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Pentagonal Tilings of the Plane

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PENTAGONAL TILINGS OF THE PLANE

A Thesis

by

ARIANA T. HINOJOSA

Submitted in Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE

Major Subject: Mathematics

The University of Texas Rio Grande Valley

May 2022

PENTAGONAL TILINGS OF THE PLANE

A Thesis
by
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May 2022

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ABSTRACT

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Tilings are mathematical objects that allow us to use geometry to visualize interaction between objects as well as to create artistic realization of mathematical objects in the plane and in the space.

We will focus on tilings of the plane that use only one type of convex pentagonal tile each, the pentagonal tilings. There are fifteen types of pentagonal tiles, with each containing their own set of restrictions. The main result of this thesis is an interactive realization of all fifteen types of pentagonal tiles using GeoGebra.

DEDICATION

I would like to dedicate this thesis to my family. A special feeling of appreciation to my loving parents, John and Ileana Hinojosa who continuously motivated and comforted me all the way until the end. My siblings, Josh, Alyssa and Anahi were always there for me when I needed them. I also dedicate this thesis to my grandparents, Nora and Martin Martinez who have always supported me and whatever I do. I could not have finished this without their continued support and encouragement.

ACKNOWLEDGMENTS

I would first like to express my sincere gratitude to my advisor, Dr. Alexey Garber, for the continuous support of my research, his countless hours of reflecting, reading, encouraging, and most of all patience through the entire process. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my study.

Besides my advisor, I would like to thank the rest of my thesis committee: Dr. Alexey Glazyrin, Dr. Oleg Musin, and Dr. Zhijun Qiao, for their encouragement, insightful comments, hard questions, and who were more than generous with their expertise and precious time.

I also want to thank my coworkers: Aldo Gonzalez, Brenda Garcia, Erika Olivares, and Brayton Wario, for all the support and continuous encouragement they provided, especially since I was the one to procrastinate the most, and for all the laughs we had at the office these last two semesters.

Lastly, I would like to thank my parents, John and Ileana Hinojosa, for giving birth to me in the first place and for always supporting me throughout my life.

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CHAPTER I

INTRODUCTION

This thesis is devoted to study of tilings. A *tiling* [5], or a tessellation, is any representation of the plane R^2 or any R^d for $d \geq 1$ as a union of compact sets that do not overlap in interior points. These compact sets are called *tiles*. Circles, for instance, will never be able to tessellate; circles have no angles, so it is clear that there cannot be multiple circles placed next to each other without containing any gaps. The word “tessellation” actually comes from Latin tessellatus, meaning “of small square stones”. This suggests that the first tilings most likely used square tiles, probably due to them having an easy shape to make patterns with. A set of *prototiles*, p , is a proposed finite set of topological discs for tiling a space S . A tiling with only a single prototile is called a *monohedral* tiling. Similarly, there are also dihedral, trihedral, tetrahedral or n -hedral for ‘ n ’ amount of prototiles. Tilings can be classified as either regular, semi-regular or non-regular (or irregular); a regular tiling is made up of regular polygons that are all meeting vertex to vertex while also being the same shape. There are only 3 regular tessellations, and they are made up of equilateral triangles, squares and hexagons. A semi-regular tiling is made up of two or more types of regular polygons that are arranged such that each vertex point is identical. There are 8 semi-regular tessellations. Lastly, a non-regular, or irregular, tiling is where there are no restrictions in regard to the shapes or the arrangements around vertices.

hexagon

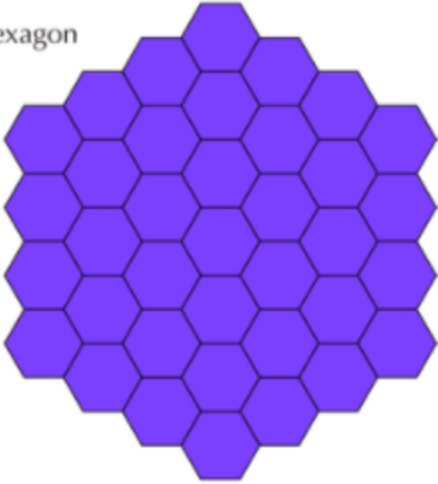


Figure 1.1: Regular tessellation example

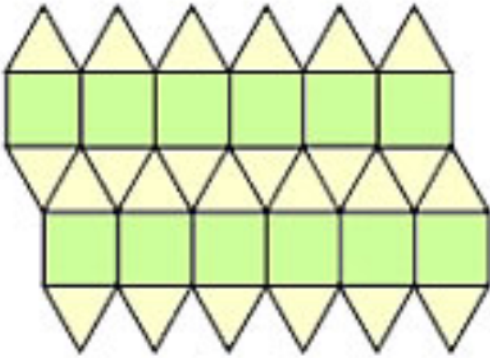


Figure 1.2: Semi-regular tessellation example



Figure 1.3: Irregular tessellation example

Tessellations have actually been connected to art in multiple cultures, with each having slight differentiations in their own patterns. Tessellations can be traced to 4,000 years BC; the Sumerians would create decorations in their homes using clay tiles. Tilings could also be found created by Egyptians, Persians, Romans, Greeks, the Japanese, and others. Like previously stated, the tilings and designs in each civilization differed from one another since they were modified according to their respective cultures and traditions. Some tiling designs are used in architecture and can be seen in mosaics by the Greeks and Romans while others, for instance, are used on balconies or roofs.

A particularly famous realization of tilings through art was implemented by Mauritus Cornelius Escher. Escher was a Dutch artist who took an interest in math. As mentioned in [1], he was friends with mathematicians like Roger Penrose, Harold Coxeter and crystallographer Friedrich Haag. While Escher had no higher knowledge of math, he would conduct his own research on tessellations for his artistic visions. It's interesting to take note that although math was not on his list of expertise, he contributed to mathematics and the concept of tessellations remarkably. He even stated "Although I am absolutely innocent of training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists." Throughout his life, Escher designed numerous tessellation patterns that even now are still admired.

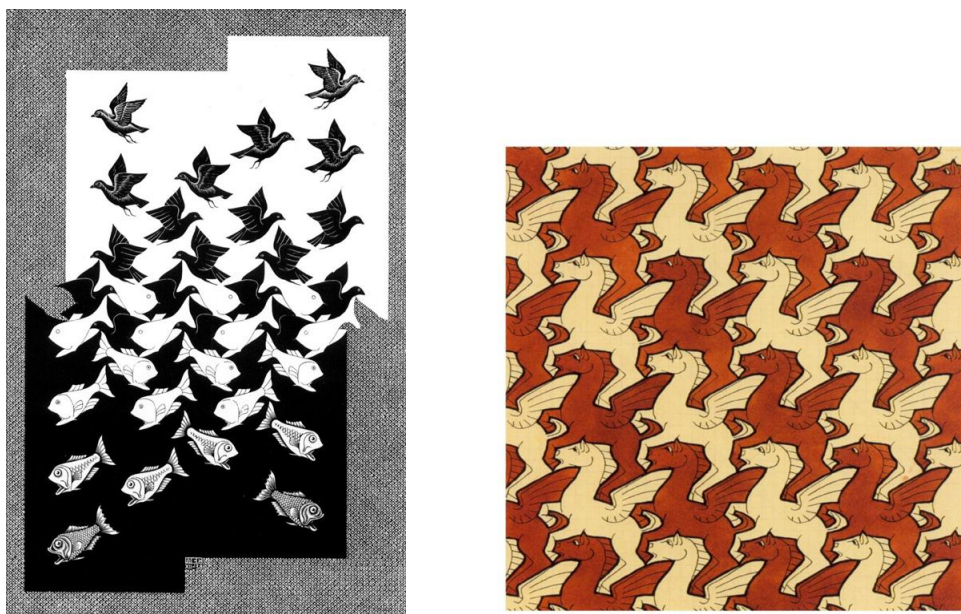


Figure 1.4: Tessellations in Art by M.C. Escher

Similarly, tessellations, particularly two-dimensional and three-dimensional, have also been linked to architecture. Tilings on the floor can be seen as tessellations that use squares as basic tiles. In other instances, tilings may be viewed on the outside of a building requiring complex computer models, or inside of a building as a mosaic or on a ceiling.



Figure 1.5: Tessellations in Architecture

When you look at a simple tiling, or tessellation, it is easy to make the observation that the original tile is just repeating in a specific pattern. One essential piece of information to understand about tessellations is the fact that they deal with symmetry. These symmetries are referred to as either plane symmetries or geometric transformations and can then be categorized into three different types. There are three types of symmetries, and they are translational symmetry, rotational symmetry, and glide reflection symmetry. *Translational symmetry* occurs when there exists a vector V such that the shift of the tiling T by V does not alter the tiling. Similarly, for *rotational symmetry*, it occurs when there exists a point P and an angle α such that the rotation of the tiling about P around α does not alter the tiling. The last symmetry, *glide reflection*, occurs when there exists a line ℓ and a vector v such that the mirror reflection about ℓ followed by the translation by v does not change the tiling; this symmetry is the only of the three that requires more than one step.

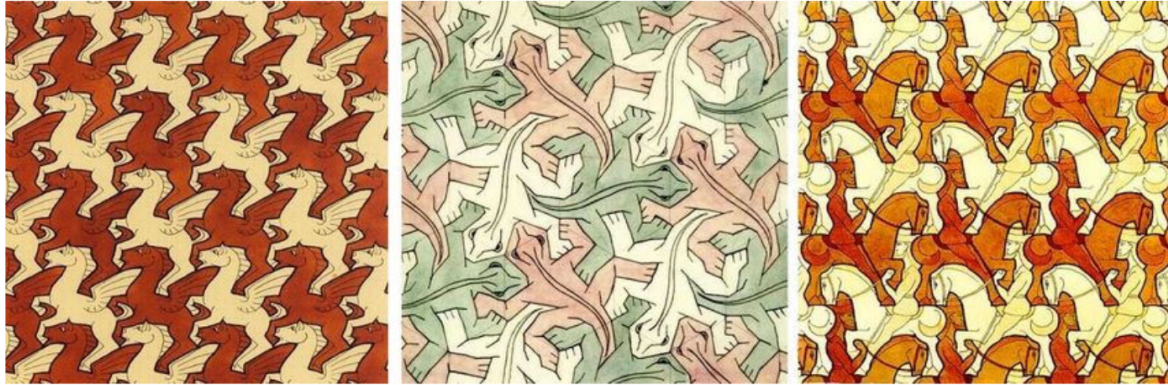


Figure 1.6: Tilings, respectively, containing translational, rotational, and glide reflection symmetry

1.1 Periodic Tilings

A tiling is *periodic* if there exist two linearly independent translational symmetries of the tiling. Conversely, an *aperiodic* tiling is a tiling that does not create translational symmetries. To determine whether or not a tiling is periodic, one method that can be utilized is to construct a lattice, a grid consisting of two sets of evenly spaced parallel lines. Using this method, one can determine whether the lattice repeats regularly in both directions.

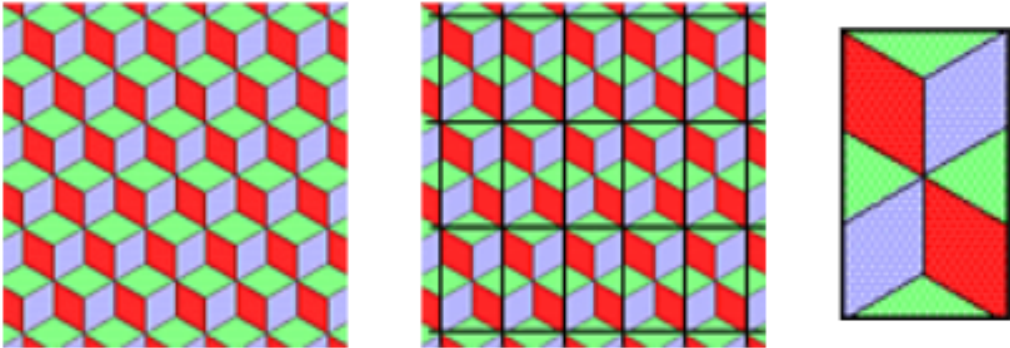


Figure 1.7: Lattice Method illustration

As previously mentioned, there are only three regular polygons able to tessellate. These are the equilateral triangles, squares and regular hexagons. Of those three, only the hexagonal tiling is considered to be *truly periodic*. Nonperiodic and aperiodic are sometimes thought to be interchangeable, but they are not. *Nonperiodic* tilings tile the plane in a non-repeating manner,

but can *also* tile it in a regular periodic manner. Aperiodic tilings cannot also be tiled in a regular periodic manner. This signifies that shifting any tiling with these shapes by any finite distance, without rotation, will not produce the same tiling. For a while it seemed when tiles could be used to create a an aperiodic tiling, then the same tiles could also be used differently to create a periodic one. In 1966, Robert Berger[2], an applied mathematician, proved that no fixed algorithm would determine whether or not a given set of arbitrary tiles would tile the plane. In doing this though, it was insinuated that there were sets of tiles that could only tile the plane nonperiodically; this property goes with a tile set that is aperiodic. That same year was when the first aperiodic set was created by Berger as well. That set actually used 20,426 prototiles; he later reduced this number to 104. Since then, mathematicians have found sets with less and less of an amount of prototiles. There are some tilings even containing one prototile that are nonperiodic. As for aperiodic, the lowest number of prototiles as far as we know is two. A well known example of an aperiodic tiling containing two protiles is the Penrose tiling, created by Roger Penrose[4],a mathematician and mathematical physicist, in 1974.

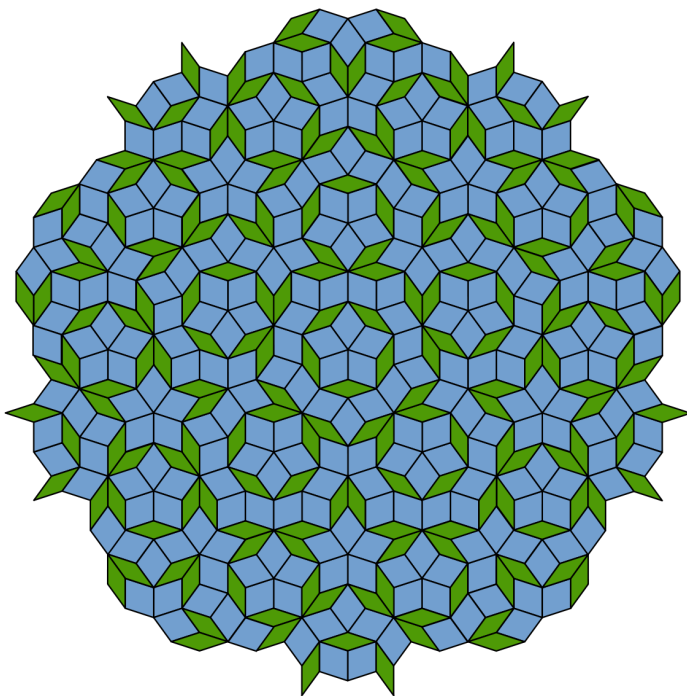


Figure 1.8: The Penrose Tiling

CHAPTER II

PENTAGONAL TILINGS

This thesis will mostly be focusing on convex pentagonal tilings that are periodic. Up until now, there have been 15 pentagonal tile patterns that have been discovered. In 1908 Karl Reinhardt [9], identified 5 types of convex pentagonal tiles in his doctoral thesis. He was uncertain if those were the only five types, but did not continue his research. Further development of the pentagonal tiles came to a stop for 5 decades. Then, in 1968, Richard Kershner discovered 3 more types of pentagonal tiles. Unlike Reinhardt, Kershner claimed that there were no other convex pentagonal tiles to exist that were able to tile the plane. Kershner's assertion came more into focus in 1975 after it appeared in Martin Gardner's [3] math column, titled "On tessellating the plane with convex polygon tiles" in *Scientific American*. An amateur mathematician, Marjorie Rice [8], came across Gardner's column and became very interested. She later identified 4 more types of tiles by the end of 1977. After she read through the column and completed her own findings, she sent her work to Gardner; Gardner then sent it to a tiling expert, Doris Schattscheider, and it was he who established that her findings were correct. In 1985, a computer programmer Richard James found another type of convex pentagonal tile. The last type found was fairly recent in 2015 by Casey Mann and collaborators.

It was Michaël Rao [10], who announced a proof that would ultimately finish the classification of tessellating convex pentagons. When Rao heard of the discovery of the fifteenth type, he decided to do an extensive search on convex pentagonal tiles able to tile the plane. With the help of computers, his proof found that there were 371 possible scenarios of how the corners of pentagons could join in a tiling, and he then proceeded to check each one of them. He first showed that if a pentagon tiles the plane, it would then be able to create a tiling with each vertex having a positive

density; it can be noted that this is a faint outline of what a periodic tiling is. In his proof, it was also indicated there was only a finite amount of potential vertex types in the revised pentagonal tessellation. He identified that only the 15 discovered pentagon families could achieve it. Mann had mentioned that he and his collaborators were planning on taking the same initiative as Rao in regards to doing a thorough search, but Rao had already achieved it. Since then, Thomas Hales, a professor of mathematics at the University of Pittsburgh has reproduced Rao's proof, which goes to show there is no flaw in his research.

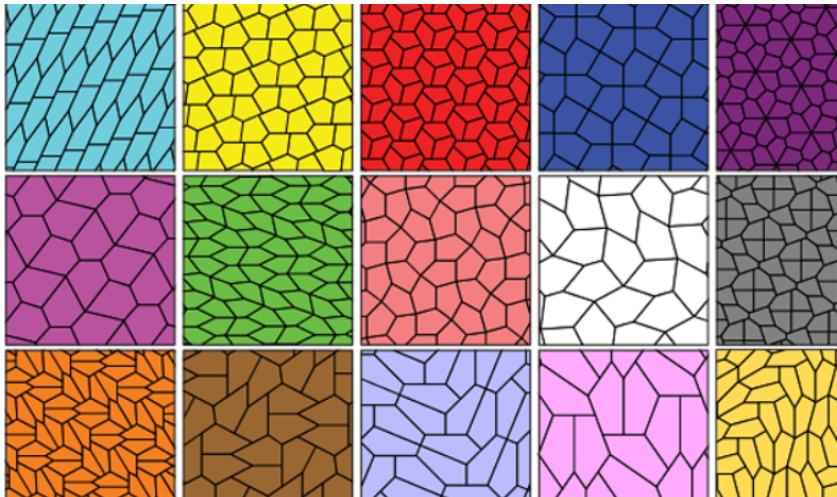


Figure 2.1: 15 Periodic Pentagonal Tilings

By Rao's proof [7], we now know that there are only 15 known types of pentagons that can create a tiling, but each family contains their own set of restrictions. The following figure depicts what conditions each type of pentagonal tile entails.

15 monohedral pentagonal tiles

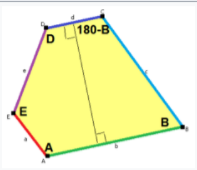
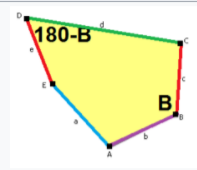
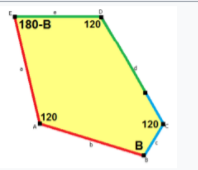
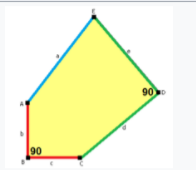

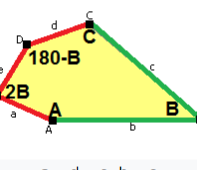
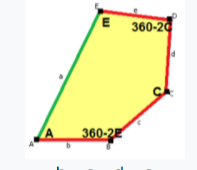
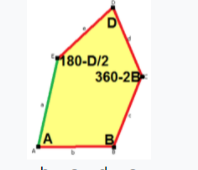
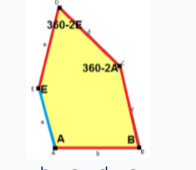
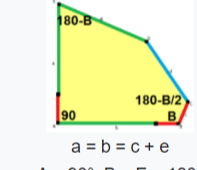
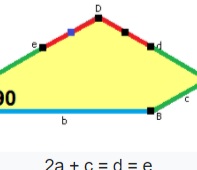
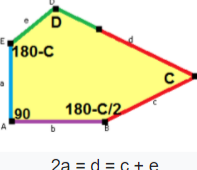
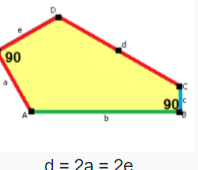
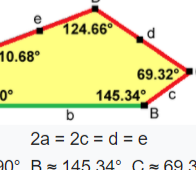
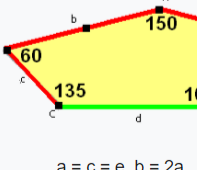
| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|--|
|  <p> $B + C = 180^\circ$ $A + D + E = 360^\circ$ </p> |  <p> $c = e$ $B + D = 180^\circ$ </p> |  <p> $a = b, d = c + e$ $A = C = D = 120^\circ$ </p> |  <p> $b = c, d = e$ $B = D = 90^\circ$ </p> |  <p> $a = b, d = e$ $A = 60^\circ, D = 120^\circ$ </p> |
| 6 | 7 | 8 | 9 | 10 |
|  <p> $a = d = e, b = c$ $B + D = 180^\circ, 2B = E$ </p> |  <p> $b = c = d = e$ $B + 2E = 2C + D = 360^\circ$ </p> |  <p> $b = c = d = e$ $2B + C = D + 2E = 360^\circ$ </p> |  <p> $b = c = d = e$ $2A + C = D + 2E = 360^\circ$ </p> |  <p> $a = b = c + e$ $A = 90^\circ, B + E = 180^\circ$ $B + 2C = 360^\circ$ </p> |
| 11 | 12 | 13 | 14 | 15 |
|  <p> $2a + c = d = e$ $A = 90^\circ, C + E = 180^\circ$ $2B + C = 360^\circ$ </p> |  <p> $2a = d = c + e$ $A = 90^\circ, C + E = 180^\circ$ $2B + C = 360^\circ$ </p> |  <p> $d = 2a = 2e$ $B = E = 90^\circ$ $2A + D = 360^\circ$ </p> |  <p> $2a = 2c = d = e$ $A = 90^\circ, B \approx 145.34^\circ, C \approx 69.32^\circ$ $D \approx 124.66^\circ, E \approx 110.68^\circ$ $(2B + C = 360^\circ, C + E = 180^\circ)$ </p> |  <p> $a = c = e, b = 2a$ $A = 150^\circ, B = 60^\circ, C = 135^\circ$ $D = 105^\circ, E = 90^\circ$ </p> |

Figure 2.2: Periodic Pentagonal Tiling conditions

CHAPTER III

GEOGEBRA CONSTRUCTIONS

The goal for this thesis was to recreate the 15 tilings using the online tool GeoGebra[6]. The purpose of this chapter is to provide an insight to the processes needed in order to reconstruct the tiles. There are only fifteen types of periodic pentagonal tiles, and each family of the fifteen tiles contains its own set of conditions. There are different procedures needed to follow to recreate each one, and also different symmetries used to tile them. Whenever a type of tile is being referred to, the conditions for that specified type can be found in Figure 2.2.

3.1 Construction for Type 1

The construction of the fifteen types of tiles have been completed, using GeoGebra; GeoGebra not only shows the tilings, but also allows to dynamically change the defining components and visualize the whole family rather than a single tiling. Some were able to be composed using basic geometric properties, like the first few types of pentagonal tiles, while others involved more identities and formulas that were utilized. The process to create the first type of pentagonal tile, for instance, was more trivial. As seen in Figure 1.10, the only conditions type 1 contains are that the sum of the angles B and C equal 180° , and that the sum of angles A , D , and E equal 360° . The first step we took for the recreation of this tile was to make two points, A and B ; then, we drew a line going between those two points that would be the side b . We then created a point D somewhere below A and B and drew a line going through D such that it was parallel to the line going through the points A and B . Once we had two parallel lines, a perpendicular bisector was created in order to ensure that this condition was also met. After, a point C was created on the line going through B where it intersected the line going through D ; since the two lines d and b are parallel to one another,

it satisfied the the condition showing that the segment from point B to C creates the sum of angles $B + C = 180^\circ$. The reason we know this is due to the fact that they are supplementary. The last point we created was point E . E could be placed anywhere between points A and D . After all the points had been formed, we could finally connect them with segments instead of lines. The total sum of angles in a pentagon is 540° . Knowing this, it is then known that no matter what the angles are for A , D , and E , they will equal 360° when added since $540^\circ - 180^\circ$ is 360° . Hence, all 3 conditions are satisfied, and the tile is complete. We can formulate a polygon using this tile so it can be simpler to tile.

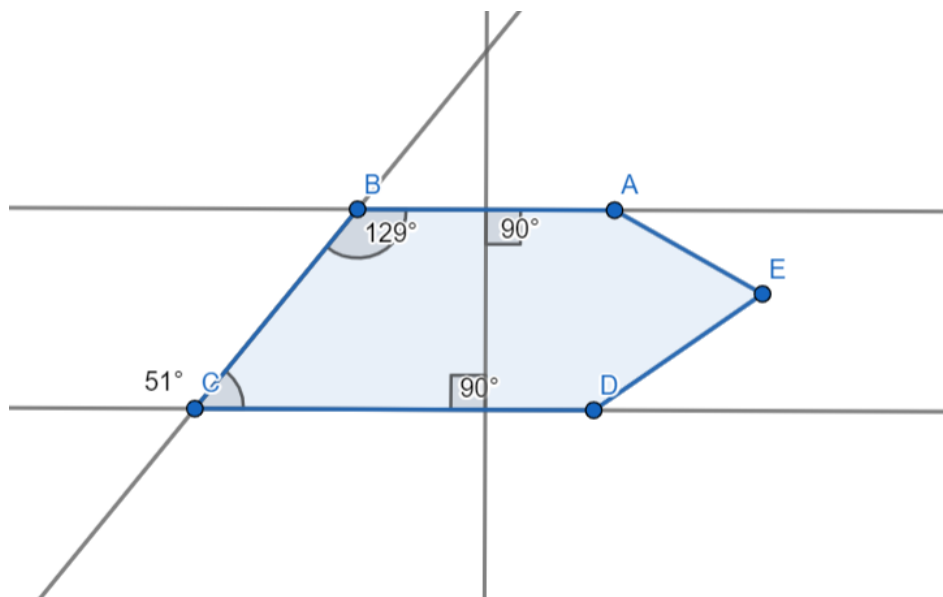


Figure 3.1: Type 1 pentagonal tile

Once the tile is done, it can be tiled to create a tessellation. For this first type of pentagonal tile, we can begin by utilizing the glide reflection symmetry. In order to do so, we can create a midpoint on segment c . Once done, GeoGebra contains an option to reflect about a point, which is the tool to accomplish this symmetry. We select the polygon for the first type of tile and choose to reflect about the midpoint we created. Once done, GeoGebra has a tool that allows to translate by vector; this tool represents translational symmetry. We then had to pick the starting point, I chose point A and place the vector at the point you would like to translate it to, I chose point D . After, we could then choose the remaining tiles and translate them using the same vector. Since points A and

D are above and below each other, the vector may only slide the tiles up and down. In order to slide the tile to the left and right, we use the same tool but change what point the vector lands on. When completed, there should be no gaps or overlaps between the tiles.

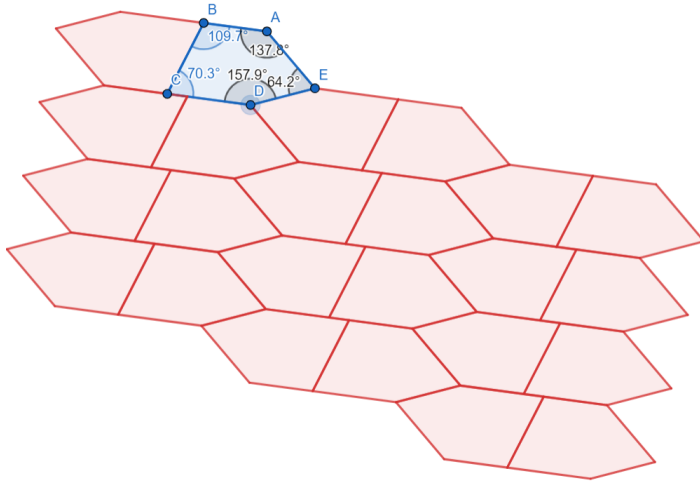


Figure 3.2: Type 1 pentagonal tiling

3.2 Construction for Type 7

The conditions for type 7 are as follows: $b = c = d = e$ and $B + 2E = 2C + D = 360^\circ$. Two points, B and C , were constructed initially. To ensure that sides c and d were equal, a circle was created going through B with center C . Point D was then placed on this circle. We let the angle C be α . Since angle D is equal to $360^\circ - 2\alpha$, the exterior angle is equal to 2α . Hence, to find the point E , we rotated C by 2α . It is rotated by a positive angle due to the fact that it is being rotated counterclockwise. We then created an angle bisector of C, D, E and a line through E perpendicular to side e . A point, X was created at the intersection of the line and bisector. A circle going through E with center X was then created to establish the distance from the center going to E, C , and A is equal. Then we created another circle going through C with center B to keep the condition of $b = c$. Point A was constructed at the intersection of those two circles. Segments were then added between the points, starting from A going to E .

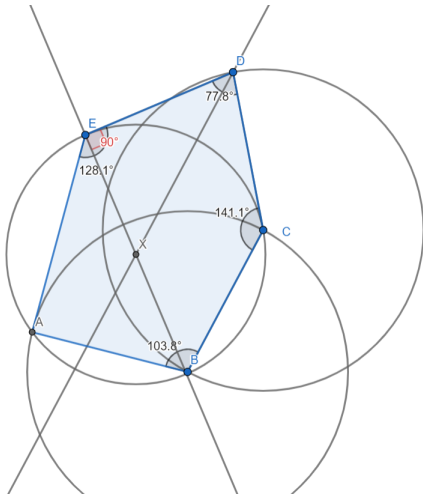


Figure 3.3: Type 7 pentagonal tile

The conditions all hold true, so we began the process to tessellate, using the polygon created. The first step was to reflect the tile about the segment AE , or side a . Then a midpoint was made on side d of the original tile and both tiles were reflected about that midpoint. Then the original tile and its reflection about the line rotated around the point C counterclockwise by the angle C . Then the two new tiles could be reflected about the midpoint on side d . In total, there were 8 tiles now made that could be used to tile the entire plane. We used the tool on GeoGebra allowing the translation by a vector. We selected all eight tiles to be translated by a vector starting at point A of the original tile and going to the last tile created on the point corresponding to D . From there, we continued using the same tool to tile in all directions.

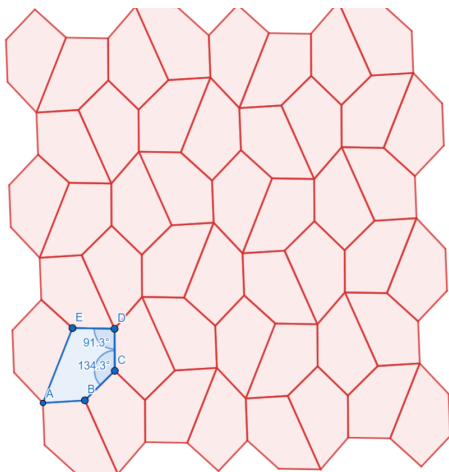


Figure 3.4: Type 7 pentagonal tiling

3.3 Construction for Type 8

For the eighth type of pentagonal tile, the conditions are $b = c = d = e$ and $2B + C = D + 2E = 360^\circ$. This tile started with the creation of two points, A and B . To guarantee that sides b and c were equivalent, a circle was formed going through point A with center B ; point C was then placed on that circle. A line was then created going through points A and B , labeled f , as well as a line through B perpendicular to f , labeled g . A circle going through B with center C , labeled k , was also created to ensure sides c and d were equivalent. Point D was placed at the intersection of the circle k and perpendicular line g . A circle, labeled e , was then formed going through point C and with center D . A segment going through A and C was created and a midpoint was placed on that segment. A circle was formed going through A and with the midpoint being the center. Point E was placed at the intersection of this circle and circle e . Each point was then made sure to be connected starting from A going to E through segments. Once the polygon was formed, the pentagonal tile was ready to be tessellated.

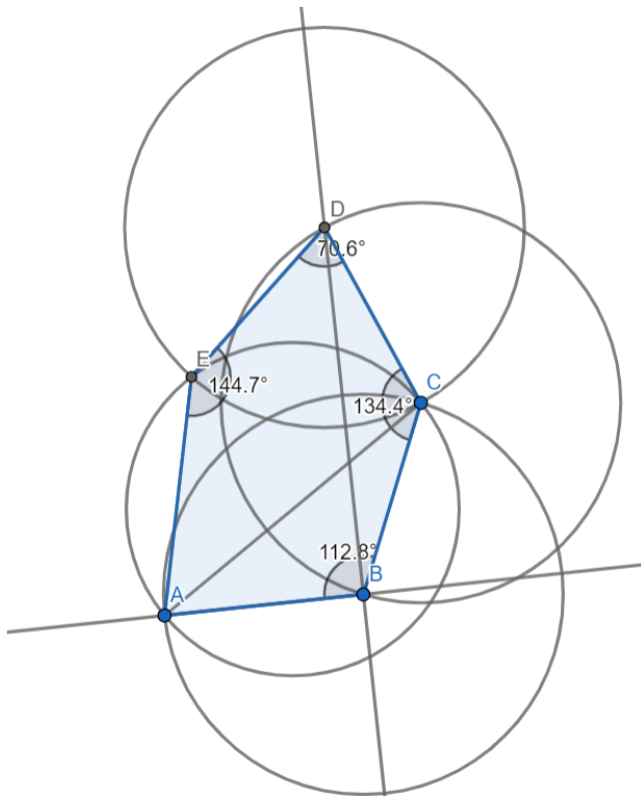


Figure 3.5: Type 8 pentagonal tile

As the conditions for this type are all met, we began to tile the polygon. The first step we took was to create a midpoint on side c and reflect the polygon around that midpoint. After, we reflected the primary polygon by the side a . On this new tile, a midpoint was placed on the side corresponding to e , and the tile was then reflected by that midpoint. The initial tile created was then rotated around the point B by angle B , and a similar process continued; a midpoint was placed on this new tile on the length corresponding to e . The new tile was then reflected by that midpoint. From there, the tile created was reflected by its side length corresponding to a . Lastly, a midpoint was made on the new tile on the side corresponding to c . In total, these eight tiles created a primitive unit that could tile the rest of the plane with the tool on GeoGebra that allows translation by a vector.

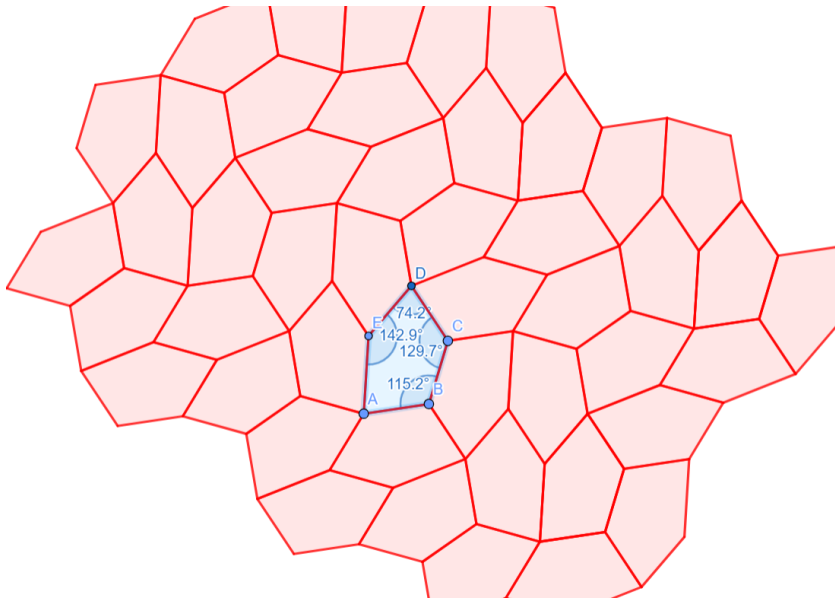


Figure 3.6: Type 8 pentagonal tiling

3.4 Construction for Type 9

Taking a look at type 9 on Figure 2.2, we see this tile has the conditions $b = c = d = e$, and the angles $2A + C = D + 2E = 360^\circ$. First two points were created, A and B . Then a circle c was created going through point A with center B , and point C was placed on the circle to ensure the sides $b = c$. A segment was made from point A to point C , and a midpoint, K was put on this segment. A circle h was created going through point A with center K . Another circle d was then formed going through point B with center C . A line going through points A and B was formed and labeled f . At

the intersection of circle d and line f a point X was placed. A circle going through point C with center X was then created and labeled e . At the intersection of circles h and e was where a point E was put. A segment was then placed going from point E to C , and point X was mirrored at this segment; the resulting point would then be point D . I then made sure each point from A to E was connected by segments, and the pentagon was then ready to be tiled on the plane.

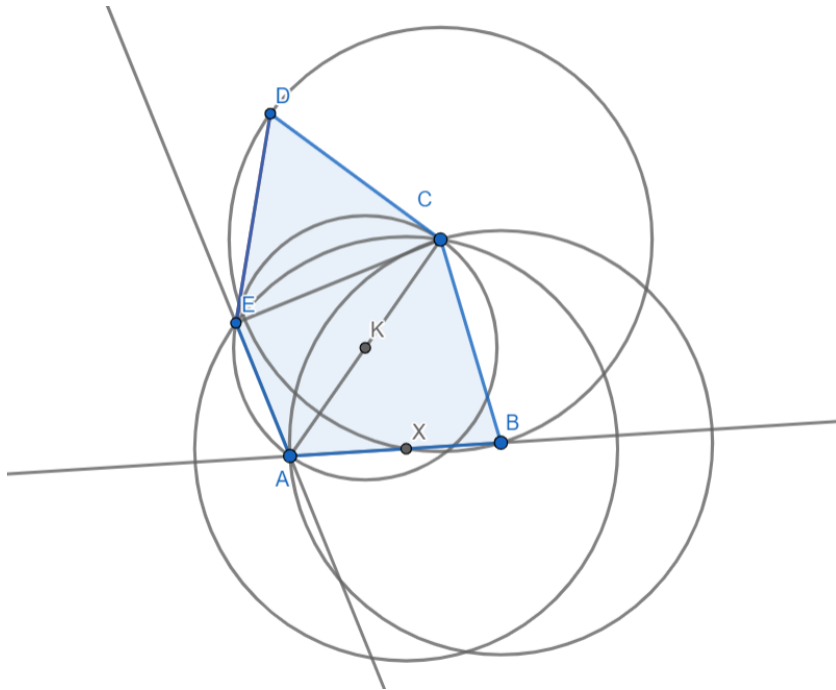


Figure 3.7: Type 9 pentagonal tile

The first step taken in order for the pentagon to tile the plane was to reflect it by the side a . On this new tile, a midpoint was placed on the side corresponding to e , and that same tile was reflected about that midpoint to form a new third tile. Then the third tile was reflected by the side corresponding to a . Then the point corresponding to B on the new fourth tile was rotated by angle B to form a fifth tile. Following the same process, the fifth tile was reflected by the side corresponding to a . Following the previous steps, a midpoint was placed on the side corresponding to e , and the tile was reflected by that midpoint. Once there was a primitive unit consisting of eight tiles, it could then be continued to tile the plane through translation by vectors.

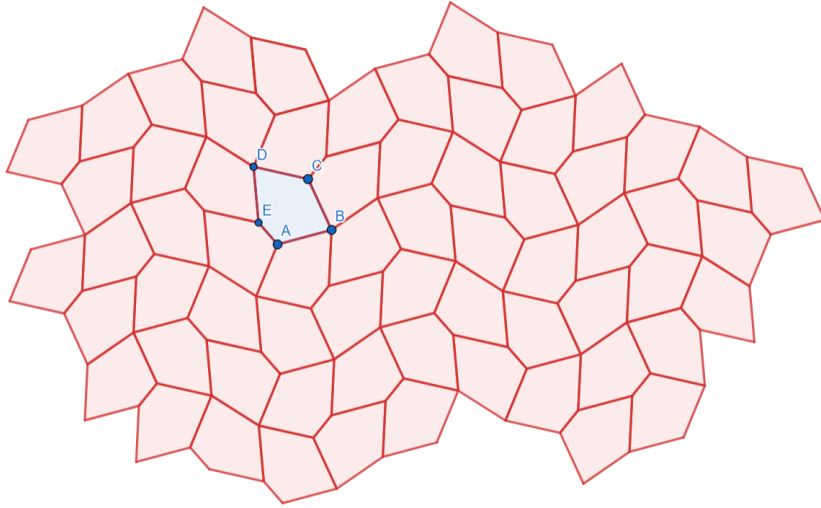


Figure 3.8: Type 9 pentagonal tiling

3.5 Construction for Type 14

If we take a look at type 14 on Figure 2.2, this tile has more conditions needed to be upheld. For instance, angle A is fixed right angle, and although angles $B, C, D,$ and E are approximations, they are also fixed. The lengths are also following certain circumstances. We can see that sides d and e are equivalent, which are also equal to twice the length of sides a and c . The first step taken to reconstruct type 14 on GeoGebra is to create two points A and B . Then, they can be connected by a segment. We can then input the formula, provided on Wikipedia, that can be used to solve for the side length a . The formula is $a = \frac{f}{\sqrt{\frac{11\sqrt{57}-25}{8}}}$ where f is the segment from A to B . Because angle A is a fixed perpendicular angle, we can rotate point B by 90° . This will create a point B' . Since we rotated B , the length from A to B would be equivalent to the length from A to B' . The conditions for type 14 show those two lengths a and b should not be equal at all times. To fix this, we can draw a line that goes through A and B' . Because we already provided to formula needed to find a , we can input a circle with center point A that contains the radius length of a . Where the line going through A and B' and the circle intersect, we can place point E . Then we can connect points A and E by a segment. Wikipedia also provided the formula needed to attain angle B . The formula given is the following: $\sin(B) = \frac{\sqrt{57}-3}{8}$. In order to construct the fixed angle for B , we have to manipulate the equation to solve for angle B . In doing so, we need to only add the inverse of sine to both sides of

the equation. We then rotate the point A by $-(180^\circ - \arcsin \frac{\sqrt{57}-3}{8})$. The reason we are subtracting it from 180° is because it contains an obtuse angle, not an acute one that \sin functions typically are. The reason for having the entire formula negative is due to the fact that the actual angle is going clockwise instead of counter-clockwise.

We know that if we fix an angle, the two sides of the angle will be equivalent, but from the conditions it is known that the sides b and c are not equal. However, a and c are, and we already know what a is. We then create a line going through points B and A' . We can then create a circle with center at point B and with radius length of a . Where this circle and the line going through points B and A' intersect, we can create point C . We know from the conditions that side d is twice the length of c . Hence, in order to meet this action we create a circle with center point C and radius length $2a$. Now we can also keep in mind that side e is twice the length of side a as well. In order to construct the distance of e , we can create another circle with center point E and radius $2a$. The intersection of these two circles is where we will place our point D . We can start adding segments between the points E and D , D and C , and C and B . We can now form this pentagonal tile into a polygon as all conditions are met.

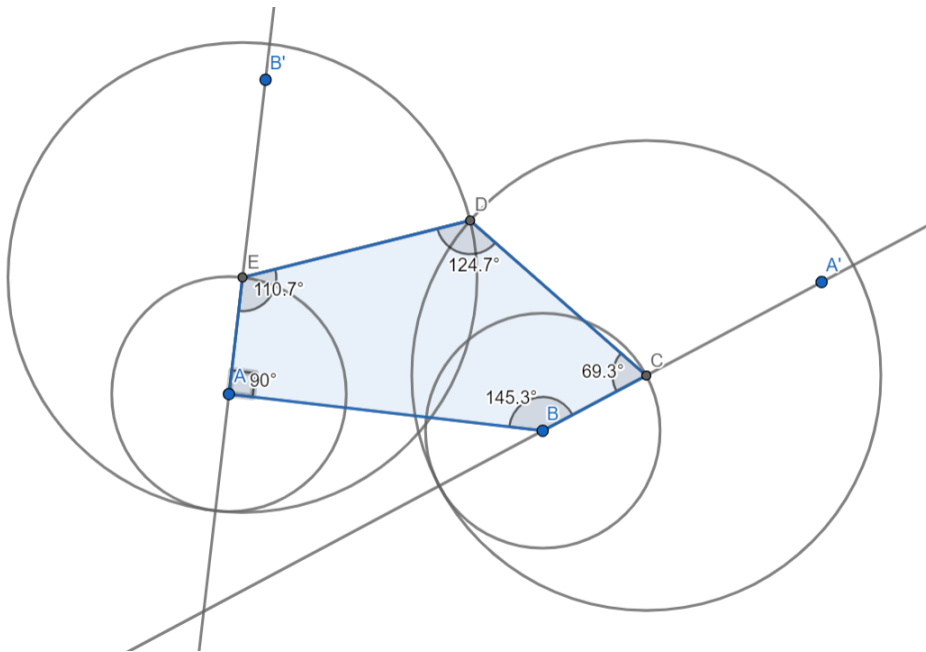


Figure 3.9: Type 14 pentagonal tile

Since the conditions have all been satisfied, we can now create the tessellation using this tile. For this type of pentagonal tile, we can begin by reflecting the tile along the side b . We can then continue by creating a midpoint on the side c to apply the glide reflection symmetry. We can then use the GeoGebra tool to reflect each polygon about the midpoint on c . This will result in two additional tessellated tiles. Subsequently, we will refer to the second tile created by the reflection of the original tile. We will then utilize the tool to rotate around a point. We will be referring to the second tile created from the original tile and rotating it around the point that was reflected by point C . The second tile will then be rotated around that point in a clockwise direction by angle D . Finally, we will use this latest tile by creating a midpoint on the side corresponding to b . Next, we once again use the tool that allows us to reflect about a point, and we can reflect the latest tile by the midpoint. Once done, the unit composed of the 6 tiles we just created can be used to tessellate the entire plane using translational symmetry. Referring to the last tile we created, we use the point corresponding to point C . We then use the tool allowing us to translate by vector. We select all 6 tiles and use that same point corresponding to point C from the last tile as the starting point and position the vector to translate to point E of the original tile. We can continue this however many times to construct one row of the tessellation. Once done, we will once again utilize the translation by vector tool using two other points now. We will be referring to the last tile translated this time, and the starting point will be the point corresponding to point E from the second tile reflected from the original. This point will be translated to the point corresponding to D . Then that translation can continue for each tile.

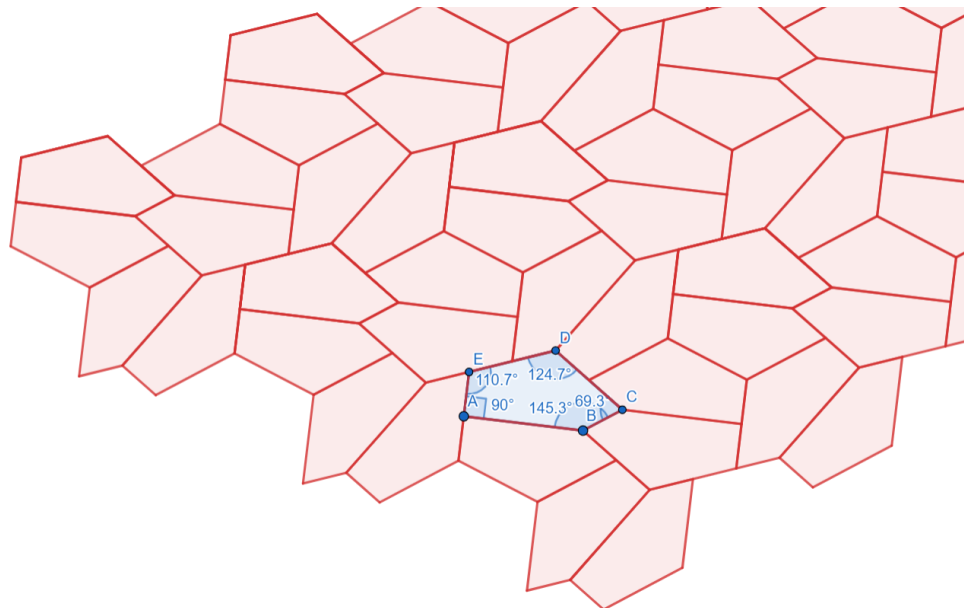


Figure 3.10: Type 14 pentagonal tiling

CHAPTER IV

CONCLUSION

In recreating the different types, I completed my study of the fifteen periodic pentagonal tilings. I gained a better understanding of how and why each tile needed a distinct process of construction. Using the models created in GeoGebra gave a more beneficial perspective on the different types of pentagonal tiles. GeoGebra allowed the interaction with the tiles and ability to view how moving the points or adjusting the dimensions did not affect the conditions being held. These interactive constructions could be potentially used to see whether other tilings exist. This thesis only covered convex periodic pentagonal tilings, but the interactive tessellations could possibly allow assistance in studying aperiodic pentagonal tilings. Other types of tilings these constructions could benefit would be substitution or pinwheel like tilings with pentagonal tiles.

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APPENDIX

APPENDIX

REMAINING TILINGS

1.1 Type 2

In order to reconstruct the second type of pentagonal tile, we start by creating 3 points E , D and C . We can create a segment from D to C , and then another going from D to E . After, we can use the angle tool to measure the angle D ; let's call this angle α . We can then create a circle with center at point C and radius being the side e . Point B can be put anywhere on that circle, and then a segment may be created from B to C . This satisfies the condition that sides c and e are equivalent to each other. We then rotate point C by $-(180^\circ - \alpha)$. It is negative due to the angle going clockwise. Then a line must be made going through B and the C' , and point A can be created anywhere on that line. We connect points A and B through a segment, and finally connect points A and E through a segment as well. The tile is complete and it can begin being tessellated.

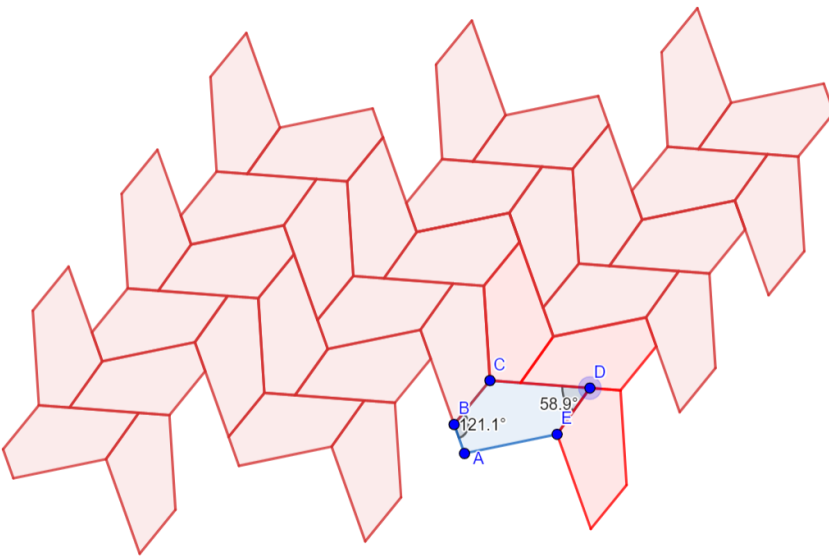


Figure A.1: Type 2 pentagonal tiling

1.2 Type 3

We can begin type 3 by constructing two points, C and D . We can make an observation that a group of three type 3 pentagonal tiles makes a hexagon. Subsequently, we can create a hexagon using the polygon tool and starting from the two points C and D . Then we can make a segment between those same points. The point A can be created at the center of the hexagon, and place point B on the side to the left of point C . We then created a circle with center C going through B and placed a point on the intersection of that circle and the segment from C to D . According to the conditions, sides a and b are equal, and angle A is 120° . Hence, we rotated point B by 120° to ensure the lengths would be the same. The intersection of that rotation and the line going through D was where we placed E . We made sure each point was connected by a segment, and it can finally be tiled.

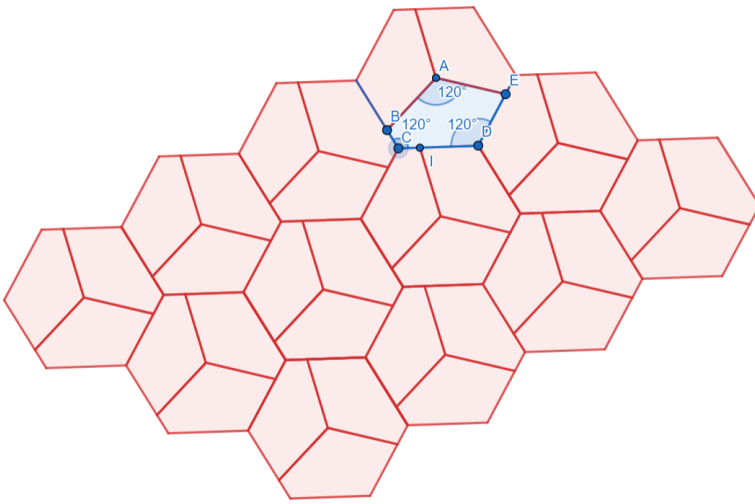


Figure A.2: Type 3 pentagonal tiling

1.3 Type 4

The conditions for the fourth type of pentagonal tile are as follows: Angles B and D are right angles and for the sides, $b = c$ and $d = e$. We started off by creating two points, A and B . We know B is a right angle and the sides b and c must be equivalent in length. Therefore, we rotated point A by -90° in order to ensure both conditions. The resulting point would be point C . The

reason we rotate it by a negative angle is due to it being rotated clockwise. We then added a point D . Since D is a right angle and sides d and e are equal, we rotate point C by -90° as well. The resulting point ended up being E . The five points could then be connected to each other through segments, and it was ready to tile after.

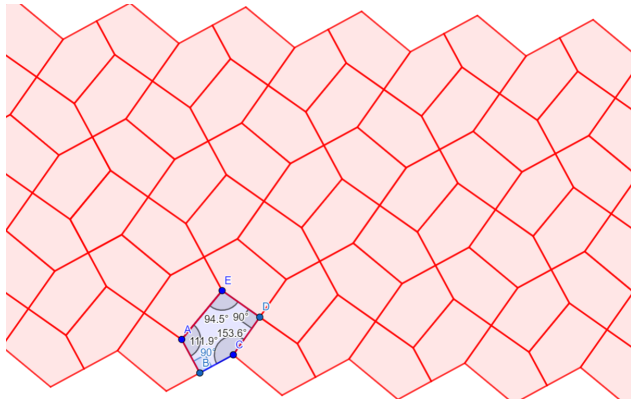


Figure A.3: Type 4 pentagonal tiling

1.4 Type 5

The necessary requirements for this tile are angle $A = 60^\circ$ and angle $D = 120^\circ$, with the sides $a = b$ and $d = e$. We first created two points, D and E . We then rotated point E by 120° to guarantee the condition of side $d = e$; this new point will be C . Point A can be added anywhere under angle D . Then we rotated point B by -60° , as it is going clockwise. Each of the points was then connected to each other by segments, and it's ready to create a tessellation.

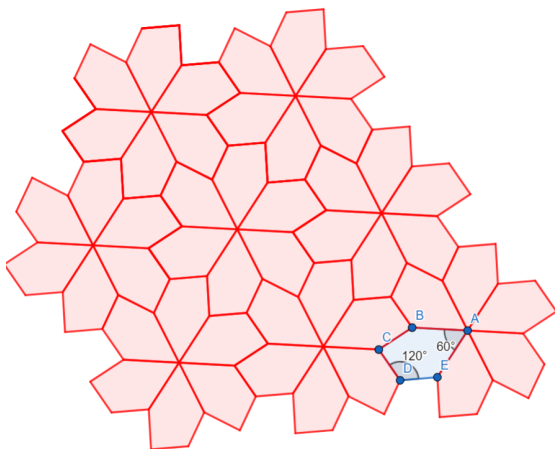


Figure A.4: Type 5 pentagonal tiling

1.5 Type 6

The conditions for this tile are sides $a = d = e, b = c$ and that angles $B + D = 180^\circ$ and $2B = E$. Two points were created first, C and D . A circle was then made going through C with center D . Point E was created on that circle. Angle D was labeled α and in order to find point A , we rotated D by angle 2α . A perpendicular bisector of AC was created. With AC , the points A, C and B would form a triangle with two equivalent sides. Hence, if angle B is β , then angles A and C must both be $90^\circ - \frac{\beta}{2}$. Point A was then rotated counterclockwise by $90^\circ - \frac{\beta}{2}$ and formed point G . A line was made going through points C and G . Point B is formed at the intersection of the perpendicular bisector and the line. The points were then connected by segments starting from A going to E , and it could then be tessellated.

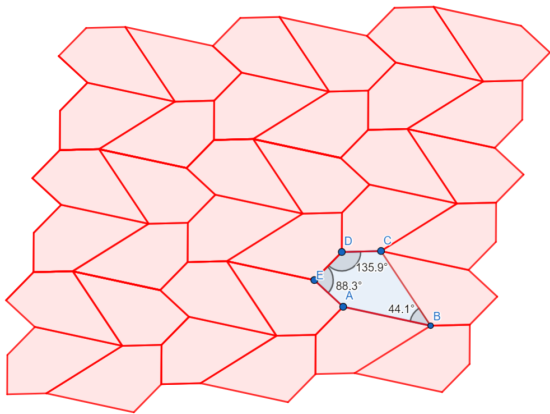


Figure A.5: Type 6 pentagonal tiling

1.6 Type 10

The conditions for this tile are that the angles $A = B = E = 90^\circ, C = D = 135^\circ$ and sides $a = b = 2c = 2e$. I first created two points, E and A . Then I rotated E by -90° to create point B and ensure $a = b$. I created segments between points A and E and A and B . I then created a midpoint on the segment going from E to A , and rotated that midpoint by 90° to create point D . I then created another midpoint on the segment from A and B and rotated that midpoint by -90° in order to create C . I then made to each point was connected to each other by segments, and it was then ready to tessellate.

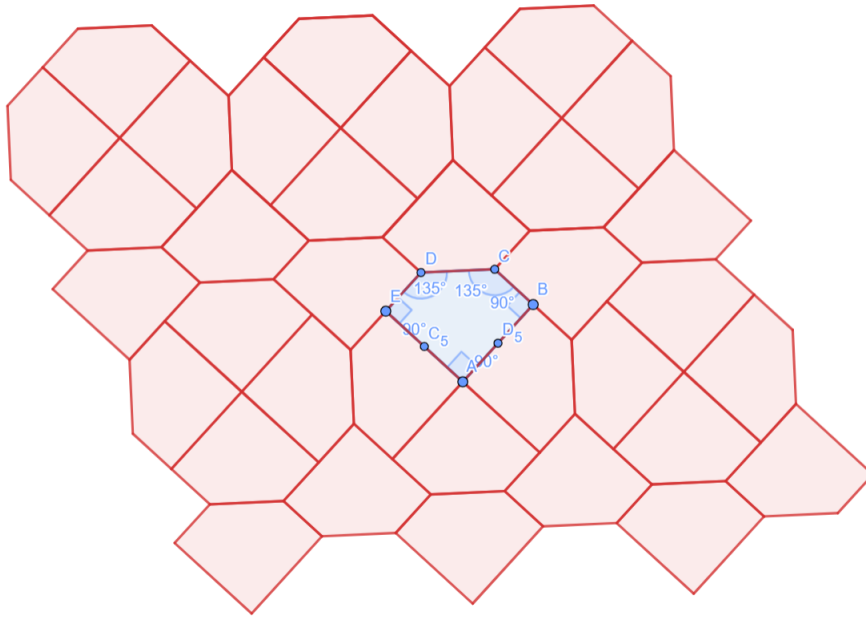


Figure A.6: Type 10 pentagonal tiling

1.7 Type 11

The conditions for this type of tile are that the angles $A = 90^\circ$, $C + E = 180^\circ$, $2B + C = 360^\circ$ and sides $2a + c = d = e$. We started by creating two points K and B , and drew a line going through those two points labeled f . Point C was then constructed somewhere beside point B . Computing the angles, we end up with side $a = \frac{2g \sin(\alpha) - g \cos(2\alpha)}{1 + 2\cos(2\alpha) - 2\sin(\alpha)}$ and define a distance $p = (g + 2a) \sin(2\alpha) + 2a \cos(\alpha)$. We created a circle with center K and radius a as well as a circle with center at K and with radius a . Point A was formed at the intersection of the circle with radius p and line f . We added another circle with center A and radius a and made a line through A perpendicular to f . Point E was then placed at the intersection of that circle and line. Two final circles were made; one contained center E and radius $2a + g$ and the other contained center C and radius $2a + g$. The intersection of the two circles would be point D . Each point starting from A going to E was then connected through segments, and it could begin to be tessellated.

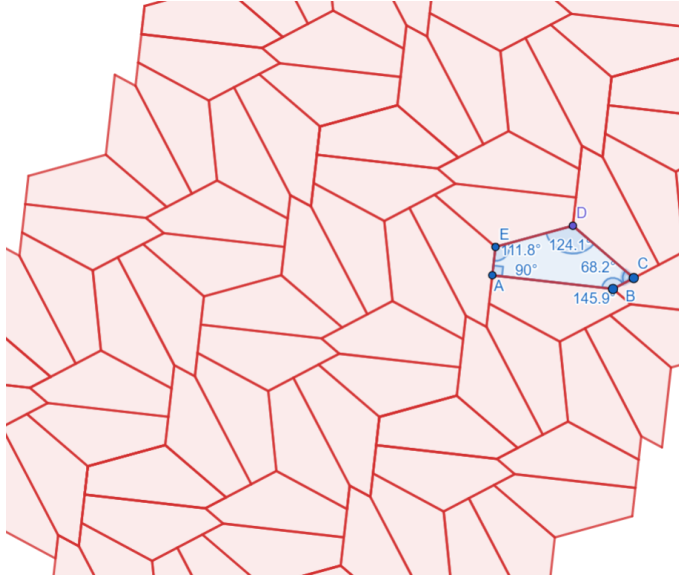


Figure A.7: Type 11 pentagonal tiling

1.8 Type 12

The conditions for this type of tile are that the angles $A = 90^\circ$, $C + E = 180^\circ$, $2B + C = 360^\circ$ and sides $2a = d = c + e$. We started by creating two points E and D . The first step was to create two points, E and D . A circle was then formed with center point D and radius being the length of side e . A point, C_1 was placed on that circle, and then a line going through points C_1 and D was formed and labeled g . The angle between points D, E and C_1 was labeled α . C_1 was then rotated by -3α . The resulting point was labeled C' . A line i going through C' and E was created. A segment connecting points E and D was created. Computing the angles using identities, we got that side $a = \frac{f(\cos(2\alpha) - \cos(4\alpha))}{4\cos(2\alpha) - 1}$ where f is the segment from E to D . I then formed a circle, e with center point C_1 and radius $2a - f$. At the intersection of circle e and line g was where point C was placed. A circle p was formed with center point C and radius being the distance from points C and C_1 . A circle d was also created with center point D and radius a . At the intersection of d and i is where point A was placed. Then a line through point A perpendicular to i was formed. At the intersection of p and j is where point B was put. I ensured each point was connected by segments going from A to E . Once the pentagon was formed, it was ready to be tiled.

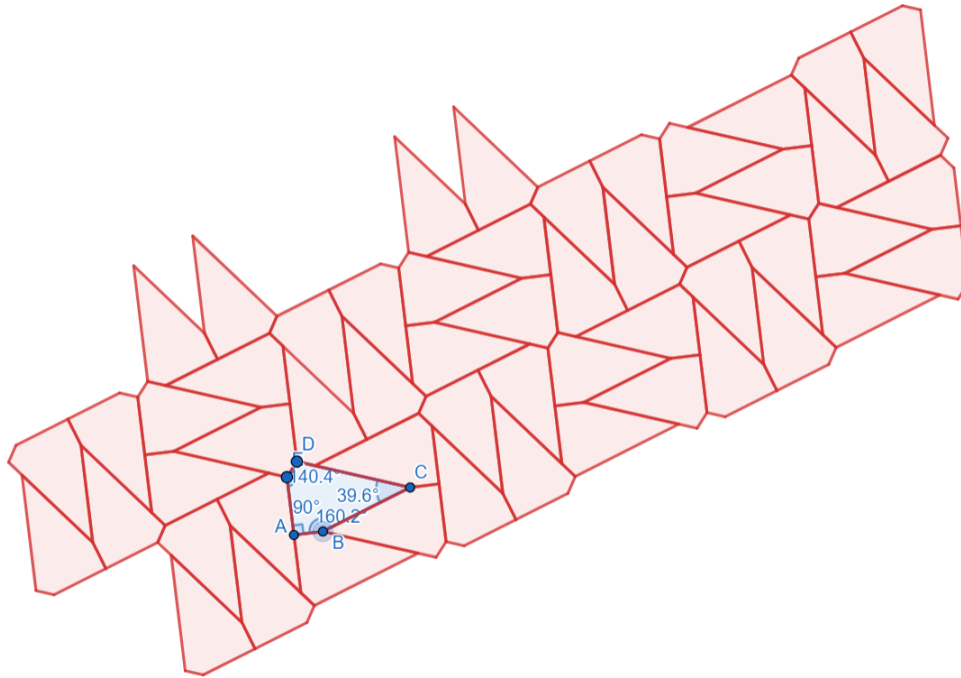


Figure A.8: Type 12 pentagonal tiling

1.9 Type 13

The conditions for this type tile are that the angles $B = E = 90^\circ$, $2A + D = 360^\circ$, and sides are $d = 2a = 2e$. We first created two points, A and E . Then we created a perpendicular line going through those same points so that angle E would be a right angle. Sides $a = e$, so a circle was created with center E going through A . Point D was then added on the intersection of that circle and the perpendicular line to ensure that condition. the radius of the circle was a , and side $d = 2a$, so point C was created on the circle with center D and radius $2a$. We adjusted the formula $2A + D = 360^\circ$ so that it would be $A = \frac{360^\circ - D}{2}$. We rotated point E by that same equation and made a line going through the resulting point and A . We added a perpendicular line going through this line and the point C . Point B was then created at the intersection of these two lines to have the condition of $B = 90^\circ$ hold. We then connected each of the points through segments before it was ready to be tessellated.

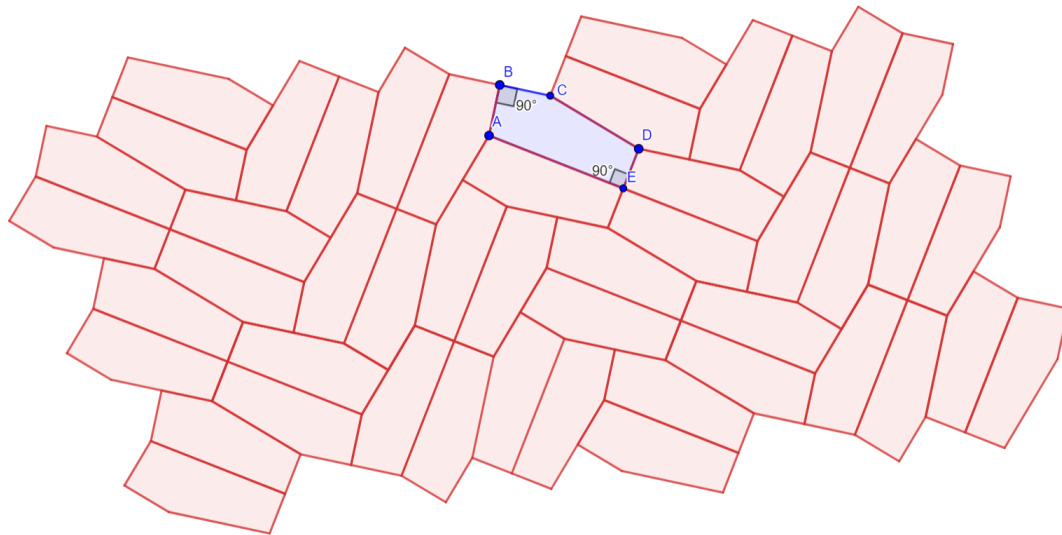


Figure A.9: Type 13 pentagonal tiling

1.10 Type 15

The conditions for this type of tile are that the angles $A = 150^\circ$, $B = 60^\circ$, $C = 135^\circ$, $D = 105^\circ$, $E = 90^\circ$ and sides are $a = c = e$ and $b = 2a$. First two points were created, D and E . E is a right angle, so we rotated point D by -90° since it is going clockwise. The resulting point would be point A . A is 150° so we then rotated point E by -150° . Since side $b = 2a$, we made a line going through the new point and A . We created a circle with center A and radius $2a$. the point intersection this line and circle was where point B was placed. Similarly, we rotate point A by -60° and created a line through this new point and B . Side $c = a$. Hence, we created a circle with center B and radius a . The intersection of the line and this circle is where point C was placed. Finally we rotate B by -135° and draw a line going through this point and C . Point D is already shown to be a point the line goes through. We then connected each point through segments before it was ready to be tessellated.

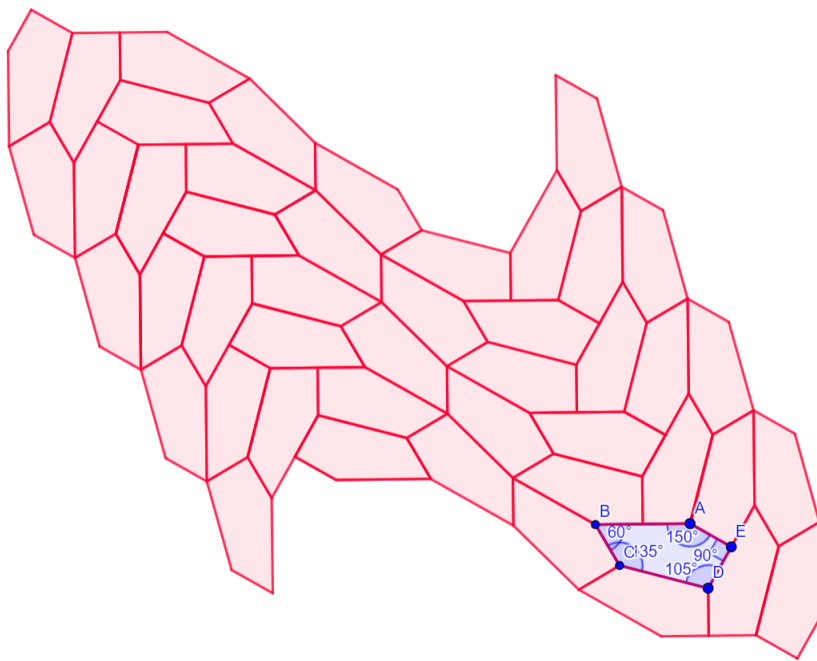


Figure A.10: Type 15 pentagonal tiling

BIOGRAPHICAL SKETCH

A 2018 graduate of UTRGV Mathematics and Science Academy, Ariana Hinojosa is an achiever who is committed to pursuing a career in lecturing at a university. Ariana received a Bachelor of Science with a major in applied mathematics in 2020; she received her Master of Science with a major in mathematics in May of 2022. She has a passion for the subject of mathematics and looks forward to sharing her knowledge of this subject with students as she enters the field of math and begins working toward her goal as a university lecturer.

Ariana worked as a graduate teaching assistant at UTRGV where she instructed a university course and helped other students master coursework and gain the study skills they needed to succeed in school. She found this work to be very rewarding and considered it to be a wonderful experience that would be beneficial in preparing her for her future career. During her high school years, Ariana served as a Mathematics tutor, helping students inside and outside the university master critical thinking, problem solving and making logical connections. She can be contacted through email at arianahinojosa23@gmail.com.