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Solitary and Periodic Wave Solutions for Several Short Wave Model Equations

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SOLITARY AND PERIODIC WAVE SOLUTIONS FOR SEVERAL SHORT WAVE MODEL EQUATIONS

A Thesis

by

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ABSTRACT

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We study the periodic and solitary wave solutions to several short wave model equations arising from a so-called β-family equation for $\beta = 1,2,4$. These are integrable cases which possess Lax pair and multi-soliton solutions. By phase plane analysis, either the loop or cuspon type solutions are predicted. Then, by introducing a hodograph, or reciprocal, transformation, a coupled system is derived for each β. Applying a travelling wave setting, we are able to find the periodic solutions exactly expressed in terms of Jacobi Elliptic functions. In the limiting cases of modulus k=1, they all converge to the known solitary waves.

DEDICATION

I dedicate this paper to my research advisor, Dr. Baofeng Feng, without whom this would not have been possible;

To my family, especially, to my dad, Vladimir A. Stukopin, who taught me the essence of mathematics;

To my wife, Mee-Lai C.W. Alvarado, who was there for me every step of the way.

ACKNOWLEDGMENTS

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I would also like to thank the committee members: Dr. Zhijun Qiao, Dr. Vesselin Vatchev and Dr. Balogh for agreeing to serve on my thesis defense committee and for teaching me a lot about Fourier Analysis, Differential Equations and Linear Algebra, respectively. I am also extremely grateful for Dr. Poletaeva's extracurricular lectures on Lie Algebras during Fall 2019.

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CHAPTER I

INTRODUCTION

This work examines several short wave model equations. The main goal is to show that the following, so-called β-family, equation,

$$u_{xt} = u + \beta u u_{xx} + u_x^2 \tag{1.1}$$

where $u = u(x,t)$ represents a scalar function of x and t, subscripts x and t denote partial differentiation, has an interesting set of solutions with structures that vary depending on the nonlinear coefficient β. From a physical standpoint, the β-family of equations appears in the description of the short-wave behavior of nonlinear systems [3]. It is known that when β is equal to 1,2, or 4, Eq.(1.1) is integrable, and, therefore is of great interest. Specifically, what we intend to do is to show that Eq.(1.1) admits the so-called loop type periodic and solitary solution when $\beta = 1$, and cuspon type periodic and solitary solutions when $\beta = 2,4$.

Back in 1978, Ostrovsky derived an equation for weakly nonlinear surface waves in a rotating ocean:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3}\right) = \gamma u$$

where c_0 is the velocity of dispersionless linear waves, α is the nonlinear coefficient, β and γ are the dispersion coefficients (for Coriolis and Bousinnesq dispersion, respectively.) This equation combines effects of small nonlinearity with weak dispersion. When $\gamma = 0$, we get the well-known KdV equation and when $\beta = 0$, we get the following equation:

$$\frac{\partial}{\partial X}\left(\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial X} + \alpha u\frac{\partial u}{\partial X}\right) = \gamma u$$

Letting $x = X - Vt$, $v = \frac{-\alpha u}{\gamma}$, we get the following:

$$\frac{\gamma}{\alpha}V v_{xx} - \frac{c_0\gamma}{\alpha}v_{xx} + \frac{\gamma^2}{\alpha}v_x^2 + \frac{\gamma}{\alpha}vv_{xx} = \frac{-\gamma^2}{\alpha}v$$

Divide by $\frac{\gamma^2}{\alpha}$:

$$\frac{V}{\gamma}v_{xx} - \frac{c_0}{\gamma}v_{xx} + v_x^2 + vv_{xx} = -v$$

Letting $W = \frac{c_0 - V}{\gamma}$, we get

$$(v - W)v_{xx} + v_x^2 + v = 0 \tag{1.2}$$

In 2005, Stepanyants provided stationary and travelling wave solutions to the above equation [7]. Shortly, Matsuno considered the following equation [5]:

$$\alpha u_x - u_{xxt} = \beta u_x u_{xx} + u u_{xxx} \tag{1.3}$$

He demonstrated that it stems from the short-wave limit of the following PDE:

$$u_t + \alpha u_x - u_{xxt} + (\beta + 1)uu_x = \beta u_x u_{xx} + u u_{xxx} \tag{1.4}$$

In fact, when $\beta = 2$, Eq.(1.4) becomes the Camassa-Holm (CH) equation, and when $\beta = 3$, it reduces to the Degasperis-Procesi (DP) equation. We notice that Eq.(1.1), upon differentiating w.r.t. x, becomes

$$u_{xxt} = u_x + (\beta + 2)u_x u_{xx} + \beta u u_{xxx}$$

which in turn can be transformed into Eq.(1.3) by the following series of transformations: $u = -\alpha\beta u$, $\partial_t = \beta\partial_t$ and $\partial_x = \frac{1}{\alpha\beta}\partial_x$. We get

$$\frac{1}{\alpha\beta}u_{xxt} = \frac{1}{\beta}u_x - \frac{\beta+2}{\beta}\frac{1}{\alpha\beta}u_x u_{xx} - \frac{1}{\alpha\beta}uu_{xxt} \implies$$
$$\implies u_{xxt} = \alpha u_x - \frac{\beta+2}{\beta}u_x u_{xx} - uu_{xxt}$$

Letting $\frac{\beta+2}{\beta} \to \beta$, yields Eq.(1.3). Therefore, Eq.(1.1) is closely related to the CH and DP equations in the sense that, for $\beta = 2$ and $\beta = 1$, it is the short-wave limit of the CH and DP equations, respectively. In 2017, Andrew N.W. Hone, Vladimir Novikov and Jing Ping Wang classified the integrable nonlinear partial differential equations of second order with quadratic and cubic nonlinear terms of the following general form:

$$u_{xt} = u + c_0 u^2 + c_1 u u_x + c_2 u u_x x + c_3 u_x^2 + d_0 u^3 + d_1 u^2 u_x + d_2 u^2 u_x x + d_3 u u_x^2 \tag{1.5}$$

namely, from the viewpoint of integrability. Their main result was proving the following theorem: If Eq.(1.5) possesses an infinite hierarchy of local symmetries, then up to rescaling $u \to \lambda u$, $x \to \mu x$, $t \to \nu t$, it is one of the following list:

$$u_{xt} = u + (u^2)_{xx} \tag{1.6}$$

$$u_{xt} = u + (u^3)_{xx} \tag{1.7}$$

$$u_{xt} = u + 4uu_{xx} + u_x^2 \tag{1.8}$$

$$u_{xt} = u + (u^2 - 4u^2 u_x)_x \tag{1.9}$$

$$u_{xt} = u + 2uu_{xx} + u_x^2 \tag{1.10}$$

$$u_{xt} = u + u^2 u_{xx} + uu_x^2 \tag{1.11}$$

$$u_{xt} = u + \alpha(2uu_{xx} + u_x^2) + \beta(u^2 u_{xx} + uu_x^2),\ \alpha\beta \neq 0 \tag{1.12}$$

It is clear that equation (1.6) can be converted into the $\beta = 1$ case of Eq.(1.1) via $u \to 2u$ rescaling. Equations (1.10) and (1.8) correspond to $\beta = 2$ and $\beta = 4$ cases, respectively and were derived in the process of studying the short-wave dynamics of surface gravity waves in [4]. Fairly recently, Matsuno published a paper [6], where he analyzed the latter equation. Solutions to the former two equations can be found in Matsuno's earlier paper that was published in 2006 [5]. In this work, we intend to obtain the periodic solutions for $\beta = 1, 2$ and $\beta = 4$. In the limiting case with period going to infinity, these solutions converge to the solutions derived by Matsuno.

CHAPTER II

SOLITARY WAVE SOLUTIONS

In this chapter, we are primarily concerned with finding solitary wave solutions to the β-family of equations. In order to do that, we will assume that the equation admits so-called traveling wave solutions. In many cases, this method allows for reduction of the problem to finding solutions of an ordinary differential equation of order one less than the order of the original partial differential equation [2]. Even though this entails a loss in terms of the number of possible solutions to the original partial differential equation, it provides a very good idea about the structure of the solutions to the equation in question. It is often impossible to directly take the integral of the dependent function and find periodic solutions. In this case, one may look for some sort of boundary conditions which would reduce the complexity of the problem and allow for finding solitary wave solutions.

2.1 Propagating Wave Solutions

We focus on propagating wave solutions to Eq.(1.1):

$$u_{xt} = u + \beta u u_{xx} + u_x^2$$

depending on one variable $u = u(\eta)$ where $\eta = x - ct$ (which is position and time dependent, c represents the speed of the wave.) We have that $u_t = -cu_\eta$, $u_x = u_\eta$. Then Eq.(1.1) becomes

$$(\beta u + c)u_{\eta\eta} + u_\eta^2 + u = 0 \tag{2.1}$$

2.1.1 Case 1: $\beta u + c > 0$

Assume $\beta u + c > 0$ and consider Eq.(2.1). We note that $(\beta u + c)u_{\eta\eta} + u_\eta^2$ can be viewed as

$$[(\beta u+c)^\alpha u_\eta]_\eta = \beta\alpha(\beta u+c)^{\alpha-1}u_\eta^2 + (\beta u+c)^\alpha u_{\eta\eta} = (\beta u+c)^{\alpha-1}[(\beta u+c)u_{\eta\eta} + \beta\alpha u_\eta^2]$$

So, it is clear that we need to have $\alpha = \frac{1}{\beta}$. Multiplying Eq.(2.1) by $(\beta u+c)^{\frac{1}{\beta}-1}$ and using the above equality, we get

$$[(\beta u+c)^{\frac{1}{\beta}} u_\eta]_\eta + u(\beta u+c)^{\frac{1}{\beta}-1} = 0$$

Now, we multiply the above equation by $(\beta u+c)^{\frac{1}{\beta}} u_\eta$ and get

$$((\beta u+c)^{\frac{1}{\beta}} u_\eta)_\eta * (\beta u+c)^{\frac{1}{\beta}} u_\eta + u u_\eta * (\beta u+c)^{\frac{2}{\beta}-1} = 0$$

Integrating w.r.t. η, we get

$$u_\eta^2 + \frac{2u-c}{2+\beta} + \frac{2D}{(\beta u+c)^{\frac{2}{\beta}}} = 0 \tag{2.2}$$

where D is a constant of integration.

2.1.2 Case 2: $\beta u + c < 0$

Assume that $\beta u + c < 0$. We multiply Eq.(2.1) by -1

$$(-\beta u - c)u_{\eta\eta} - u_\eta^2 - u = 0 \tag{2.3}$$

We rewrite $(-\beta u - c)u_{\eta\eta} - u_\eta^2$ as

$$[(-\beta u-c)^\alpha u_\eta]_\eta = -\beta\alpha(-\beta u-c)^{\alpha-1}u_\eta^2 + (-\beta u-c)^\alpha u_{\eta\eta} = (-\beta u-c)^{\alpha-1}[(-\beta u-c)u_{\eta\eta} - \beta\alpha u_\eta^2]$$

So, it is clear that we need to have $\alpha = \frac{1}{\beta}$. Multiplying Eq.(2.3) by $(-\beta u-c)^{\frac{1}{\beta}-1}$ and using the above equality, we get

$$[(-\beta u-c)^{\frac{1}{\beta}} u_\eta]_\eta - u(-\beta u-c)^{\frac{1}{\beta}-1} = 0$$

Now, we multiply the above equation by $(-\beta u-c)^{\frac{1}{\beta}}u_\eta$ and get

$$((-\beta u-c)^{\frac{1}{\beta}}u_\eta)_\eta*(-\beta u-c)^{\frac{1}{\beta}}u_\eta-uu_\eta*(-\beta u-c)^{\frac{2}{\beta}-1}=0$$

Integrating w.r.t. η, we get

$$u_\eta^2+\frac{2u-c}{2+\beta}+\frac{2D'}{(-\beta u-c)^{\frac{2}{\beta}}}=0 \tag{2.4}$$

where D' is a constant of integration (not necessarily the same as D.)

2.2 Solitary Wave Solution for $\beta = 1$

Letting $\beta=1$ and $D=D'$, equations (2.2) and (2.4) become the same:

$$u_\eta^2+\frac{2u-c}{3}+\frac{2D}{(u+c)^2}=0$$

By fixing $c=-1$, the phase plane plot is given in Fig. 2.1 for different values of $D=-1/6,-1/10$. The blue line corresponds to a solitary wave solution while the red line predicts a periodic solution. We will give more explanations as follows. We consider the case, when the period of the propagating wave becomes very large, which allows us to view it as a solitary wave. In other words, we take the following boundary conditions: as $\eta\to\pm\infty$, we have that $u,u_\eta\to 0$. Then, from the above equation, we get:

$$\frac{-c}{3}+\frac{2D}{c^2}=0\,,$$

which gives $D=\frac{c^3}{6}$. Then, the above equation becomes:

$$u_\eta^2+\frac{2u-c}{3}+\frac{c^3}{3(u+c)^2}=0 \tag{2.5}$$

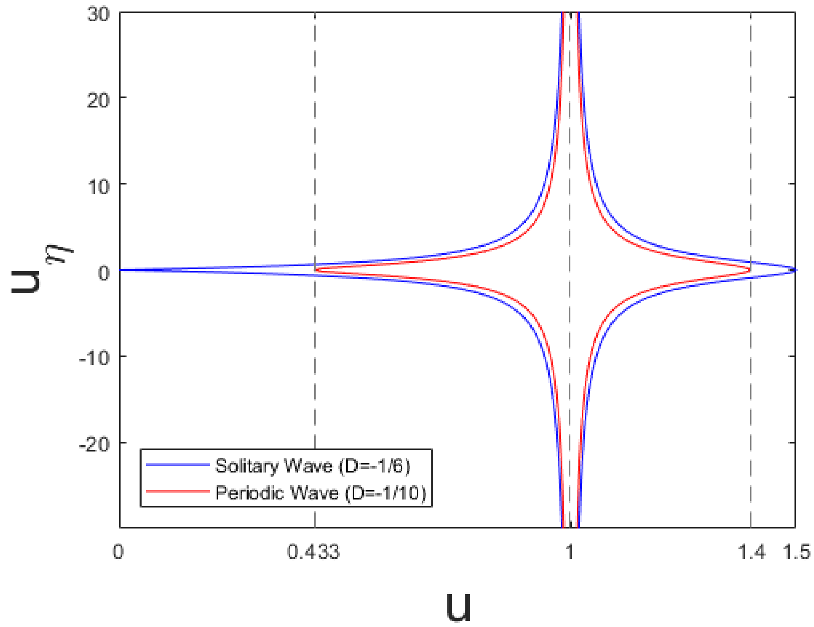

Figure 2.1: The profile of u_η for $\beta = 1$, $c = -1$. The blue line represents a solitary wave solution. The red line represents a periodic wave solution.

Now, we notice that (2.5) is separable. We let $u = cv$, then equation (2.5) becomes:

$$cv_\eta^2 + \frac{2v-1}{3} + \frac{1}{3(v+1)^2} = 0 \tag{2.6}$$

After performing the necessary algebraic operations, we arrive at

$$\int \frac{dv}{\sqrt{\frac{-2v+1}{3} - \frac{1}{3(v+1)^2}}} = \pm\frac{1}{\sqrt{c}}(\eta + \eta_0)$$

Integrating and letting $\eta_0 = 0$, yields

$$\frac{v\left(\sqrt{3}(2v+3) - 2\sqrt{2v+3}\cdot\tanh^{-1}\left(\sqrt{\frac{2v}{3}+1}\right)\right)}{(v+1)\sqrt{\frac{-v^2(2v+3)}{(v+1)^2}}} = \pm\frac{\eta}{\sqrt{c}}$$

Now, assuming $v \neq -1$ and simplifying, we have that

$$\sqrt{-c(6v+9)} - \sqrt{-4c} \cdot \tanh^{-1}\left(\sqrt{\frac{2v}{3}+1}\right) = \pm\eta \tag{2.7}$$

Substituting $v = u/c$ back into (2.7), yields an implicit solution for u in terms of η. Fig. 2.2 shows how the implicit solution looks like, which is a loop type solitary wave solution.

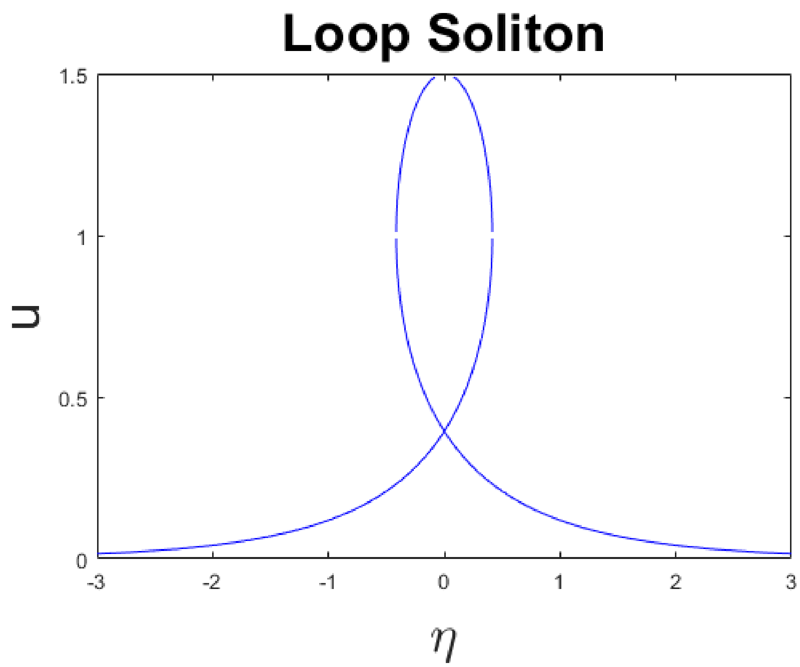

Figure 2.2: The profile of u for $\beta = 1$, $c = -1$.

2.3 Solitary Wave Solution for $\beta = 2$

We proceed to the case $\beta = 2$. It is obvious that the equations (2.2) and (2.4) to be consistent by putting $D' = -D$. Then the two cases become one and we have:

$$u_\eta^2 + \frac{2u-c}{4} + \frac{2D}{(2u+c)} = 0 \tag{2.8}$$

Fixing the value of $c = -1$, the phase plane plot is shown in Fig. 2.3 for different values of $D = 1/8, 1/10$. Same as the case of $\beta = 1$, the blue line predicts the solitary wave while the red line predicts the periodic wave. Both of them have singularity at $u = 0.5$ (cuspon). The solitary wave will be confirmed below.

We use equation (2.8) to plot u_η as a function of u implicitly:

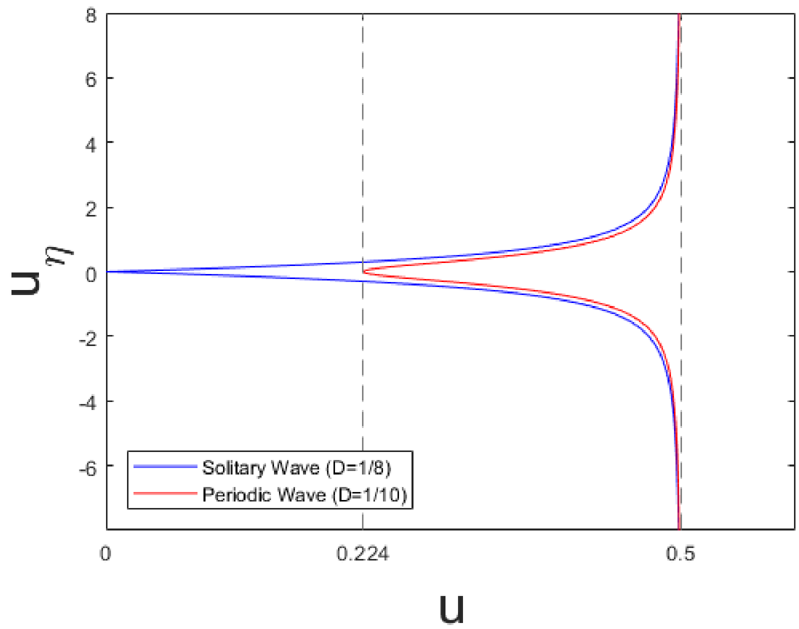

Figure 2.3: The profile of u_η for $\beta = 2$, $c = -1$. The blue line represents a solitary wave solution. The red line represents a periodic wave solution.

In view of the boundary conditions $u, u_\eta \to 0$ as $\eta \to \pm\infty$ for a solitary wave solution, we must have

$$\frac{-c}{4} + \frac{2D}{c} = 0$$

which gives $D = \frac{c^2}{8}$, then equation (2.8) becomes:

$$u_\eta^2 + \frac{2u-c}{4} + \frac{c^2}{4(2u+c)} = 0 \tag{2.9}$$

Again, we notice that (2.9) is separable and let $u = cv$. After performing the necessary algebraic operations, we arrive at

$$\int \frac{dv}{\sqrt{\left(\frac{-2u+1}{4}\right) - \frac{1}{4(2v+1)}}} = \pm\frac{1}{\sqrt{|c|}}(\eta+\eta_0)$$

Integrating and letting $\eta_0 = 0$, yields

$$\frac{2\sqrt{\frac{-v^2}{2v+1}}\left(-2v+\sqrt{2v+1}\cdot\tanh^{-1}\left(\sqrt{2v+1}\right)-1\right)}{v} = \pm\frac{\eta}{\sqrt{c}}$$

Now, we assume that $v \neq -1/2$ and simplify the above equation to get:

$$-\sqrt{4c(2v+1)}+\sqrt{-4c}\cdot\tanh^{-1}\left(\sqrt{2v+1}\right) = \pm\eta \tag{2.10}$$

We substitute $v = u/c$ back into equation (2.10) and plot η versus u:

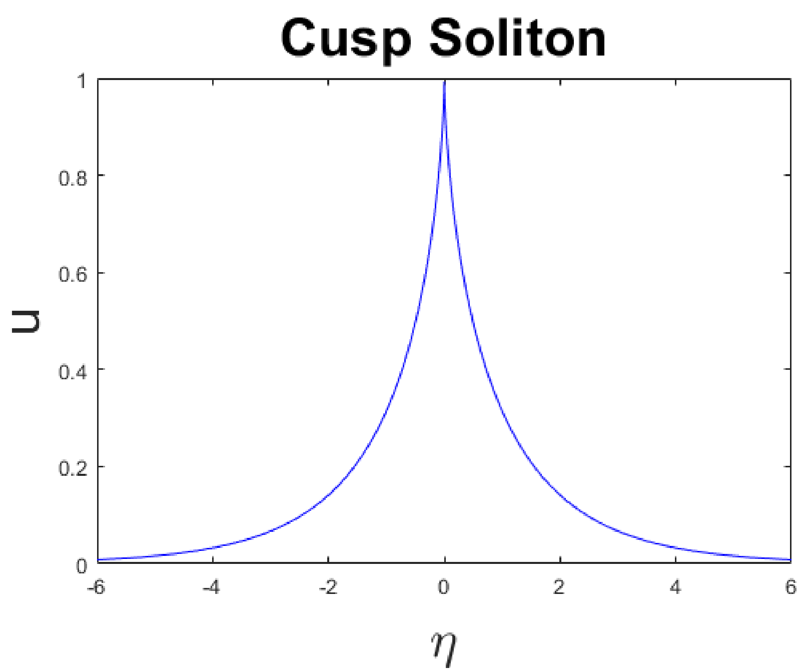

Figure 2.4: The profile of u for $\beta = 2$, $c = -2$.

We see that the solution, u, has a cusp-shaped form at $\eta = 0$. Since $u \to 0$ as $\eta \to \pm\infty$, it is a one-soliton solution.

2.4 Solitary Wave Solution for $\beta = 4$

For $\beta = 4$, equation (1.1) admits periodic and solitary wave solutions when $\beta u + c < 0$. To this end, we substitute $\beta = 4$ into equation (2.4):

$$u_\eta^2 + \frac{2u - c}{6} + \frac{2D'}{(-4u - c)^{\frac{1}{2}}} = 0 \tag{2.11}$$

From Fig.2.5, one can see that by fixing $c = -1$, a solitary wave solution is anticipated for $D' = -1/12$ and a periodic one for $D' = -1/18$. Both of the solutions are cuspon type with singularity at $u = 0.25$. Then, for a solitary wave, we have

$$\frac{-c}{6} + \frac{2D'}{(-c)^{1/2}} = 0$$

which implies $D' = \frac{c(-c)^{1/2}}{12}$. When plugged into (2.11), it gives us the following equation:

$$u_\eta^2 + \frac{2u - c}{6} + \frac{c(-c)^{1/2}}{6(-4u - c)^{1/2}} = 0 \tag{2.12}$$

Plotting u_η as a function of u implicitly, we have

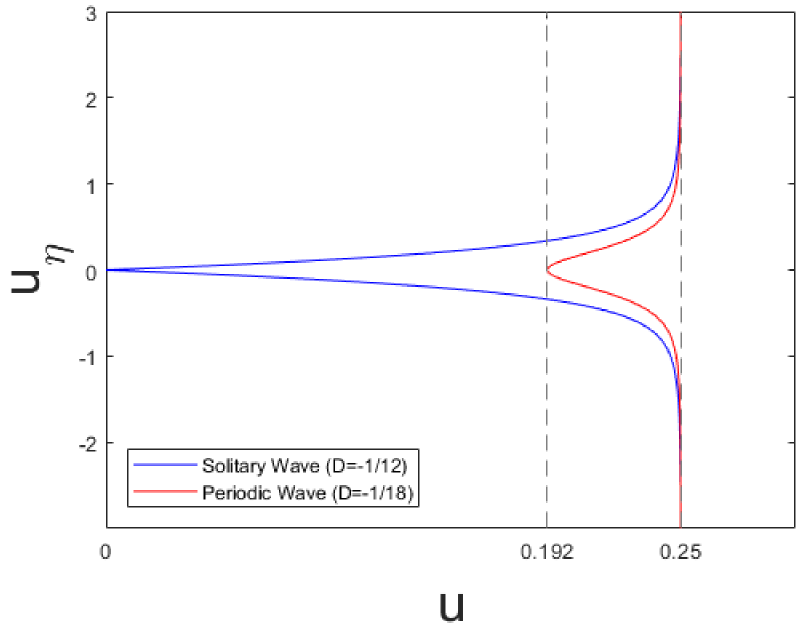

Figure 2.5: The profile of u_η for $\beta = 4$, $c = -1$. The blue line represents a solitary wave solution. The red line represents a periodic wave solution.

Now, equation (2.12) is separable. Let $u = cv$, then we have:

$$\int \frac{dv}{\sqrt{\frac{-2v+1}{6c} - \frac{1}{6c\sqrt{4v+1}}}} = \pm(\eta + \eta_0)$$

After quite a hairy integration, letting $\eta_0 = 0$, we get

$$\pm\eta = \Bigg[\sqrt{6}\sqrt{4v+1} - 1\sqrt{\frac{8v^2 - 2v + \sqrt{4v+1} - 1}{4v+1}} \cdot \ln\left((\sqrt{4v+1} - 1)^2\right) + 12\sqrt{4v+1}v - 6\sqrt{v+1} + 6-$$
$$-\sqrt{6}(\sqrt{4v+1} - 1)\sqrt{\frac{8v^2 - 2v + \sqrt{4v+1} - 1}{4v+1}} \cdot \ln\left(-8v + \sqrt{4v+1} - \sqrt{24v+6}\sqrt{2v + \frac{1}{\sqrt{4v+1}} - 1} - 1\right)\Bigg] \Bigg/$$
$$\Bigg/\left[\sqrt{6}(\sqrt{4v+1} - 1)\sqrt{\frac{-(2v+1) + \frac{1}{\sqrt{4v+1}} - 1}{c}}\right]$$

We substitute $v = u/c$ back into the above equation and plot η versus u:

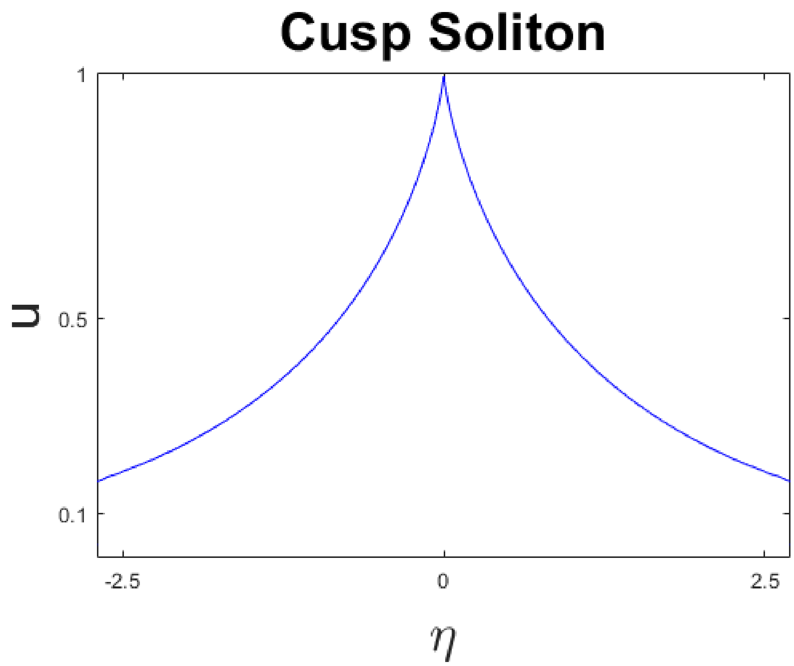

Figure 2.6: The profile of u for $\beta = 4$, $c = -4$.

We see that $u \to 0$ as $\eta \to \pm\infty$. So we have a one-soliton cuspon solution.

CHAPTER III

PERIODIC WAVE SOLUTIONS

In this chapter, we are after a more difficult task - finding periodic solutions to the β-family of equations. In order to do so, we will need to calibrate our equation via several transformations in order to be able to "intelligently guess" the form in which the solutions may appear. This task seems rather abstract, but once the equation takes its final shape, more information could be derived from it, and then the matter of guessing the right form of the solutions will be entirely replaced by the matter of choosing the right form.

3.1 Hodograph Transformation

Recall that equation (1.1) is given by:

$$u_{xt} = u + \beta u u_{xx} + u_x^2$$

Multiplying (1.1) by -1 and making use of the following transformation: $u \to -u$, we have

$$u_{xt} = u - \beta u u_{xx} - u_x^2 \tag{3.1}$$

Let us establish an equivalence between (3.1) and the following system, which was first introduced in [1]:

$$\begin{cases} m_t + m_x u + b m u_x = 0 & (3.2) \\ m = 1 - u_{xx} & (3.3) \end{cases}$$

Substituting (3.3) into (3.2), we get

$$-u_{xxt} + (-u_{xxx}u) + b(1 - u_{xx})u_x = 0 \implies$$

$$\implies u_{xxt} = bu_x - u_{xxx}u - bu_{xx}u_x \tag{3.4}$$

Let $b = \frac{\beta+2}{\beta}$; $\partial_t = \frac{1}{\beta}\partial_t$; $\partial_x = (\beta+2)\partial_x$; $u = (\beta+2)^{-1}u$.

Then, (3.4) becomes

$$\frac{(\beta+2)^2}{\beta}\frac{1}{\beta+2}u_{xxt} = \frac{\beta+2}{\beta}u_x - \frac{(\beta+2)^3}{(\beta+2)^2}uu_{xxx} - \frac{\beta+2}{\beta}\frac{(\beta+2)^3}{(\beta+2)^2}u_{xx}u_x \implies$$

$$\implies \frac{\beta+2}{\beta}u_{xxt} = \frac{\beta+2}{\beta}u_x - (\beta+2)uu_{xxx} - \frac{(\beta+2)^2}{\beta}u_{xx}u_x \implies u_{xxt} = u_x - \beta uu_{xxx} - (\beta+2)u_{xx}u_x$$

Integrating w.r.t. x, we get

$$u_{xt} = u - \beta uu_{xx} - u_x^2$$

Thus, equation (3.1) is equivalent to the system of equations (3.2) and (3.3). Now, consider Eq.(3.2):

$$m_t + m_x u + bmu_x = 0$$

Multiply equation (3.2) by $\frac{1}{b}m^{\frac{1}{b}-1}$. We get:

$$\frac{1}{b}m^{\frac{1}{b}-1}m_t + \frac{1}{b}m^{\frac{1}{b}-1}m_x u + m^{\frac{1}{b}}u_x = 0$$

or

$$(m^{\frac{1}{b}})_t + (m^{\frac{1}{b}}u)_x = 0 \tag{3.5}$$

Let $m^{\frac{1}{b}} = p$. This implies $m = p^b$. Substituting the former into (3.5), we have:

$$p_t + (pu)_x = 0 \tag{3.6}$$

Above equation implies a conservative law, so we can introduce a hodograph transformation

$$dy = pdx - pudt, \quad ds = dt \tag{3.7}$$

Such that $\partial_x = p\partial_y$; $\partial_t = \partial_s - up\partial_y$. Substitute into (3.6)

$$\begin{aligned} p_s - upp_y + p(p_y u + u_y p) = 0 &\implies p_s - upp_y + upp_y + u_y p^2 = 0 \\ &\implies p_s + u_y p^2 = 0 \implies \frac{p_s}{p^2} = -u_y \\ &\implies \left(\frac{1}{p}\right)_s = u_y \end{aligned}$$

Also, substituting $m = p^b$ into (3.3) and using the above mentioned transformation: $\partial_x = p\partial_y$, we get

$$p^b = 1 - u_{xx} = 1 - p(pu_y)_y$$

Thus, we have the following system:

$$\begin{cases} \left(\frac{1}{p}\right)_s = u_y & (3.8) \\ p^b = 1 - p(pu_y)_y & (3.9) \end{cases}$$

We note that the inverse mapping $(y,s) \to (x,t)$ is given by the following conditions:

$$\frac{\partial x}{\partial y} = \frac{1}{p} \tag{3.10}$$

$$\frac{\partial x}{\partial s} = u \tag{3.11}$$

3.2 Elliptic Functions

In order to proceed with finding periodic solutions to the β-family of equations, we need to give a couple of definitions and several basic identities. The trigonometric form of the incomplete elliptic integral of the first kind is given by

$$F(\phi,k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

Now, letting $t = \sin\theta$ and $z = \sin\phi$ gives the Legendre normal form:

$$F(z;k) = \int_0^{z} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

Then, the Jacobi elliptic sine and cosine along with the delta amplitude are given as follows:

$$\mathrm{sn}(u) = \sin\phi = z \tag{3.12}$$

$$\mathrm{cn}(u) = \cos\phi = \sqrt{1-\mathrm{sn}^2(u)} \tag{3.13}$$

$$\mathrm{dn}(u) = \sqrt{1-k^2\mathrm{sn}^2(u)} \tag{3.14}$$

Some basic identities:

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1 \tag{3.15}$$

$$\mathrm{dn}^2(u) + k^2\mathrm{sn}^2(u) = 1 \tag{3.16}$$

The derivatives of the Jacobi elliptic functions are given as follows:

$$(\mathrm{sn}(u))' = \mathrm{cn}(u)\mathrm{dn}(u) \tag{3.17}$$

$$(\mathrm{cn}(u))' = -\mathrm{sn}(u)\mathrm{dn}(u) \tag{3.18}$$

$$(\mathrm{dn}(u))' = -k^2\mathrm{sn}(u)\mathrm{cn}(u) \tag{3.19}$$

Now, the Weierstrass elliptic functions have a second-order pole at $u = 0$ and, therefore, in order to be completely specified, need either their half-periods: ω_1, ω_2 or so-called elliptic invariants: g_2,g_3. As far as this research is concerned, we need to define the Weierstrass elliptic functions only via their Jacobi counterparts:

$$\wp(u; g_2, g_3) = e_3 + (e_1 - e_3)\mathrm{sn}^{-2}\left(u\sqrt{e_1 - e_3}, \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}\right) \tag{3.20}$$

where e_1, e_2 and e_3 are the roots of $p(\mu) = 4\mu^3 - g_2\mu - g_3$.

3.3 Periodic Wave Solution for $\beta = 1$

When $\beta = 1$, we have that $b = \frac{\beta+2}{\beta} = 3$. Substitute $b = 3$ into equation (3.9). Then, we get the following system:

$$\begin{cases} \left(\dfrac{1}{p}\right)_s = u_y & (3.21) \\ p^3 = 1 - p(pu_y)_y & (3.22) \end{cases}$$

Now, we let $w = p^{-1}$ and assume that $w(y,s) = w(\eta)$ as well as $u(y,s) = u(\eta)$, where $\eta = y - cs$. (3.21) yields that

$$-cw_\eta = u_\eta$$

Substituting the above into (3.22), we get that

$$\begin{aligned} &w^{-3} = 1 - w^{-1}(w^{-1}u_\eta)_\eta \implies \frac{1}{w^3} = 1 + \frac{c}{w}\left(\frac{w_\eta}{w}\right)_\eta \implies \\ &\implies \frac{w_\eta}{w^3} = w_\eta + c\frac{w_\eta}{w}\left(\frac{w_\eta}{w}\right)_\eta \implies \frac{-1}{2}w^{-2} + E = w + \frac{c}{2}\left(\frac{w_\eta}{w}\right)^2 \implies \\ &\implies w_\eta^2 = \frac{-1}{c}(2w^3 - 2Ew^2 + 1) = \frac{-2}{c}(w^3 - Ew^2 + \frac{1}{2}) \end{aligned}$$

So, we have a cubic expression in the dependent variable, w, on the right-hand side which can be factored out in the following way:

$$w_\eta^2 = \frac{-2}{c}(w-\alpha)(w-\beta)(w-\gamma) \tag{3.23}$$

Collecting and comparing the coefficients of w^0 and w^1, yields:

$$w^0 : \ \frac{-1}{2} = \alpha\beta\gamma \tag{3.24}$$

$$w^1 : \ 0 = \alpha\gamma + \beta\gamma + \beta\alpha \tag{3.25}$$

We consider the case $c < 0$.

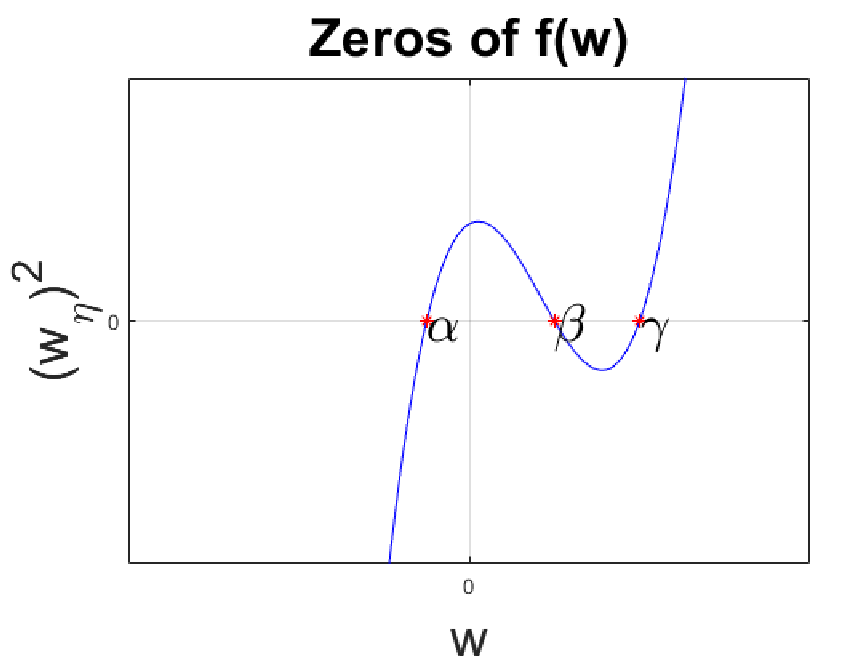

Figure 3.1: The presumptive profile of the zeros of $(w_\eta)^2$ for $b = 3$, $c < 0$.

In the figure above, we assumed, without loss of generality, that $\alpha < 0 < \beta < \gamma$. We see that in order for $(w_\eta)^2$ to be non-negative, we must have $\alpha \le w \le \beta$. So, we let

$$w = \alpha + (\beta - \alpha)\sin^2(t) \tag{3.26}$$

A very similar idea was used by Toda in his book [8] for the KdV equation. Now, equation (3.26) can be used to derive the following several equations:

$$dw = 2(\beta-\alpha)\sin(t)\cos(t)dt \tag{3.27}$$

$$w-\alpha = (\beta-\alpha)\sin^2(t) \tag{3.28}$$

$$w-\beta = -(\beta-\alpha)\cos^2(t) \tag{3.29}$$

$$w-\gamma = -(\gamma-\alpha)(1-k^2\sin^2(t)) \tag{3.30}$$

Where $k^2 = \frac{\beta-\alpha}{\gamma-\alpha}$. We substitute (3.28), (3.29) and (3.30) into (3.23) and get that:

$$\left(\frac{dw}{d\eta}\right)^2 = \frac{-2}{c}(\gamma-\alpha)(\beta-\alpha)^2\sin^2(t)\cos^2(t)(1-k^2\sin^2(t))$$

$$\implies \frac{dw}{d\eta} = \pm\sqrt{\frac{-2}{c}(\gamma-\alpha)(1-k^2\sin^2(t))}\cdot(\beta-\alpha)\sin(t)\cos(t)$$

$$\implies \pm(\eta+\eta_0) = \int \frac{dw}{\sqrt{\frac{-2}{c}(\gamma-\alpha)(1-k^2\sin^2(t))}\cdot(\beta-\alpha)\sin(t)\cos(t)} =$$

$$= \int \frac{2(\beta-\alpha)\sin(t)\cos(t)dt}{\sqrt{\frac{-2}{c}(\gamma-\alpha)(1-k^2\sin^2(t))}\cdot(\beta-\alpha)\sin(t)\cos(t)} =$$

$$= \sqrt{\frac{-2c}{\gamma-\alpha}}\int^t \frac{dt}{\sqrt{(1-k^2\sin^2(t))}}$$

We let $z = \sin(t)$ and $\eta_0 = 0$, then we have that:

$$\pm\eta = \sqrt{\frac{-2c}{\gamma-\alpha}} \int^{z} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} =$$
$$= \sqrt{\frac{-2c}{\gamma-\alpha}} \mathrm{sn}^{-1}(z) \implies z = \sin(t) = \mathrm{sn}\left(\pm\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)$$
$$\implies w = \alpha + (\beta-\alpha)\sin^2(t) = \alpha + (\beta-\alpha)\mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)$$

Since $w = \alpha + (\beta-\alpha)\mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)$, then w varies between $\alpha < 0$ and $\beta > 0$, so we always have the periodic loop-shaped waves. In the limiting case $k \to 1$, we have $\beta = \gamma$, then we solve for $\beta = \gamma = 1$ and $\alpha = -1/2$, so

$$w = 1 - \frac{3}{2}\mathrm{sech}^2\left(\eta\sqrt{\frac{3}{-4c}}\right) \tag{3.31}$$

We use the fact that that $-cw_\eta = u_\eta$, which implies that:

$$u = A + \frac{3c}{2}\mathrm{sech}^2\left(\eta\sqrt{\frac{3}{-4c}}\right) \tag{3.32}$$

for some constant A. Also, recall that $w = 1/p = \partial x/\partial y$. Then, setting $s = 0$ and $c = -1$, we get the following:

$$x = \int 1 - \frac{3}{2}\mathrm{sech}^2\left(y\frac{\sqrt{3}}{2}\right) dy = y - \sqrt{3}\tanh\left(y\frac{\sqrt{3}}{2}\right) \tag{3.33}$$

Now, equations (3.32) and (3.33) give us a parametric solution for u in terms of x:

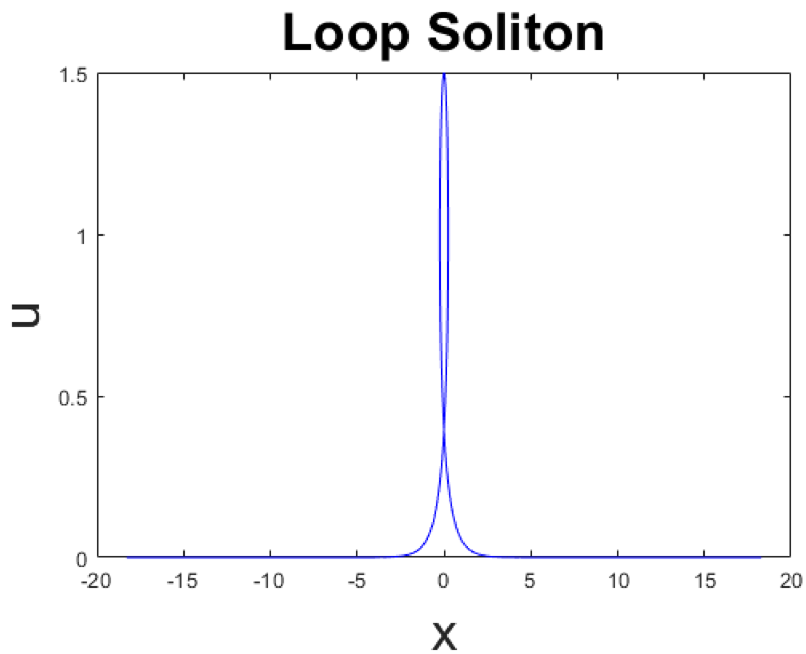

Figure 3.2: The profile of u for $b = 3$, $c = -1$, $\alpha = -0.5$, $\beta = \gamma$, $k = 1$, $A = -1$ at $s = 0$.

Which is consistent with our result from Chapter 2. Now, let us obtain the periodic solution. Using the fact $\partial x/\partial y = w$, we have :

$$x = \alpha y + (\beta - \alpha) \int^{y} \mathrm{sn}^2 \left(\eta \sqrt{\frac{\gamma - \alpha}{-2c}} \right) dy + K(s)$$

We recall: $-cw_\eta = u_\eta$ which implies that:

$$u = A - c\alpha - c(\beta - \alpha)\mathrm{sn}^2 \left(\eta \sqrt{\frac{\gamma - \alpha}{-2c}} \right) \qquad (3.34)$$

for some constant A. Now, we solve for $K(s)$ by using the fact $\partial x/\partial s = u$:

$$\frac{\partial x}{\partial s} = u \implies \frac{\partial}{\partial s}\left(\alpha y + (\beta-\alpha)\int^{y} \mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)dy + K(s)\right) = A - c\alpha - c(\beta-\alpha)\mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)$$

$$\implies \frac{\partial}{\partial y}\left(-c(\beta-\alpha)\int^{y} \mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)dy\right) + K'(s) = A - c\alpha - c(\beta-\alpha)\mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)$$

$$\implies -c(\beta-\alpha)\mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right) + K'(s) = A - c\alpha - c(\beta-\alpha)\mathrm{sn}^2\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right)$$

$$\implies K'(s) = A - c\alpha \implies K(s) = (A-c\alpha)s + d$$

So, we have that:

$$x = \alpha y + \frac{\beta-\alpha}{k^2}y - \frac{\beta-\alpha}{k^2}\sqrt{\frac{-2c}{\gamma-\alpha}}E\left(\eta\sqrt{\frac{\gamma-\alpha}{-2c}}\right) + (A-c\alpha)s + d \qquad (3.35)$$

where $E(u) = \int_0^u dn^2(v)dv$. Conditions (3.24) and (3.25) imply:

$$-\frac{1}{2} \leq \alpha,\ \alpha \neq 0$$
$$\beta = \frac{\pm(\sqrt{8a^3+1})+1}{4a^2}$$
$$\gamma = \frac{\mp(\sqrt{8a^3+1})+1}{4a^2}$$

In light of the above equations, (3.34) and (3.35), setting $d = 0$, we plot x vs.$-u$ (due to the transformation we made at the beginning: $u \to -u$) at $s = 0$ (note that y becomes η):

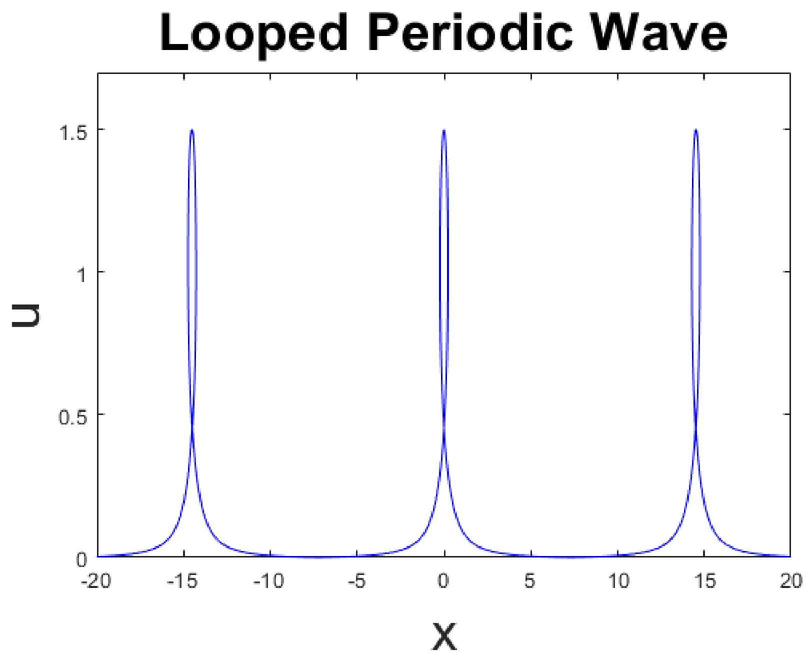

Figure 3.3: The profile of the periodic loop solution for $b = 3$, $c = -1$, $k \to 1$ $(k \neq 1)$.

The closer α is to -0.5 and, therefore, k is to 1, the larger the period of the wave is. However, it is clear that as long as $k \neq 1$, we will have a periodic solution.

3.4 Periodic Wave Solution for $\beta = 2$

When $\beta = 2$, we have that $b = \frac{\beta+2}{\beta} = 2$. Substitute $b = 2$ into equation (3.9), we get the following system:

$$\begin{cases} \left(\dfrac{1}{p}\right)_s = u_y & (3.36) \\ p^2 = 1 - p(pu_y)_y & (3.37) \end{cases}$$

We let $w = p^{-1}$ and assume that $w(y,s) = w(\eta)$ as well as $u(y,s) = u(\eta)$, where $\eta = y - cs$. As before, we have that: $-cw_\eta = u_\eta$. Substituting this into (3.37), we get

$$w^{-2} = 1 - w^{-1}(w^{-1}u_\eta)_\eta \implies \frac{1}{w^2} = 1 + \frac{c}{w}\left(\frac{w_\eta}{w}\right)_\eta \implies$$
$$\implies \frac{w_\eta}{w^2} = w_\eta + c\frac{w_\eta}{w}\left(\frac{w_\eta}{w}\right)_\eta \implies \frac{-1}{w} + D = w + \frac{c}{2}\left(\frac{w_\eta}{w}\right)^2 \implies$$
$$\implies w_\eta^2 = \frac{-2}{c}(w^3 - Dw^2 + w) = \frac{-2}{c}w(w^2 - Dw + 1)$$

Factoring $w^2 - Dw + 1$, we have that

$$w_\eta^2 = \frac{-2}{c}w(w - \alpha_1)(w - \alpha_2) \tag{3.38}$$

such that $\alpha_1\alpha_2 = 1$ and $\alpha_1 + \alpha_2 = D$. This means that either $c < 0$ and $\alpha_1, \alpha_2 < 0$, or $c > 0$ and $\alpha_1, \alpha_2 > 0$. We consider the case $c < 0$.

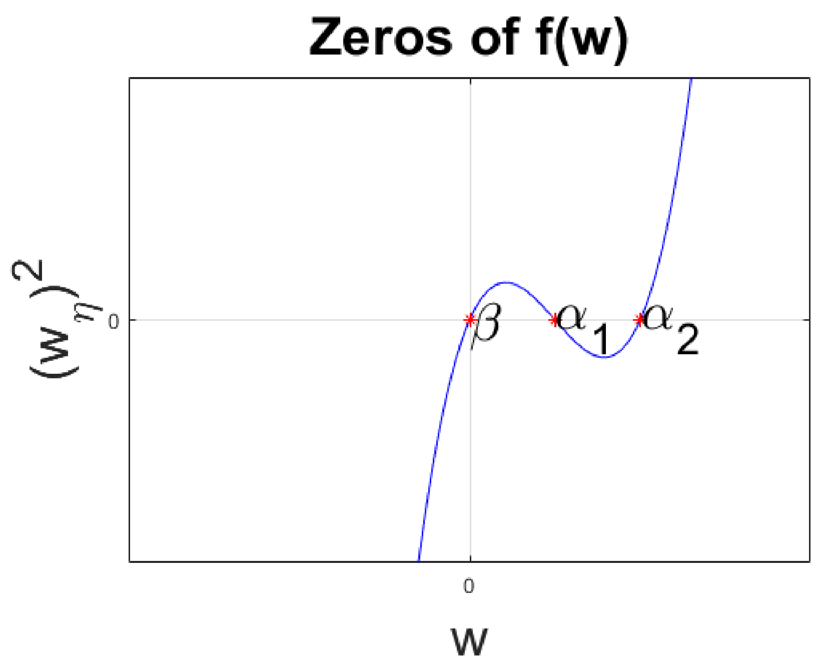

Figure 3.4: The presumptive profile of the zeros of $(w_\eta)^2$ for $b = 2$, $c < 0$.

We assume $0 < \alpha_1 < \alpha_2$ and let

$$w = \alpha_1 \sin^2(t) \tag{3.39}$$

Then, we have that

$$dw = 2\alpha_1 \sin(t)\cos(t)dt \tag{3.40}$$

$$w - \alpha_1 = -\alpha_1 \cos^2(t) \tag{3.41}$$

$$w - \alpha_2 = -\alpha_2(1 - k^2 \sin^2(t)) \tag{3.42}$$

where $k^2 = \frac{\alpha_1}{\alpha_2}$. Following the footsteps of the case $b = 3$, we substitute (3.39), (3.41) and (3.42) into (3.38) and get that:

$$w = \alpha_1 \mathrm{sn}^2\left(\eta\sqrt{\frac{\alpha_2}{-2c}}\right) = \alpha_1 - \alpha_1 \mathrm{cn}^2\left(\eta\sqrt{\frac{\alpha_2}{-2c}}\right) \tag{3.43}$$

Since w varies between 0 and α_1, we have a cuspon solution. Consider the limiting case of $k = 1$. In this case, we get $\alpha_1 = \alpha_2 = 1$. Then we have

$$w = 1 - \mathrm{sech}^2\left(\frac{\eta}{\sqrt{-2c}}\right) \tag{3.44}$$

We use the fact that that $-cw_\eta = u_\eta$ which implies that:

$$u = c \cdot \mathrm{sech}^2\left(\frac{\eta}{\sqrt{-2c}}\right) \tag{3.45}$$

Note that the integration constant was set to 0 as we are considering the limiting case. Also, recall that $w = 1/p = \partial x/\partial y$. Then, setting $s = 0$ and $c = -1$ and integrating with respect to y, we have:

$$x = y - \sqrt{2}\tanh\left(\frac{y}{\sqrt{2}}\right) \tag{3.46}$$

Equations (3.45) and (3.46) give us a parametric solution for u in terms of x:

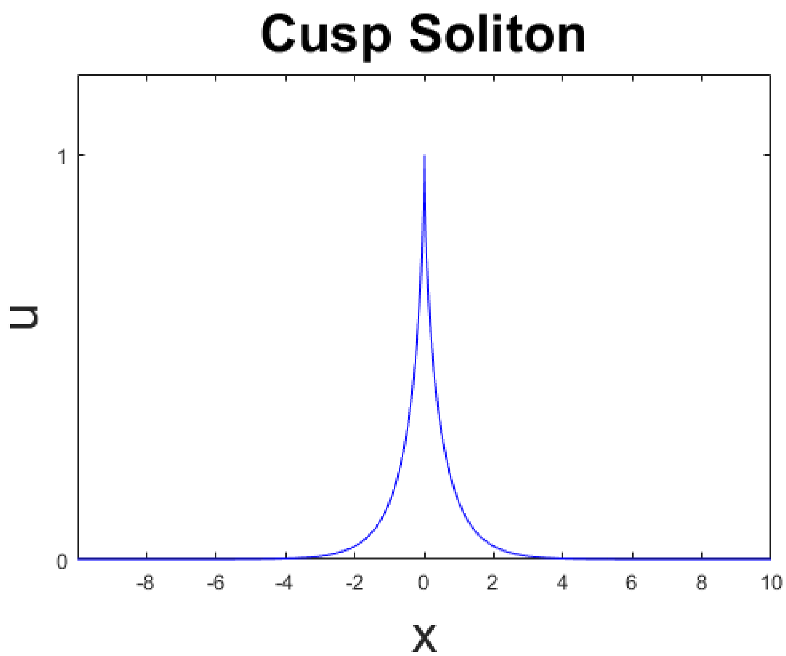

Figure 3.5: The profile of u for $b = 2$, $c = -1$, $\alpha_1 = \alpha_2 = 1$, $k = 1$ at $s = 0$.

The result is consistent with the one form Chapter 2. Now, we construct a periodic solution explicitly using the fact $w = 1/p = \partial x/\partial y$. We have :

$$x = \alpha_1 \int^y \mathrm{sn}^2\left(\eta\sqrt{\frac{\alpha_2}{-2c}}\right) dy + K(s)$$

We solve for $K(s)$ by using the fact $\partial x/\partial s = u$:

$$K(s) = As + d$$

We recall that $-cw_\eta = u_\eta$ which implies that

$$u = A - c\alpha_1 \mathrm{sn}^2\left(\eta\sqrt{\frac{\alpha_2}{-2c}}\right) \tag{3.47}$$

Similarly to the case $c < 0$ for $b = 3$, we have that:

$$x = \frac{\alpha_1}{k^2}y - \frac{\alpha_1}{k^2}\sqrt{\frac{-2c}{\alpha_2}}E\left(\eta\sqrt{\frac{\alpha_2}{-2c}}\right) + As + d \tag{3.48}$$

Keeping the conditions $\alpha_1\alpha_2 = 1$, $\alpha_1 < \alpha_2$ as well as $k = \sqrt{\alpha_1/\alpha_2}$ in mind and setting $d = 0$, we plot x vs.$-u$ at $s = 0$:

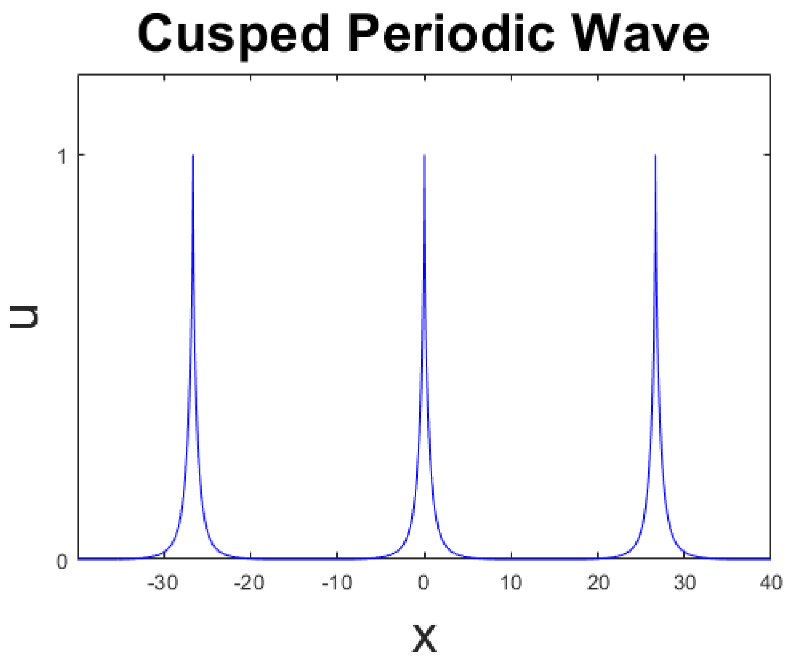

Figure 3.6: The profile of the periodic loop solution for $b = 2$, $c = -1$, $k \to 1$ ($k \neq 1$).

3.5 Periodic Wave Solution for $\beta = 4$

When $\beta = 4$, we have that $b = \frac{\beta+2}{\beta} = \frac{4+2}{4} = \frac{3}{2}$. Substitute $b = \frac{3}{2}$ into equation (3.9), we get the following system:

$$\begin{cases} \left(\dfrac{1}{p}\right)_s = u_y & (3.49) \\ p^{\frac{3}{2}} = 1 - p(pu_y)_y & (3.50) \end{cases}$$

Let $p^{-1/2} = w$, then $p = w^{-2}$, $u_\eta = -2cww_\eta$, substituting the first equation, we have

$$w^{-3} = 1 + 2cw^{-2}(w^{-1}w_\eta)_\eta \tag{3.51}$$

Multiplying both sides by ww_η and integrating, we have

$$-w^{-1} + D = \frac{1}{2}w^2 + c(w^{-1}w_\eta)^2 \tag{3.52}$$

or

$$2c(w_\eta)^2 = -(w^4 - 2Dw^2 + 2w) = -(w-\alpha)w(w-\beta)(w-\gamma) \tag{3.53}$$

where D is an integration constant. We consider the case $c < 0$ and assume $\alpha < 0 < \beta < \gamma$ with the relations $\alpha + \beta + \gamma = 0$ and $\alpha\beta\gamma = -2$.

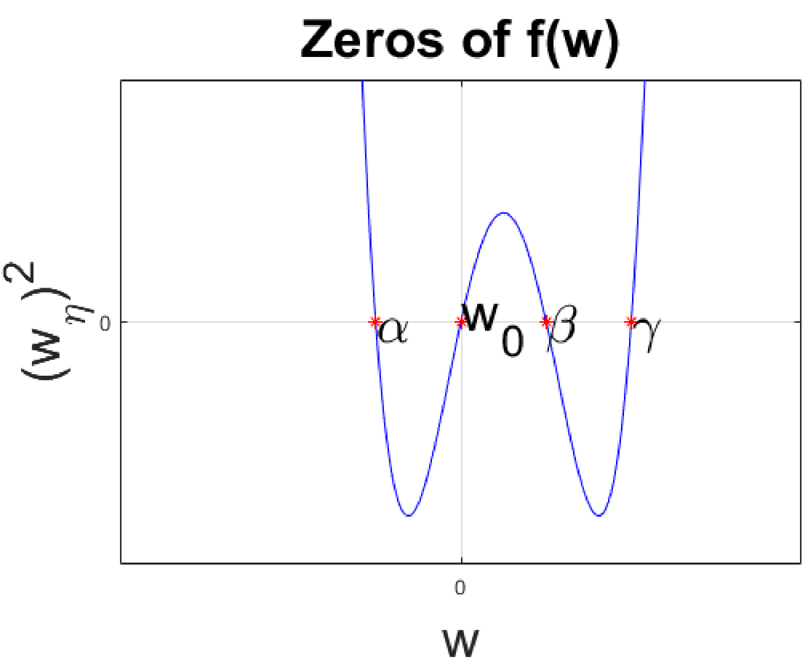

Figure 3.7: The presumptive profile of the zeros of $(w_\eta)^2$ for $b = \frac{3}{2}$, $c < 0$.

Let $z = \eta/\sqrt{-2c}$, then we have

$$z = \int_{w_0}^{w} \frac{dw}{\sqrt{w^4 - 2Dw^2 + 2w}} \tag{3.54}$$

Let $f(w) = w^4 - 2Dw^2 + 2w$. It was shown in [9] that if we choose $w_0 = 0$, so that $f'(w_0) = 2$ and $f''(w_0) = -4D$, then we have that the above integral admits the solution

$$w = w_0 + \frac{1}{4} f'(w_0) \cdot \left(\wp(z; g_2, g_3) - \frac{1}{24} f''(w_0) \right)^{-1} \implies \tag{3.55}$$

$$w = \frac{1}{2} \left(\wp(z; g_2, g_3) + \frac{D}{6} \right)^{-1} \tag{3.56}$$

where $g_2 = \frac{1}{3}D^2$, $g_3 = \frac{1}{27}D^3 - \frac{1}{4}$ and

$$\wp(z; g_2, g_3) = e_3 + (e_1 - e_3)\mathrm{ns}^2(\sqrt{(e_1 - e_3)}z, k) \tag{3.57}$$

is the Weierstrass elliptic function with $e_1 > e_2 > e_3$ being three roots of the cubic polynomial defined by:

$$p(\mu) = 4\mu^3 - g_2\mu - g_3 = 4\mu^3 - \frac{1}{3}D^2\mu - \frac{1}{27}D^3 + \frac{1}{4} = 4(\mu - e_1)(\mu - e_2)(\mu - e_3) \tag{3.58}$$

Thus, we see that the following conditions must hold:

$$e_1 e_2 e_3 = \frac{1}{108}D^3 - \frac{1}{16} \tag{3.59}$$

$$e_1 e_2 + e_1 e_3 + e_2 e_3 = -\frac{1}{12}D^2 \tag{3.60}$$

$$e_1 + e_2 + e_3 = 0 \tag{3.61}$$

Then, equation (3.56) becomes

$$w = \frac{1}{(2e_3 + D/3) + 2(e_1 - e_3)\mathrm{ns}^2(\sqrt{(e_1 - e_3)}z, k)} \tag{3.62}$$

In the limiting case of $k=1$, we can solve for $D=3/2$, $e_1=e_2=1/4$, $e_3=-1/2$, so

$$w=\frac{1-\text{sech}(\sqrt{3}z)}{1+2\text{sech}(\sqrt{3}z)}=\frac{\cosh(\sqrt{3/(-2c)}\eta)-1}{\cosh(\sqrt{3/(-2c)}\eta)+2} \tag{3.63}$$

Here we have used the formula

$$\text{ns}^2(z,k)=(\text{sn}^2(z,k))^{-1}=\frac{1+\text{dn}(2z,k))}{1-\text{cn}(2z,k))}\to\frac{1+\text{sech}(2z)}{1-\text{sech}(2z)} \tag{3.64}$$

We let $\sqrt{3/(-2c)}=\kappa$, then $\sqrt{3/(-2c)}\eta=\xi=\kappa y+\frac{3}{2}\kappa^{-1}s$. Consider the boundary conditions: $u\to 0$, $p^{-1}=w^2\to 1$ as $y\to\infty$. Then, we have:

$$u=c(1-w^2)=-\frac{9}{2\kappa^2}\frac{2\cosh\xi+1}{(\cosh\xi+2)^2} \tag{3.65}$$

Also, we use $dx/dy=w^2$ to solve for x. Setting $s=0$ and $c=-1$, we get:

$$x=\int\left(\frac{\cosh(\sqrt{3/(2)}y)-1}{\cosh(\sqrt{3/(2)}y)+2}\right)^2 dy=\frac{2y-\sqrt{6}\sinh\sqrt{\frac{3}{2}}y+y\cosh\sqrt{\frac{3}{2}}y}{\cosh\sqrt{\frac{3}{2}}y+2} \tag{3.66}$$

The above results are consistent with those obtained by Matsuno in [6]. We use equations (3.65) and (3.66) to plot u expressed in terms of x:

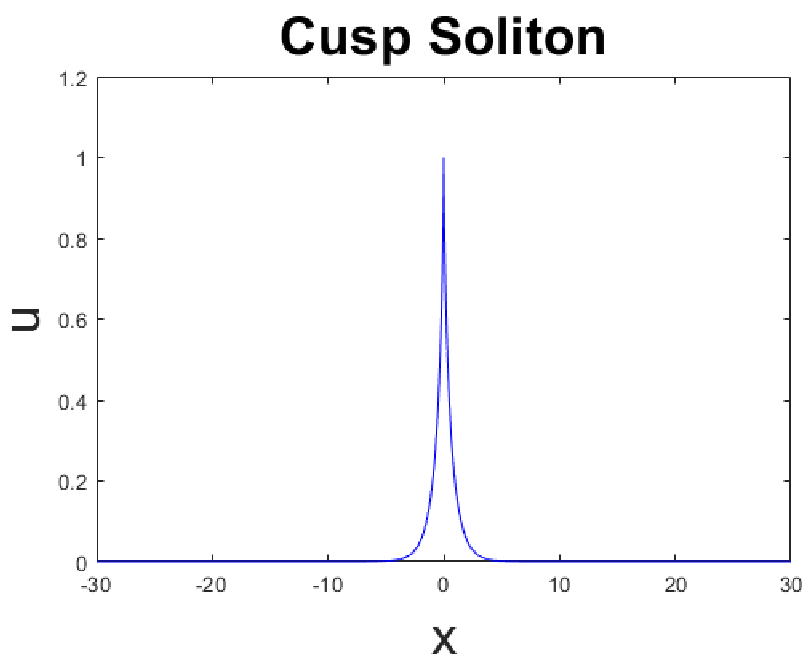

Figure 3.8: The profile of u for $b = \frac{3}{2}$, $c = -1$, $e_3 = -0.5$, $e_1 = e_2 = 0.25$, $D = 1.5$, $k = 1$, $A = -1$ at $s = 0$.

The above profile is consistent with our result from Chapter 2. One can use equation (3.62) to get a numerical solution for x by solving the following first order ODE:

$$\frac{dx}{dy} = \left(\frac{1}{(2e_3 + D/3) + 2(e_1 - e_3)\mathrm{ns}^2(\sqrt{(e_1 - e_3)}z, k)} \right)^2 \tag{3.67}$$

Recall that $u_\eta = -2cww_\eta \implies u = A - cw^2$ where A is a constant of integration w.r.t. η. Therefore,

$$u = A - c\left(\frac{1}{(2e_3 + D/3) + 2(e_1 - e_3)\mathrm{ns}^2(\sqrt{(e_1 - e_3)}z, k)} \right)^2 \tag{3.68}$$

Above two relations constitute the following parametric solution (at $s = 0$):

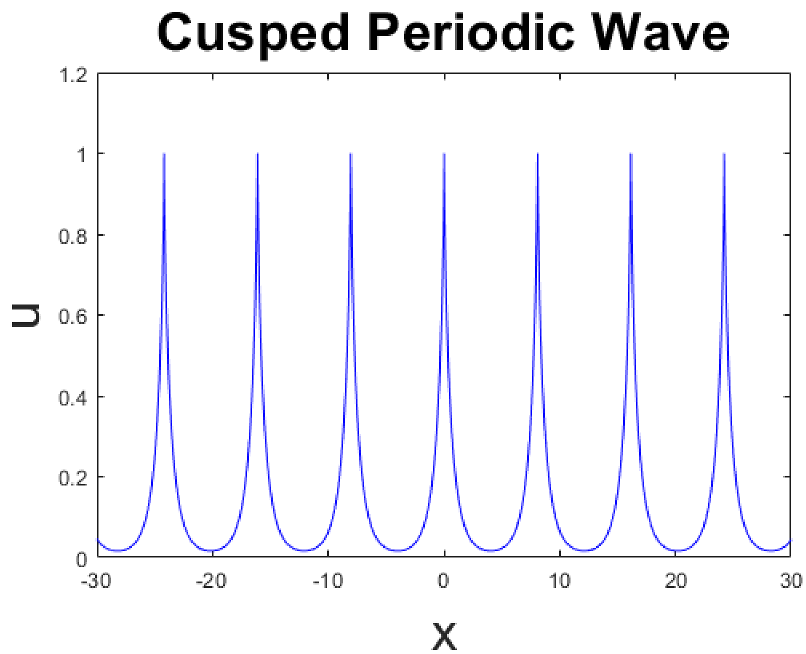

Figure 3.9: The profile of u for $b = \frac{3}{2}$, $c = -1$, $A = -1$, $e_2 \to e_1$, $k \to 1$ $(k \neq 1)$.

We see that the solution takes the form of a cusped periodic wave when $k < 1$. The closer k is to 1, the larger the period of the wave is. This is consistent with the fact that we get a cusp soliton profile for u when $k = 1$.

CHAPTER IV

CONCLUDING REMARKS

As we were primarily interested in the structure of the solutions to the β-family of equations, we showed in Chapter 2 that the one-soliton solutions take the forms of loop and cusp solitons. In Chapter 3, we showed that the loop and cusp shaped solutions are preserved for the periodic wave solutions. As for the methods, we first used the traveling wave method along with direct integration of the resulting "to-be-determined" function under solitary wave boundary conditions to determine the structure of the solitary wave solutions to the β-family of equations. After that, to confirm our results for the case of periodic waves, we applied a hodograph transformation to the β-family equation in order to convert it into a form that could be integrated. We managed to find the right form and express the solutions as Jacobi elliptic functions. Overall, we have presented parametric solutions to the β-family of equations by means of a traveling-wave method combined with the hodograph transformation and have shown that the β-family of equations has loop solitary as well as periodic wave solutions when $\beta = 1$ and cuspon solutions when $\beta = 2, 4$.

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BIOGRAPHICAL SKETCH

Andrey V. Stukopin was born in Oblivskaya, Rostov Oblast', Russia on June 2, 1994. He is the second child of Mr. Vladimir Stukopin and Mrs. Marina Stukopina. Andrey was raised in Rostov-on-Don, a relatively big city in South-Western part of Russia, where he attended elementary and middle school 105. When he turned 13 years old, his parents and he decided that it was best for him to move to Moscow to study in a boarding school, called "Moscow Specialized College of Olympic Reserve", in order to excel in chess. In 2014, Andrey became a Grandmaster in chess and was invited to attend the University of Texas at Brownsville (before it became part of UTRGV) to represent the university in the realm of chess. 4 years later, Andrey graduated from the University of Texas Rio Grande Valley with a Bachelor's degree in Applied Mathematics. During his last year of Undergraduate School, Andrey started working with Dr. Baofeng Feng and did his Final Project on the Inverse Scattering Transform for the KdV equation as well as presented the research at the 2018 College of Sciences Annual Research Conference. In 2018, Andrey was awarded with the Dean's Research Assistanship and continued to work under the supervision of Dr. Baofeng Feng towards his Master's degree. During his first year of Graduate School, he made a presentation on the KP (Kadomstev-Petviashvili) hierarchy at the 2019 College of Sciences Annual Research Conference. During his last year, Andrey was given the opportunity to work as a Teaching Assistant. During Fall 2019, he assisted Dr. Mikhail Bouniaev with his Geometry class and got some teaching experience with a College Algebra stand-alone course under the supervision of Dr. Mikhail Bouniaev. Lastly, in Spring 2020, Andrey was assisting Dr. Alexey Glazyrin with his Differential Equations and Mathematics of Chess (Special Topics in Math) courses. In May 2020, Andrey Stukopin earned his Master of Science degree in Mathematics (MS) from The University of Texas Rio Grande Valley.

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