$. Because of the same reasoning as given in the proof of Proposition 5.2.4, we have the distortion error as

$$V(P;\alpha_9) = 2\Big(\int_{-\frac{\sqrt{3}}{2}}^{a_1} \rho((x,0),(\cos\theta_1,\sin\theta_1))\,dP(x) + \sum_{i=1}^{3}\int_{a_i}^{a_{i+1}} \rho((x,0),(\cos\theta_{i+1},\sin\theta_{i+1}))\,dP(x) + \int_{a_4}^{0} \rho((x,0),(0,-1))\,dP(x)\Big). \tag{9}$$

The canonical equations are

$$\rho((a_i,-\frac{1}{2}),(\cos\theta_i,\sin\theta_i)) - \rho((a_i,-\frac{1}{2}),(\cos\theta_{i+1},\sin\theta_{i+1})) = 0 \text{ for } i=1,2,3,4.$$

Solving the canonical equations, we obtain the values of a_i for $1 \leq i \leq 4$. Putting the values of a_i in (9), we see that $V(P;\alpha_9)$ is a function of θ_i for $i = 1, 2, 3, 4$. Since $V(P;\alpha_9)$ is optimal we have

$$\frac{\partial}{\partial\theta_i}V(P;\alpha_9) = 0 \text{ for } i=1,2,3,4.$$

Solving the above four equations, we obtain the values of θ_i for which $V(P;\alpha_9)$ is minimum as

$$\theta_1 = 3.73841, \theta_2 = 3.92497, \theta_3 = 4.15435, \theta_4 = 4.42182,$$

Due to symmetry $\theta_6, \theta_7, \theta_8, \theta_9$ can also be obtained. Recall that θ_i represents the element $(\cos\theta_i, \sin\theta_i)$. Hence, we obtain the optimal set of nine-points as mentioned in the proposition with quantization error $V_9 = 0.122546$ (see Figure 1). Thus, the proof of the proposition is complete. $\square$

6. Constrained quantization when the support lies on a line segment outside of a circle and the optimal elements lie on the circle

In this section, our goal is to investigate the optimal sets of n-points and the nth constrained quantization errors for a Borel probability measure P on $\mathbb{R}^2$ such that P is uniform on its support which is a closed interval $[a, b]$ on a line $y = mx + c$, and the optimal elements lie on a circle $(x-h)^2 + (y-k)^2 = r^2$ which does not have any point of intersection with the line. Notice that the circle has rotational symmetry, so instead of considering the line $y = mx + c$, we can take the line as $y = c$. Then, by giving some affine transformations, we can reduce the system so that the equation of the circle becomes of the form $x^2 + y^2 = r^2$ and the equation of the line is $y = k$ for some real $k \in \mathbb{R}$. For simplicity and definiteness sake, in this section, we investigate the optimal sets of n-points and the nth constrained quantization errors for a Borel probability measure P on $\mathbb{R}^2$ such that P is uniform on its support which is the closed interval $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } y = -2\}$, and the elements in the optimal sets lie on the circle $C := x^2 + y^2 = 1$. Notice that the parametric representation of the unit circle is $C := \{(x, y) : x = \cos\theta,\ y = \sin\theta \text{ for } 0 \leq \theta \leq 2\pi\}$, and P is a Borel probability measure on $\mathbb{R}^2$ with probability density function f such that

$$f(x, y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \text{ and } y = -2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, on the line $y = -2$, we have $dP(x, y) = dP(x) = P(dx) = f(x, -2)dx$. If $(\cos\theta, \sin\theta)$ is an element on the circle, we will identify the element by its parameter θ. The lines joining the elements $(\cos\theta, \sin\theta)$ on the circle and the center $(0, 0)$ intersect the line segment L at the elements $(-2\cot\theta, -2)$ for $0 \leq -2\cot\theta \leq 1$, i.e., if $\frac{3\pi}{2} \leq \theta \leq 2\pi - \cot^{-1}\frac{1}{2}$.

Proposition 6.1. *The set* $\{(0.242536, -0.970143)\}$ *forms an optimal set of one-point with quantization error* $V_1 = 1.21023$.

Proof. Let $\alpha := \{(\cos\theta, \sin\theta)\}$ be an optimal set of one-point. The distortion error due to the set α is given by

$$\begin{aligned} V(P; \alpha) &= \int_{\mathbb{R}} \min_{a \in \alpha} \rho((x, -2), (\cos\theta, \sin\theta))\, dP(x) \\ &= \int_0^1 \rho((x, -2), (\cos\theta, \sin\theta))\, dx = 4\sin\theta - \cos\theta + \frac{16}{3}, \end{aligned}$$

the minimum value of which is 1.21023 and it occurs at $\theta = 4.95737$. Hence, the optimal set of one-point is $\{(0.242536, -0.970143)\}$ with quantization error $V_1 = 1.21023$ (see Figure 2), which is the proposition. $\square$

Proposition 6.2. *The set* $\{(0.120535, -0.992709), (0.348179, -0.937428)\}$ *forms an optimal set of two-points with quantization error* $V_2 = 1.18174$.

Proof. Let $\alpha := \{\theta_1, \theta_2\}$ be an optimal set of two-points such that $\frac{3\pi}{2} \leq \theta_1 < \theta_2 \leq 2\pi - \cot^{-1}\frac{1}{2}$. Let the boundary of their Voronoi regions intersects the support of P at the element $(a, -2)$. The distortion error is given by

$$V(P; \alpha) = \int_0^a \rho((x, -2), (\cos\theta_1, \sin\theta_1))\, dP(x) + \int_a^1 \rho((x, 0), (\cos\theta_2, \sin\theta_2))\, dP(x). \tag{10}$$

The canonical equation is

$$\rho((a, -2), (\cos\theta_1, \sin\theta_1)) - \rho((a, -2), (\cos\theta_2, \sin\theta_2)) = 0.$$

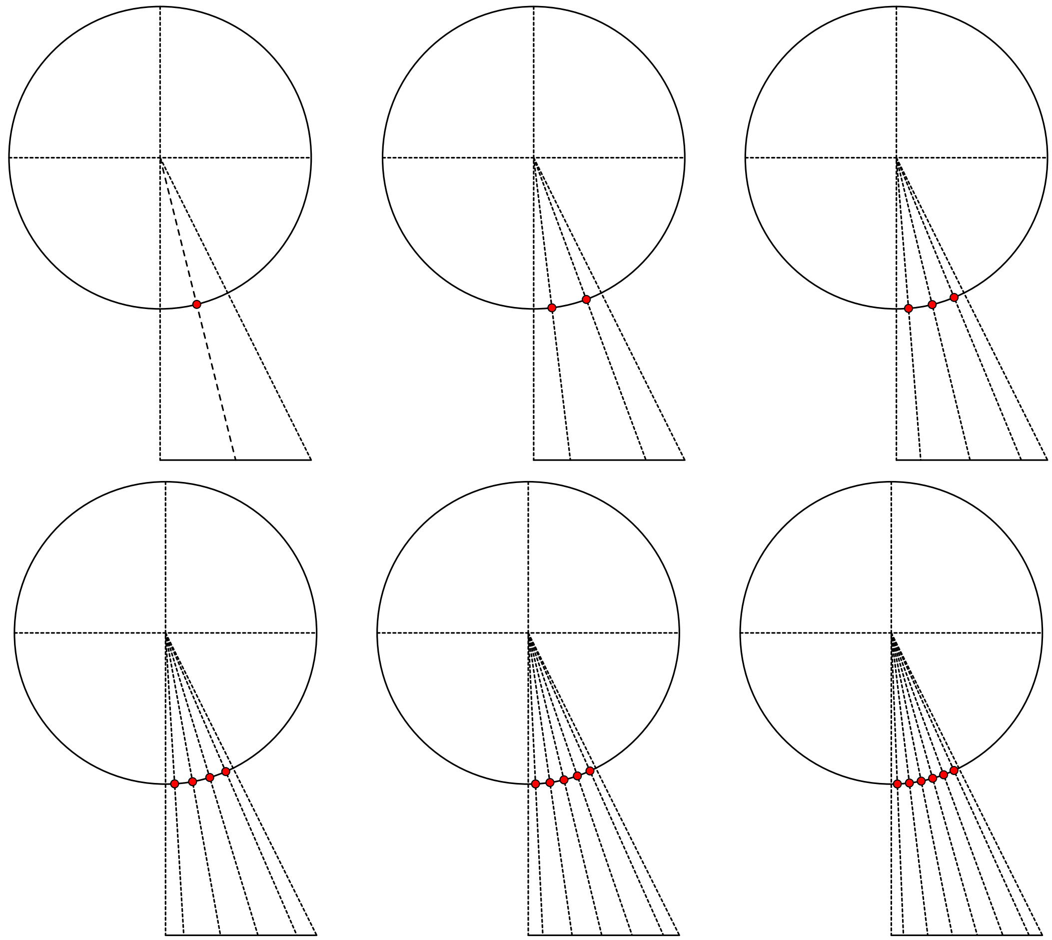

FIGURE 2. Optimal configuration of n elements for $1 \leq n \leq 6$.

Solving the canonical equation, we have

$$a = -2\cot\frac{1}{2}(\theta_1 + \theta_2).$$

Putting the values of a in (10), we see that $V(P;\alpha)$ is a function of θ_i for $i = 1, 2$. Since $V(P;\alpha)$ is optimal, we have

$$\frac{\partial}{\partial\theta_i}V(P;\alpha) = 0 \text{ for } i = 1, 2.$$

Solving the above two equations, we obtain the values of θ_i for which $V(P;\alpha)$ is minimum as

$$\theta_1 = 4.83322 \text{ and } \theta_2 = 5.06802.$$

Thus, we obtain the optimal set of two-points as mentioned in the proposition with quantization error $V_2 = 1.18174$ (see Figure 2). Thus, the proof of the proposition is complete. □

Remark 6.2.1. The above two propositions give the optimal sets of one-point and two-points. In the following, we calculate the optimal set of six-points. Following a similar technique, we can calculate all the optimal sets of n-points and the nth quantization errors for all $n \in \mathbb{N}$.

Proposition 6.2.2. *The optimal set of six-points is given by*

$$\{(0.0401095, -0.999195), (0.119815, -0.992796), (0.198005, -0.980201), (0.273754, -0.9618),$$
$$(0.346252, -0.938142), (0.414838, -0.909895)\}$$

with quantization error $V_6 = 1.17345$.

Proof. Let $\alpha := \{\theta_1, \theta_2, \cdots \theta_6\}$ be an optimal set of six-points such that $\frac{3\pi}{2} \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 < \theta_6 \leq 2\pi - \cot^{-1}\frac{1}{2}$. Let the boundaries of the Voronoi regions of θ_i and θ_{i+1} intersect the support of P at the elements $(a_i, -2)$, where $1 \leq i \leq 5$. The distortion error is given by

$$V(P;\alpha) = \Big(\int_0^{a_1} \rho((x,0),(\cos\theta_1, \sin\theta_1))\, dP(x) + \sum_{i=1}^{4}\int_{a_i}^{a_{i+1}} \rho((x,0),(\cos\theta_{i+1}, \sin\theta_{i+1}))\, dP(x) \qquad (11)$$
$$+ \int_{a_5}^{1} \rho((x,0),(\cos\theta_6, \sin\theta_6))\, dP(x)\Big).$$

The canonical equations are

$$\rho((a_i,-2),(\cos\theta_i,\sin\theta_i)) - \rho((a_i,-2),(\cos\theta_{i+1},\sin\theta_{i+1})) = 0 \text{ for } 1 \leq i \leq 5.$$

Solving the canonical equations, we have

$$a_i = -2\cot\frac{\theta_i + \theta_{i+1}}{2} \text{ for } 1 \leq i \leq 5.$$

Putting the values of a_i in (11), we see that $V(P;\alpha)$ is a function of θ_i for $1 \leq i \leq 6$. Since $V(P;\alpha)$ is optimal we have

$$\frac{\partial}{\partial\theta_i} V(P;\alpha) = 0 \text{ for } 1 \leq i \leq 6.$$

Solving the above six equations, we obtain the values of θ_i for which $V(P;\alpha)$ is minimum as

$$\theta_1 = 4.75251,\ \theta_2 = 4.83249,\ \theta_3 = 4.91171,\ \theta_4 = 4.98968,\ \theta_5 = 5.06596,\ \theta_6 = 5.14015.$$

Recall that θ_i represents the element $(\cos\theta_i, \sin\theta_i)$. Thus, we obtain the optimal set of six-points as mentioned in the proposition with quantization error $V_6 = 1.17345$ (see Figure 2). Hence, the proof of the proposition is complete. □

7. Quantization dimensions and quantization coefficients

Let P be a Borel probability measure on $\mathbb{R}^k$ equipped with a metric, and let $r \in (0, +\infty)$. In unconstrained quantization (see [GL]), the numbers

$$\underline{D}_r(P) := \liminf_{n\to\infty} \frac{r\log n}{-\log V_{n,r}(P)} \text{ and } \overline{D}_r(P) := \limsup_{n\to\infty} \frac{r\log n}{-\log V_{n,r}(P)}, \qquad (12)$$

are called the *lower* and the *upper quantization dimensions* of the probability measure P of order r, respectively. If $\underline{D}_r(P) = \overline{D}_r(P)$, the common value is called the *quantization dimension* of P of order r and is denoted by $D_r(P)$. In unconstrained quantization (see [GL]) for any $\kappa > 0$, the two numbers $\liminf_n n^{\frac{r}{\kappa}} V_{n,r}(P)$ and $\limsup_n n^{\frac{r}{\kappa}} V_{n,r}(P)$ are, respectively, called the κ-*dimensional lower* and *upper quantization coefficients* for P. The quantization coefficients provide us with more accurate information about the asymptotics of the quantization error than the quantization dimension. In unconstrained case, it is known that for an absolutely continuous probability measure, the quantization dimension always exists and equals the Euclidean dimension of the underlying object, and the quantization coefficient exists as a finite positive number (see [BW]). If the κ-dimensional quantization coefficient for P exists as a finite positive number, then κ equals the quantization dimension of P (see [GL]).

Unconstrained quantization error $V_{n,r}(P)$ goes to zero as n tends to infinity (see [GL]). This is not true in the case of constrained quantization. Constrained quantization error $V_{n,r}(P)$ can approach to any nonnegative number as n tends to infinity, and it depends on the constraint S that occurs in the definition of constrained quantization error as given in (1). In this regard, we give the following examples:

Let P be a Borel probability measure on $\mathbb{R}^2$ such that P is uniform on its support $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2 \text{ and } y = 0\}$. Let $V_n(P) := V_{n,2}(P)$ be its constrained quantization error. If the elements in the optimal sets lie on the line $y = 1$ between the two elements $(\frac{1}{2}, 1)$ and $(\frac{3}{2}, 1)$, then by Theorem 3.6, for $n \geq 3$,

$$V_n(P) = \frac{25n^2 - 50n + 26}{24(n-1)^2} \text{ implying } \lim_{n\to\infty} V_n(P) = \frac{25}{24}. \tag{13}$$

If the elements in the optimal sets lie on the line $y = 1$ between the two elements $(0, 1)$ and $(\frac{28}{15}, 1)$, then by Theorem 3.8, for $n \geq 8$,

$$V_n(P) = \frac{7(5788(n-1)n + 3015)}{10125(1-2n)^2} \text{ implying } \lim_{n\to\infty} V_n(P) = \frac{10129}{10125}. \tag{14}$$

On the other hand, if the elements in the optimal sets lie on the line $y = \sqrt{3}x$ between the two elements $(0, 0)$ and $(2, 2\sqrt{3})$, then by Corollary 3.2, for $n \geq 2$,

$$V_n(P) = \frac{144n^2 + 196n - 576}{48n^3} \text{ implying } \lim_{n\to\infty} V_n(P) = 0. \tag{15}$$

Moreover, notice that if P is a uniform distribution on a unit circle, and if the elements in an optimal set of n-points lie on a concentric circle with radius a, then by Theorem 4.3, for $n \geq 3$,

$$V_n(P) = a^2 + 1 - \frac{2an}{\pi} \sin\frac{\pi}{n} \text{ implying } \lim_{n\to\infty} V_n(P) = (a-1)^2, \tag{16}$$

which is a nonnegative constant depending on the values of a.

Let us now give the following definition.

Definition 7.1. Let P be a Borel probability measure on $\mathbb{R}^k$ equipped with a metric d, and let $r \in (0, +\infty)$. Let $V_{n,r}(P)$ be the nth constrained quantization error of order r for a given S that occurs in (1). Then, the nth constrained quantization error $V_{n,r}(P)$ is a decreasing sequence and converges to its exact lower bound, which is a nonnegative constant. Set

$$V_{\infty,r}(P) := \lim_{n\to\infty} V_{n,r}(P).$$

Then, $(V_{n,r}(P) - V_{\infty,r}(P))$ is a decreasing sequence of nonnegative real numbers such that

$$\lim_{n\to\infty}(V_{n,r}(P) - V_{\infty,r}(P)) = 0.$$

Write

$$\begin{cases} \underline{D}_r(P) := \liminf\limits_{n\to\infty} \frac{r\log n}{-\log(V_{n,r}(P) - V_{\infty,r}(P))}, \text{ and} \\ \overline{D}_r(P) := \limsup\limits_{n\to\infty} \frac{r\log n}{-\log(V_{n,r}(P) - V_{\infty,r}(P))}. \end{cases} \tag{17}$$

$\underline{D}_r(P)$ and $\overline{D}_r(P)$ are called the *lower* and the *upper constrained quantization dimensions* of the probability measure P of order r, respectively. If $\underline{D}_r(P) = \overline{D}_r(P)$, the common value is called the *constrained quantization dimension* of P of order r and is denoted by $D_r(P)$. The constrained quantization dimension measures the speed at which the specified measure of the constrained quantization error converges as n tends to infinity. For any $\kappa > 0$, the two numbers

$$\liminf_n n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P)) \text{ and } \limsup_n n^{\frac{r}{\kappa}}(V_{n,r}(P) - V_{\infty,r}(P))$$

are, respectively, called the *κ-dimensional lower* and *upper constrained quantization coefficients* for P. If the *κ-dimensional lower* and *upper constrained quantization coefficients* for P exists, and are equal, then we call it the *κ-dimensional constrained quantization coefficient* for P. If the κ-dimensional constrained quantization coefficient for P exists as a finite positive number, then κ equals the constrained quantization dimension of P.

Let $V_{n,2}(P)$ be the nth constrained quantization error of order 2. Then, (13) implies that

$$\lim_{n\to\infty} \frac{2\log n}{-\log(V_{n,2}(P) - V_{\infty,2}(P))} = 1 \text{ and } \lim_{n\to\infty} n^2(V_{n,2}(P) - V_{\infty,2}(P)) = \frac{1}{24}, \tag{18}$$

(14) implies that

$$\lim_{n\to\infty} \frac{2\log n}{-\log(V_{n,2}(P) - V_{\infty,2}(P))} = 1 \text{ and } \lim_{n\to\infty} n^2(V_{n,2}(P) - V_{\infty,2}(P)) = \frac{2744}{10125}, \tag{19}$$

(15) implies that

$$\lim_{n\to\infty} \frac{2\log n}{-\log(V_{n,2}(P) - V_{\infty,2}(P))} = 2 \text{ and } \lim_{n\to\infty} n(V_{n,2}(P) - V_{\infty,2}(P)) = 3, \tag{20}$$

and (16) implies that

$$\lim_{n\to\infty} \frac{2\log n}{-\log(V_{n,2}(P) - V_{\infty,2}(P))} = 1 \text{ and } \lim_{n\to\infty} n^2(V_{n,2}(P) - V_{\infty,2}(P)) = \frac{\pi^2 a}{3}. \tag{21}$$

7.2. **Observations and Conclusions.**

(1) In unconstrained quantization, the elements in an optimal set are the conditional expectations in their own Voronoi regions. It is not true in constrained quantization, for example, for the probability measure P, defined in Corollary 3.2, the optimal set of two-points is obtained as $\{(\frac{1}{8}, \frac{1}{8}\sqrt{3}), (\frac{3}{8}, \frac{3}{8}\sqrt{3})\}$, and the set of conditional expectations of the Voronoi regions is $\{(\frac{1}{2}, 0), (\frac{3}{2}, 0)\}$, i.e., the two sets are different.

(2) In unconstrained quantization if the support of P contains infinitely many elements, then an optimal set of n-points contains exactly n elements. This is not true in constrained quantization. For example, from Subsection 5.1, we see that if a Borel probability measure P on $\mathbb{R}^2$ has support the diameter of a circle and the constraint S is the circle, then the optimal sets of n-points for all $n \geq 2$ always contain exactly two-elements although the support has infinitely many elements.

(3) In unconstrained quantization, the quantization dimension for an absolutely continuous probability measure exists and equals the Euclidean dimension of the support of P. This fact is not true in constrained quantization, as can be seen from the expressions (18), (19), and (20). Each of the probability measures has support the closed interval $[0, 2]$ on a line, but the quantization dimensions are different, i.e., the quantization dimension in constrained quantization depends on the constraint S that occurs in the definition of constrained quantization. The quantization dimension, in the case of unconstrained quantization, if it exists, measures the speed at which the specified measure of the error goes to zero as n tends to infinity, on the other hand, in the case of constrained quantization, if it exists, measures the speed at which the specified measure of the error converges as n tends to infinity.

(4) In unconstrained quantization, the quantization coefficient for an absolutely continuous probability measure exists as a unique finite positive number. In constrained quantization, the quantization coefficient for an absolutely continuous probability measure also exists, but it is not unique, and can be any nonnegative number as can be seen from the expressions of quantization coefficients in (18), (19), (20), and (21), i.e., the quantization coefficient in constrained quantization depends on the constraint S that occurs in the definition of constrained quantization.

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