5-2022

Iterated Rascal Triangles

Jena M. Gregory

The University of Texas Rio Grande Valley

ITERATED RASCAL TRIANGLES

A Thesis

by

JENA M. GREGORY

Submitted in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

Major Subject: Mathematics

The University of Texas Rio Grande Valley

May 2022

ITERATED RASCAL TRIANGLES

A Thesis
by
JENA M. GREGORY

COMMITTEE MEMBERS

Dr. Brandt Kronholm
Chair of Committee

Dr. Jacob White
Committee Member

Dr. Timothy Huber
Committee Member

Dr. Sergey Grigorian
Committee Member

May 2022

ABSTRACT

Gregory, Jena M., Iterated Rascal Triangles. Master of Science (MS), May, 2022, 27 pp., 3 figures, references, 6 titles.

We introduce a sequence of number triangles, $\{R_i\}_{i=0}^{\infty}$, such that the entries of each share a common generalized recurrence relation. R_1 is the Rascal triangle and as i grows large, R_i becomes Pascal's triangle. For all i, we provide a combinatorial interpretation and find closed-term formulas for the entries of R_i, denoted by $\binom{n+d}{d}_i$. Our proofs rely on generating functions and other combinatorial arguments.

DEDICATION

To my husband Lance, for patiently riding this roller coaster I call mathematics.

ACKNOWLEDGMENTS

I want to acknowledge my children Alex, Susanna, Alyssa, and Ximena for all the ups and downs that was their mother earning a MS in Mathematics.

I'd like to acknowledge my parents Harold and Lois Christensen, who allowed me to dream big. I am grateful for my five brothers, Wynn, Wyatt, Jared, Jacob and Wade Christensen who were there in the middle of the night texting me as I struggled, encouraging me to never give up. I appreciate my in-laws Garald and Nyla Gregory for always making me feel like one of their own children.

I would like thank the math department at Frontier Junior High in Eagle Mountain, Utah. They are an amazing group of people that inspired, motivated and made me feel safe to be myself. I thoroughly enjoyed my time there.

I appreciate Dr. Jacob White for his help writing this thesis and being an excellent professor for several classes throughout my time here at UTRGV.

Lastly, I'd like to acknowledge Dr. Brandt Kronholm and Dr. Becky Koehler who took me under their wing and nurtured my talents and mathematical abilities into what they are today. I cannot even begin to count how many words they have heard me say or write, but each word they listened to was highly appreciated.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Mathematicians are very familiar with Pascal's triangle and have studied it extensively. It can be generated with generating functions and has a great many other properties and identities. Each entry of Pascal's triangle is calculated from previous entries using the recurrence relation $\binom{n+d}{d} = \binom{n-1+d}{d} + \binom{n+d-1}{d-1}$. This relation is often described as South = East + West. But what happens when non-mathematicians encounter Pascal's triangle? In 2010, middle school students Alif Anggoro, Eddy Liu, and Angus Tulloch [1] were challenged with determining the next row of numbers in an incomplete Pascal's triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Instead of writing the next row of Pascal's Triangle as expected, 1 4 6 4 1, they presented a new row, 1 4 5 4 1. They came up with this new row using what they called the “diamond formula"

$$\text{South} = \frac{(\text{West} \times \ \text{East}) + 1}{\text{North}} \tag{1.1}$$

where North, South, East, and West represent the locations on the number triangle as follows:

North

West East

South

With this formula, Anggoro, Liu, and Tulloch were able to construct a number triangle different than Pascal's triangle. They called it "the Rascal triangle".

The Rascal Triangle (R_1)

$$
\begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&&1&&&&&&&&&&& \\
&&&&&&&&&&1&&1&&&&&&&&&& \\
&&&&&&&&&1&&2&&1&&&&&&&&& \\
&&&&&&&&1&&3&&3&&1&&&&&&&& \\
&&&&&&&1&&4&&5&&4&&1&&&&&&& \\
&&&&&&1&&5&&7&&7&&5&&1&&&&&& \\
&&&&&1&&6&&9&&10&&9&&6&&1&&&&& \\
&&&&1&&7&&11&&13&&13&&11&&7&&1&&&& \\
&&&1&&8&&13&&16&&17&&16&&13&&8&&1&&& \\
&&1&&9&&15&&19&&21&&21&&19&&15&&9&&1&& \\
&1&&10&&17&&22&&25&&26&&25&&22&&17&&10&&1& \\
1&&11&&19&&25&&29&&31&&31&&29&&25&&19&&11&&1 \\
&&&&&&&&&&&\vdots&&&&&&&&&&&
\end{array}
$$

We see the first two diagonals of the Rascal triangle are the same as Pascal's triangle. The students realized they needed to prove all the entries in their triangle were integers. This was done with a short proof using basic algebra.

The Rascal triangle turns out to be sequence A077028 in the Online Encyclopedia of Integer Sequences [5].

A mathematics for liberal arts class, taught by Julian Fleron at Westfield State University, discovered a diamond pattern of their own while studying the Rascal triangle in 2015 [2].

They found that

$$\text{South} = \text{East} + \text{West} - \text{North} + 1. \tag{1.2}$$

Later we will show the diamond formula (1.1) equals Fleron's recurrence relation (1.2).

Our motivation for this thesis was the question, "is the Rascal Triangle part of a larger family of triangles?" Phil Hotckiss of Westfield State University and Fleron have presented one such family, Generalized Rascal Triangles [4], as discussed at the end of this thesis in "Continuing Research: Generalized Rascal Triangles". We offer a second answer and introduce a sequence of number triangles $\{R_0, R_1, R_2, \ldots\}$ in which the Rascal triangle, denoted R_1, is the second in the sequence. We denote the entries of R_i by $\binom{n+d}{d}_i$, where n is the n^{th} entry, d is the d^{th} diagonal, and i is the i^{th} iteration of triangles. We say this as "$n+d,\ i$, choose d".

CHAPTER II

GENERATING FUNCTIONS

Since we wondered if the Rascal triangle could be part of a larger family of number triangles, we experimented, just as Anggoro, Liu, and Tulloch did, and came up with a generating function in order to create such a family. Our result is as follows:

Definition 2.0.1. *Let d, i, n be non-negative integers. We define the generating function for the n^{th} entry of the d^{th} diagonal of the i^{th} number triangle R_i to be*

$$\sum_{n=0}^{\infty} \binom{n+d}{d}_i q^n = \frac{\sum_{j=0}^{i} \binom{j+d-(i+1)}{d-(i+1)} q^j}{(1-q)^{i+1}}. \tag{2.1}$$

Utilizing Definition 2.0.1 we present the next iteration of triangle, R_2.

Triangle R_2

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		16		19		16		6		1	
1		7		21		31		31		21		7		1

Note, the first three diagonals of R_2 are the same as Pascal's triangle, however, the rest of the diagonals are new sequences different than the Rascal triangle.

It is staight forward to see that the diagonals of the Rascal triangle are given by the following generating functions, For example,

For R_1 and $d = 1$

$$\sum_{n=0}^{\infty} \binom{n+1}{1}_1 q^n = 1 + 2q + 3q^2 + 4q^3 \cdots = \frac{1}{(1-q)^2}. \tag{2.2}$$

For $d = 2$

$$\sum_{n=0}^{\infty} \binom{n+2}{2}_1 q^n = 1 + 3q + 5q^2 + 7q^3 \cdots = \frac{1+q}{(1-q)^2} \tag{2.3}$$

For $d = 3$

$$\sum_{n=0}^{\infty} \binom{n+3}{3}_1 q^n = 1 + 4q + 7q^2 + 10q^3 \cdots = \frac{1+2q}{(1-q)^2} \tag{2.4}$$

In this manner, we can generalize the generating function for any diagonal d in R_1 as

$$\sum_{n=0}^{\infty} \binom{n+d}{d}_1 q^n = \frac{\binom{d-2}{d-2} + \binom{d-1}{d-2} q}{(1-q)^2} = \frac{1+(d-1)q}{(1-q)^2}. \tag{2.5}$$

In R_2 and for $d = 3$,

$$\sum_{n=0}^{\infty} \binom{n+3}{3}_2 q^n = 1 + 4q + 10q^2 + 19q^3 + 31q^4 \cdots = \frac{1+q+q^2}{(1-q)^3} \tag{2.6}$$

where the coefficients of the numerator are the first three entries of the 0^{th} diagonal of Pascal's Triangle 1,1,1.

For $d = 4$,

$$\sum_{n=0}^{\infty} \binom{n+4}{4}_2 q^n = 1 + 5q + 31q^2 + 53q^3 + 81q^4 \cdots = \frac{1+2q+3q^2}{(1-q)^3} \tag{2.7}$$

where the coefficients of the numerator are the first three entries of the 1^{st} diagonal of Pascal's Triangle 1, 2, 3. Hence, in R_2, the d^{th} coefficients of the numerator are the first three entries of the $(d-3)^{th}$ diagonal of Pascal's triangle. We see the generating function of the d^{th} diagonal in R_2 is:

$$\sum_{n=0}^{\infty} \binom{n+d}{d}_2 q^n = \frac{\binom{d-3}{d-3} + \binom{d-2}{d-3} q + \binom{d-1}{d-3} q^2}{(1-q)^3} \tag{2.8}$$

Motivated by lines (2.2) through (2.8), we established Definition 2.0.1,

CHAPTER III

RECURRENCE RELATION

We have found a recurrence relation similar to Fleron's Rascal triangle recurrence relation (1.2) for all $\binom{n+d}{d}_i$.

For any number triangle R_i in this sequence, each entry $\binom{n+d}{d}_i$, this rule established as follows: Set

$$South_i = \binom{n+d}{d}_i$$

so that

$$East_i = \binom{n-1+d}{d}_i$$

$$West_i = \binom{n+d-1}{d-1}_i$$

$$North_i = \binom{n-1+d-1}{d-1}_i$$

and

$$North_{i-1} = \binom{n-1+d-1}{d-1}_{i-1}.$$

Theorem 3.0.1. *For integers n, d, and, i,*

$$\binom{n+d}{d}_i = \binom{n-1+d}{d}_i + \binom{n+d-1}{d-1}_i - \binom{n-1+d-1}{d-1}_i + \binom{n-1+d-1}{d-1}_{i-1}. \quad (3.1)$$

If $n, d,$ or $i = 0$, then $\binom{n+d}{d}_i = 1$.

This has the appearance of Fleron's recurrence relation with the integer 1 replaced by $\binom{n-1+d-1}{d-1}_{i-1}$.

Proof. We consider the associated generating functions and apply Definition 2.0.1.

$$\sum_{n=0}^{\infty}\binom{n+d}{d}_i q^n = \sum_{n=0}^{\infty}\binom{n-1+d}{d}_i q^n + \sum_{n=0}^{\infty}\binom{n+d-1}{d-1}_i q^n - \sum_{n=0}^{\infty}\binom{n-1+d-1}{d-1}_i q^n + \sum_{n=0}^{\infty}\binom{n-1+d-1}{d-1}_{i-1} q^n \quad (3.2)$$

$$= q\sum_{n=0}^{\infty}\binom{n+d}{d}_i q^n + \sum_{n=0}^{\infty}\binom{n+d-1}{d-1}_i q^n - q\sum_{n=0}^{\infty}\binom{n+d-1}{d-1}_i q^n + q\sum_{n=0}^{\infty}\binom{n+d-1}{d-1}_{i-1} q^n. \quad (3.3)$$

$$= \frac{q\sum_{j=0}^{i}\binom{j+d-(i+1)}{d-(i+1)}q^j}{(1-q)^{i+1}} + \frac{\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+2)}q^j}{(1-q)^{i+1}} - \frac{q\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+2)}q^j}{(1-q)^{i+1}} + \frac{q\sum_{j=0}^{i-1}\binom{j+d-(i+1)}{d-(i+1)}q^j}{(1-q)^{i}} \quad (3.4)$$

where

$$\frac{q\sum_{j=0}^{i-1}\binom{j+d-(i+1)}{d-(i+1)}q^j}{(1-q)^{i}} = \frac{\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+1)}q^j}{(1-q)^{i}}. \quad (3.5)$$

Continuing from (3.4) and finding a common denominator of $(1-q)^{i+1}$, we have

$$= \frac{q\sum_{j=0}^{i}\binom{j+d-(i+1)}{d-(i+1)}q^j}{(1-q)^{i+1}} + \frac{\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+2)}q^j}{(1-q)^{i+1}} - \frac{q\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+2)}q^j}{(1-q)^{i+1}} + \frac{\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+1)}q^j}{(1-q)^{i+1}} - \frac{q\sum_{j=0}^{i}\binom{j+d-(i+2)}{d-(i+1)}q^j}{(1-q)^{i+1}}. \quad (3.6)$$

Collecting like terms in (3.6) to make use of the binomial identity $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$, we have

$$\frac{q\sum_{j=0}^{i}\left(\binom{j+d-(i+1)}{d-(i+1)} - \binom{j+d-(i+2)}{d-(i+2)} - \binom{j+d-(i+2)}{d-(i+1)}\right)q^j}{(1-q)^{i+1}} + \frac{\sum_{j=0}^{i}\left(\binom{j+d-(i+2)}{d-(i+2)} + \binom{j+d-(i+2)}{d-(i+1)}\right)q^j}{(1-q)^{i+1}} = \frac{\sum_{j=0}^{i}\binom{j+d-(i+1)}{d-(i+1)}q^j}{(1-q)^{i+1}}. \quad (3.7)$$

Thus,

$$\binom{n+d}{d}_i = \binom{n-1+d}{d}_i + \binom{n+d-1}{d-1}_i - \binom{n-1+d-1}{d-1}_i + \binom{n-1+d-1}{d-1}_{i-1}. \quad (3.8)$$

□

CHAPTER IV

A POLYNOMIAL FORMULA

As the entries of the Rascal triangle are not binomial coefficients, we wanted a method using only binomial coefficients in which to calculate all $\binom{n+d}{d}_i$. The result is as follows:

Proposition 4.0.1. *For n, d, and* $i > 0$ *each rascal coefficient*

$$\binom{n+d}{d}_i = \sum_{m=0}^{i} \binom{m+d-(i+1)}{d-(i+1)} \binom{n+i-m}{i}. \tag{4.1}$$

Proof.

$$\sum_{n=0}^{\infty} \binom{n+d}{d}_i q^n = \frac{\sum_{j=0}^{i} \binom{j+d-(i+1)}{d-(i+1)} q^j}{(1-q)^{i+1}} = \sum_{j=0}^{i} \binom{j+d-(i+1)}{d-(i+1)} q^j \times \sum_{k=0}^{\infty} \binom{k+i}{i} q^i \tag{4.2}$$

After distributing,

$$\begin{aligned} &= \sum_{k=0}^{\infty} \binom{k+i}{i} q^k + \sum_{k=0}^{\infty} \binom{1+d-(i+1)}{d-(i+1)} \binom{k+i}{i} q^{k+1} \\ &\qquad + \sum_{k=0}^{\infty} \binom{2+d-(i+1)}{d-(i+1)} \binom{k+i}{i} q^{k+2} + \\ &\qquad\qquad \cdots + \sum_{k=0}^{\infty} \binom{d-1}{d-(i+1)} \binom{k+i}{i} q^{k+i} \end{aligned} \tag{4.3}$$

Re-indexing (4.3) gives us the following,

$$= \sum_{k=0}^{\infty} \binom{k+i}{i} q^k + \sum_{k=0}^{\infty} \binom{1+d-(i+1)}{d-(i+1)} \binom{k+i-1}{i} q^k$$
$$+ \sum_{k=0}^{\infty} \binom{2+d-(i+1)}{d-(i+1)} \binom{k+i-2}{i} q^k + \sum_{k=0}^{\infty} \cdots + \binom{d-1)}{d-(i+1))} \binom{k}{i} q^k \quad (4.4)$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{i} \binom{m+d-(i+1)}{d-(i+1)} \binom{k+i-m}{i} q^n. \quad (4.5)$$

Then we have,

$$\sum_{k=0}^{\infty} \binom{n+d}{d}_i q^n = \sum_{k=0}^{\infty} \sum_{m=0}^{i} \binom{m+d-(i+1)}{d-(i+1)} \binom{k+i-m}{i} q^n. \quad (4.6)$$

Thus,

$$\binom{n+d}{d}_i = \sum_{m=0}^{i} \binom{m+d-(i+1)}{d-(i+1)} \binom{n+i-m}{i}. \quad (4.7)$$

□

CHAPTER V

DIAMOND PATTERN AND FLERON'S RECURRENCE RELATION

Proposition 5.0.1. *The triangles generated by the diamond formula* (1.1) *and Fleron's recurrence relation are equal* (1.2)*.*

Utilizing Proposition (4.7) we consider following example. For $i = 1$, the formula for a Rascal triangle coefficient is

$$\binom{n+d}{d}_1 = 1 + dn \tag{5.1}$$

Proof. From the formula in (4.7) we re-write the diamond formula (1.1) as

$$\frac{\left(\binom{n+d-1}{d-1}_1 \times \binom{n-1+d}{d}_1\right) + 1}{\binom{n-1+d-1}{d-1}_1}$$
$$= \frac{(1+dn-n)\ \times (dn-d+1)+1}{-n+dn-d+2} = 1 + dn = \binom{n+d}{d}_1 \tag{5.2}$$

We rewrite Fleron's recurrence relation (1.2) as

$$\binom{n-1+d}{d}_1 + \binom{n+d-1}{d-1}_1 - \binom{n-1+d-1}{d-1}_1 + 1$$
$$= [dn-d+1] + [1+dn] + [-n+dn-d+2]) + [1] = 1 + dn = \binom{n+d}{d}_1 \tag{5.3}$$

Thus, the triangles generated by the diamond formula (1.1) and Fleron's recurrence relation are equal (1.2). □

CHAPTER VI

LIMIT TO PASCAL'S TRIANGLE

We will show that as i grows large, the limiting number triangle in the sequence $\{R_i\}_{i=0}^{\infty}$ is Pascal's triangle.

We want to show that for any diagonal $d, n \leqslant i$, each entry of $\{R_i\}_{i=0}^{\infty}$ is a binomial coefficient. This will make each diagonal $d \leqslant i$ a "Pascal diagonal" and as $i \to \infty$, $\{R_i\}_{i=0}^{\infty}$ will limit to Pascal's triangle. First we show the following,

Proposition 6.0.1. *For $d \leqslant i$ or $n \leqslant i$ then $\binom{n+d}{d}_i = \binom{n+d}{d}$.*

We require the following result.

Proposition 6.0.2. *[6] For negative n and integer k, we have*

$$\binom{n}{k} = \begin{cases} (-1)^k \binom{-n+k-1}{k} & \text{if } k \geqslant 0 \\ (-1)^{n-k} \binom{-k-1}{n-k} & \text{if } k \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

We will prove Proposition 6.0.1 using Proposition 6.0.2 and the following identities:

$$\binom{a}{b}\binom{a-b}{c} = \binom{a}{c}\binom{a-c}{b}. \tag{6.2}$$

Chu-Vandermonde Identity

$$\binom{a+b}{c} = \sum_{k=0}^{c} \binom{a}{k}\binom{b}{c-k}. \tag{6.3}$$

Principle of Inclusion-Exclusion

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \delta_{i,j}. \tag{6.4}$$

Where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise.

Proof. Consider $d \leqslant i$. Then by (4.7) and Proposition (6.0.1)

$$\binom{n+d}{d}_i = \sum_{m=0}^{i}\binom{m+d-(i+1)}{d-(i+1)}\binom{n+i-m}{i}$$
$$= \sum_{m=0}^{i}(-1)^m\binom{i-d}{m}\binom{n+i-m}{i} = \sum_{m=0}^{i-d}(-1)^m\binom{i-d}{m}\binom{n+i-m}{i} \tag{6.5}$$

We compute:

$$\sum_{m=0}^{i-d}(-1)^m\binom{i-d}{m}\binom{n+i-m}{i} = \sum_{m=0}^{i-d}(-1)^m\binom{i-d}{m}\sum_{k=0}^{i}\binom{i-d-m}{k}\binom{n+d}{i-k} \tag{6.6}$$
$$= \sum_{m=0}^{i-d}\sum_{k=0}^{i}\binom{n+d}{i-k}(-1)^m\binom{i-d}{m}\binom{i-d-m}{k} \tag{6.7}$$
$$= \sum_{k=0}^{i}\sum_{m=0}^{i-d}\binom{n+d}{i-k}(-1)^m\binom{i-d}{k}\binom{i-d-k}{m} \tag{6.8}$$
$$= \sum_{k=0}^{i}\binom{n+d}{i-k}\binom{i-d}{k}\sum_{m=0}^{i-d}(-1)^m\binom{i-d-k}{m} \tag{6.9}$$
$$= \sum_{k=0}^{i}\binom{n+d}{i-k}\binom{i-d}{k}\delta_{k,i-d} \tag{6.10}$$
$$\tag{6.11}$$

$$= \binom{n+d}{i}\binom{i-d}{0}\delta_{0,i-d} + \cdots + \binom{n+d}{d}\binom{i-d}{i-d}\delta_{i-d,i-d} + \ldots$$
$$+ \binom{n+d}{0}\binom{i-d}{i}\delta_{i,0} = \binom{n+d}{d} \tag{6.12}$$

Line (6.6) uses the Chu-Vandermonde identity. Line (6.7) uses identity (6.2) to swap the roles of m and k, which also swaps the summations. Line (6.8) moves the left summation back to the middle, and line (6.9) uses the Principle of Inclusion-Exclusion. We see in (6.12) there is only one non-zero term in the sum, and it is $\binom{n+d}{d}$. Thus, for $d \leqslant i$

$$\binom{n+d}{d}_i = \binom{n+d}{d}. \tag{6.13}$$

□

Remark: We observe from (6.13) as each rascal coefficient in a diagonal $d \leqslant i$ is a binomial coefficient, then the entire diagonal is a Pascal diagonal. In this manner, as $i \to \infty$, every diagonal becomes a Pascal diagonal. Therefore, as i grows arbitrarily large, then the limit of $\{R_i\}_{i=0}^{\infty}$ becomes Pascal's triangle.

CHAPTER VII

COMBINATORIAL INTERPRETATION

The standard combinatorial interpretation of the binomial coefficient $\binom{n+d}{d}$ is that it counts the number of ways to choose an unordered subset of d elements from a set of $n+d$ elements. In this section we explore one possibility of what the Rascal coefficient $\binom{n+d}{d}_i$ counts.

We require a familiar definition.

Definition 7.0.1. *A* partition λ *of a positive number k is a finite non-increasing sequence of positive integers* $\lambda_1, \lambda_2, \ldots \lambda_r$ *such that* $\sum_{i=1}^{r} \lambda_i = k$.

Another interpretation of $\binom{n+d}{d}$ is the size of the collection of partitions of all non-negative integers not larger than $n \times d$ into at most d parts with each part not larger than n.

Definition 7.0.2. *Let* $Par(n,k)$ *be the set of integer partitions of at most n and parts of size at most k.*

We note the fact $|\mathrm{Par}(n,k)|$ is a binomial coefficient. Our combinatorial interpretation of $\binom{n+d}{d}_i$ when $i < d$ is as follows.

Definition 7.0.3. *For non-negative integers* d,i,n *with* $i < d$, *a* rascal partition *is any partition* λ *that satisfies the following:*

$$\lambda_1 \leqslant n \text{ and } \lambda_{d-i} \geqslant n-i, \tag{7.1}$$

where the number of parts is at most d and the largest part has size of at most n.

Proposition 7.0.4. *The rascal coefficients count the number of rascal partitions.*

We will give a combinatorial proof of this result, but first require the following definitions.

Definition 7.0.5. *Let $R_{i,d,n}$ be the set of rascal partitions with $\lambda_1 \leqslant n$ and at most d parts.*

For integer m, with $0 \leqslant m \leqslant i$, let

$$R_{i,d,n,m} = \{\lambda : \lambda \in R_{i,d,n}, \lambda_{d-i} = n-m\}. \tag{7.2}$$

Then

$$R_{i,d,n} = \bigcup_{m=0}^{i} R_{i,d,n,m} \tag{7.3}$$

and the sets $R_{i,d,n,m}$ appearing in the union are pairwise disjoint.

Definition 7.0.6. *Given a rascal partition $\lambda \in R_{i,d,n,m}$, define $f(\lambda) = (\mu_1, \ldots, \mu_{d-i-1})$ by the equation $\mu_j = \lambda_j - (n-m)$ for all j.*

We see that $f(\lambda)$ is obtained by deleting $n-m$ from the first $d-i-1$ parts of λ, and that $f(\lambda)$ has largest part at most m.

Definition 7.0.7. *Define $g(\lambda) = (\nu_1, \ldots, \nu_i)$ by $\nu_j = \lambda_{d-i+j}$ for all j. Thus, we are removing the first $d-i$ parts from λ, and $g(\lambda)$ has largest part at most $n-m$.*

Proof. Consider the function $F : R_{i,d,n,m} \to Par(m, d-i-1) \times Par(n-m, i)$ given by $F(\lambda) = (f(\lambda), g(\lambda))$. We show here that this function is a bijection.

We need to describe the inverse function $F^{-1} : Par(m, d-i+1) \times Par(n-m, i) \to R_{i,d,n,m}$. Given $\mu \in Par(m, d-i+1)$ and $\nu \in Par(n-m, i)$, let $\lambda = F^{-1}(\mu, \nu)$ be given by:

$$\lambda_j = \begin{cases} \mu_j + n - m & j < d-i \\ n-m & j = d-i \\ \nu_{j-d-i} & j > d-i. \end{cases} \tag{7.4}$$

Since $F_m(\lambda) = (\mu, \nu)$ is a bijection and since $|\text{Par}(n,k)| = \binom{n+k}{k}$, we conclude

$$|R_{i,d,n,m}| = |\text{Par}(m, d-i-1)||\text{Par}(n-m, i)|$$
$$= \binom{m+d-(i+1)}{d-(i+1)}\binom{n+i-m}{i}. \quad (7.5)$$

Putting it all together, and making use of (4.7), we obtain

$$|R_{i,d,n}| = \sum_{m=0}^{i} |R_{i,d,n,m}| = \sum_{m=0}^{i} \binom{m+d-(i+1)}{d-(i+1)}\binom{n+i-m}{i}$$
$$= \binom{n+d}{d}_i. \quad (7.6)$$

Thus, the rascal coefficients count the number of rascal partitions. □

Example 7.0.8. *Consider rascal partition (6,5,5,3,1) with $i = 2, d = 5$, and $n = 6$. The resulting Fererrs diagram is:*

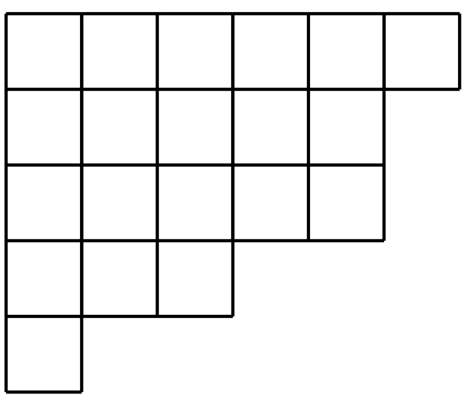

Figure 7.1: The Ferrers diagram of rascal partition (6,5,5,3,1)

To find m, we use the calculation,

$$\lambda_{d-i} = \lambda_{5-2} = \lambda_3 = 5$$

$$5 = n - m = 6 - m$$

$$m = 1.$$

Since $d-i=3$, we will look at the 3rd row and remove the rectangle generated by that row. As $m=1$, the shaded rectangle will be 1 box from the right.

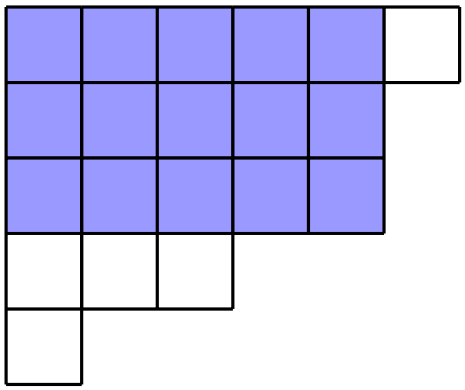

Figure 7.2: Rectangle generated by the 3^{rd} row in partition (6,5,5,3,1)

After removing the rectangle generated by the 3rd row, we are left with two partitions, μ and ν.

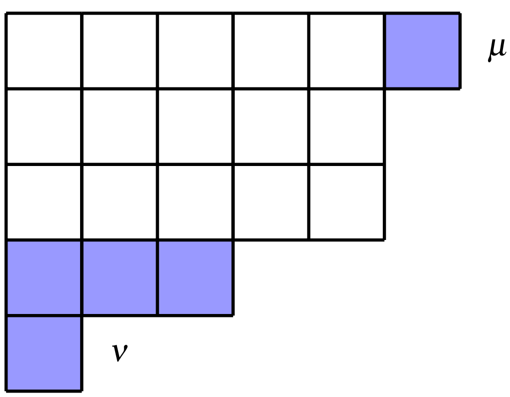

Figure 7.3: The two remaining partitions, μ and ν in partition (6,5,5,3,1)

Notice that $\mu_1 \leqslant n-(n-m)=m=1$, so that μ_1 has size of at most $m=1$, where the entire Ferrers diagram is part of the larger d x n, or 5 x 6 grid.

Example 7.0.9. *Consider the rascal coefficient $\binom{4+3}{3}_1 = 13$ where $n=4$, $d=3$ and $i=1$. We are looking at partitions of at most $d=3$ parts; $(\lambda_1,\lambda_2,\lambda_3)$, or (λ_1,λ_2), or (λ_1). Since $\lambda_{d-i} \geqslant n-i$,* *this means we will have $\lambda_2 \geqslant 3$, which means $\lambda_1 \geqslant 3$*

As $\lambda_1 \leqslant n = 4$ then $3 \leqslant \lambda_1 \leqslant 4$.

The corresponding set of rascal partitions is

$$\{(3,3),(3,3,1),(3,3,2),(3,3,3),(4,3),(4,3,1),(4,3,2),$$
$$(4,3,3),(4,4),(4,4,1),(4,4,2),(4,4,3),(4,4,4)\}. \quad (7.7)$$

Hence, there are 13 rascal partitions satisfying Definition (7.0.3) where the corresponding rascal coefficient $\binom{4+3}{3}_1 = 13$.

Example 7.0.10. *Consider the rascal coefficient $\binom{4+3}{3}_2 = 31$ where $n = 4$, $d = 3$, and $i = 2$. We are looking at the partitions of at most $d = 3$ parts; $(\lambda_1, \lambda_2, \lambda_3)$, or (λ_1, λ_2), or (λ_1). Since $\lambda_{d-i} \geqslant n - i$, this means we will have $\lambda_1 \geqslant 2$, As $\lambda_1 \leqslant n = 4$, then $2 \leqslant \lambda_1 \leqslant 4$.*

The corresponding set of rascal partitions is

$$\{(2),(2,1),(2,1,1),(2,2),(2,2,1),(2,2,2),(3),(3,1),(3,1,1),(3,2),$$
$$(3,2,1),(3,2,2),(3,3),(3,3,1),(3,3,2),(3,3,3),(4),(4,1),$$
$$(4,1,1),(4,2),(4,2,1),(4,2,2),(4,3),(4,3,1),(4,3,2),(4,3,3),$$
$$(4,4),(4,4,1),(4,4,2),(4,4,3),(4,4,4)\}. \quad (7.8)$$

Hence, there are 31 rascal partitions satisfying Definition (7.0.3) where the corresponding rascal coefficient $\binom{4+3}{3}_2 = 31$.

Example 7.0.11. *Let $i = 3$, $d = 3$, and $n = 4$, with $d \leqslant i$ and $k \leqslant d$. Then the corresponding rascal partitions are required to have largest part at most n, and to satisfy*

$$\{\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0 : \lambda_1 \leqslant n\}. \quad (7.9)$$

The following 35 corresponding rascal partitions are:

$$\{(0),(1),(1,1),(1,1,1),(2),(2,1),(2,1,1),(2,2),(2,2,1),$$
$$(2,2,2),(3),(3,1),(3,1,1),(3,2),(3,2,1),(3,2,2),(3,3),(3,3,1),$$
$$(3,3,2),(3,3,3),(4),(4,1),(4,1,1),(4,2),(4,2,1),(4,2,2),(4,3),$$
$$(4,3,1),(4,3,2),(4,3,3),(4,4),(4,4,1),(4,4,2),$$
$$(4,4,3),(4,4,4)\}. \quad (7.10)$$

Hence, when $d \leqslant i$, the rascal coefficient $\binom{4+3}{3}_3 = \binom{4+3}{3} = 35$.

CHAPTER VIII

CONTINUING RESEARCH: GENERALIZED RASCAL TRIANGLES

Phil Hotchkiss of Westfield State University, defines a Generalized Rascal Triangle (GRT) [4] as any number triangle whose diagonals are arithmetic sequences.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 3 & & & & & & & & \\
 & & & & & & & 3 & & 3 & & & & & & & \\
 & & & & & & 3 & & 4 & & 3 & & & & & & \\
 & & & & & 3 & & 5 & & 5 & & 3 & & & & & \\
 & & & & 3 & & 6 & & 7 & & 6 & & 3 & & & & \\
 & & & 3 & & 7 & & 9 & & 9 & & 7 & & 3 & & & \\
 & & 3 & & 8 & & 11 & & 12 & & 11 & & 8 & & 3 & & \\
 & 3 & & 9 & & 13 & & 15 & & 15 & & 13 & & 9 & & 3 & \\
3 & & 10 & & 15 & & 18 & & 19 & & 18 & & 15 & & 10 & & 3 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

We have extended our research into GRT's and found a similar infinite family of triangles for which we have a generating function, a recurrence relation, and a limiting triangle.

Definition 8.0.1. *The generating function for the d^{th} diagonal of any Generalized Rascal Triangle is*

$$\sum_{n=0}^{\infty} \binom{n+d}{d}_{i,(k,c)} q^n = \frac{c\sum_{j=0}^{i-1} \binom{j+d-(i+1)}{d-(i+1)} q^j - c\binom{d-1}{d-i} q^i + k\binom{d}{d-i} q^i}{(1-q)^{i+1}}. \tag{8.1}$$

The recurrence relation for GRT's is identical to the Rascal triangle recurrence relation. When $k = c = 1$, we have the Rascal triangle. For any symmetric Generalized Rascal Triangle we have

$$\binom{n+d}{d}_{i,(k,c)} = \binom{n-1+d}{d}_{i,(k,c)} + \binom{n+d-1}{d-1}_{i,(k,c)} - \binom{n-1+d-1}{d-1}_{i,(k,c)} + \binom{n-1+d-1}{d-1}_{i-1,(k,c)} \quad (8.2)$$

We can also show the limiting number triangle for any GRT $R_{i,(k,c)}$ is as follows:

Theorem 8.0.2.

$$\lim_{i\to\infty} \binom{n+d}{d}_{i,(k,c)} = c \times \binom{n+d}{d}$$

In other words, for any symmetric Generalized Rascal Triangle $R_{i,(k,c)}$, the limiting number triangle is a "constant multiple of Pascal's triangle", where each entry is $c \times \binom{n+d}{d}$.

$$\lim_{i\to\infty} R_{i,(k,c)} = \text{"}c \times \textit{Pascal's triangle}\text{"}.$$

The limiting triangle is "3×Pascal's triangle", where each Pascal's triangle entry is multiplied by $c = 3$.

$R_{5,(1,3)}$

$$\begin{array}{c}
3\\
3 \quad 3\\
3 \quad 6 \quad 3\\
3 \quad 9 \quad 9 \quad 3\\
3 \quad 12 \quad 18 \quad 12 \quad 3\\
3 \quad 15 \quad 30 \quad 30 \quad 15 \quad 3\\
3 \quad 18 \quad 45 \quad 60 \quad 45 \quad 18 \quad 3\\
3 \quad 21 \quad 63 \quad 105 \quad 105 \quad 63 \quad 21 \quad 3\\
3 \quad 24 \quad 84 \quad 168 \quad 210 \quad 168 \quad 84 \quad 24 \quad 3\\
3 \quad 27 \quad 108 \quad 252 \quad 378 \quad 378 \quad 252 \quad 108 \quad 27 \quad 3\\
\vdots
\end{array}$$

CHAPTER IX

CONCLUSION

The Rascal triangle was discovered when three middle school students were given an incomplete number triangle and asked to figure out the next row and unexpectedly discovered a new number triangle. They were expected to find Pascal's triangle, but instead created something brand new and previously not considered.

As mathematicians, we often get comfortable with mathematics, especially with results that we are familiar with. These students showed that with new perspective and some creativity, mathematics can still have unexpected results. Hotchkiss' and Fleron's students have also studied the Rascal triangle and come up with interesting and strange patterns and formulas.

One such pattern is the "T-Meg" rule [3]

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 5 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 7 & & 7 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 9 & & 10 & & 9 & & 6 & & 1 & & \\
 & 1 & & 7 & & 11 & & 13 & & 13 & & 11 & & 7 & & 1 & \\
1 & & 8 & & 13 & & 16 & & 17 & & 16 & & 13 & & 8 & & 1 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

where $\binom{n+d}{d}_1 = \binom{n-1+d-1}{d-1}_1 + \binom{n+1+d-3}{1}_1 + \binom{n+2+d-3}{0}_1$. For example $\binom{4+4}{4}_1 = \binom{3+3}{3}_1 + \binom{5+1}{1}_1 +$

$\binom{6+1}{0}_1 = 10 + 6 + 1 = 17.$

The “T-Meg" rule only works in R_1 or $R_{1,(k,c)}$. Is there a “T-Meg" rule lurking in R_2, R_3 or beyond ? As i grows large, does “T-Meg" turn into a familiar Pascal identity? Is there another unknown pattern in the Rascal triangle? As we can see, iterated Rascal triangles have considerable possibility for future research.

REFERENCES

[1] A. ANGGORO, E. LIU, AND A. TULLOCH, *The rascal triangle*, The College Mathematics Journal, 41 (2010), pp. 393–395.

[2] J. FLERON, *Fresh perspectives bring discoveries*, Math Horiz., (2017).

[3] P. K. HOTCHKISS, *Student inquiry and the rascal triangle*, arXiv:1907.07749.

[4] P. K. HOTCHKISS, *Generalized rascal triangles*, Journal of Integer Sequences, 23 (2020).

[5] OEIS FOUNDATION INC., *The on-line encyclopedia of integer sequences*, 2022.

[6] R. SPRUGNOLI, *Negation of binomial coefficients*, Discrete Math, (2008), pp. 5070–5077.

BIOGRAPHICAL SKETCH

Jena M. Gregory was born in Murray, Utah. After graduating from American Fork High School in 1992, she attended Brigham Young University. She graduated in 1996 with a Bachelors in Art in Mathematics Education and was an educator in the Utah public school system for 16 years. Jena is licensed for grades 6-12 with an endorsement to teach any secondary mathematics course.

While teaching in 2017, she attended the University of Texas Rio Grande Valley and graduated in 2022 with a Masters of Science in Mathematics.

Permanent address: 3437 Country Club Dr. N Edinburg, TX 78542

Email address: mathmom27@gmail.com