5-2022

A Decomposition Formula for the Multi-Soliton Solutions to the 'Good' Boussinesq Equation

Aldo Gonzalez
The University of Texas Rio Grande Valley

A DECOMPOSITION FORMULA FOR THE MULTI-SOLITON SOLUTIONS TO THE ‘GOOD’ BOUSSINESQ EQUATION

A Thesis

by

ALDO GONZALEZ

Submitted in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

Major Subject: Mathematics

The University of Texas Rio Grande Valley

May 2022

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COMMITTEE MEMBERS

Dr. Vesselin Vatchev
Chair of Committee

Dr. Sergey Grigorian
Committee Member

Dr. Dambaru Bhatta
Committee Member

Dr. Zhijun Qiao
Committee Member

May 2022

ABSTRACT

Gonzalez, Aldo, A Decomposition Formula for the Multi-Soliton Solutions to the 'good' Boussinesq Equation. Master of Science (MS), May, 2022, 36 pp., 3 figures, references, 21 titles.

In this thesis, we relate multi-soliton waves generated by the 'good' Boussinesq equation to the distribution functions in the classical linear Schrödinger equation. The linear Schrödinger equation describes the distribution of a particle or particles in a particular environment. The Schrödinger equation is linear, the superposition principle of the solutions, especially the eigenfunctions is nonlinear and we will show that we may observe similar behavior in the solutions of the Boussinesq equations for soliton waves. The work extends the study of two-soliton solutions to the Boussinesq equation to the case of three-soliton solutions. The methods can be easily extended to n-solitons.

DEDICATION

This thesis is dedicated to my family. My heartfelt gratitude goes out to my wonderful parents, Ramon Gonzalez and Patricia Moreno who motivated nonstop and comforted me all the way until the end. My dear sister, Debanhi, who was a role model in furthering my education. I could not have made this far without them. Forever in my heart, forever in these pages.

ACKNOWLEDGMENTS

First and foremost, I'd want to convey my heartfelt thanks to my mentor, Dr. Vesselin Vatchev, for his unwavering support of my studies, his numerous hours of reflection, reading, and encouragement, and most importantly, his patience throughout the whole process. Throughout the research and writing of my thesis, his advice was invaluable. For my studies, I could not have asked for a more qualified advisor and mentor.

Besides my advisor, I would like to thank the rest of my thesis committee: Dr. Sergey Grigorian, Dr. Dambaru Bhatta, and Dr. Zhijun Qiao, for their encouragement, insightful comments, hard questions, and who were more than generous with their wisdom and precious time.

Lastly, I would like thank my friends and coworkers: Ariana Hinojosa, Brenda Garcia, Brayton Wario, and Erika Olivares. They were a bright light when I needed them most throughout the dark times in my graduate studies.

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CHAPTER I

INTRODUCTION

One-dimensional weakly nonlinear dispersive water waves are described using the Boussinesq equation. In other words, we have a non-linear equation that can be used to study non-linear waves in shallow water. Further, there are various applications in different branches in physics in which the Boussinesq equation is used. Another model for water waves propagation is the nonlinear Schrödinger equation. It models evolution of a one-dimensional packet of surface waves on sufficiently deep water [2]. So we have a wave equation related to the classical Schrödinger equation of quantum mechanics with which we can predict the future behavior of a dynamic system. In other words, we have a wave equation in terms of the wavefunction Ψ which can predict the distribution of the results.

In our work, one of the goals is to relate the classical linear Schrödinger equation to models describing behavior of multi-soliton waves generated by the 'good' Boussinesq equation. The linear Schrödinger equation describes the distribution of a particle or particles in a particular environment. The Schrödinger equation is linear, the superposition principle of the solutions, especially the eigenfunctions is nonlinear and we will show that we may observe similar behaviour in the solutions of the Boussinesq equations for soliton waves. The probability density is as follows

$$|\Psi(x,t)|^2 = \psi_1^2 + \psi_2^2 + 2\psi_1\psi_2 cos((\omega_2 - \omega_1)t).$$

Boussinesq equation describes the propagation of surface waves in a narrow channel. We consider the ‘good’ Boussinesq equation which will be referred as "gB equation" moving forward [12].

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3}u_{xxx} = 0.$$

gB has a multi-soliton solutions. A soliton concept refers to a particle-like nature when referring to waves. What makes these surface waves so unique is that after colliding with one another they are able to preserve their respective shapes and speeds after collision [13]. In this work we will discuss relation between the multi-soliton solutions of gB, represented in a form like energy functions and the classical energy functions for the Schrondinger equation.

In chapter 2, we introduce the classical linear Schrödinger equation. In chapter 3, we introduce the ‘good’ Boussinesq equation and discuss multi-soliton solutions obtained by the Wronskian method. Advanced results from linear algebra are included in chapter 4. The main results of the thesis are reported in chapter 5.

CHAPTER II

THE CLASSICAL SCHRÖDINGER EQUATION

2.1 The Schrödinger Equation

The nonlinear Schrödinger equation can model evolution of a one-dimensional packet of surface waves on sufficiently deep water [2]. So we have a wave equation of quantum mechanics [3] with which we can use to predict the future behavior of a dynamic system. In other words, we have a wave equation in terms of the wavefunction Ψ which can predict the distribution of the results. In our work, one of the goals is to relate the classical linear Schrödinger equation to models describing behavior of multi-soliton waves generated by the 'good' Boussinesq equation. The linear Schrödinger equation describes the distribution of a particle or particles in a particular environment. The Schrödinger equation is linear, the superposition principle of the solutions, especially the eigenfunctions is nonlinear and we will show that we may observe similar behaviour in the solutions of the Boussinesq equations for soliton waves.

Time Dependent Schrödinger Equation for one spatial dimension defined in [15]:

$$\frac{-\hbar^2}{2m}\frac{\partial\Psi(x,t)}{\partial x^2}+V(x)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t} \tag{2.1}$$

where we have Planck's constant denoted by $\hbar$ and divided by 2π, $i=\sqrt{-1}$. our mass of particle m, and V as the potential energy of the particle.

In addition, the time independent equation for Schrödinger equation as defined in [15],

$$\text{Time Evolution } \; H\Psi = i\hbar\frac{\partial \Psi}{\partial t} \tag{2.2}$$

where H is a Hamiltonian operator. The Hamiltonian operator, or simply Hamiltonian, is related to the total energy of the system.

$$\text{Time Independent Equation } \; \frac{-\hbar^2}{2m}\frac{\partial^2\Psi(x)}{\partial x^2} + V(x)\Psi(x) = E\Psi(x) \tag{2.3}$$

Which can be solved by performing separation of variables. For the time-independent Schrödinger equation solutions do in fact exist; however, only for certain values of energy. In order to find the specific values for E we must operate on the wavefunction with the Hamiltonian operator. (2.3) is part of the SE, where Schrödinger considers a moment t for which the time derivative equals the second derivative (2.2) has time derivative.

2.2 Superposition Principle

A particle can be in a superposition of states that have different energies. The superposition mentioned will still be a solution of the time-dependent Schrödinger equation. However, it should be noted that it is not a solution of the time-independent Schrödinger equation since we have two different E's Looking at the time dependence, we consider two energy states in an infinite well where the superposition is as follows [5]:

$$\Psi(x,t) = \psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t} \tag{2.4}$$

where we have that $\omega_1 = \frac{E_1}{\hbar}$ and $E_1 = \frac{\hbar^2}{8mL^2}$ as well as $\omega_2 = \frac{E_2}{\hbar}, E_2 = 4E_1$ where the frequencies of the two terms mentioned are different, and oscillate in and out of phase. Additionally, ψ_1 and ψ_2 are solutions of (2.3) for different values of E.

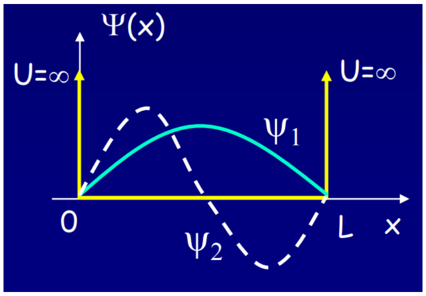

Figure 2.1 Infinite Well Superposition.

The probability density is as follows [5]

$$|\Psi(x,t)|^2 = \psi_1^2 + \psi_2^2 + 2\psi_1\psi_2 cos((\omega_2 - \omega_1)t). \tag{2.5}$$

2.3 Schrödinger Equation for the Hydrogen Atom

This is a particular example where the focus is on the hydrogen atom, which is regarded as the simplest amongst all the atoms. To describe this singular atom, it is simply an atom that contains one proton and one electron. There we can see an electron moving around a proton. From there, we can take a look at its electron in the spherical polar coordinates when developing the Schrödinger Equation. For this, we have the potential energy term, V, so that we may describe the attraction between proton and electron. For this, the following Coulomb term is used [9]:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r} \tag{2.6}$$

where we have that r is the distance between said electron and proton. Leading to the following form of the Schrödinger equation for the hydrogen atom [14]

$$\left[\frac{-\hbar^2}{2m}\nabla^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}\right]\Psi = E\Psi. \tag{2.7}$$

As well as another form of the Schrödinger Equation for the hydrogen atom [9]

$$\hat{H}(r,\theta,\varphi)\Psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi). \tag{2.8}$$

CHAPTER III

BOUSSINESQ EQUATION

One-dimensional weakly nonlinear dispersive water waves are described using the Boussinesq equation. In other words, we have a non-linear equation that can be used to study non-linear waves in shallow water. Further, there are various applications in different branches in physics in which the Boussinesq equation is used.

Boussinesq equation describes the propagation in a narrow channel under some disturbance, and we consider multi-soliton solutions. These multi-soliton solutions will represent in a form like energy functions for the Schrödinger equation. The Boussinesq equation is derived from the Euler's equations as it describes water waves. Note that their is a difference between deep water versus shallow water: large wavelength versus small wavelength [4]. Also, waves do behave differently in deep or shallow water where the ocean bottom does not affect the wave while it does in shallow water [16]. The Euler's equations without surface tension [17],[6],

$$
\begin{aligned}
\text{Free surface condition: } p = 0, \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w, \text{on } \ z = \eta(x,y,t),\\
\text{Momentum equation: } \frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + g\hat{z} = 0,\\
\text{Continuity equation: } \operatorname{div}(\mathbf{u}) = \nabla \cdot \mathbf{u} = 0,\\
\text{Bottom boundary condition: } \mathbf{u}\cdot\nabla(z + h(x,y)) = 0, \text{on } \ z = -h(x,y),
\end{aligned}
\tag{3.1}
$$

where pressure and vertical displacement of free surface are denoted by p and η, respectively. As well as $\mathbf{u} = (u, v, w)$ as the three-dimensional velocity, ρ as density, g acceleration from gravity, and $h(x,y)$ the bottom topography.

Furthermore, we also consider a solitary waves. For nonlinear partial differential equations that describes wave propagation, we have solitary waves. A solitary wave may be thought of as a local containment of the wave field's energy [7]. As well as being a nonlinear wave with the following properties:

- Localized wave propagates without change of its properties (shape, velocity, etc.)
- Localized waves are stable against mutual collisions and retain their identities

This soliton concept refers to a particle-like nature when referring to waves. What makes these surface waves so unique is that after colliding with one another they are able to preserve their respective shapes and speeds after collision or interaction [13],[8]. Further, not all nonlinear partial differential equations have soliton solutions. Although finding one definite definition of what a soliton is difficult, it remains an intriguing topic. A soliton is smooth, bounded, and with asympthotic decay in the space variable at infinity. Where an example of the following can be seen below for Nonlinear Schrödinger equation [20].

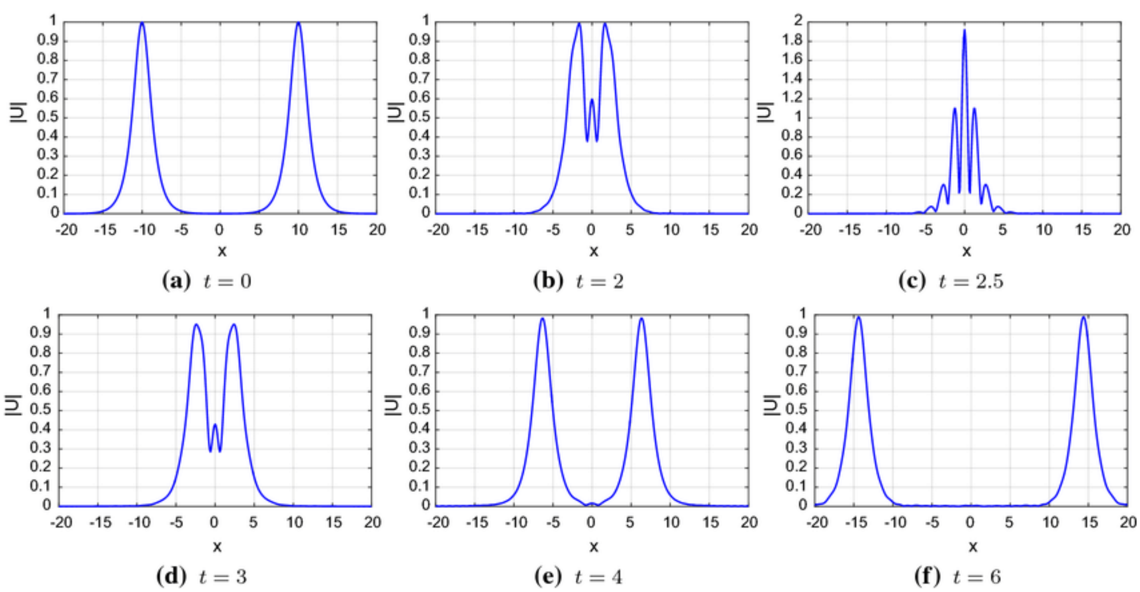

Figure 3.1 Example of soliton-solution at different times *t* for a Nonlinear Schrödinger Equation.

3.1 The Boussinesq Equation

The general form of Boussinesq equation that is derived from Euler's equation has the following form [12]

$$u_{tt} + a_1 u_{xx} + a_2 (u^2)_{xx} + a_3 u_{xxx} = 0, \tag{3.2}$$

where a_i, $i = 1,2,3$ are real numbers and $a_2 a_3 \neq 0$. In the case $a_3 > 0$ it is known as the 'good' Boussinesq equation and is equivalent to

$$v_{tt} + (v^2)_{xx} + v_{xxx} = 0, \tag{3.3}$$

under the transformation

$$u(x,t) = -\frac{a_1}{2a_2} + \frac{a_3}{a_2} v(x)\sqrt{a_3 t}. \tag{3.4}$$

It should be noted that (3.4) looks like the “real" Schrödinger equation.

Thus, we can consider the 'good' Boussinesq equation which will be referred as "gB equation" moving forward.

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3} u_{xxx} = 0. \tag{3.5}$$

which has the following Lax pair, thus we have an integrable system.

$$\phi_{xxx} + u\phi_x + w\phi = \lambda\phi, \tag{3.6}$$

$$\phi_t = -\phi_{xx} - u\phi. \tag{3.7}$$

Next, we discuss how to obtain multi-soliton solutions to gB.

3.2 The Wronskian Method for Solutions

In this section, we are interested in multi-soliton solutions, which can be obtained by the general method of the Wronskian determinant. The Wronskian is regarded as an important mathematical instrument that provides explicit solutions and a better understanding. Where such properties from the explicit solutions can allow us to compute the solitons, negatons, positons, and complexitons for Boussinesq equation [12]. However, the main focus of this thesis will be solely solitons as mentioned in chapter 3. Thus, we use the following notation [12]

$$W(\phi_1,\phi_2,...,\phi_N)=\begin{vmatrix}\phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)}\\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)}\\ \vdots & \vdots & \ddots & \vdots\\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)}\end{vmatrix} \tag{3.8}$$

where $\phi_i^{(0)}=\phi_i$, $\phi_i^{(j)}=\frac{\partial^j}{\partial x^j}\phi_i$, $j\geq 1$, $1\leq i\leq N$, and then the solution of gB is [12]

$$v=2(\log W)_{xx}.$$

In the case of two solitons, the solution can be defined by the following Wronskian determinant

$$W(f,g)=\begin{vmatrix} f & g\\ f_x & g_x\end{vmatrix}=fg_x-f_xg, \tag{3.9}$$

with two generating functions f and g defined as

$$f(x,t)=e^{p_1x-k_1t}+e^{-p_1x-k_1t},$$

$$g(x,t)=e^{p_2x-k_2t}+e^{-p_2x-k_2t}.$$

for real p_1,p_2,k_1,k_2. In chapter 5, we consider the case n=3, i.e. three-soliton solution.

3.3 Properties of 2-Soliton solutions for the Good Boussinesq Equation

Next, for the 'good' Boussinesq equation the functions f and g have the following form:

$$
\begin{aligned}
f(x,t) &= e^{m_1(x-m_1t)} + e^{n_1(x-n_1t)}, \\
g(x,t) &= e^{m_2(x-m_2t)} + e^{n_2(x-n_2t)},
\end{aligned}
\tag{3.10}
$$

where we have that m_1, m_2 are solutions to the equation $\mu^3 - \frac{3}{4}\mu = \lambda_j$, $\lambda_j \in R$ and $|\lambda_j| \leq \frac{1}{4}$. First, we consider a result for the KdV equation,

$$
u_t - 6uu_x + u_{xxx} = 0
$$

where the equation looks like a real version of the Schrödinger equation dependent on time. The following lemma is for the KdV equation.

Lemma A1 (Lemma 2.1, [18]). *If* $\Psi_1 = \frac{s_2}{K}$ *and* $\Psi_2 = \frac{c_1}{K}$ *then*

$$
k = 2(lnK)_{xx} = 2p_1^2(p_2^2 - p_1^2)\Psi_1^2 + 2p_2^2(p_2^2 - p_1^2)\Psi_2^2
$$

and Ψ_i *are solutions of the eigenvalue problem* $\Psi_{xx} = (\lambda^2 - k)\Psi$ *for the eigenvalues* p_1 *and* p_2 *correspondingly.*

It should be noted that the results for the KdV were added because the equation for the KdV looks like a a real version of the Schrödinger equation independent of time and the solution resembles the wavefunctions and the distribution (2.7).

Further, the results for the two-soliton solution for the Boussinesq equation has a similar structure as the superposition for two particles.

Theorem A2 (Theorem 3.1, [18]). *For two-soliton solution of gB the following result is reported in [18]. For functions f and g, defined in (3.10), a two soliton solution $b = 2(lnV)_{xx}$ for the gB has the following characterization.*

Let the functions $\phi_1 = \frac{g}{V}$ and $\phi_2 = \frac{f}{V}$ be two different solutions to the wave equation $\phi_t = \phi_{xx} + v\phi$ then

$$v = 8p_1^2 e^{-2\theta_1}(D\frac{g_x}{g} + L)\phi_1^2 - 8\varepsilon p_2^2 e^{-2\theta_2}(D\frac{f_x}{f} + L)\phi_2^2 \tag{3.11}$$

This is similar to the Superposition Principle for the time varying Schrödinger Equation because the functions ϕ_1 and ϕ_2 are solutions of the real version of the Schrödinger equation. Furthermore:

The functions $\Phi_i = e^{-\theta_i}\phi_i$ are solutions of the transport-eigenvalue problem $\Phi_t + 2R\Phi_x = \Phi_{xx}(-\lambda^2 + v)\Phi$ with R=M and $\lambda = p_1$ for Φ_1, and $R = \frac{n_1+n_2}{2}$ and $\lambda = p_2$ for Φ_2

$$b = 8p_1^2(D\frac{g_x}{g} + L)\Phi_1^2 - 8\varepsilon p_2^2(D\frac{f_x}{f} + L)\Phi_2^2 \tag{3.12}$$

The similarity to the Schrödinger equation is further extended in the following result from [18].

Theorem A3 (Theorem 3.2, [18]). *] Let $\sigma = \sqrt{P_{11}P_{12}}$, $b_2 = \frac{p_2^2}{2}(\sqrt{|P_{22}|} - \sqrt{|P_{21}|})^2$ and in case of chasing solitons, $\varepsilon = 1$, $b_1 = b_1^c = \frac{p_1^2}{2}(\sqrt{P_{12}} - \sqrt{P_{11}})^2$ and in case of solitons traveling in opposite directions, $\varepsilon = -1$, $b_1 = b_1^o = \frac{p_1^2}{2}(\sqrt{P_{12}} + \sqrt{P_{11}})^2$ then*

$$b = 2\sigma p_1^2\left(\frac{s_2^*}{B}\right)^2 + 2\sigma p_2^2\left(\frac{c_1^*}{B}\right)^2 + (b_2 - b_1)\frac{1}{B^2}. \tag{3.13}$$

The goal of the preceding research was to study the properties of 2-soliton solutions. The two-soliton solution for the Boussinesq looks like the superposition for two eigenstates in the Schrödinger equation. Next, the results will be extended to 3-soltions where the same idea will be followed. We will develop the method for 3-solitons in chapter 5. Because of the usage of the Wronskian determinants, we will first consider some results from linear algebra.

CHAPTER IV

PROPERTIES OF DETERMINANTS

The following properties from linear algebra were required in order to prove the properties of 3-soliton solutions for the gB equation. Each section in this chapter pertains to a specific technique or method used to get a better understanding of a determinant—e.g., differentiating or expanding. The methods and/or identities that follow had they're own specific purpose in furthering the proof which will be addressed later.

4.1 Generalized Laplace Expansion

Where D is a determinant of the order n and $1 \leq r \leq n$, by $D\begin{bmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{bmatrix}$ will be denoted the minor of the order r lying in the intersection of i_1-th, i_2-th,...,i_r-th rows and j_1-th, j_2-th,...,j_r-th columns of D and by $\bar{D}\begin{bmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{bmatrix}$ will be denoted its complement minor, that is, the minor of the order $n-r$ lying in the intersection of remaining $n-r$ rows and columns of D. If $n=r$ we then put $\bar{D}=1$ [10]. For the set $\{i_1,...,i_s\}$ of natural numbers we use abbreviation $I_s = i_1+\cdots+i_s$. For two natural numbers j and k we define

$$j(k) = \begin{cases} j, & j \leq k, \\ j-1, & j > k \end{cases}.$$

Which leads us to the following theorem, where the determinant of D of the order n may be expanded along i_1-th,...,i_r-th rows, where $1 \leq i_1 < \cdots < i_r$ are arbitrary.

Theorem B1 (Theorem 2.1, [10])**.** *Let D be a determinant of the order n, then for every $1 \leq i_1 < i_2 \cdots < i_r \leq n$ holds*

$$D = \sum_{1 \leq j_1 < \cdots < j_r \leq n} (-1)^{I_r + J_r} \begin{bmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{bmatrix} \cdot \bar{D} \begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_r \end{bmatrix},$$

where $j_1, \ldots, j_r$ run through all subsets of $1, 2, \ldots, n$ with exactly r elements. The determinant D of the order n may be expanded along i_1-th,...,i_r-th rows, where $1 \leq j_1 < \ldots < i_r \leq n$ are arbitrary.

4.2 Sylvester's Determinantal Identity

When evaluating specific sorts of determinants, the Sylvester's determinantal identity is a useful tool. The identity was first proposed in 1851 by an English mathematician named James Sylvester. It is incredibly useful as it allows one to express a determinant that is made up of bordering determinants in terms of the original one [11]. In order for this to work we must consider this a square matrix $M = \left(a_{ij}\right)$, of order n over K, a commutative field . Its determinant is then denoted as $\det M$, for an integer $t, 0 \leq t \leq n-1$, and with the following integers i, j with $t < i, j \leq n$, so that we can define our determinant $a_{i,j}^{(t)}$ as follows

$$a_{i,j}^{(t)} = \left| \begin{array}{ccc|c} a_{11} & \cdots & a_{1t} & a_{1j} \\ a_{21} & \cdots & a_{2t} & a_{2j} \\ \vdots & & \vdots & \vdots \\ a_{t1} & \cdots & a_{tt} & a_{tj} \\ \hline a_{i1} & \cdots & a_{it} & a_{ij} \end{array} \right|, \text{ for } 0 < t \leq n-1 \text{ and } a_{i,j}^{(0)} = a_{ij} \tag{4.1}$$

which is a determinant of order $t+1$ from matrix M by extending its leading principal submatrix of order t with the row i and the column j of M [11].

Theorem B2 (Theorem 1, Sylvester's Identity, [11]). *Let M be a square matrix of order n and t an integer, $0 \leq t \leq n-1$. Then, the following identity holds*

$$detM \cdot \left[a_{t,t}^{(t-1)}\right]^{n-t-1} = \begin{vmatrix} a_{t+1,t+1}^{(t)} & \cdots & a_{t+1,n}^{(t)} \\ \vdots & & \vdots \\ a_{n,t+1}^{(t)} & \cdots & a_{n,n}^{(n)} \end{vmatrix},$$

with $a_{0,0}^{(-1)} = 1$.

4.3 Plucker's Relation

Next, we take a look at Plucker's relation.

Corollary B3 (Corollary 2.2, [21]). *Let M be a $n \times (n-2)$ matrix and a, b, c and d four n-order column vectors. then*

$$\begin{vmatrix} M & a & b \end{vmatrix} \cdot \begin{vmatrix} M & c & d \end{vmatrix} - \begin{vmatrix} M & a & c \end{vmatrix} \cdot \begin{vmatrix} M & b & d \end{vmatrix} + \begin{vmatrix} M & a & d \end{vmatrix} \cdot \begin{vmatrix} M & b & c \end{vmatrix} = 0$$

$$M = n \times n - 2$$

$$\left|\begin{array}{cc|cccc} c_{1,1} & c_{1,n-2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ c_{n-2,1} & c_{n-1,n-2} & 0 & 0 & \cdots & 0 \\ c_{n-1,1} & c_{n-1,n-2} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & a_{1,1} & a_{1,2} & \cdots & \\ \vdots & \vdots & a_{2,1} & \cdots & \cdots & \\ 0 & 0 & a_{n,1} & a_{n,2} & \cdots & \end{array}\right| \tag{4.2}$$

Case $n = 3$

$$\begin{vmatrix} c_{1,1} & 0 & 0 & 0 & 0 & 0 \\ c_{2,1} & 0 & 0 & 0 & 0 & 0 \\ c_{3,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{1,1} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & c_{2,1} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & c_{3,1} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{vmatrix} \tag{4.3}$$

where by expanding with respect to the 1^{st} *row, the result for the determinant is equal to 0.*

Explicit relation for Plucker's for three matrices when n=3.

Similarly, using (4.2) we can get the following determinant when $n = 4$ *as well as* $n = 5$.

Case $n = 4$

$$\begin{vmatrix} c_{1,1} & c_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{2,1} & c_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{3,1} & c_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{4,1} & c_{4,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{1,1} & c_{1,2} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 0 & c_{2,1} & c_{2,2} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & c_{3,1} & c_{3,2} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & 0 & c_{4,1} & c_{4,2} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix} \tag{4.4}$$

Case $n = 5$

$$\begin{vmatrix} c_{1,1} & c_{1,2} & c_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{2,1} & c_{2,2} & c_{2,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{4,1} & c_{4,2} & c_{4,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{5,1} & c_{5,2} & c_{5,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{1,1} & c_{1,2} & c_{1,3} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 0 & 0 & c_{2,1} & c_{2,3} & c_{2,3} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & 0 & c_{3,1} & c_{3,2} & c_{3,3} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & c_{4,1} & c_{4,2} & c_{4,3} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \\ 0 & 0 & 0 & c_{5,1} & c_{5,2} & c_{5,3} & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} \end{vmatrix} \tag{4.5}$$

Note that the equation above is the simplest case of the Plucker relations, which plays an important role in nonlinear dynamics and soliton theory. It shows that many of the differential and difference equations in mathematical physics are merely disguised versions of the Plucker relations.

Theorem B4 (Theorem 3.1, [21]). *Let $A = (a_{ij})_{n\times n}$ be a n-order matrix. Denoted by M^{ij}_{kl} the cofactor of the minor determinant* $\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix}$*, then*

$$M^{ij}_{kl}M^{ij}_{sr} - M^{ij}_{ks}M^{ij}_{lr} + M^{ij}_{kr}M^{ij}_{ls} = 0.$$

Proof. It is no less of generality to consider the case of $i < j$ and $k < l < s < r$. Denoted by M the $(n-1)\times(n-4)$ submatrix obtained by eliminating the i-th and j-th rows and the k-th, l-th, s-th and r-th columns from A. The four $(n-2)$-order column vectors obtained by eliminating the i-th and j-th components from the k-th, l-th, s-th and r-th column vectors in A are denoted by a, b, c and d respectively. Then it is easy to see that [21]

$$M^{ij}_{kl} = (-1)^{k+l}\begin{vmatrix} M & \mathbf{a} & \mathbf{b} \end{vmatrix}; M^{ij}_{sr} = (-1)^{s+r}\begin{vmatrix} M & \mathbf{c} & \mathbf{d} \end{vmatrix}$$

$$M_{ks}^{ij} = (-1)^{k+s} \begin{vmatrix} M & \mathbf{a} & \mathbf{c} \end{vmatrix}; M_{lr}^{ij} = (-1)^{l+r} \begin{vmatrix} M & \mathbf{b} & \mathbf{d} \end{vmatrix}$$

$$M_{kr}^{ij} = (-1)^{k+r} \begin{vmatrix} M & \mathbf{a} & \mathbf{d} \end{vmatrix}; M_{ls}^{ij} = (-1)^{l+s} \begin{vmatrix} M & \mathbf{b} & \mathbf{c} \end{vmatrix}.$$

Utilizing Plucker relation from before, we get

$$\begin{aligned} & M_{kl}^{ij} M_{sr}^{ij} - M_{ks}^{ij} M_{lr}^{ij} + M_{kr}^{ij} M_{ls}^{ij} \\ &= (-1)^{k+l+r+s} \Big(\begin{vmatrix} M & \mathbf{a} & \mathbf{b} \end{vmatrix} \cdot \begin{vmatrix} M & \mathbf{c} & \mathbf{d} \end{vmatrix} - \begin{vmatrix} M & \mathbf{a} & \mathbf{c} \end{vmatrix} \cdot \begin{vmatrix} M & \mathbf{b} & \mathbf{d} \end{vmatrix} \\ &\quad + \begin{vmatrix} M & \mathbf{a} & \mathbf{d} \end{vmatrix} \cdot \begin{vmatrix} M & \mathbf{b} & \mathbf{c} \end{vmatrix} \Big) \\ &= 0 \end{aligned}$$

$\square$

4.4 Differentiating Determinants

Next, we will see how to differentiate the determinant of a matrix. For this, we will work with the following 2×2 determinant, $r(x)$

$$\text{Let } r(x) = \begin{vmatrix} a(x) & b(x) \\ e(x) & d(x) \end{vmatrix}. \tag{4.6}$$

Defining each element, $a(x), b(x), e(x)$ and $d(x)$ as functions of x. Then, the differentiation of the determinant

$$r'(x) = \begin{vmatrix} a'(x) & b(x) \\ e'(x) & d(x) \end{vmatrix} + \begin{vmatrix} a(x) & b'(x) \\ e(x) & d'(x) \end{vmatrix} \tag{4.7}$$

where we differentiate each column while leaving the remaining columns untouched. Each column that has been differentiated has been denoted by its respective derivative —e.g., $a'(x)$ and $e'(x)$ in the first column [1].

Similarly, if were to differentiate with respect to the rows of a determinant using $r(x)$

$$r(x) = \begin{vmatrix} a(x) & b(xt) \\ e(x) & d(x) \end{vmatrix} \tag{4.8}$$

where we differentiate each row while leaving the remaining rows untouched. Each row that has been differentiated has similarly been denoted by its respective derivative —e.g., $a'(x)$ and $b'(x)$ in the first row.

$$r'(x) = \begin{vmatrix} a'(x) & b'(x) \\ e(x) & d(x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) \\ e'(x) & d'(x) \end{vmatrix} \tag{4.9}$$

It is important to have a complete understanding of the properties of determinants as they were incredibly helpful. Additionally, the main work of this thesis mainly dealt with linear algebra and applications. Their importance in proving the main results found in the following chapter is undeniable so that we may be able to extend the method of 2-solitons to 3-soltions. Next, we will be computing for 3-soliton solutions for the Boussinesq equation.

CHAPTER V

3-SOLITON SOLUTIONS FOR 'GOOD' BOUSSINESQ EQUATION

5.1 Wronskian Solution

In this chapter we extend the results for two-soliton solutions of gB reported in [18] to the case of three-solitons. The method used for three can also extend the results to n solitons. However, we will keep the proof to three for simplicity's sake. As mentioned in a previous chapter, it can be done for n functions, we will focus only on three functions. Using three functions for the 'good' Boussinesq equation we will have f_1, f_2, and f_3.

$$\begin{aligned} f_1(x,t) &= e^{m_1(x-m_1t)} + e^{n_1(x-n_1t)} \\ f_2(x,t) &= e^{m_2(x-m_2t)} + e^{n_2(x-n_2t)} \\ f_3(x,t) &= e^{m_3(x-m_3t)} + e^{n_3(x-n_3t)} \end{aligned} \tag{5.1}$$

The results hold, considering plus or minus case

$$f_j(x,t) = e^{m_jx-m_j^2t} + \varepsilon_j e^{n_jx-n_j^2t}, \tag{5.2}$$

where $\varepsilon_j = \pm 1$ but for concise position we will consider every $\varepsilon = 1$.

For the previous equations, f_1, f_2, and f_3 we have that m_1, m_2, and m_3 are solutions of the cubic equation for different λ_j and n_j. The solutions can be obtained the same way for two functions using the Wronskian Determinant.

We will follow the Wronskian method described in ch.4, in this case the Wronskian Determinant is

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix}. \tag{5.3}$$

In what follows, we consider W, $W > 0$, for necessary conditions see [19]. We also defined the reduced Wronskians as

$$W_j = W(f_1, ..., f_{j-1}, f_{j+1,...,f_N}) \tag{5.4}$$

and in particular

$$W_1 = \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix}, W_2 = \begin{vmatrix} f_1 & f_3 \\ f_{1,x} & f_{3,x} \end{vmatrix}, W_3 = \begin{vmatrix} f_1 & f_2 \\ f_{1,x} & f_{2,x} \end{vmatrix}, \tag{5.5}$$

which are the Wronskian determinants, $W_1 = W(f_2, f_3)$, $W_2 = W(f_1, f_3)$ and $W_3 = W(f_1, f_2)$.

From section 3, we have that the solution of the general equation for gB in this case is

$$v = 2(\log(W))_{xx}. \tag{5.6}$$

Next figure is typical graph of 3-solitons

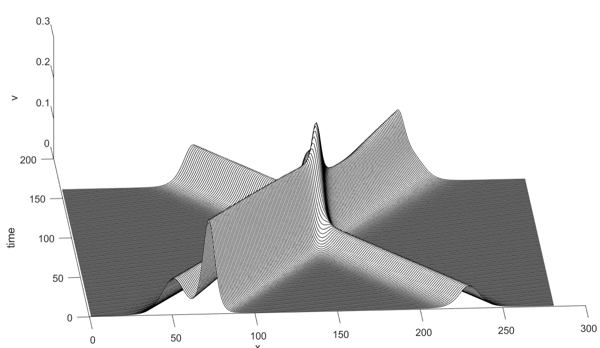

Figure 5.1 Example of 3-Solitons.

Next we consider properties involving W_j, $j = 1, 2, 3$.

5.2 Properties of the Components

We define these functions, similarly, to the case of 2-soltions. These functions have special properties and Theorem 1 is one of the main results in this thesis.

Let

$$\phi_1 = \frac{W_1}{W},\ \phi_2 = \frac{W_2}{W},\ \phi_3 = \frac{W_3}{W} \tag{5.7}$$

Then we show that ϕ_1 , ϕ_2 , and ϕ_3 can be considered as the steady states of the real Schrödinger equation.

Theorem 1. *Let $v = 2(\log W)_{xx}$ where W is defined in (5.3), then the functions ϕ_1 , ϕ_2 , and ϕ_3 satisfy the following partial differential equation*

$$\phi_{j,t} - \phi_{j,xx} - v\phi = 0, j = 1,2,3. \tag{5.8}$$

Proof. The proof is based on an identity similar to Plucker's identity. We consider only $\phi_1 = \frac{W_1}{W}$, as the proof for ϕ_2 and ϕ_3 is similar. For $\phi_1 = \frac{W_1}{W}$ and v defined as above (5.6), we show that (5.8) is equivalent to $W_{1,t}W - W_1W_t - W_{1,xx}W - W_1W_{xx} + 2W_xW_{1,x} = 0$

Furthermore, since $\phi_1 = \frac{W_1}{W}$, then its derivative with respect to t is as follows

$$\phi_t = \frac{W_{1,t}W - W_1W_t}{W^2}, \tag{5.9}$$

$W_{1,t}$ is the partial derivative of W_1 with respect to the variable t.

Finding the second derivative results in the following

$$\phi_{xx} = \left(\frac{W_{1,x}W - W_1W_x}{W^2}\right)_x =$$

$$\frac{(W_{1,xx}W + W_{1,x}W_x - W_1W_{xx} - W_{1,x}W_x)W^2 - (W_{1,x}W - W_1W_x)2WW_x}{W^4} =$$

$$\frac{W_{1,xx}W^2 - W_1W_{xx}W - 2W_{1,x}W_xW + 2W_1W_x^2}{W^3}. \tag{5.10}$$

Substituting the above into equation (5.8) lead to

$$u_t - u_{xx} - vu = \frac{W_{1,t}W - W_1W_t}{W^2} - \frac{W_1W_{xx}W^2 - W_1W_{xx}W - 2W_{1,x}W_xW + 2W_1W_x^2}{W^3} - \frac{W_1}{W}\frac{2W_{xx}W - 2W_x^2}{W^2} \tag{5.11}$$

$$= \frac{W_{1,t}W - W_1W_t}{W^2} - \frac{W_1W_{xx}W^2 - W_1W_{xx}W - 2W_{1,x}W_xW + 2W_1W_x^2 + 2W_1W_{xx}W - 2W_1W_x^2}{W^3} \tag{5.12}$$

$$= \frac{W_{1,t}W - W_1W_t - W_{1,xx}W - W_1W_{xx} + 2W_{1,x}W_x}{W^2}. \tag{5.13}$$

Let $E(x,t) = W_{1,t}W - W_1W_t - W1,xxW - W_1W_{xx} + 2W_xW_{1,x}$, then $E(x,t) = 0$. Indeed, rewriting as

$$(W_{1,t} - W_{1,xx})W - (W_t + W_{xx})W_1 + 2W_xW_{1,x} = 0. \tag{5.14}$$

Next by using results from chapter 4 we derived some identities for determinants. It should be noted that $f_{1,t} = -f_{1,xx}$ for the following determinants.

Using W_1

$$W_1 = \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix}. \tag{5.15}$$

Differentiating Determinants as per section 4.4

$$W_{1,t} = \begin{vmatrix} f_{2,x} & f_{3,x} \\ f_{2,xx} & f_{3,xx} \end{vmatrix} - \begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.16}$$

$$W_{1,x} = \begin{vmatrix} f_2 & f_3 \\ f_{2,xx} & f_{3,xx} \end{vmatrix}. \tag{5.17}$$

Second derivative for W_1

$$W_{1,xx} = \begin{vmatrix} f_{2,x} & f_{3,x} \\ f_{2,xx} & f_{3,xx} \end{vmatrix} + \begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix}. \tag{5.18}$$

From $W(f_1, f_2, f_3)$ we get the following derivatives:

$$W_x = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.19}$$

$$W_{xx} = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxxx} & f_{2,xxxx} & f_{3,xxxx} \end{vmatrix}, \tag{5.20}$$

$$W_t = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} - \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxxx} & f_{2,xxxx} & f_{3,xxxx} \end{vmatrix}. \tag{5.21}$$

And substituting the terms from (5.14) individually

$$W_{1,t} - W_{1,xx} = -2 \begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.22}$$

$$W(W_{1,t-W_{1,xx}}) = -2 \begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix}, \tag{5.23}$$

$$W_t + W_{xx} = 2 \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.24}$$

$$-(W_t+W_{xx})W_1=-2\begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.25}$$

Lastly,

$$W_xW_{1,x}=2\begin{vmatrix} f_2 & f_3 \\ f_{2,xx} & f_{3,xx} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} \tag{5.26}$$

Therefore, we get the following,

$$E(x,t)=-2\begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix}-2\begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}+ \tag{5.27}$$

$$2\begin{vmatrix} f_2 & f_3 \\ f_{2,xx} & f_{3,xx} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}.$$

and

$$\frac{1}{2}[(W_{1,t}-W_{1,xx})W-(W_t+W_{xx})W_1+2W_xW_{1,x}]= \tag{5.28}$$

$$\begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix}-\begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}+ \tag{5.29}$$

$$\begin{vmatrix} f_2 & f_3 \\ f_{2,xx} & f_{3,xx} \end{vmatrix}\begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}.$$

First lets notice the following identity where we will show that the R.H.S. is zero

$$\begin{vmatrix} f_2 & f_3 & f_1 & f_2 & f_3 \\ -f_2 & -f_3 & f_1 & f_2 & f_3 \\ f_{2,x} & f_{3,x} & f_{1,x} & f_{2,x} & f_{3,x} \\ f_{2,xx} & f_{3,xx} & f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{2,xxx} & f_{3,xxx} & f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = 0 \tag{5.30}$$

which can be seen when subtracting the 1^{st} column from the 4^{th} column, as well as the 2^{nd} column from the 5^{th} column so the determinant above equals 0.

We can compute (5.30) by using the Laplace expansion, section 4.1, with respect to the first two columns. Therefore, we get the following when selecting the rows and columns. By section 4.1, the first two columns when expanding are fixed, leads to J,then $J = 1+2$ for all columns. However, it should be noted that when selecting 1^{st} and 2^{nd} rows the result is a zero determinant.

$$\begin{vmatrix} f_2 & f_3 \\ -f_2 & -f_3 \end{vmatrix} = 0 \tag{5.31}$$

By using the notations from section 4.1, we introduce the determinants

$$R^3_{i_1,i_2} = (-1)^{(I+J)} D \begin{vmatrix} i_1 & i_2 \\ 1 & 2 \end{vmatrix} \tag{5.32}$$

where $i_1 < i_2$ and $i_1, i_2 \in \{1,2,3,4,5\}$ are the number of rows selected and D is a determinant created by the intersection of rows and columns when expanding.

The following identities are apparent when observing (5.30). For, $R^3_{3,4}, R^3_{3,5}, R^3_{4,5}$, we take a look at their respective adjoint created when selecting their respective rows and the first two columns that are fixed. For example, when selecting the 1^{st} and 2^{nd} columns with the 3^{rd} and 4^{th} rows it results in the following adjoint for $R^3_{3,4}$:

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1 & f_2 & f_3 \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = 0,$$

by the properties of determinants, when two rows are identical then the determinant is equal to zero. The results for $R^3_{3,5}$ and $R^3_{4,5}$ also result in a zero determinant as they also have two identical rows in their respective adjoints.

Therefore,

$$R^3_{1,2} = R^3_{3,4} = R^3_{3,5} = R^3_{4,5} = 0. \tag{5.33}$$

Selecting 1^{st} and 3^{rd} rows where $I = 1+3 = 4$ and $J = 1+2 = 3$ for all columns

$$R^3_{1,3} = (-1)^{(3+4)} \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.34}$$

selecting 1^{st} and 4^{th} rows where $I = 1+4 = 5$

$$R^3_{1,4} = (-1)^{(3+5)} \begin{vmatrix} f_2 & f_3 \\ f_{2,xx} & f_{3,xx} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix}, \tag{5.35}$$

and selecting 1^{st} and 5^{th} rows where $I = 1+5 = 6$

$$R^3_{1,5} = (-1)^{(3+6)} \begin{vmatrix} f_2 & f_3 \\ f_{2,xxx} & f_{3,xxx} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix}. \tag{5.36}$$

Now, selecting 2^{nd} and 3^{rd} rows where $I = 2+3 = 5$, and factoring out the negative sign from the determinant below we get the following

$$R^3_{2,3} = (-1)^{(3+5)} \begin{vmatrix} -f_2 & -f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = \tag{5.37}$$

$$(-1)^{(3+5+1)} \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = R^3_{1,3}.$$

Similary, selecting 2^{nd} and 4^{th} rows where $I = 2+4 = 6$

$$R^3_{2,4} = (-1)^{(3+6)} \begin{vmatrix} -f_2 & -f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = \tag{5.38}$$

$$(-1)^{(3+6+1)} \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = R^3_{1,4},$$

and selecting 2^{nd} and 5^{th} rows where $I = 2+5 = 7$

$$R^3_{2,5} = (-1)^{(3+7)} \begin{vmatrix} -f_2 & -f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix} = \tag{5.39}$$

$$(-1)^{(3+7+1)} \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix} = R^3_{1,5}.$$

Adding the determinants from above respectively, such as $R^3_{1,3}$ and $R^3_{2,3}$, which are equal that leads to twice the sum. This sum equals zero which results in the identity from above. □

Next we present the decomposition formula for three-soliton solution of gB.

5.3 General Decomposition

Theorem 2. *If f_j are as in (5.1), u_j as in (5.7), and W as in (5.3), then $v = 2(\log W)_{xx}$ is a three-solion solution of the gB and has the following decomposition*

$$v(x,t) = A_1(x,t)\phi_1^2(x,t) + A_2(x,t)\phi_2^2(x,t) + A_3(x,t)\phi_3^2(x,t), \tag{5.40}$$

where

$$A_1 = 2\frac{W(f_1, f_{1,x}, f_2, f_3)}{W(f_2, f_3)}, A_2 = -2\frac{W(f_1, f_2, f_{2,x}, f_3)}{W(f_1, f_3)}, A_3 = 2\frac{W(f_1, f_2, f_3, f_{3,x})}{W(f_1, f_2)}. \tag{5.41}$$

Proof. We notice that

$$\frac{v}{2} = (logW)_{xx} = \left(\frac{W_x}{W}\right)_x$$

and let $T = \frac{W_x}{W}$

$$T = \frac{W(f_{1,x}, f_2, f_3)}{W(f_1, f_2, f_3)} + \frac{W(f_1, f_{2,x}, f_3)}{W(f_1, f_2, f_3)} + \frac{W(f_1, f_2, f_{3,x})}{W(f_1, f_2, f_3)}$$

denoting each numerator as $T_1 = W(f_{1,x}, f_2, f_3), T_2 = W(f_1, f_{2,x}, f_3)$, and $T_3 = W(f_1, f_2, f_{3,x})$ where we have that

$$\frac{v}{2} = T_x = \sum_{j=1}^{3}\left(\frac{T_j}{W}\right)_x = \sum_{j=1}^{3}\frac{T_{j,x}W - T_jW_x}{W^2}$$

Let us consider $\left(\frac{T_1}{W}\right)_x = \frac{T_{1,x}W - T_1W_x}{W^2}$

$$\left(\frac{T_1}{W}\right)_x = \frac{W_x(f_{1,x}, f_2, f_3)W(f_1, f_2, f_3) - W(f_{1,x}, f_2, f_3)W_x(f_1, f_2, f_3)}{W^2} = \frac{W_1}{W_2}W(f_1, f_{1,x}, f_2, f_3). \tag{5.42}$$

Let us denote

$$L_1 = W_x(f_{1,x}, f_2, f_3)W(f_1, f_2, f_3) - W(f_{1,x}, f_2, f_3)W_x(f_1, f_2, f_3) = \tag{5.43}$$

$$\begin{vmatrix} f_{1,x} & f_2 & f_3 \\ f_{1,xx} & f_{2,x} & f_{3,x} \\ f_{1,xxxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix} - \begin{vmatrix} f_{1,x} & f_2 & f_3 \\ f_{1,xx} & f_{2,x} & f_{3,x} \\ f_{1,xxx} & f_{2,xx} & f_{3,xx} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & f_3 \\ f_{1,x} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{2,xx} & f_{3,xx} \end{vmatrix}. \tag{5.44}$$

Then by using Sylvester's Identity,4.2, for $W(f_1, f_{1,x}, f_2, f_3)$, and applying it along the last two rows and first two columns we get the identity

$$L_1 = \begin{vmatrix} f_2 & f_3 \\ f_{2,x} & f_{3,x} \end{vmatrix} \begin{vmatrix} f_1 & f_{1,x} & f_2 & f_3 \\ f_{1,x} & f_{1,xx} & f_{2,x} & f_{3,x} \\ f_{1,xx} & f_{1,xxx} & f_{2,xx} & f_{3,xx} \\ f_{1,xxx} & f_{1,xxxx} & f_{2,xxx} & f_{3,xxx} \end{vmatrix} = W_1 W(f_1, f_{1,x}, f_2, f_3). \tag{5.45}$$

In a similar manner, the coefficients A_2 and A_3 can be derived. $\square$

CHAPTER VI

CONCLUSION

6.1 Conclusions

In this thesis we extended results,[18], regarding similarities of distribution functions for the classical Schrodinger equations and multi-soliton solutions of gB to the case of three-solitons. The decomposition formula stated in Theorem 2 indicates that the soliton can be considered as a joint distribution with interactions. In Theorem 1, we showed that the functions ϕ_i, $i = 1,2,3$, can be interpreted as distribution functions of a single soltion-particle. The coefficients A_i in (5.41) have a complex structure which indicates that they carry interaction information as well. Further work is needed to determine the interaction coefficients.

The main tools in the proofs of the new results, Theorem 1, Theorem 2, were determinant identities, chapter 4. The methods can be successfully used to extend the results to any number of solitons.

Another interesting observation was that the functions ϕ_i are solutions to the evolution equation of the Lax pair of gB.

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BIOGRAPHICAL SKETCH

A Mexican-American student, Aldo Gonzalez, is a 2016 graduate of Brownsville Early College High school. He received a Bachelor of Science with a major in Mathematics in December 2019 from UTRGV. Aldo decided to pursue knowledge in the field of mathematics and obtained his Master of Science degree with a major in mathematics in May 2022 at UTRGV.

To discover limits and definitions that surround all aspects of life was his goal. A goal with an unkown destination. Wherever that may take him. To infinity and beyond.

By attending various educational institutions, knowledge was always at his fingertips. Knowledge that would quench his thirst for wisdom. Wisdom that would guide him down a path filled with unknown variables.

Variables have instilled his life in ways still unknown to him. The unknown; however, does not frighten him nor deter him. Even he does not know what may deter him, but like a little engine he knows he can. He may be reached by email at aldog628@gmail.com.