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Dynamical Study and Optimal Harvesting of a Two-species Amensalism Model Incorporating Nonlinear Harvesting

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Abstract

This study proposes a two-species amensalism model with a cover to protect the first species from the second species, with the assumption that the growth of the second species is governed by nonlinear harvesting. Analytical and numerical analyses have both been done on this suggested ecological model. Boundedness and positivity of the solutions of the model are examined. The existence of feasible equilibrium points and their local stability have been discussed. In addition, the parametric conditions under which the proposed system is globally stable have been determined. It has also been shown, using the Sotomayor theorem, that under certain parametric conditions, the suggested model exhibits a saddle-node bifurcation. The parametric conditions for the existence of the bionomic equilibrium point have been obtained. The optimal harvesting strategy has been investigated utilising the Pontryagins Maximum Principle. The potential phase portrait diagrams have been provided to corroborate the acquired findings.

Keywords: Amensalism model; Stability; Harvesting; Saddle-node bifurcation; Phase portrait; Bionomic equilibrium; Pontryagins Maximum Principle

MSC 2010 No.: 92B05, 35B35, 34C23

1. Introduction

Species interactions are common in the natural system. From an ecological and biological point of view, these interations have been found to be one of the most complex and challenging phenomenon. These interactions take several forms, one of which is amensalism. It refers to a species-to-species relationship in which one species does damage to another without gaining any benefits or costs. Some examples of amensalism relations between the species are discussed in Rambabu et al. (2012). In this paper, they proposed and analyzed a harvested amensalism model having two species. In 2016, Wu et al. proposed a two-species amensalism model employing the Holling II type functional response. To protect the first species from the second species, a cover was also considered. They found that both the species can coexist whenever the cover is sufficient to accumulate the first species. Xie et al. (2016) developed and analyzed the local stability as well as global stability for a two-species amensalism model in the presence of a cover to save the first species. R. Wu (2018) studied the local and global stability of a nonlinear amensalism model. In 2019, Guan and Chen proposed and analyzed a two species amensalism model that incorporate Beddington DeAngelis functional response. They also considered that the growth of the second species is governed by the Allee effect. Zhao et al. (2020) proposed and studied an amensalism system in which Allee effect governs the growth rate of the first species, while Wei et al. (2020) proposed and investigated the dynamical behavior of an amensalism model in which the growth of the second species is influenced by the weak Allee effect.

The harvesting of species plays a vital role, which is required for the optimum utilization of the renewable biological resources for the survival of the human population. The first ecological model incorporating harvesting was proposed in Clark (1976). In general, three kinds of harvesting exist in mathematical ecology, known as, constant harvesting (Huange et al. (2013); Hu and Cao (2005)), linear harvesting (Liu et al. (2018); Zhang et al. (2011)), and nonlinear harvesting (Gupta and Chandra (2013); Hu and Cao (2017); Yu et al. (2019)). The nonlinear harvesting is biologically and economically more realistic (Gupta and Chandra (2015)). A number of scholars (Das et al. (2009); Kar et al. (2012); Gupta and Chandra (2013); Hu and Cao (2017); Singh and Bhadauria (2020)) have analyzed the impact of nonlinear harvesting on various predator-prey systems and found some very important outcomes to protect the ecological systems. Hu and Cao (2017) investigated the dynamical complexity for a predator-prey model including nonlinear predator harvesting.

Recently, researchers have started to investigate the influence of harvesting on the amensalism system. Chen (2018) studied a non-selective harvesting amensalism model incorporating partial closure for the populations is proposed and studied in this paper. They investigated the local and global stability of the feasible equilibria and concluded that harvesting enhances the dynamic behaviors of the system. Liu et al. (2018) formulated a amensalism ecological system that incorporates a cover for the first species. They assumed that the growth of the first species is subjected to non-linear harvesting. In this paper, they discussed the local stability, saddle-node bifurcation, and transcritical bifurcation.

The above discussion draws our attention to the fact that the dynamical behaviors of an amensalism ecological model in the presence of a cover are interesting and realistic. Human actions in the ecosystem are increasing day by day, which affects the environment very badly. Hence, it is important to examine the dynamical complexity of an amensalism model equipped with non-linear harvesting. In many biological systems, there is a time delay (Xu et al. (2021b); Xu et al. (2022c); Xu et al. (2022b); Xu et al. (2022a); Xu et al. (2021a)). In this research article, we have considered a two-species amensalism system in the presence of a cover for the first species and non-linear harvesting for the second species.

2. Mathematical Model

G. C. Sun (2003) proposed a mathematical model, the first effort to investigate the behavior of a two-species amensalism system. A two-species amensalism model where the first species has a partial cover to shield it from the second species was taken into consideration by Xie et al. (2016).

$$\begin{cases} \frac{dx}{d\tau} = a_1x - b_1x^2 - c_1(1-k)xy, \\ \frac{dy}{d\tau} = a_2y - b_2y^2. \end{cases} \tag{1}$$

To ensure that the above model is ecologically well-posed, we have assumed that all parameters a_1, a_2, a_3, a_4, c_1, and k are positive. a_1, b_1, and c_1 reflect the intrinsic growth rate, rate of drop in density due to inadequate food, rate of fall in density due to inhibition for the first species, respectively. Additionally, a_2 and b_2 indicate the intrinsic growth rate and rate of decline in density due to inadequate food for the second species, respectively. The positive parameter k $(0 < k < 1)$, is a cover to protect the species x from the species y.

The above model can be rewritten as

$$\begin{cases} \frac{dx}{d\tau} = r_1x(1-\frac{x}{k_1}) - c_1(1-k)xy, \\ \frac{dy}{d\tau} = r_2y(1-\frac{y}{k_2}), \end{cases} \tag{2}$$

where $r_1 = a_1$ and $r_2 = a_2$ denote for the maximum intrinsic growth rate of the x species and y species, respectively; $k_1 = \frac{a_1}{b_1}$ and $k_2 = \frac{a_2}{b_2}$ denote for environmental carrying capacities of the x species and y species, respectively.

Introducing the non-linear harvesting term in the y equation of system (2), it becomes

$$\begin{cases} \frac{dx}{d\tau} = r_1x(1-\frac{x}{k_1}) - c_1(1-k)xy, \\ \frac{dy}{d\tau} = r_2y(1-\frac{y}{k_2}) - \frac{qEy}{m_1E+m_2y}, \end{cases} \tag{3}$$

where $m_i, \ i = 1, 2$ are positive constants, and E, q are positive parameters stand for the harvesting effort and catchability coefficient, respectively.

To make the analysis more straightforward, we adopted the following non-dimensionalization strategy. Consider

$$\phi : \bar{\psi} \times \bar{R} \to \psi \times R,$$

where

$$\phi(u,v,t)=\Big(\frac{x}{k_1},\frac{c_1(1-k)y}{r_1},r_1\tau\Big).$$

Consider $a=\frac{r_2}{r_1},\ b=\frac{r_1}{c_1(1-k)k_2},h=\frac{qE(1-k)c_1}{m_2r_1r_2},\ c=\frac{m_1E(1-k)c_1}{m_2r_1}$. The system (3) transforms into a non-dimensionalised state

$$\begin{cases}\frac{du}{dt}=u(1-u-v),\\ \frac{dv}{dt}=av(1-bv-\frac{h}{c+v}),\end{cases} \tag{4}$$

with $u(0)>0,\ v(0)>0$.

3. Positivity and Boundedness of Solution

Lemma 3.1.

(a) If $t\geq 0$, then every solution of the system (4) corresponding to the initial conditions is positive.
(b) If $t\geq 0$, then every solution of system (4) corresponding to the initial conditions is bounded.

Proof:

(a) The prey equation of the system (4) gives

$$u(t)=u(0)exp\Big[\int_0^t\Big(1-u(\tau)-v(\tau)\Big)d\tau\Big]; \tag{5}$$

similarly, the predator equation of the system (4) gives

$$v(t)=v(0)exp\Big[\int_0^t a\Big(1-bv(\tau)-\frac{h}{c+v(\tau)}\Big)d\tau\Big]. \tag{6}$$

Since $u(0)>0$ and $v(0)>0$, therefore, $u(t)>0$ and $v(t)>0$. This implies that every solution trajectory beginning at any point of the first quadrant of uv-plane remain positive for all future time.

(b) The first equation of system (4) gives

$$\frac{du}{dt}\leq u(1-u).$$

Employing the Lemma (2.2) of Lin and Ho (2006) and positivity of variable u, we get

$$u(t)\leq max\{1,u_0\}\equiv N_1.$$

Similarly,

$$v(t)\leq max\left\{\frac{1}{b},v_0\right\}\equiv N_2.$$

This follows the proof. ■

4. Existence, Stability and Bifurcation of Feasible Equilibria

The feasible equilibria of the system (4) are the nonnegative solutions of the system

$$\begin{cases} u(1-u-v)=0, \\ v(1-bv-\frac{h}{c+v})=0. \end{cases} \tag{7}$$

There are two types of equilibria: boundary equilibria and interior equilibria.

4.1. Boundary Equilibria

Evidently, the system (4) always confesses the boundary equilibrium points $E_0(0,0)$ and $E_1(1,0)$, while other feasible boundary equilibrium points exist under certain parametric conditions.

If $u=0, v\neq 0$, then we get other boundary equilibria whose ordinates are given by

$$bv^2-v(1-bc)+h-c=0.$$

On solving the above equation, we get

$$v_{12}=\frac{(1-bc)-\sqrt{\Delta}}{2b}, \quad v_{13}=\frac{(1-bc)+\sqrt{\Delta}}{2b},$$

where, $\Delta=(1-bc)^2-4b(h-c)$.

If $\Delta>0$ and $h>c$, then the equilibria $E_{12}(0,v_{12})$ and $E_{13}(0,v_{13})$ both will exist, if $1>bc$. If $\Delta=0$, the two roots v_{12} and v_{13} will collide with each other and hence a unique equilibrium point $E_{11}(0,v_{11})=(0,\frac{1-bc}{2b})$ will materialize, if $1>bc$. If $h<c$, then a unique equilibrium point $E_{13}(0,v_{13})$ will exist.

Theorem 4.1.

(i) The equilibrium point E_0 is a saddle point if $h>c$ and it is an unstable point if $h<c$.
(ii) The equilibrium point E_1 is asymptotically stable if $h>c$ and it is a saddle point if $h<c$.
(iii) The equilibrium point E_{12} is an unstable point if $1>v_{12}$ and it is a saddle point if $1<v_{12}$.
(iv) The equilibrium point E_{13} is a saddle point if $1>v_{13}$ and it is a stable point if $1<v_{13}$.

Proof:

(i) The community matrix of the system (4) evaluated at E_0 is

$$J_{E_0}=\begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{c}(c-h) \end{bmatrix}.$$

The eigenvalues of the matrix J_{E_0} are $\lambda_1=1>0$ and $\lambda_2=\frac{a}{c}(c-h)$. Thus, the equilibrium point E_0 is a saddle point if $h>c$ and it is an unstable point if $h<c$.

(ii) The community matrix of the system (4) evaluated at $E_1(1,0)$ is

$$J_{E_1}=\begin{bmatrix} -1 & -1 \\ 0 & \frac{a}{c}(c-h) \end{bmatrix}.$$

The eigenvalues of the matrix J_{E_1} are $\lambda_1 = -1 < 0$ and $\lambda_2 = \frac{a}{c}(c-h)$. Thus, the equilibrium point E_1 is asymptotically stable if $h > c$ and it is a saddle point if $h < c$.

(iii) The community matrix of the system (4) evaluated at $E_{12}(0, v_{12})$ is

$$J_{E_{12}} = \begin{bmatrix} 1 - v_{12} & 0 \\ 0 & \frac{av_{12}\sqrt{\Delta}}{c+v_{12}} \end{bmatrix}.$$

The eigenvalues of the matrix $J_{E_{12}}$ are $\lambda_1 = 1 - v_{12}$ and $\lambda_2 = \frac{av_{12}\sqrt{\Delta}}{c+v_{12}}$. Thus, the equilibrium point E_{12} is an unstable point if $1 > v_{12}$ and it is a saddle point if $1 < v_{12}$.

(iv) The community matrix of the system (4) evaluated at $E_{13}(0, v_{13})$ is

$$J_{E_{13}} = \begin{bmatrix} 1 - v_{13} & 0 \\ 0 & -\frac{av_{13}\sqrt{\Delta}}{c+v_{13}} \end{bmatrix}.$$

The eigenvalues of the matrix $J_{E_{13}}$ are $\lambda_1 = 1 - v_{13}$ and $\lambda_2 = -\frac{av_{13}\sqrt{\Delta}}{c+v_{13}}$. Thus, the equilibrium point E_{13} is a saddle point if $1 > v_{12}$ and it is a stable point if $1 < v_{13}$. ■

4.2. Interior Equilibria

The ordinates of interior equilibria of system (4) are the positive roots of the quadratic equation

$$bv^{*2} - v^*(1 - bc) + h - c = 0, \tag{8}$$

whereas, the abscissa are given by $u^* = 1 - v^*,\ v^* < 1$.

On solving the quadratic equation (8), we get

$$v_2^* = \frac{1 - bc - \sqrt{\Delta}}{2b},\ \ v_3^* = \frac{1 - bc + \sqrt{\Delta}}{2b},$$

where $\Delta = (1 - bc)^2 - 4b(h - c)$. The following three cases arise:

4.2.1. *Case I:* $h > c$

It is simple to demonstrate from algebraic theory that $i)$ if $\Delta < 0$, then no interior equilibria will exist; $ii)$ if $\Delta = 0$, a unique positive equilibrium point $E_1^*(u_1^*, v_1^*) = \left(1 - v_1^*, \frac{1-bc}{2b}\right)$ will exist, if $1 > bc$; $iii)$ if $\Delta > 0$, then two distinct interior equilibria $E_2^* = (u_2^*, v_2^*)$ and $E_3^* = (u_3^*, v_3^*)$ will exist, if $1 > bc$.

Theorem 4.2.

The interior equilibrium point $E_2^*(u_2^*, v_2^*)$ is always a saddle point, while the interior equilibrium point $E_3^*(u_3^*, v_3^*)$ is always an asymptotically stable point.

Proof:

The community matrix of the system (4) evaluated at the interior equilibria $E_i^*(u_i^*, v_i^*)$, $i = 2, 3$ is

$$J_{E_i^*} = \begin{bmatrix} -u_i^* & -u_i^* \\ 0 & av_i^*(-b + \frac{h}{(c+v_i^*)^2} \end{bmatrix}.$$

The determinant of the matrix $J_{E_i^*}$ at $E_i^*(u_i^*, v_i^*)$ is det $J_{E_i^*} = au_i^* v_i^* \left(b - \frac{h}{(c+v_i^*)^2}\right)$ and the trace of the matrix $J_{E_i^*}$ at $E_i^*(u_i^*, v_i^*)$ is tr $J_{E_i^*} = -\left(av_i^*\left(b - \frac{h}{(c+v_i^*)^2}\right) + u_i^*\right)$.

Now, det $J_{E_2^*} = -\frac{au_2^* v_2^* \sqrt{\Delta}}{c+v_2^*} < 0$. This implies that the point E_2^* is a saddle point. Further, we have det $J_{E_3^*} = \frac{au_3^* v_3^* \sqrt{\Delta}}{c+v_3^*} > 0$ and tr $J_{E_3^*} = -\left(u_3^* + \frac{av_3^* \sqrt{\Delta}}{c+v_3^*}\right) < 0$. Thus, the point E_3^* is always an asymptotically stable point. ■

It is easy to check that determinant of the community matrix calculated at the point E_1^* is zero. This implies that point E_1^* is a non-hyperbolic point. As a result, a bifurcation may occur at this point. It has been stated that when $\Delta > 0$, the model (4) has two distinct feasible interior equilibria E_2^* and E_3^* and when $\Delta = 0$, these two equilibria meet with each other and materialize a unique interior equilibrium E_1^*. Further, whenever $\Delta < 0$ there exist no interior equilibria. This change in the number of feasible interior equilibria of the system (4) maybe because of the occurrence of saddle-node bifurcation. We shall apply the Sotomayor's theorem (Perko (2001)) to ascertain the occurrence of the bifurcation.

Theorem 4.3.

The system (4) exhibits a saddle-node bifurcation at the equilibrium point $E_1^* = (u_1^*, v_1^*)$ with respect to the parameter $h = h^{[sn]} = \frac{(1-bc)^2+4bc}{4b}$.

Proof:

The community matrix for the system (4) at the interior equilibrium point E_1^*, is

$$J_{E_1^*} = \begin{bmatrix} -u_1^* & -u_1^* \\ 0 & 0 \end{bmatrix}.$$

Evidently, one eigenvalue of the community matrix $J_{E_1^*}$ is zero and other is non zero. Let X and Y be the eigenvectors for the zero eigenvalue for the matrices $J_{E_1^*}$ and $J_{E_1^*}^T$, respectively. A simple computation implies

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consider

$$f(u, v, h) = \begin{pmatrix} 1 - u - v \\ 1 - bv - \frac{h}{(c+v)} \end{pmatrix}. \tag{9}$$

Thus, $f_h\Big(E_1^*, h^{[sn]}\Big) = \begin{bmatrix} 0 \\ -\frac{1}{(c+v_1^*)} \end{bmatrix}, \quad D^2 f\Big(E_1^*, h^{[sn]}\Big)(X, X) = \begin{pmatrix} 0 \\ -\frac{2h}{(c+v_1^*)^3} \end{pmatrix}.$

Now

$$Y^T.f_h\Big(E_1^*, h^{[sn]}\Big) = -\frac{1}{(c+v_1^*)} < 0,$$

$$Y^T.D^2 f\Big(E_1^*, h^{[sn]}\Big)(X, X) = -\frac{2h}{(c+v_1^*)^3} < 0.$$

The above two conditions ("the transversality condition for saddle-node bifurcation") ascertain the occurrence of saddle-node bifurcation for the proposed system (4). ∎

4.2.2. *Case II:* $h < c$.

In this case, the system (4) has only one feasible interior equilibrium point $E^* = (u^*, v^*)$, where $u^* = 1 - v^*$ and $v^* = \frac{1-bc+\sqrt{\Delta}}{2b}$.

Theorem 4.4.

The feasible point E^* is always an asymptotically stable point.

Proof:

The community matrix of the model (4) evaluated at the interior equilibrium point E^* is

$$J_{E^*} = \begin{bmatrix} -u^* & -u^* \\ 0 & av^*\Big(-b + \frac{h}{(c+v^*)^2}\Big) \end{bmatrix}.$$

The eigenvalues of the matrix J_{E^*} are $\lambda_1 = -u^*$ and $\lambda_2 = -av^*\left(b - \frac{h}{(c+v^*)^2}\right) = -\frac{av^*\sqrt{\Delta}}{c+v^*} < 0$. Thus, the point E^* is always an asymptotically stable point. ∎

Theorem 4.5.

The feasible point E^* is globally asymptotically stable if it is locally asymptotically stable and $4a(b - \frac{h}{(c+v)(c+v^*)}) > 1$.

Proof:

Consider a Lyapunov function

$$G(u, v) = (u - u^*) - u^* \log\left(\frac{u}{u^*}\right) + (v - v^*) - v^* \log\left(\frac{v}{v^*}\right).$$

Evidently, the function G is positive for all values of u, v except at equilibrium point E^*. The time derivative of the function G along the trajectories of model (4) is

$$\frac{dG}{dt} = \Big(\frac{u-u^*}{u}\Big)\frac{du}{dt} + \Big(\frac{v-v^*}{v}\Big)\frac{dv}{dt} = (u-u^*)[1-u-v] + a\Big[\Big(v-v^*)(1-bv-\frac{h}{c+v}\Big)\Big]. \quad (10)$$

We have equilibrium equations

$$1 - u^* - v^* = 0, \tag{11}$$

and

$$1 - bv^* - \frac{h}{c + v^*} = 0, \tag{12}$$

corresponding to the steady state $E^*(u^*, v^*)$. From Equations (10), (11) and (12), we can write

$$\frac{dG}{dt} = (u - u^*)[1 - u - v - 1 + u^* + v^*] + a(v - v^*)\left[1 - bv - \frac{h}{c+v} - 1 + bv^* + \frac{h}{c+v^*}\right]$$
$$= -[(u - u^*)^2 + (u - u^*)(v - v^*) + (v - v^*)^2 a(b - \frac{h}{(c+v)(c+v^*)})].$$

Thus, $\frac{dG}{dt}$ can be written as a quadratic form in the variables $(u - u^*)$ and $(v - v^*)$ which is negative definite if the matrix

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & a\left(b - \frac{h}{(c+v)(c+v^*)}\right) \end{bmatrix}$$

is positive definite.

Evidently, $\frac{dG}{dt} < 0$ if $4a\left(b - \frac{h}{(c+v)(c+v^*)}\right) > 1$. Hence, the result follows. ■

4.2.3. *Case III:* $h = c$.

In this case, the system (4) has only one feasible interior equilibrium point $E_* = (u_*, v_*)$, where $u_* = (1 - v_*)$ and $v_* = \frac{1-bc}{b}$, if $1 > bc$ and $v_* < 1$.

Theorem 4.6.

The feasible point E_* is always an asymptotically stable point.

Proof:

The community matrix of the model (4) evaluated at the interior equilibrium point E^* is

$$J_{E_*} = \begin{bmatrix} -u_* & -u_* \\ 0 & av_*\Big(-b + \frac{h}{(c+v_*)^2}\Big) \end{bmatrix}.$$

The eigenvalues of the matrix J_{E_*} are $\lambda_1 = -u_*$ and $\lambda_2 = -av_*\left(b - \frac{h}{(c+v_*)^2}\right) = -abv_*(1 - bc) < 0$. Thus, the point E^* is always an asymptotically stable point. ■

Theorem 4.7.

The equilibrium point E_* is globally asymptotically stable if $4ab\left(1 - \frac{h}{c+v}\right) > 1$.

Proof:

The proof is analogous to the proof of Theorem (4.5). ■

5. Bionomic Equilibrium Point

A point of the model (3) which is economic equilibrium point as well as biological equilibrium point, is referred to as the bionomic equilibrium point of the proposed model (3). Geometrically, it is an intersection point of the zero growth isoclines and zero profit line.

Let p and C, respectively, be the constant price per unit biomass of the second species and the harvesting cost per unit effort. Consequently, the net profit is represented by

$$\pi(x,y,E) = \Big(\frac{pqy}{m_1E+m_2y} - C\Big)E,$$

where x stands for the first species, y stands for the second species and E stands for the effort applied to harvest. The bionomic equilibrium point $(x_\infty, y_\infty, E_\infty)$ is the real solution of the curves $\frac{dx}{d\tau}=0$, $\frac{dy}{d\tau}=0$ and $\pi(x,y,E)=0$, that is, the real solution of the following system

$$\begin{cases} \frac{dx}{d\tau} = r_1(1-\frac{x}{k_1}) - c(1-k)y = 0, \\ \frac{dy}{d\tau} = r_2(1-\frac{y}{k_2}) - \frac{qE}{m_1E+m_2y} = 0, \\ \frac{pqy}{m_1E+m_2y} - C = 0. \end{cases} \tag{13}$$

The real solution of system (13) is

$$(x_\infty, y_\infty, E_\infty) = \left(\frac{r_1k_1 - c(1-k)k_1y_\infty}{r_1}, \frac{k_2r_2m_1p + ck_2m_2 - pqk_2}{r_2m_1p}, \frac{(pq-cm_2)y_\infty}{cm_1}\right),$$

provided, $\frac{cm_2}{p} < q < \frac{k_2r_2m_1p+ck_2m_2}{p}$ and $r_1 > c(1-k)$.

6. Optimal Harvesting

The objective of this section is to maximize the current value of continuous time stream of revenues which is given by

$$J(x,y,E,\tau) = \int_0^\infty \pi(x,y,E,t)e^{-\eta\tau}d\tau, \tag{14}$$

where ζ represents the instantaneous annual discount rate. Using Pontryagin's maximum principle, the problem reduces to

Maximize J

subject to

$$\begin{cases} \frac{dx}{d\tau} = \Big(r_1(1-\frac{x}{k_1}) - c(1-k)y\Big)x = 0, \\ \frac{dy}{d\tau} = \Big(r_2(1-\frac{y}{k_2}) - \frac{qE}{m_1E+m_2y}\Big)y = 0, \\ 0 \leq E \leq E_{max}. \end{cases} \tag{15}$$

The Hamiltonian associated to the problem is

$$\begin{aligned} H =& \Big(\frac{pqy}{m_1E+m_2y}\Big)Ee^{-\eta\tau} + \lambda_1\Big(\Big(r_1(1-\frac{x}{k_1}) - c(1-k)y\Big)x\Big) \\ &+ \lambda_2\Big(\Big(r_2(1-\frac{y}{k_2}) - \frac{qE}{m_1E+m_2y}\Big)y\Big), \end{aligned} \tag{16}$$

where $\lambda_1(\tau)$ and $\lambda_2(\tau)$ are known as adjoint variables. The maximization condition of H yields

$$\lambda_2 e^{\eta\tau} = p - \frac{C(m_1E+m_2y)^2}{qm_2y^2}, \tag{17}$$

where $\lambda_2 e^{\eta\tau}$ is the usual shadow price.

The adjoint equations $\frac{d\lambda_1}{d\tau} = -\frac{\partial H}{\partial x}, \frac{d\lambda_2}{d\tau} = -\frac{\partial H}{\partial y}$ are

$$\frac{d\lambda_1}{d\tau} = \lambda_1\Big(\frac{r_1}{k_1}\Big), \tag{18}$$

and

$$\frac{d\lambda_2}{d\tau} = -\Big(\frac{pqm_1E^2}{(m_1E+m_2y)^2}\Big)e^{-\eta t} + \lambda_1\Big(c(1-k)\Big)x + \lambda_2\Big(\frac{r_2}{k_2} - \frac{qEm_2}{(m_1E+m_2y)^2}\Big)y, \tag{19}$$

respectively.

The appearance of $e^{-\eta\tau}$ in the above equation confirms that steady state is not possible for the this system. Consider the transformation

$$\lambda_i(\tau) = \mu_i(\tau)e^{-\eta\tau}, \ \ i = 1, 2, \tag{20}$$

where μ_i represents the present value of the adjoint variable λ_i. From Equations (17), (18) and (20), we have

$$\frac{d\mu_1}{d\tau} - \eta\mu_1 = -R(y), \tag{21}$$

where, $R(y) = \frac{r_1}{k_1}\Big(\frac{C(m_1E+m_2y)^2}{qm_2y^2} - p\Big)$. The shadow prices $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_2$ do not vary over time in singular equilibrium to satisfy the transversality conditions at ∞ (i.e., $\lim_{t\to\infty} \lambda_i(\tau) = 0$, for $i = 1, 2$). Hence, we get the solution of differential equation (21) as

$$\mu_1 = \frac{R(y)}{\eta}.$$

Employing this value of μ_1, Equation (19) can be expressed in term of μ_2 as follows:

$$\frac{d\mu_2}{d\tau} - (\eta + S_1(y))\mu_2 = -S_2(y), \tag{22}$$

where $S_1(y) = (\frac{r_2}{k_2} - \frac{qEm_2}{(m_1E+m_2y)^2})y$ and $S_2(y) = \frac{pqm_1E^2}{(m_1E+m_2y)^2} - \frac{c(1-k)xR(y)}{\eta}$. The solution of the differential equation (22), which is satisfying the transversality condition at ∞, is

$$\mu_2 = \frac{S_2(y)}{\eta + S_1(y)}. \tag{23}$$

From Equations (17) and (23), we have

$$\frac{C(m_E + m_2y)^2}{qm_2y^2} + \frac{S_2}{\eta + S_1} = p. \tag{24}$$

Equation (24) gives the desired solution.

7. Numerical Simulations

In this section, numerical simulations are carried out to validate analytical conclusions. The diagrams have been sketched by MATHEMATICA 7.0 software.

1) Consider $a = 5,\ b = 0.615,\ c = 0.9$ and $h > c$. If $h = 0.98$, then the system (4) has six feasible equilibria $E_0 = (0,0)$, $E_1 = (1,0)$, $E_{12} = (0, 0.34855)$, $E_{13} = (0, 0.404162)$, $E_2^* = (0.678145, 0.34855)$ and $E_3^* = (0.595838, 0.404162)$. The equilibria $E_0,\ E_{13}$ and E_2^* are saddle points, the equilibrium point E_{12} is an unstable point, and equilibria E_1 and E_3^* are asymptotically stable points (see Figure 1(a)). If $h = 0.981042$, then the model (4) has four feasible equilibria points $E_0 = (0,0),\ E_1 = (1,0),\ E_{0,11} = (0, 0.363008)$ and $E_1^* = (0.636992, 0.363008)$. The point E_0 is saddle, point E_1 is always asymptotically stable and equilibria E_{11}, and E_1^* are saddle-node points (see Figure 1(b)). If $h = 0.99$, then the system (4) has two axial equilibria $E_0(0,0)$ and $E_1(1,0)$. The equilibrium point E_0 is saddle while the equilibrium E_1 is always asymptotically stable (see Figure 1(d)).

2) Consider $a = 5,\ b = 0.615,\ c = 0.91$, satisfying $h < c$ if $h = 0.9$, then the system (4) has four feasible equilibria points $E_0(0,0),\quad E_1(1,0),\ E_{13}(0, 0.738048)$ and $E^* = (u^*, v^*) = (0.261952, 0.738048)$. The point E_0 is unstable, points $E_1,\ E_{13}$ are saddle while E^* is asymptotically stable point (see Figure 2).

3) Consider $a = 5,\ b = 0.615,\ c = 1.3$, satisfying $h = c$ if $h = 1.3$, then the system (4) has four feasible equilibria points $E_0(0,0),\quad E_1(1,0)$, $E_{13}(0, 0.326016)$ and $E_* = (u_*, v_*) = (0.673984, 0.326016)$. The point E_0 is unstable and points $E_1,\ E_{13}$ are saddle point while E^* is asymptotically stable (see Figure 3).

8. Conclusion

In this paper, a two-species amensalism model is made more realistic by introducing non-linear harvesting in the second species. It has been observed that the behavior of the proposed model is

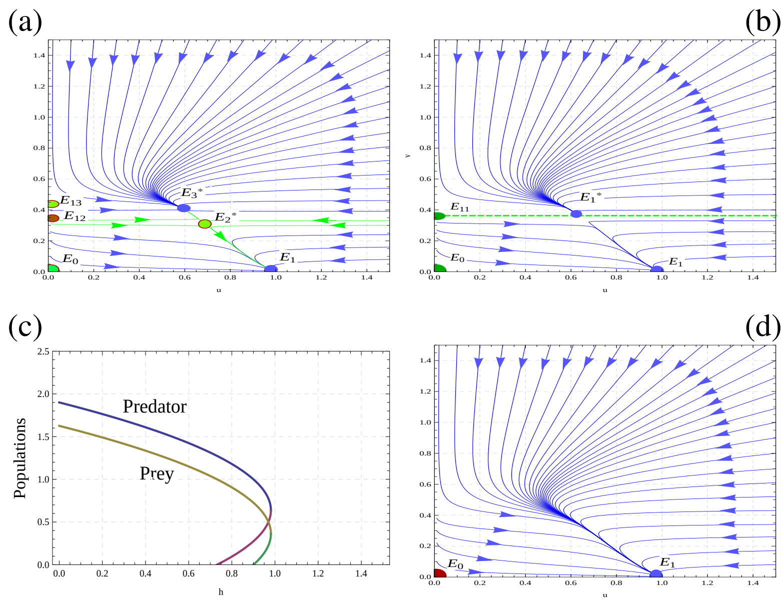

Figure 1. $a = 5,\ b = 0.615,\ c = 0.9$. Phase portrait diagram of system (4) (a) $h = 0.98$. (b) $h = 0.981042$, obtained from bifurcation diagram (c). (d) $h = 0.99$.

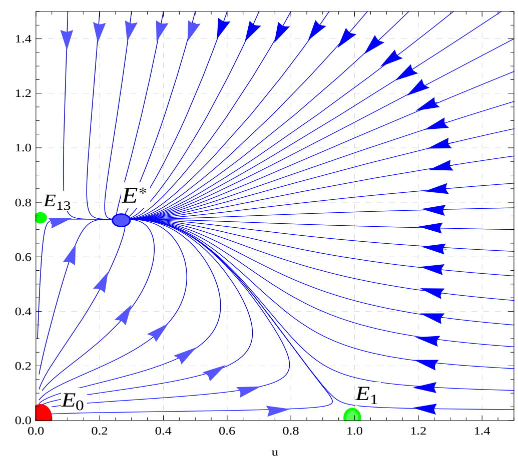

Figure 2. $a = 5,\ b = 0.615,\ c = 0.91$. Phase portrait diagram of system (4) when $h = 0.9$.

more complicated since it has more equilibrium points, and the number of equilibrium points may vary when the harvesting parameter changes.

It has been shown that, for certain parametric conditions the proposed system has at most six feasible equilibria including two positive interior equilibria. Moreover, a separatrix divides the feasible area into two parts. The solutions converge to the coexisting equilibrium point if the starting values are above this separatrix, while the solutions converge to the second species free equilibrium point if the initial values are below the separatrix. Ecologically, the solutions are highly dependent on the initial values. It has also been found that the number of interior equilibrium points varies from two to zero as the bifurcation parameter crosses a certain value. Sotomayor's theorem has been used to show the presence of saddle-node bifurcation. Ecologically, below a certain value of the bifurcation parameter, both species can coexist, but above that, the second species suffer extinction.

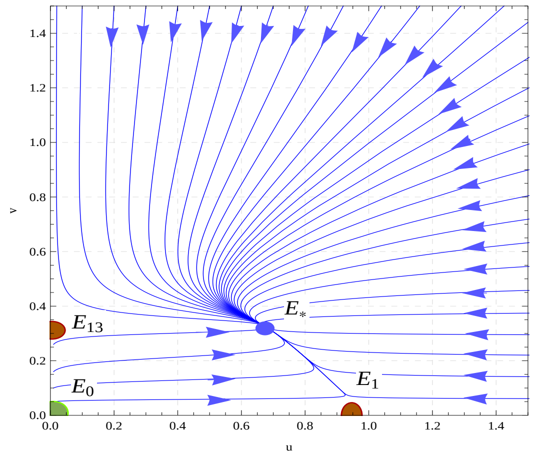

Figure 3. $a = 5,\ b = 0.615,\ c = 1.3$. Phase portrait diagram of system (4) when $h = 1.3$

Further, it is shown that for certain parametric conditions the proposed system has at most four feasible equilibria including a unique interior equilibrium point exist. This interior equilibrium point is globally asymptotically stable for certain parametric conditions.

The parametric conditions under which a bionomic equilibrium point exists have been established. We have also discussed the optimal harvesting policy in which Pontryagin's maximum principle is used to maximize the amount of revenues.

The mathematical study leads us to the ecological conclusion that harvesting has no adverse effects on the first species but may result in the extinction of the second species. Additionally, since the origin is never stable, the amensalism system will never collapse for any values of the parameter. In light of non-linear harvesting, the amensalism model will thus be more useful and applicable in actual circumstances.

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REFERENCES

Chen, B. G. (2018). Dynamic behaviors of a non-selective harvesting Lotka-Volterra amensalism model incorporating partial closure for the populations, Adv. Differ. Equ., Vol. 2018, No. 1, pp. 1–14.

Clark, C.W. (1976). *Mathematical Bionomics: The Optimal Management of Renewable Resources*, Wiley, New York.

Das, T., Mukherjee, R. N. and Chaudhuri, K. S. (2009). Harvesting of a prey-predator fishery in the presence of toxicity. Appl. Math. Model., Vol. 33, No. 5, pp. 2282–2292.

Guan, X. and Chen, F. (2019). Dynamical analysis of a two species amensalism model with Beddington-DeAngelis functional response and Allee effect on the second species, Nonlinear Anal. Real World Appl., Vol. 48, pp. 71–93.

Gupta, R. P. and Chandra, P.(2013). Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting, J. Math. Anal. Appl., Vol. 398, No. 1, pp. 278–295.

Gupta, R. P., Chandra, P. and Banerjee, M. (2015). Dynamical complexity of a Predator-prey model with nonlinear predator harvesting, Dis. Con. Dyn. Sys. Ser., Vol. 20, No. 2, pp. 423–443.

Han, R., Xue, Y., Yang, L. and Chen, F. (2015). On the existence of positive periodic solution of a Lotka-Volterra amensalism model, J. Rongyang Univ., Vol. 33, No. 2, pp. 22–26.

Hu, D. and Cao, H. (2017). Stability and bifurcation analysis in a predator-prey system with Michaelis-Menten type predator harvesting, Nonlinear Anal. Real World Appl., Vol. 33, pp. 58–82.

Huang, J., Gong, Y. and Ruan, S. (2013). Bifurcation analysis in a predator-prey model with constant-yield predator harvesting, Discrete Contin. Dyn. Syst., Ser., Vol. 18, No. 8, pp. 2101–2121.

Kar, T.K. and Ghosh, B. (2012). Sustainability and optimal control of an ex- ploited prey-predator system through provision of alternative food to predator, Biosystems, Vol. 109, 220–232.

Liu, Y., Xie, X. and Lin, Q. (2018). Permanence, partial survival, extinction, and global attractivity of a nonautonomous harvesting Lotka-Volterra commensalism model incorporating partial closure for the populations, Adv. Differ. Equ., Vol. 2018, No. 1, pp. 1–16.

Lin, C. M. and Ho, C. P. (2006). Local and global stability for a predator-prey model of modified Leslie-Gower and Holling-type II with time-delay, Tunghai Sci., Vol. 8, pp. 33–61.

Liu, Y., Zhao, L., Huang, X. and Deng, H. (2018). Stability and bifurcation analysis of two species amensalism model with Michaelis-Menten type harvesting and a cover for the first species, Adv. Differ. Equ., Vol. 2018, No. 1, pp. 1–19.

Perko, L. (1996). *Differential Equations and Dynamical Systems*, Springer, New York.

Rambabu, B.S., Narayan, K. L. and Bathul, S. (2012). A mathematical study of two species amensalism model with a cover for the first species by homotopy analysis method, Adv. Appl. Sci. Res., Vol. 3, No. 3, pp. 1821–1826.

Singh, M. K. and Bhadauria, B. S. (2020). The impact of nonlinear harvesting on a ratio-dependent Holling-Tanner predator-prey system and optimum harvesting, Appl. Appl. Math., Vol. 15, No. 1, pp. 117–148.

Sun, G. C. (2003). Qualitative analysis on two populations amensalism model, J. Jiamusi Univ. (Natl. Sci. Ed.), Vol. 21, No. 3, pp. 283–286.

Wei, Z. Xia, Y. and Zhang, T. (2020) Stability and bifurcation analysis of an Amensalism model with weak Allee effect, Qual. Theory Dyn. Syst, Vol. 19, No. 1, pp. 1–15.

Wu, R. (2018). Dynamic behaviors of a nonlinear amensalism model, Adv. Differ. Equ., Vol. 2018, No. 1, pp. 1–13.

Wu, R. X., Zhao, L. I. A. N. G. and Lin, Q. X. (2016). Stability analysis of a two species amensalism model with Holling II functional response and a cover for the first species, J. Nonlinear

Funct. Anal., Vol. 46, pp. 1–15.

Xiao, D. and Jennings, L. S. (2005). Bifurcations of a ratio-dependent predator-prey system with constant rate harvesting, J. App. Math, Vol. 65, No. 3, pp. 737–753.

Xie, X., Chen, F. D. and He, M. X. (2016). Dynamic behaviors of two species amensalism model with a cover for the first species, J. Math. Comput. Sci., Vol. 16, pp. 395–401.

Xu, C. Liao, M. Li, P. and Yuan, S. (2021a). Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks, Chaos, Solitons and Fractals, Vol. 142, 110535.

Xu, C. Liu, Z. Aouiti, C. Li, P. Yao, L. and Yan, J. (2022a). New exploration on bifurcation for fractional-order quaternion-valued neural networks involving leakage delays, Cogn. Neurodyn, Vol. 16, No. 5, 1233–1248.

Xu, C. Liu, Z. Liao, M. and Yao, L. (2022b). Theoretical analysis and computer simulations of a fractional order bank data model incorporating two unequal time delays, Expert Syst. Appl., Vol. 199, 116859.

Xu, C. Zhang, W. Aouiti, C. Liu, Z. Liao, M. and Li, P. (202b1). Further investigation on bifurcation and their control of fractional-order bidirectional associative memory neural networks involving four neurons and multiple delays, Math. Meth. in the App. Sci., Vol. 46, No. 3, 3091–3114.

Xu, C. Zhang, W. Aouiti, C. Liu, Z. and Yao, L. (2022c). Further analysis on dynamical properties of fractional-order bi-directional associative memory neural networks involving double delays, Math. Meth. in the App. Sci., Vol. 145, No. 17, 11736–11754.

Yu, L., Guan, X., Xie, X. and Lin, Q. (2019). On the existence and stability of positive periodic solution of a nonautonomous commensal symbiosis model with Michaelis-Menten type harvesting, Commun. Math. Biol. Neurosci, Vol. 2019, No. 2, pp. 1–20.

Zhang, N., Chen, F., Su, Q. and Wu, T. (2011). Dynamic behaviors of a harvesting Leslie-Gower predator-prey model, Discrete Dyn. Nat. Soc., Vol. 2011, Article ID 473949.

Zhao, M. and Du, Y. (2020). Stability and bifurcation analysis of an amensalism system with Allee effect, Adv. Differ. Equ., Vol. 2020, No. 1, 1–13.