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Modeling Functions into an Angular Displacement of an Elastic Pendulum

Brenda Lee Garcia

The University of Texas Rio Grande Valley

MODELING FUNCTIONS INTO AN ANGULAR DISPLACEMENT OF AN ELASTIC PENDULUM

A Thesis

by

BRENDA LEE GARCIA

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COMMITTEE MEMBERS

Dr. Vesselin Vatchev
Chair of Committee

Dr. Zhijun Qiao
Committee Member

Dr. Sergey Grigorian
Committee Member

Dr. Andras Balogh
Committee Member

May 2022

ABSTRACT

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In this thesis we study the relation between analytic signals and a variety of pendulum systems. The representation of a signal as a pair of time varying amplitude and phase has been well studied and often related to linear mass spring systems. The differential equations describing pendulum systems are nonlinear and we provide analytical and numerical results regarding interpretation about the amplitude and the phase of signals in different pendulum settings. We report an explicit solution of the Elastic Pendulum problem in the case of linear phase. We develop an experimental procedure to piece-wise approximate bounded functions on a partition of a finite interval. On each sub-interval the function is approximated by a solution of a Pendulum system. The parameters of the corresponding differential equations are determined by optimization on each sub-interval. The smoothness of the approximation is controlled by the initial conditions provided by the given function.

DEDICATION

I would like to dedicate this thesis to my wonderful and supportive parents, Maria S. and Ruben Garcia, who sacrificed everything they had in order for me to receive an education. I would also like to dedicate this thesis to my lovely sister, Gabriela Garcia, and my grandma, Guadalupe Peña, who have both been my ray of sunshine at all times. Last but not least, I would like to dedicate this thesis to my chihuahua, Jumbie, who has been by my side at all times, even when I am up at 2 in the morning working on homework.

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First and foremost, I would like to acknowledge my advisor, Dr. Vesselin Vatchev, for the continuous support and patient guidance throughout the past two years. This thesis would not have been accomplished without his assistance and dedicated involvement in every step of the process. I would also like to acknowledge Dr. Zhijun Qiao for allowing me the opportunity to be a student in the program, and for his constant feedback and support of my research. Additionally, I would like to acknowledge my committee members, Dr. Sergey Grigorian and Dr. Andras Balogh.

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CHAPTER I

INTRODUCTION

The goal of this work is to model a bounded C^2 function $\theta(t)$ on a finite interval as the angular displacement of a pendulum. The procedure we present can be used for denoising of solutions of the Pendulum Equation and extends similar result for a mass spring system studied in [8]. The experimental work helps to gain better understanding about the pendulum system.

1.1 Simple Pendulum

A simple pendulum (shown below in figure 1.1) as defined in [7], is a body suspended from a fixed support so that it swings freely back and forth under the influence of gravity. We use the method of Lagrange's equations to find the equation of motion of the simple pendulum for a body with a mass of 1 as derived similarly in [4].

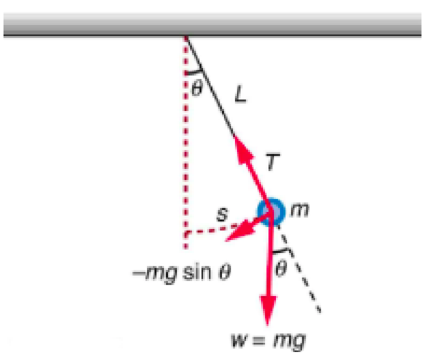

Figure 1.1: Simple Pendulum [7].

Our system consists of a single independent variable t, therefore, the corresponding Lagrange's Equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{1.1}$$

The symbol q in the equation symbolizes generalized coordinates. A generalized coordinate is a coordinate in which we don't really mind about what direction we use, we only care about the magnitude. Mechanical energy is conserved in a simple pendulum with no friction. Kinetic and gravitational potential energy combine to make total mechanical energy. There is a continual interchange between kinetic energy and gravitational potential energy as the pendulum swings back and forth. The Kinetic energy as defined below by T is the kinetic energy along the direction of the pendulum and the energy involved in the swinging motion of the pendulum. The Kinetic energy is given by the equation $T = \frac{1}{2}mv^2$ where m is the mass of the pendulum, and v is the velocity in the tangential direction θ due to the swaying of the pendulum. The velocity in the tangential θ direction is $\ell\dot{\theta}$. Hence, the kinetic energy of the simple pendulum is

$$T = \frac{1}{2}m\ell^2\dot{\theta}^2. \tag{1.2}$$

The potential energy of the simple pendulum can be modeled by the gravitational potential energy equation $V_g = mgh$ where g is the acceleration due to gravity and h is the height. This equation is used to model objects of free fall, the pendulum, on the other hand, is restrained by the rod or string and does not fall freely. As a result, we must express the height in terms of, the angle, and ℓ, the pendulum's length. Hence, the potential energy for the simple pendulum is

$$V = mg\ell(1 - \cos\theta). \tag{1.3}$$

When we take the potential energy (energy of position) away from the kinetic energy (energy of motion) we end up with an equation called the Lagrangian of the system, denoted below in equation

$$L = T - V = \frac{1}{2}mv^2 - [mg\ell(1 - \cos\theta)] \tag{1.4}$$

Now we substitute L into Lagrange's equation (1.1) and take the derivative with respect to the only generalized coordinate $q_1 = \theta$.

$$\frac{\partial L}{\partial \theta} = -mg\ell \sin\theta$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = m\ell^2\ddot{\theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m\ell^2\ddot{\theta} - (-mg\ell\sin\theta) = 0 \tag{1.5}$$

After simplification and setting $m = 1$ we get the following equation of motion for the simple pendulum

$$\ddot{\theta} + \frac{g}{\ell}sin\theta = 0, \tag{1.6}$$

with g as the magnitude of the gravitational field, ℓ is the length of the rod or cord, and θ is the angle from the equilibrium position to the pendulum. A pendulum is a simple harmonic oscillator with a restoring force F that acts to bring the body to its equilibrium position and undergoes a simple harmonic motion with a constant amplitude and frequency around the equilibrium point for minor displacements (which does not depend on the amplitude) [7].

1.2 Analytic Signal

The equation of motion for a pendulum has only 1 degree of freedom, angle of displacement. Given an analytic signal as defined in [2], is a complex-valued function with no negative frequency components. It is defined below as a two-dimensional signal with real and imaginary parts that are real-valued functions related by the Hilbert transform as

$$U(t) = u(t) + iH[u(t)]. \tag{1.7}$$

The Hilbert transform is defined as a specific linear operator that takes a real-valued function $u(t)$ and produces another real-valued function $H(u(t))$.

The Hilbert transform of the function $u(t)$ is

$$H[u(t)] = \tilde{u}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau}\, d\tau, \tag{1.8}$$

which we will use the notation $\tilde{u}(t)$ in this paper. The mathematical integral definition does not provide much insight into the undertstanding and application of the HT. The physical meaning of the HT, on the other hand, allows us to obtain a far deeper understanding of the transformation. The HT is physically equivalent to a special kind of linear filter in which all of the spectral components' amplitudes are left unchanged but their phases are shifted by $\pi/2$.

We consider the analytic signal in the form

$$F(t) = f(t) + i\tilde{f}(t), \tag{1.9}$$

where $\tilde{f}(t) = H[f(t)]$ is defined as

$$F(t) = x(t)\cos\theta(t) + ix(t)\sin\theta(t). \tag{1.10}$$

We will model $x(t)$ as an amplitude and $\theta(t)$ is an angular displacement in Pendulum systems. A physical system with an additional degree of freedom can be considered if the rod has a variable length.

CHAPTER II

ELASTIC PENDULUM

An elastic pendulum (shown below in Figure 2.1) is a physical system where a mass is suspended from a fixed point by a light spring, which can stretch but not bend, so that the resulting motion contains elements of both a simple pendulum and a one-dimensional spring-mass system. We use the method of Lagrange's equations to find the equations of motion as derived similarly in [3] and [9].

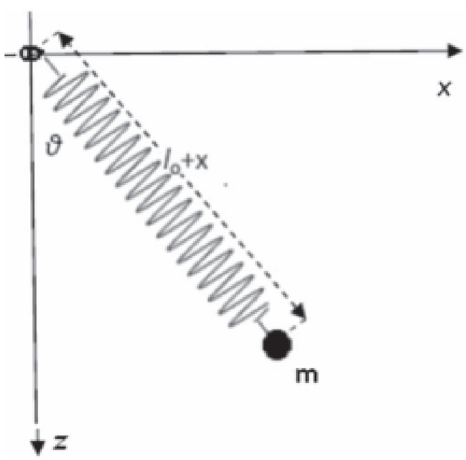

Figure 2.1: Elastic Pendulum [1].

2.1 Equations of Motion

Our system consists of a single independent variable t, therefore, the corresponding Lagrange's Equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.1}$$

The symbol q in the equation symbolizes generalized coordinates. A generalized coordinate is a coordinate in which we don't really mind about what direction we use, we only care about the magnitude. The system we are using only uses the generalized coordinates of extension $x(t)$ (which is the length of the string) and angle $\theta(t)$ (the angle of the string from vertical). Let us consider a pendulum made of a light spring with a mass m attached at the end as shown above in Figure 2.1. The equilibrium length of the spring is ℓ_0. Let the spring have length $\ell(t) = \ell_0 + x(t)$ when stretched by $x(t)$, and let its angle with the vertical be $\theta(t)$. In this case there are two energy's present: kinetic energy and potential energy.

Kinetic energy as defined below by T is the kinetic energy along the direction of the spring and the energy involved in the swinging motion of the pendulum. The Kinetic energy is given by the equation $T = \frac{1}{2}mv^2$, where m is the mass of the pendulum, and velocity v, in the case of the elastic pendulum, is a vector since it has two components, one in the radial direction r due to the spring displacement, and the other in the tangential direction θ due to the swaying of the pendulum. The velocity is a vector squared and can be rewritten as $v^2 = v_r^2 + v_\theta^2$, because it is a vector, hence, we square each component of the vector. The velocity in the radial direction is $\dot{x}$ and the velocity in the tangential θ direction is $\ell\dot{\theta} = (\ell_0 + x)\dot{\theta}$. Hence, the kinetic energy for the elastic pendulum is

$$T = \frac{m}{2}(\dot{x}^2 + (\ell_0 + x)^2\dot{\theta}^2). \tag{2.2}$$

Potential energy, the stored energy of position, defined below by V is simply given by gravitational potential energy and elastic potential energy. The amount of gravitational potential energy is dependent to the mass m and height h of the object. The gravitational potential energy equation is $V_g = mgh$ where g is the gravitational field strength (also known as the acceleration generated by gravity) and has a value of 9.8 N/kg. The gravitational potential energy for the elastic pendulum is $V_g = -mg(\ell_0 + x)\cos\theta$. Now, because the elastic restoring force is not constant, for this reason, we integrate Hooke's Law $F = kx$ with respect to x and get $V_k = \frac{1}{2}kx^2$ which is the elastic potential energy of the spring where k is the spring constant, and x is the spring stretch.

Hence, the potential energy for the Elastic Pendulum is

$$V = -mg(\ell_0 + x)\cos\theta + \frac{1}{2}kx^2. \tag{2.3}$$

When we take the potential energy (energy of position) away from the kinetic energy (energy of motion) we end up with an equation called the Lagrangian of the system, denoted below in equation (2.4) as L

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\ell_0 + x)^2\dot{\theta}^2 + mg(\ell_0 + x)\cos\theta - \frac{1}{2}kx^2. \tag{2.4}$$

Now, we substitute L into Lagrange's equation (2.1) and in each case take the derivative with respect to the associate coordinate. For the generalized coordinate $q_1 = x$ we get

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = m(\ell_0 + x)\dot{\theta}^2 + mg\cos\theta - kx$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} - (m(\ell_0 + x)\dot{\theta}^2 + mg\cos\theta - kx) = 0, \tag{2.5}$$

and for the generalized coordinate $q_2 = \theta$ we get

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = 2m(\ell_0 + x)\dot{x}\dot{\theta} + m(\ell_0 + x)^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mg(\ell_0 + x)\sin\theta$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 2m(\ell_0 + x)\dot{x}\dot{\theta} + m(\ell_0 + x)^2\ddot{\theta} - (-mg(\ell_0 + x)\sin\theta) = 0. \tag{2.6}$$

We simply equations (2.5) and (2.6) and get the equations of motion for the Elastic pendulum system below:

$$\ddot{x} - (\ell_0 + x)\dot{\theta}^2 + \frac{k}{m}x - g\cos\theta = 0 \tag{2.7}$$

$$\ddot{\theta} + \frac{2\dot{x}}{\ell_0 + x}\dot{\theta} + \frac{g}{\ell_0 + x}\sin\theta = 0 \tag{2.8}$$

The parameters in the equations of motion as defined in [9] are the rest length ℓ_0 which can be stretched by length $x(t)$, g is the gravitational acceleration, k/m is the stiffness of mass ratio, and the angle of oscillation of the pendulum to its equilibrium position is $\theta(t)$. The equations (2.7) and (2.8) cannot be solved explicitly. The two equations are linear in $x(t)$, but nonlinear in $\theta(t)$ because of $\cos\theta$ and $\sin\theta$. In the next section, we will eliminate $x(t)$ from the system to obtain equation for $\theta(t)$.

2.2 Solving for the Spring Length

We will label the first equation of the elastic pendulum system as E1 and the second equation of the elastic pendulum system as E2. We let $\ell_0 = 1$, $\frac{k}{m} = b$ and multiply E2 by $\frac{1+x}{2\dot{\theta}}$ and get the following

$$E1 : \ddot{x} + (b - \dot{\theta}^2)x - \dot{\theta}^2 - g\cos\theta = 0 \tag{2.9}$$

$$E2 : \dot{x} + \frac{\ddot{\theta}}{2\dot{\theta}}x + \frac{\ddot{\theta}}{2\dot{\theta}} + \frac{g\sin\theta}{2\dot{\theta}} = 0. \tag{2.10}$$

We consider solution for the system above for general $\theta(t)$ and $x(t)$. Although both equations are linear for $x(t)$, the first equation is second order for $x(t)$ and the second equation is first order for $x(t)$ so we choose to solve equation (2.10) for $x(t)$ and we get the result below. We can solve for $x(t)$ by the integrating factor method. The first step is to rewrite E2 (2.10) in standard form

$$E2 : \dot{x} + \frac{\ddot{\theta}}{2\dot{\theta}}x = -\frac{\ddot{\theta}}{2\dot{\theta}} - \frac{g\sin\theta}{2\dot{\theta}}. \tag{2.11}$$

Now we solve for our integrating factor

$$v = e^{\frac{1}{2}\int \frac{\ddot{\theta}}{\dot{\theta}}dt} = e^{\frac{1}{2}ln|\dot{\theta}|} = \sqrt{|\dot{\theta}|}.$$

We multiply it to both sides and integrate, then solve for $x(t)$

$$\sqrt{|\dot\theta|}\left(\dot x + \frac{\ddot\theta}{2\dot\theta}x = -\frac{\ddot\theta}{2\dot\theta} - \frac{g\sin\theta}{2\dot\theta}\right) \tag{2.12}$$

$$\int (\sqrt{|\dot\theta|}x)'dt = -\int \sqrt{|\dot\theta|}\left(\frac{\ddot\theta + g\sin\theta}{2\dot\theta}\right)dt \tag{2.13}$$

$$\sqrt{|\dot\theta|}x = -\frac{1}{2}\int \sqrt{|\dot\theta|}\left(\frac{\ddot\theta + g\sin\theta}{\dot\theta}\right)dt + c \tag{2.14}$$

$$x = -\frac{1}{2\sqrt{|\dot\theta|}}\left(\int \sqrt{|\dot\theta|}\left(\frac{\ddot\theta + g\sin\theta}{\dot\theta}\right)dt + c\right) \tag{2.15}$$

for a constant c. We label our solution $x_2(t)$ respectively

$$x_2(t) = -\frac{1}{2\sqrt{|\dot\theta|}}\left(\int \sqrt{|\dot\theta|}\left(\frac{\ddot\theta + g\sin\theta}{\dot\theta}\right)dt + c\right). \tag{2.16}$$

Next is to find $x_1(t)$ using E1 (2.9). The first step is to get rid of $\ddot x$ and we do it by $\frac{2\dot\theta}{\ddot\theta}(\frac{d}{dt}E2 - E1)$ and after simplifying we get

$$\dot x + \left(\frac{\dddot\theta\,\dot\theta - \ddot\theta^2}{\ddot\theta\dot\theta} - \frac{2\dot\theta}{\ddot\theta}(b - \dot\theta^2)\right)x + \left(\frac{\dddot\theta\,\dot\theta - \ddot\theta^2}{\ddot\theta\dot\theta} + \frac{3\dot\theta g\cos\theta}{\ddot\theta} - \frac{g\sin\theta}{\dot\theta} + \frac{2\dot\theta^3}{\ddot\theta}\right) = 0. \tag{2.17}$$

Now in order to get rid of $\dot x$ we subtract E2 from the above equation and get

$$\left(\frac{\dddot\theta\,\dot\theta - \ddot\theta^2}{\ddot\theta\dot\theta} - \frac{2\dot\theta}{\ddot\theta}(b - \dot\theta^2) - \frac{\ddot\theta}{2\dot\theta}\right)x + \left(\frac{\dddot\theta\,\dot\theta - \ddot\theta^2}{\ddot\theta\dot\theta} + \frac{3\dot\theta g\cos\theta}{\ddot\theta} - \frac{g\sin\theta}{\dot\theta} + \frac{2\dot\theta^3}{\ddot\theta} - \frac{\ddot\theta}{2\dot\theta} - \frac{g\sin\theta}{2\dot\theta}\right) = 0. \tag{2.18}$$

Lastly, we solve for x and after simplifying we get

$$x_1(t) = -1 + \frac{3g\ddot\theta\sin\theta - 6g\dot\theta^2\cos\theta - 4b\dot\theta^2}{2\,\dddot\theta\,\dot\theta - 2\ddot\theta^2 + 4\dot\theta^4 - 4b\dot\theta^2 - \ddot\theta^2}. \tag{2.19}$$

The next step is to solve for $\theta(t)$ by setting

$$x_1(t) = x_2(t),$$

and we get

$$-1+\frac{3g\ddot{\theta}\sin\theta-6g\dot{\theta}^2\cos\theta-4b\dot{\theta}^2}{2\,\dddot{\theta}\,\dot{\theta}-2\ddot{\theta}^2+4\dot{\theta}^4-4b\dot{\theta}^2-\ddot{\theta}^2}=-\frac{1}{2\sqrt{|\dot{\theta}|}}\left(\int\sqrt{|\dot{\theta}|}\left(\frac{\ddot{\theta}+g\sin\theta}{\dot{\theta}}\right)dt+c\right), \tag{2.20}$$

which is nonlinear third order integro-differential equation. We run into complications in equation (2.20) above: The absolute value of $\dot{\theta}$ might result in a break point where it is not smooth. The nonlinearity is a big problem, the equation is very nonlinear. To remove the integral on the right side of the equation our method would be to differentiate the equation, but then we are left with a very complicated nonlinear fourth order differential equation in terms of $\theta(t)$.

If we compare, the solution to the pendulum equation as solved in [6] and expressed by elliptic functions as the following

$$\phi(t)=2\tan^{-1}\left(\frac{\tan\frac{\phi_0}{2}cn(i\omega t|m)-\frac{i}{2\omega}\dot{\phi}_0\sec^2(\frac{\phi_0}{2})dn(i\omega t|m)sn(i\omega t|m)}{1+\left(\frac{\dot{\phi}_0^2}{2\omega^2(\cos(\phi_0)+1)}-1\right)sn(i\omega t|m)^2}\right), \tag{2.21}$$

$$m=\frac{1}{4}\left(2(1+\cos(\phi_0))-\frac{\dot{\phi}_0^2}{\omega^2}\right), \tag{2.22}$$

we can see that this solution is very complicated for a practical use. Hence, solving for $\theta(t)$ for the elastic pendulum system will require more work. So in our case, we consider fixing $\theta(t)$ to find a particular class of solutions.

2.3 Linear Angular Displacement

We consider a simple but nontrivial example by fixing $\theta(t)$ to be a linear function.

Theorem 1. *Let $\theta(t) = \alpha t + \beta$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0$, then for $b = 4\alpha^2$ and*

$$x(t) = \frac{g}{2\alpha^2}\cos(\alpha t + \beta) + \frac{1}{3} \tag{2.23}$$

the pair $\theta(t)$ and $x(t)$ is a solution of the Elastic Pendulum system.

Proof. We want to proof that $x(t)$ and $\theta(t)$ is a solution of the Elastic Pendulum system below

$$E1 : \ddot{x} + (b - \dot{\theta}^2)x - \dot{\theta}^2 - g\cos\theta = 0 \tag{2.24}$$

$$E2 : \dot{x} + \frac{\ddot{\theta}}{2\dot{\theta}}x + \frac{\ddot{\theta}}{2\theta} + \frac{g\sin\theta}{2\dot{\theta}} = 0 \tag{2.25}$$

If $\theta(t) = \alpha t + \beta$ is fixed when we compute its derivatives with respect to t

$$\dot{\theta}(t) = \alpha$$

$$\ddot{\theta}(t) = 0$$

and substitute them into equations (2.24) and (2.25) we get

$$E1 : \ddot{x} + (b - \alpha^2)x - \theta^2 - g\cos(\alpha t + \beta) = 0 \tag{2.26}$$

$$E2 : \dot{x} = \frac{-g}{2\alpha}\sin(\alpha t + \beta) \tag{2.27}$$

From from E2 in (2.27) we integrate with respect to t and solve for $x(t)$ and we get the following solution

$$x(t) = \frac{g}{2\alpha^2}\cos(\alpha t + \beta) + c. \tag{2.28}$$

for a constant c.

Using the solution $x(t)$ in (2.28) we calculate $\dot{x}$ and $\ddot{x}$

$$\dot{x}(t) = \frac{-g}{2\alpha}\sin(\alpha t + \beta) \tag{2.29}$$

$$\ddot{x}(t) = \frac{-g}{2}\cos(\alpha t + \beta) \tag{2.30}$$

and substitute it into E1 in (2.26) to solve for the unknowns c and b

$$-\frac{g}{2}\cos(\alpha t + \beta) + (b - \alpha^2)(\frac{g}{2\alpha^2}\cos(\alpha t + \beta) + c) - \alpha^2 - g\cos(\alpha t + \beta) = 0 \tag{2.31}$$

$$\Rightarrow (-3 + \frac{b}{\alpha^2} - 1)\frac{g}{2}\cos(\alpha t + \beta) + (b - \alpha^2)c - \alpha^2 = 0 \tag{2.32}$$

For equation (2.32) to equal to zero, we need $(-4 + \frac{b}{\alpha^2}) = 0$ and $(b - \alpha^2)c - \alpha^2 = 0$, hence we need

$$\Rightarrow b = 4\alpha^2 \text{ and} \Rightarrow c = \frac{1}{3}.$$

Therefore, when $b = 4\alpha^2$, for a linear $\theta(t) = \alpha t + \beta$, and

$$x(t) = \frac{g}{2\alpha^2}\cos(\alpha t + \beta) + \frac{1}{3}, \tag{2.33}$$

the pair $\theta(t)$ and $x(t)$ is a class of solutions of the Elastic Pendulum system. □

If we consider $x(t)\cos\theta(t)$ being the signal $f(t)$ then $H[f(t)] = \tilde{f}(t) = x(t)\sin\theta(t)$ and since $\theta(t)$ and $x(t)$ are analytic, this means that $x(t)$ and $\theta(t)$, being the amplitude and frequency for this analytic signal, are also the angular displacement and the varying spring length in the Elastic Pendulum. In the next chapter we reformulate the elastic pendulum system in "rectangular coordinates" with respect to f and $\tilde{f}$.

CHAPTER III

ELASTIC PENDULUM IN RECTANGULAR COORDINATES AND ANALYTIC SIGNAL

Why Hilbert Transform? The Hilbert transform (HT), unlike other integral transforms such as Fourier or Laplace, is not a domain transform meaning it does not involve change of domain. Because our functions are of time domain signal, it is clear that the HT of our signal would be another time-domain signal. And if our signal is real value, then the HT of our signal is also real valued [2].

3.1 Analytic Representation

For our system of the Elastic pendulum, the analytic signal

$$F(t) = f(t) + i\tilde{f}(t) \tag{3.1}$$

where $\tilde{f}(t) = H[f(t)]$ is defined as

$$F(t) = (\ell_0 + x(t))\cos\theta(t) + i(\ell_0 + x(t))\sin\theta(t) \tag{3.2}$$

where $x(t)$ is the amplitude and $\theta(t)$ is the frequency around the equilibrium point. The hilbert transform of $f(t)$ is simply a phase shift of $\pi/2$ which gives us $\tilde{f}(t) = (\ell_0 + x(t))\sin\theta(t)$.

3.2 Polar Notation

A real vibration process $u(t)$, detected by a transducer, is merely one of many potential projections (the real component) of some analytic signal $U(t)$, according to analytic signal theory. The hilbert transform will then conjugate the second projection of the same signal (the imaginary component) $\tilde{u}(t)$ as defined in equation (3.2).

The geometrical representation of an analytic signal is a phasor rotating in the complex plane.

A phasor can be represented as a vector at the origin of the complex plane having a length $A(t)$ (which is the relation to the length of the string of the elastic pendulum) and an angle, or angular displacement, $\psi(t)$ (the relation to the angle of displacement of the elastic pendulum). The initial real signal is defined by $u(t) = A(t)\cos\psi(t)$ as a projection on the real axis. The analytic signal in exponential representation is of the form

$$U(t) = |U(t)|[\cos\psi(t) + i\sin\psi(t)] = A(t)e^{i\psi(t)}, \tag{3.3}$$

its instantaneous amplitude can be determined as

$$A(t) = |U(t)| = \sqrt{u^2(t) + \tilde{u}^2(t)} \tag{3.4}$$

and its instantaneous phase as

$$\psi(t) = arctan\frac{\tilde{u}(t)}{u(t)}. \tag{3.5}$$

The change of coordinates from rectangular $(u, \tilde{u})$ to polar (A, ψ) produces $u(t) = A(t)cos\psi(t)$, $\tilde{u}(t) = A(t)sin\psi(t)$ [2].

3.3 Rectangular Coordinates Transformation

In Theorem 1 and the comments after it, we showed that for linear phase θ and the corresponding amplitude x, the signal $f(t) = x(t)cos\theta(t)$ is analytic and in polar representation. Next, we formulate a system which corresponds to this observation i.e. a necessary condition for analytic signal $f + i\tilde{f} = xe^{i\theta}$ to be such that x and θ are solutions for the elastic pendulum.

Lemma 2. *Let f and $\tilde{f}$ be two real valued functions and $f + i\tilde{f} = (l_0 + x)\cos\theta(t) + i(\ell_0 + x(t))\sin\theta(t)$, then if $\theta(t)$ and $x(t)$ are solutions of the elastic pendulum system the pair of functions*

f and $\tilde{f}$ are solutions of the following system:

$$ff'' + \tilde{f}\tilde{f}'' + \frac{k}{m}(f^2 + \tilde{f}^2 - \ell_0\sqrt{f^2 + \tilde{f}^2}) - g\tilde{f} = 0 \tag{3.6}$$

$$f\tilde{f}'' + \tilde{f}f'' + gf = 0 \tag{3.7}$$

Proof. By direct computation, we consider the Elastic Pendulum system

$$\ddot{x} - (\ell_0 + x)\dot{\theta}^2 + \frac{k}{m}x - g\cos\theta = 0 \tag{3.8}$$

$$\ddot{\theta} + \frac{2\dot{x}}{\ell_0 + x}\dot{\theta} + \frac{g}{\ell_0 + x}\sin\theta = 0. \tag{3.9}$$

with the analytic signal defined previously as

$$F(t) = f(t) + i\tilde{f}(t) = (\ell_0 + x(t))\cos\theta(t) + i(\ell_0 + x(t))\sin\theta(t). \tag{3.10}$$

From definition (3.4) we define the amplitude as

$$x(t) = \sqrt{f^2 + \tilde{f}^2} \tag{3.11}$$

and from definition (3.5) we define the phase as

$$\theta(t) = tan^{-1}(\frac{\tilde{f}}{f}). \tag{3.12}$$

We compute the derivatives for $x(t)$

$$\dot{x}(t) = \frac{ff' + \tilde{f}\tilde{f}'}{\sqrt{f^2 + \tilde{f}^2}}, \tag{3.13}$$

$$\ddot{x}(t) = \frac{f^3 f'' + f^2 \tilde{f}'^2 + \tilde{f} f^2 \tilde{f}'' - 2f\tilde{f}f'\tilde{f}' + f\tilde{f}^2 f'' + \tilde{f}^2 f'^2 + \tilde{f}^3 \tilde{f}''}{(f^2+\tilde{f}^2)^{3/2}}. \tag{3.14}$$

We compute the derivatives for $\theta(t)$

$$\dot{\theta}(t) = \frac{\tilde{f}' f - \tilde{f} f'}{f^2 + \tilde{f}^2}, \tag{3.15}$$

$$\ddot{\theta}(t) = \frac{2f\tilde{f}f'^2 + f\tilde{f}^2\tilde{f}'' - 2f\tilde{f}\tilde{f}'^2 - f^2\tilde{f}f'' - 2f^2 f'\tilde{f}' + 2\tilde{f}^2 f'\tilde{f}' - \tilde{f}^3 f'' + f^3 \tilde{f}''}{(f^2+\tilde{f}^2)^2}. \tag{3.16}$$

We substitute the respective derivatives into the Elastic Pendulum system and note that from $f(t) = (\ell_0 + x(t))\cos\theta(t)$ and $\tilde{f}(t) = (\ell_0 + x(t))\sin\theta(t)$ we solve for $\cos\theta(t)$ and $\sin\theta(t)$ and substitute that as well. After simplification we get the following equations below

$$ff'' + \tilde{f}\tilde{f}'' + \frac{k}{m}(f^2 + \tilde{f}^2 - \ell_0\sqrt{f^2+\tilde{f}^2}) - g\tilde{f} = 0 \tag{3.17}$$

$$f\tilde{f}'' + \tilde{f}f'' + gf = 0. \tag{3.18}$$

□

We have the system in rectangular coordinates, and it requires additional study, which we will leave for future investigation. Next, we consider a different pendulum system to expand the class of solutions in order to achieve the goal of modeling a larger class of functions into an angular displacement of an elastic pendulum.

CHAPTER IV

PENDULUM WITH VARYING PARAMETERS

Next, we will model a given function $\theta(t)$ by a simple pendulum system with friction. We consider the pendulum with a fixed length for a particular interval, and in the next interval the length will change, and again on the next interval the length will change, and it'll continue changing in each interval. It will not be a smooth change, meaning it is not elastic all the time, it is elastic by the sudden jumps. Hence, we want to see how the process will work if we fix the length of the pendulum to be piece-wise constant.

4.1 Numerical Approach

The second equation of motion (2.8) can be approximated by setting $\gamma(t) = \frac{2\dot{x}}{\ell_0+x}$ (which defines friction in the pendulum) and $\omega_0(t) = \frac{g}{\ell_0+x}$ (which is related to $x(t)$, the length of the string/spring) and we get

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0 sin\theta = 0. \tag{4.1}$$

This will be our model to investigate, which is the second equation of the Elastic Pendulum system with parameters γ and ω_0 as defined above, and is also a generalization of the Pendulum equation with an external force.

Let a partition P be defined by the points, $t_0 = 0 < t_1 < ... < t_n = 1$ and intervals $I_j = (t_{j-1}, t_j)$ for $j = 1, 2, 3, .., n$. We are looking for a numerical solution to the pendulum system described above on each I_j intervals with fixed γ_j and ω_{0_j}.

We can relate the process to the elastic pendulum. Once we find γ_j and ω_{0_j} we can obtain an estimate for x and $\dot{x}$. Using $\omega_0(t)$ we solve for $x(t)$ and get equation (4.2) as shown below.

We substitute equation (4.2) into $\gamma(t)$ for $x(t)$ and solve for $\dot{x}(t)$ and we get equation (4.3) as shown below

$$x(t_j) = \frac{g}{\omega_{0_j}} - \ell_0 \tag{4.2}$$

$$\dot{x}(t_j) = \frac{g}{2} \cdot \frac{\gamma_j}{\omega_{0_j}}. \tag{4.3}$$

In our setting, we are given $\theta(t)$, hence the procedure that we use to determine the optimal parameters γ_j and ω_{0_j} on each interval I_j for equation (4.1) can be the least square method.

4.2 Methodology

We are looking for coefficients γ_j and ω_{0_j} that will minimize the L2 norm or the integral in (4.4) on each interval I_j. Let a partition P be defined by the points, $t_0 = 0 < t_1 < ... < t_n = 1$ and intervals $I_j = (t_{j-1}, t_j)$ for $j = 1, 2, 3, .., n$ then

$$L = min(\gamma_j, \omega_{0_j}) \int_{I_j} (\ddot{\theta} + \gamma_j \dot{\theta} + \omega_{0_j} sin\theta)^2 dt. \tag{4.4}$$

To simplify the expression we use definitions such as $\|\theta\|_2 = (\int_{I_j} \theta^2 dt)^{\frac{1}{2}}$ and $\langle \dot{\theta}, sin\theta \rangle = \int_{I_j} \dot{\theta} sin\theta dt$ and get

$$L = \|\ddot{\theta}\|_2^2 + 2\langle \ddot{\theta}, \dot{\theta} \rangle \gamma + 2\langle \ddot{\theta}, sin\theta \rangle \omega_0 + \gamma^2 \|\dot{\theta}\|_2^2 + 2\gamma \langle \dot{\theta}, sin\theta \rangle \omega_0 + \omega_0^2 \|sin\theta\|_2^2. \tag{4.5}$$

Now we minimize (4.5) with respect to the unknowns as shown below for γ

$$\frac{\partial L}{\partial \gamma} = 2\langle \ddot{\theta}, \dot{\theta} \rangle + 2\gamma \|\dot{\theta}\|_2^2 + 2\langle \dot{\theta}, sin\theta \rangle \omega_0 = 0 \tag{4.6}$$

and ω_0

$$\frac{\partial L}{\partial \omega_0} = 2\langle \ddot{\theta}, sin\theta \rangle + 2\omega \langle \dot{\theta}, sin\theta \rangle + 2\omega_0 \|sin\theta\|_2^2 = 0. \tag{4.7}$$

Then (4.6) and (4.7) gives us the following system for a partition P on I_j

$$\begin{bmatrix} 2\|\dot{\theta}\|_2^2 & 2\langle\dot{\theta}, sin\theta\rangle \\ 2\langle\dot{\theta}, sin\theta\rangle & 2\|sin\theta\|_2^2 \end{bmatrix} \begin{bmatrix} \gamma_j \\ w_{0_j} \end{bmatrix} = \begin{bmatrix} -2\langle\ddot{\theta}, \dot{\theta}\rangle \\ -2\langle\ddot{\theta}, sin\theta\rangle \end{bmatrix}. \tag{4.8}$$

There is a condition that $\omega_0 > 0$, since it is related to the length of the string, hence, it cannot be negative. We can check if the condition follows by solving for ω_0. We set the system in (4.8) as

$$Ax = b$$

and use inverse matrix to solve for x

$$x = A^{-1}b$$

and we get the following below

$$\begin{bmatrix} \gamma \\ \omega_0 \end{bmatrix} = \begin{bmatrix} \frac{\|sin\theta\|_2^2}{2(\|\dot{\theta}\|_2^2)(\|sin\theta\|_2^2)-2(\langle\dot{\theta},sin\theta\rangle)(\langle\dot{\theta},sin\theta\rangle)} & \frac{-\langle\dot{\theta},sin\theta\rangle}{2(\|\dot{\theta}\|_2^2)(\|sin\theta\|_2^2)-2(\langle\dot{\theta},sin\theta\rangle)(\langle\dot{\theta},sin\theta\rangle)} \\ \frac{-\langle\dot{\theta},sin\theta\rangle}{2(\|\dot{\theta}\|_2^2)(\|sin\theta\|_2^2)-2(\langle\dot{\theta},sin\theta\rangle)(\langle\dot{\theta},sin\theta\rangle)} & \frac{\|\dot{\theta}\|_2^2}{2(\|\dot{\theta}\|_2^2)(\|sin\theta\|_2^2)-2(\langle\dot{\theta},sin\theta\rangle)(\langle\dot{\theta},sin\theta\rangle)} \end{bmatrix} \begin{bmatrix} -2\langle\ddot{\theta}, \dot{\theta}\rangle \\ -2\langle\ddot{\theta}, sin\theta\rangle \end{bmatrix}. \tag{4.9}$$

Now we solve for ω_0 and obtain the following solution

$$\omega_0 = \frac{\langle\dot{\theta}, sin\theta\rangle\langle\ddot{\theta}, \dot{\theta}\rangle}{\|\dot{\theta}\|_2^2\|sin\theta\|_2^2 - \langle\dot{\theta}, sin\theta\rangle\langle\dot{\theta}, sin\theta\rangle} - \frac{\|\dot{\theta}\|_2^2\langle\ddot{\theta}, sin\theta\rangle}{\|\dot{\theta}\|_2^2\|sin\theta\|_2^2 - \langle\dot{\theta}, sin\theta\rangle\langle\dot{\theta}, sin\theta\rangle}. \tag{4.10}$$

As mentioned before we want ω_0 to be positive. Therefore, one can use the equation (4.10) to check if the condition $\omega_0 > 0$ is not satisfied.

4.3 Algorithm Outline

The least square method is applied on partition P on each interval I_j to find γ_j and ω_{0_j}. On I_1, we solve the initial value problem

$$\ddot{y}_1 + \gamma_1 \dot{y}_1 + \omega_{0_1} \sin y_1 = 0 \tag{4.11}$$

with initial conditions

$$y_1(t_1) = f(t_0) \tag{4.12}$$

$$\dot{y}_1(t_1) = \dot{f}(t_0). \tag{4.13}$$

Inductively, we repeat the process for $j = 2, 3, ..., n$ with initial conditions on I_j

$$y_j(t_j) = y_{j-1}(t_{j-1}) \tag{4.14}$$

$$\dot{y}_j(t_j) = \dot{y}_{j-1}(t_{j-1}). \tag{4.15}$$

Hence, I_1 is the first interval. We solve it on the first interval with initial conditions coming from the given function f, the given function is $\theta(t)$, this is defined in equations (4.12) and (4.13). Then, on the next intervals, the initial conditions won't be from $\theta(t)$, it'll be equations (4.14) and (4.15) on y_{j-1} for $j = 2, 3, ..., n$ which are the initial conditions from the function before. We inductively repeat this process on each interval until we have done so for all intervals. We have two different examples for our function $\theta(t)$ in the next section where we present the numerical results using the process explained.

CHAPTER V

NUMERICAL RESULTS

The goal is to consider functions that are not solutions to the pendulum and that are standard functions in signal processing. The numerical work is done by using Matlab, the codes can be found in Appendix. We present two numerical results in the following sections.

5.1 Elliptic Cosine

The first example is the function $\theta(t)$ as an approximation of an elliptic cosine [5],

$$\theta(t) = \frac{\sqrt{1+L}cos(\sqrt{1+L})t}{\sqrt{1+Lcos(\sqrt{1+L}t)}}. \tag{5.1}$$

The time variable $t \in [0,1]$. For reference we include the pendulum with a fixed length rod on one interval for $L = 80$ shown in figure 5.1.

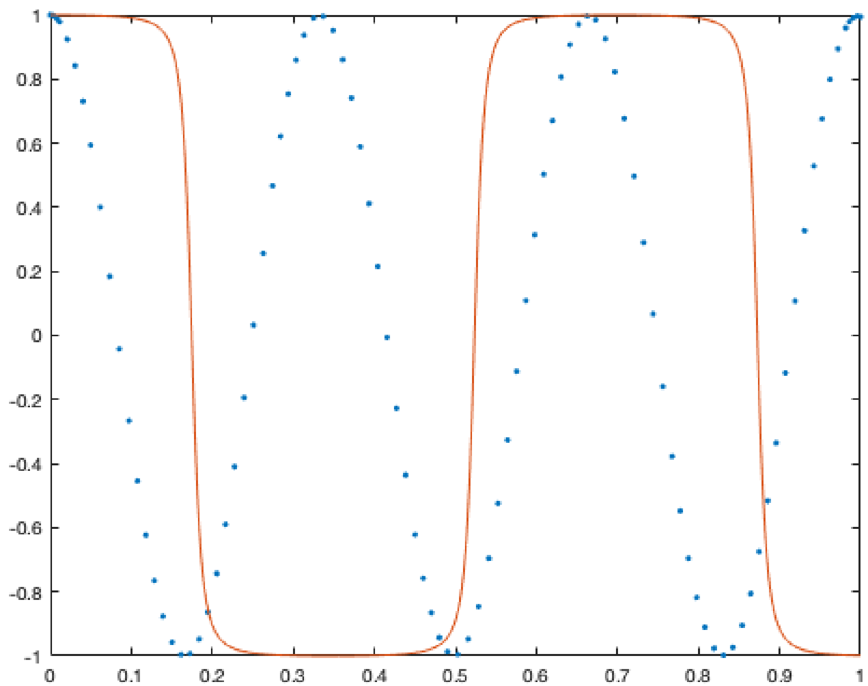

Figure 5.1: Signal approximation 1.

In Figure 5.1 the continuous line is the function $\theta(t)$ and the dotted line is the approximation obtained by the algorithm described above on one interval with a constant length. We see that the two lines have nothing in common, so the length being constant will not work. Now we consider a pendulum with varying length:

The time variable $t \in [0,1]$ in partition P where the partitions are considered equidistant. For $L = 80$ and 30 sub-intervals the results are listed in Figures 5.2, 5.3 and 5.4.

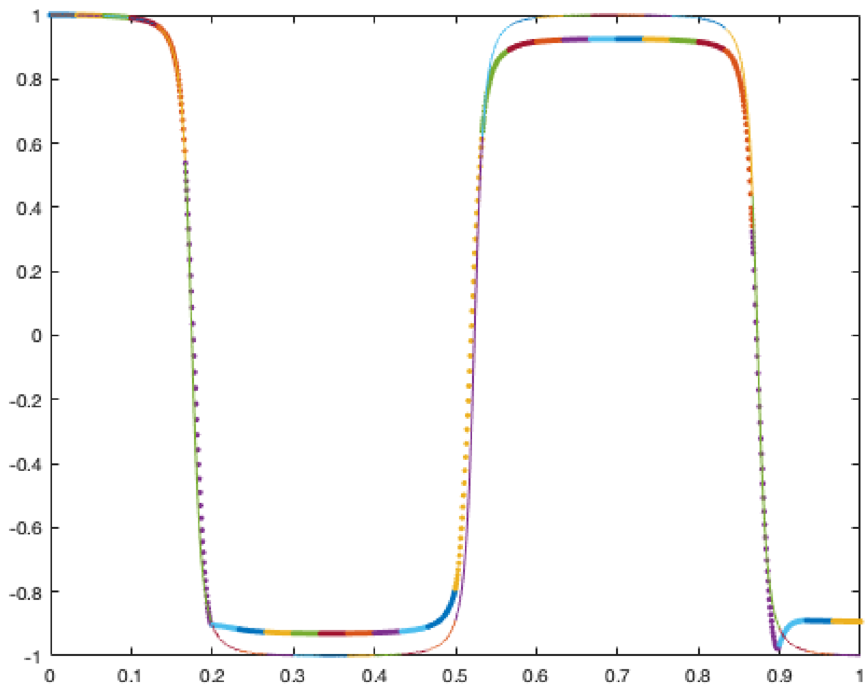

Figure 5.2: Signal approximation 2.

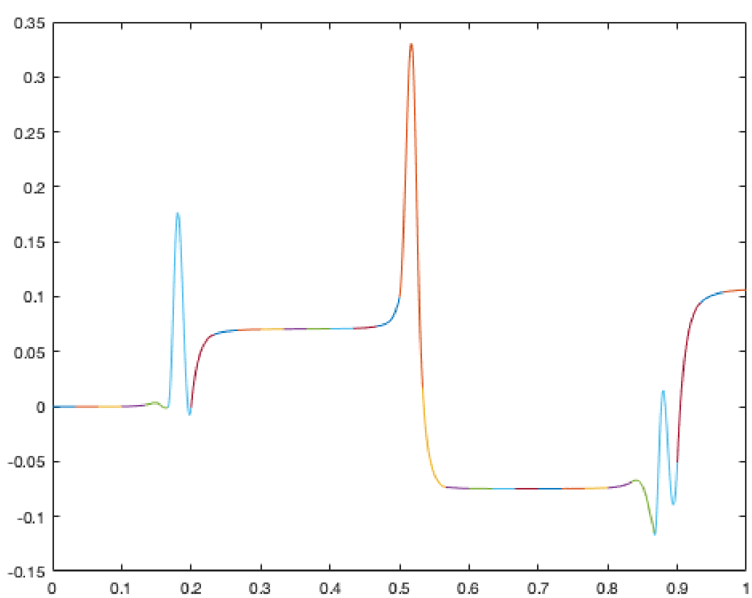

Figure 5.3: Error 1.

In Figure 5.2 the continuous line is the approximation of the elliptic cosine function $\theta(t)$ and the dotted line is the approximation obtained by the algorithm described above. The colorful lines in the dotted lines are the solutions on different intervals. Figure 5.3 shows the difference between the function θ and the approximant. We see that our procedure preserves the general characteristics of the function and is a very good approximation. We used 30 sub-intervals, and although our solution is a good approximation, we have error at the end, in which we hope to improve by using more sub-intervals.

Now we take a look at amplitude approximation of the function in Figure 5.4.

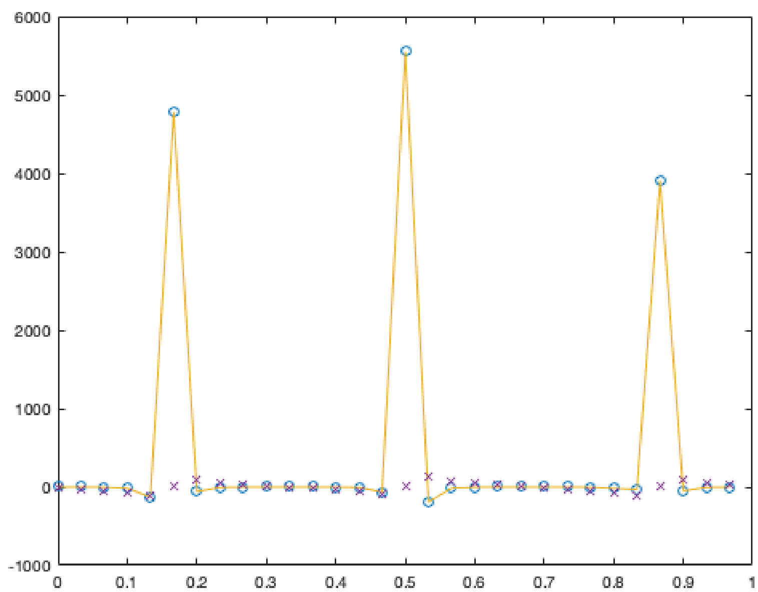

Figure 5.4: Amplitude approximation 1.

In Figure 5.4 the continuous line is the interpolated w_0. The x (crosses) are the γ values, and the circles are the w_0 values on the 30 sub intervals. We can conclude from this graph that these peaks in space correspond to where the original function has rapid changes as displayed in Figure 5.2. Hence, it is necessary for the length of the string to have these varying changes. From this example, we conclude that with small modifications, the approximation of elliptic cosine as our function $\theta(t)$ and a pendulum with varying length, we might be able to model it as a solution of the elastic pendulum.

5.2 Chirp Type Function

Now, we consider a second example as the function $\theta(t)$

$$\theta(t) = e^t cos(10t^2). \tag{5.2}$$

We have the same setting as before, the time variable $t \in [0, 1]$ in partition P where the partitions are considered equidistant. For $L = 80$ and 30 sub-intervals the results are shown below in Figures 5.5, 5.6, and 5.7.

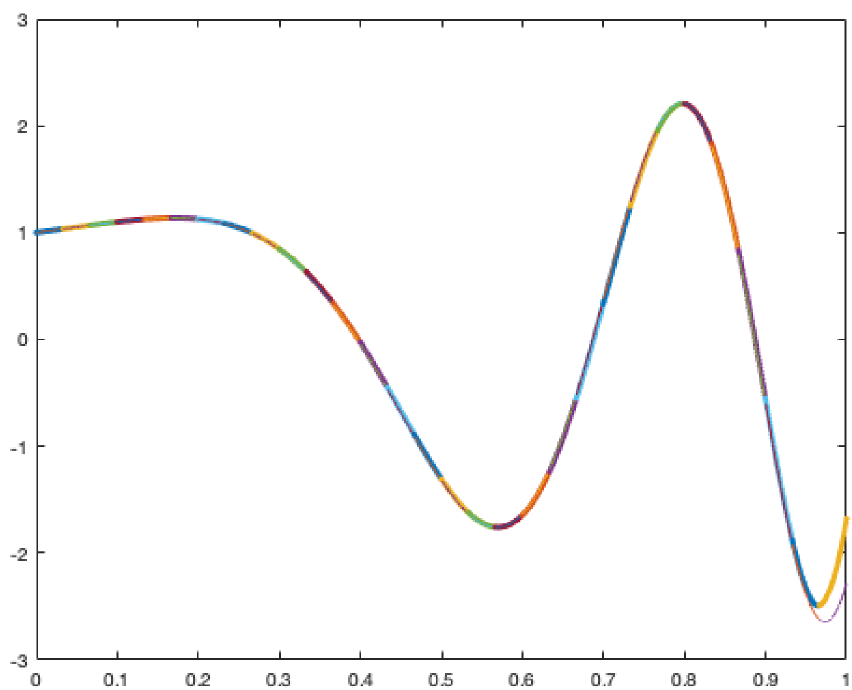

Figure 5.5: Signal approximation 3.

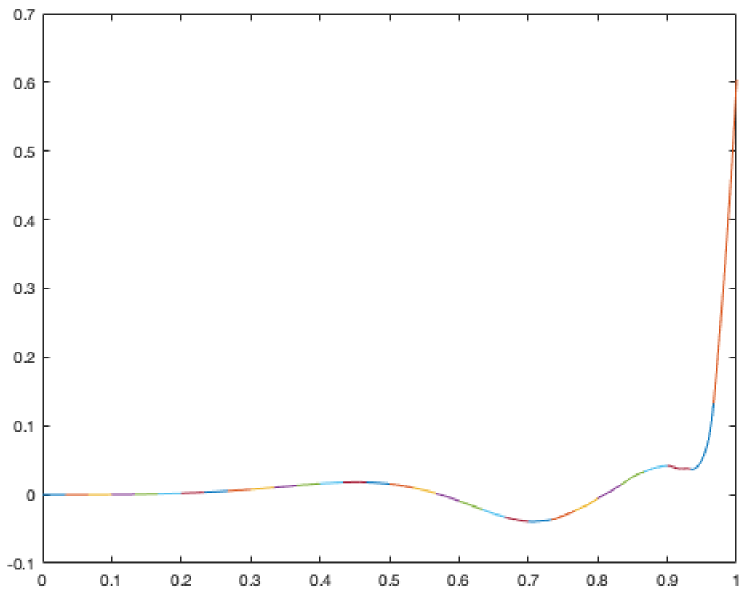

Figure 5.6: Error 2.

In Figure 5.5 the continuous line is the chirp type function $\theta(t)$ and the dotted line is the approximation obtained by the algorithm described above. The colorful lines in the dotted lines are the solutions on different intervals. Figure 5.6 shows the difference between the function θ and the approximant. We see that our procedure preserves the general characteristics of the function and is an almost perfect approximation. We used 30 sub-intervals, and although our solution is a good approximation, we have error at the end, in which we hope will improve by using more sub-intervals.

Now we take a look at amplitude approximation of the function in Figure 5.7.

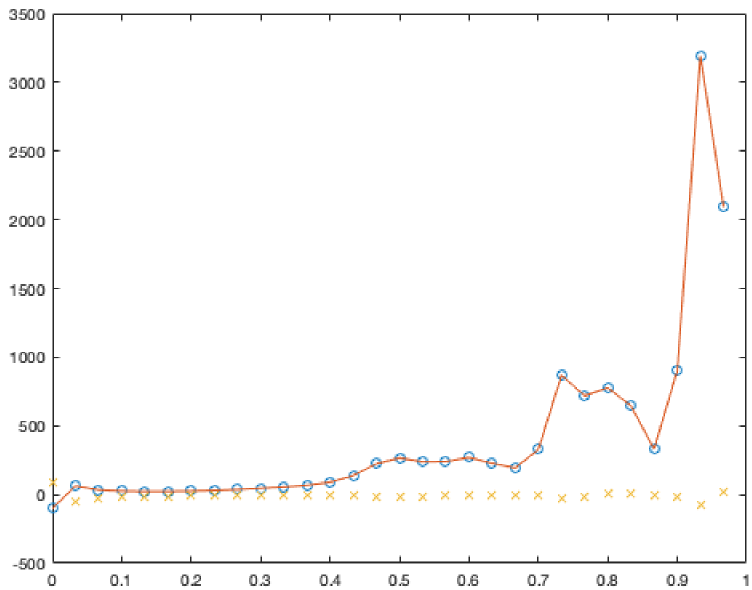

Figure 5.7: Amplitude approximation 2.

In Figure 5.7 the continuous line is the interpolated w_0. The x (crosses) are the γ values, and the circles are the ω_0 values on the 30 sub intervals. Again, we come to the same conclusion from this graph these peaks in space correspond to where the original function has rapid changes as displayed in Figure 5.5. Hence, it is necessary for the length of the string to have these varying changes. From this example, we conclude that with small modifications, the chirp type function $\theta(t)$ and a pendulum with varying length, we might be able to model it as a solution of the elastic pendulum. Note, that in figures 5.4 and 5.7 we visualize what is the behavior of the elastic string (spring) to model it in this physical setting, and we can see clearly that it requires the string (spring) part to change rapidly, hence, we need elasticity.

CHAPTER VI

CONCLUSION

In this thesis we investigated the relations between the analytic signals and different pendulum systems. The pendulum systems considered were the elastic pendulum and the simple pendulum with varying coefficients. For the elastic pendulum we found analytic solutions in the case of the linear phase and reformulated the system in signal processing setting. In chapter IV we considered a numerical procedure for modeling a given function $\theta(t)$ as a simple pendulum with friction. We developed an experimental procedure to piece-wise approximate bounded functions on a partition of a finite interval. On each sub-interval the function is approximated by a solution of a Pendulum system. The parameters of the corresponding differential equations are determined by optimization on each sub-interval. The smoothness of the approximation is controlled by the initial conditions provided by the given function.

This work has a variety of potential future investigation. The most important problem is to solve the fourth order nonlinear integro-differential equation from chapter II. By solving this equation we will obtain a solution of the elastic pendulum system. As well as solving for f and $\tilde{f}$ from the elastic pendulum system in rectangular coordinates that we derived in chapter IV. Solving for f and $\tilde{f}$ will give us the solution of the system in signal processing setting. Although we successfully modeled a wide class of functions as a simple pendulum with friction, the question we can further study is what kind of functions can be modeled? Can all functions be modeled as a simple pendulum with friction, or only certain functions? This is a future problem that requires more investigation.

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APPENDIX

APPENDIX

MATLAB CODES

In this appendix we list the Matlab procedures used in chapter V.

Least square method (LSM)

The function leastsq takes as input the interval $[a,b]$, a step dt and a function f. The outputs are the two parameters k and w, corresponding to ....

```
function [k w] = leastsq(a,b,t,dt,f)

% compute the derivatives of f.
dz=diff(f)/dt;df(end+1)=df(end);x
ddf=diff(df)/dt;ddf(end+1)=ddf(end);

%compute the coefficients in the LS matrix.
R=cumsum(df.*df)*dt;M(1,1)=R(end);
R=cumsum(df.*sin(f))*dt;M(2,1)=R(end);M(1,2)=M(2,1);
R=cumsum(sin(f).*sin(f))*dt;M(2,2)=R(end);
R=cumsum(ddf.*df)*dt;B(1,1)=-R(end);
R=cumsum(ddf.*sin(f))*dt;B(2,1)=-R(end);
```

% compute the parameters k, w.

```
r=M B;
k=r(1);w=r(2);

end;
```

Pendulum Equation with friction (PEF)

Once the parameters k, w are obtained from leastsq.m the solution of the corresponding *PEF* is obtained by using *ODE*45 with the following code.

```
function dy = pendul(t,y,k,w)
dy=zeros(2,1);
dy(1) = y(2);
dy(2)=-k*y(2)-w*sin(y(1));

end;
```

Main Procedure

The main function is elpend.m. The inpput parameters are the number of partition N, and the parameter in the elliptic cosine L.

```
function elpend=elpend(N,L)

%elliptic cosine function initial conditions
```

```
i0=1;
i1=0;
%chirp type function phase initial conditions
i0=1;
i1=1;

ind=1;
st=1/N;
for a=0:st:(1-st)
b=a+st;
dt=(b-a)/1000;t=a:dt:b;

% elliptic cosine function
% f=sqrt(1+L)*cos(sqrt(1+L)*t)./sqrt(1+L*cos(sqrt(1+L)*t).^2)+0*cos(17*t);

% chirp type function
% f=exp(t).*cos(10*t.^2);

[k w] =leastsq(a,b,t,dt,f);
[T Y] = ode45(@(t,y)pendul(t,y,k,w),[a b],[i0 i1]);
% Figure 1 progressively shows the original function f and the approximant Y.
hold on;figure(1);plot(T,Y(:,1),'.',t,f);
% The initial conditions for the next iteration are set next.
gam(ind)=k;len(ind)=w;ind=ind+1;
i0=Y(end,1);i1=Y(end,2);
pause;
end;
```

```
t=0:st:1-st;
% Figure 2 shows the parameters k_j and w_j.
figure(2);plot(t,len,'o',t,len,t,gam,'x');
elpend=len;
end;
```

BIOGRAPHICAL SKETCH

Brenda Lee Garcia was born in Brownsville, Texas. She attended James Pace Early College High School and after graduating in 2015 she entered The University of Texas Rio Grande Valley in fall 2015. She received a Bachelor of Science in Mathematics with a teaching concentration in secondary Math in Spring 2020. In Fall 2020 she entered the Mathematics graduate program with a concentration in Applied Mathematics at The University of Texas Rio Grande Valley. She worked for the School of Mathematical and Statistical Sciences as a graduate research assistant her first year, and a graduate teaching assistant her second year. She received a Master of Science in Mathematics on May 2022. She can be contacted by email: brendalee0828@gmail.com.