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A Brief Note on Space Time Fractional Order Thermoelastic Response in a Layer

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Abstract

In this study, a one-dimensional layer of a solid is used to investigate the exact analytical solution of the heat conduction equation with space-time fractional order derivatives and to analyze its associated thermoelastic response using a quasi-static approach. The assumed thermoelastic problem was subjected to certain initial and boundary conditions at the initial and final ends of the layer. The memory effects and long-range interaction were discussed with the help of the Caputo-type fractional-order derivative and finite Riesz fractional derivative. Laplace transform and Fourier transform techniques for spatial coordinates were used to investigate the solution of the temperature distribution and stress functions. Numerical investigations are also shown graphically for non-dimensional temperature and stress for different space and time fractional derivative values, respectively. In addition, some applicable limiting cases are discussed for standard equations, such as wave equation, Laplace equation, and diffusion equation.

Keywords: Heat Equation; Caputo fractional derivative; Riesz fractional derivative; Integral transformation; Generalized Mittag-Leffler function; Temperature; Thermal stresses

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1. Introduction

In recent years, numerous fractional calculus applications have been observed in various fields, such as mechanics, physics, chemistry, biology, and engineering. Different forms of non-integer or fractional differential operators are mainly described by Caputo-type fractional derivatives, Riemann-Liouville derivatives, and Riesz fractional derivatives (Herrmann (2011), Hilfer (2000), Miller and Ross (1993), Oldham and Spanier (1974), and Podlubny (1999)).

The thermal equation is a parabolic form in classical thermoelasticity theory, which propagates a heat wave's infinite velocity. Such an approximation fails to present the fundamental physical properties, mainly in problems related to shocks. This problem has been eliminated in the last few years by introducing several theories based on a non-classical approach. Such ideas replace the classical Fourier's law of heat conduction with a modified form and involve hyperbolic form in place of parabolic heat transfer equations with thermal pulses of finite velocity. In modified theories, thermal propagation behaves as a wavefront and diffusion phenomenon.

Few authors, like Povstenko (2011a, b, c, and 2015), Warbhe et al. (2018), and Kumar and Kamdi (2020), studied the thermal effect in solid bodies by considering the time-fractional derivative heat conduction equation. Lamba (2022) studied the thermosensitive analysis of a F.G. cylinder by considering the time fractional derivative theory.

The available literature shows that only limited work has been done on space fractional-order derivatives. Such work is essential in designing new, novel materials with real physical properties reflecting long-range interaction. Fil'Shtinskii et al. (2013) discussed and analyzed the one-dimensional fractional space-time problem of half-space. Sayed and Gaber (2006) derived finite Riesz-type fractional derivatives using left and right finite Caputo derivatives. Pawar et al. (2015) determined the thermal stresses in a hollow, thick sphere of functionally graded material subjected to non-uniform internal heat generation. Tripathi et al. (2017) discussed the quasi-static uncoupled theory of thermoelasticity based on the time-fractional heat conduction equation for a thin circular plate.

Other works related to time-fractional order derivatives have been noted in Benchohra et al. (2008), Ciesielski and Leszczynski (2006), Gorenflo et al. (2002), Jiang and Xu (2010), Molliq et al. (2009), Momani (2006), Murio (2008), Povstenko (2009), Povstenko (2008), and Yang et al. (2010).

In the present research article, we assume a layer of a one-dimensional solid and discuss the space-time fractional thermoelastic model using a quasi-static approach. The results are illustrated analytically by the integral transform technique and numerical illustrations have been done with the help of graphical representation.

In this paper, we have formulated the analytical and mathematical (physical) model of a given solid with the help of space-time fractional order theory and illustrated the thermal stress numerically using space-time temperature distribution.

2. Formulation of the Problem

We have formulated a mathematical model of a one-dimensional layer of a solid occupying the space $0 \le x \le l$. The layer's initial end is kept at zero temperature while the final end is subjected to a prescribed jump function $P(t)$. With the help of temperature distribution, we have discussed the thermoelasticity of a finite length layer and studied the thermoelasticity with the impact of space-time fraction order derivatives numerically. In this problem, limiting cases for weak, moderate and superconductivity have been discussed.

2.1 Temperature Distribution

In this problem, we have modified the work of heat conduction studied by Fil'Shtinskii et al. (2013) within the framework of the space-time fractional order derivative for a one-dimensional finite-length layer in range $0 \le x \le l$ as defined as:

$$\frac{\partial^{\beta} T(x,t)}{\partial x^{\beta}} = \frac{1}{a}\frac{\partial^{\alpha} T(x,t)}{\partial t^{\alpha}}, \tag{1}$$

$$0<\alpha<2\,,\ 1<\beta<2\,,\ 0\le x\le l,\ t>0.$$

with the initial and boundary conditions as

$$T(x,t)\big|_{x=0} = 0, \tag{2}$$

$$\left.\frac{\partial T(x,t)}{\partial x}\right|_{x=l} = P(t), \tag{3}$$

$$T(x,t)\big|_{t=0} = 0 \quad ; \quad 0<\alpha\le 1, \tag{4}$$

$$\left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0} = 0 \quad ; \quad 1<\alpha<2. \tag{5}$$

Here, $T(x,t)$ denotes the temperature, a represents the thermal diffusivity coefficient, $\partial^{\beta}/\partial x^{\beta}$ denotes the Riesz space fractional order derivative of order β, $\partial^{\alpha}/\partial t^{\alpha}$ denotes Caputo type time-fractional derivative of order α and $P(t)$ refers as a jump function.

We have discussed the thermoelasticity of a given solid with the help of the temperature distribution of a finite length layer defined as above.

2.2 Thermal displacements and thermal stress

For a one-dimensional layer of a solid, the stress, displacement, and temperature relationship is as

$$\sigma_{22} = \sigma_{33} = -2\mu\frac{\partial u}{\partial x} = -\frac{2\mu\gamma}{\lambda+2\mu}T, \tag{6}$$

$$\sigma_{11} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0.$$

where u is the displacement potential defined as:

$$u = \frac{\partial \phi}{\partial x}, \tag{7}$$

where λ, μ are the Lames constants, $\gamma = 3K\alpha_T$, α_T is the linear expansion coefficient, K is the bulk modulus and σ_{ij} is the stress components.

Thus, to analyze the stress functions in the assumed domain, it is necessary to first determine the temperature distribution function from equation (1) with conditions (2) to (5).

The above equations (1) to (7) constitute the mathematical formulation of the problem under consideration.

3. Solution of the problem

3.1 Solution of the heat conduction problem

On applying Laplace and cos-Fourier transformations and taking their inversion, the required analytical expression of the temperature distribution from equation (1) is obtained as

$$T(x,t) = \frac{2a}{l}\sum_{\zeta=1}^{\infty}(-1)^{\zeta}\,\zeta^{\beta-2}\left\{\int_0^t \tau^{\alpha-1} E_{\alpha,\beta}\left(\frac{-a\zeta^{\beta}\pi^2}{l^2}\tau^{\alpha}\right)P(t-\tau)\,d\tau\right\}\cos\frac{\zeta\pi x}{l}, \quad \text{for } x>0,\ t>0. \tag{8}$$

where $E_{\alpha,\beta}$ denotes the generalized Mittag-Leffler function,

$$L^{-1}\left\{\frac{1}{s^{\alpha}+\frac{a\zeta^{\beta}\pi^2}{l^2}}\right\} = t^{\alpha-1}E_{\alpha,\beta}\left(\frac{-a\zeta^{\beta}\pi^2}{l^2}t^{\alpha}\right); \quad \alpha>0,\ \frac{a\zeta^{\beta}\pi^2}{l^2}>0.$$

3.2 Solution of the thermal stress problem

The expressions of displacement potential and stress functions are obtained by using Equations (6), (7), and (8), respectively, as

$$\phi(x) = \frac{-m}{4\pi}\int_0^l \frac{1}{(x-\xi)}\left[\frac{2a}{l}\sum_{\zeta=1}^{\infty}(-1)^{\zeta}\,\zeta^{\beta-2}\left\{\int_0^t \tau^{\alpha-1}E_{\alpha,\beta}\left(\frac{-a\zeta^{\beta}\pi^2}{l^2}\tau^{\alpha}\right)P(t-\tau)\,d\tau\right\}\times\cos\frac{\zeta\pi\xi}{l}\right]d\xi. \tag{9}$$

$$\sigma_{22} = \sigma_{33} = -2\mu m\frac{2a}{l}\sum_{\zeta=1}^{\infty}(-1)^{\zeta}\,\zeta^{\beta-2}\left\{\int_0^t \tau^{\alpha-1}E_{\alpha,\beta}\left(\frac{-a\zeta^{\beta}\pi^2}{l^2}\tau^{\alpha}\right)P(t-\tau)\,d\tau\right\}\cos\frac{\zeta\pi x}{l}. \tag{10}$$

4. Numerical Analysis

The mathematical model has been prepared for the different parameters and functions as a special case. Following are the non-dimensional parameters used for the numerical analysis of the problem:

$$\tau = \left(\frac{a}{L^{\beta}}\right)^{\frac{1}{\alpha}} t,\ \xi = \frac{x}{L},\ \theta = \frac{T}{T_0},\ \sum_{22} = -\frac{\sigma_{22}}{2\mu m}. \tag{11}$$

where L denotes the scale of length, T_0 is initially distributed temperature, θ and $\sum_{22}$ stand for non-dimensional distribution of temperature and stress function. The following figures show the behaviours of non-dimensional temperature variation and the stress distribution for various values of α and β.

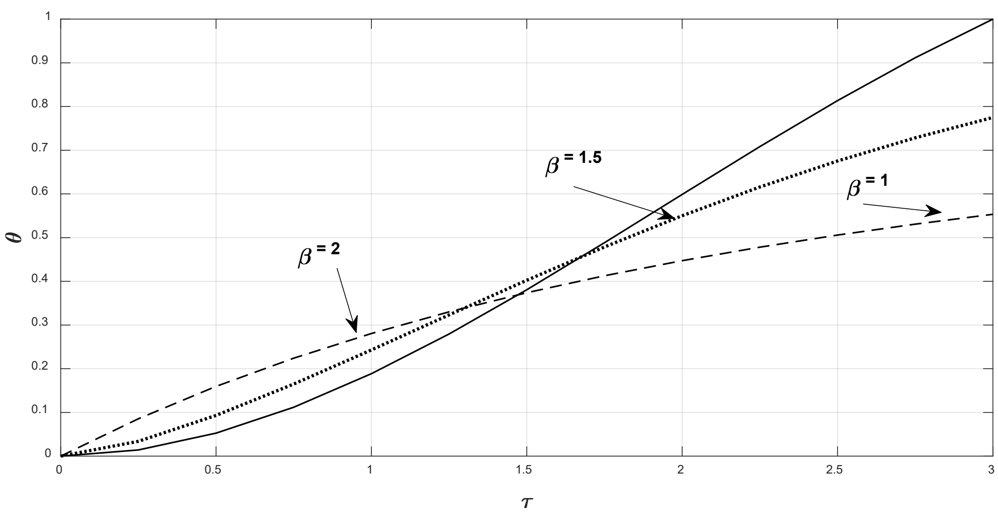

Figure 1. Non-dimensional temperature (stress) flow with respect to non-dimensional time at $\xi = 1$ for different $\beta(\alpha = 1)$

Figure 1 shows the non-dimensional temperature (stress) variation for a time at $\xi = 1$ different values $\beta(\alpha = 1)$. From the plotting, it is observed that the variation of temperature depends on both the neighbourhood of selected points and on mesh points. It is noted that the temperature (stress) distribution shows significant variance for large values of time or spatial variables. The following cases are also observed.

Case 1: For $\alpha = 1$ and $\beta = 2$, equation (1) is a diffusion equation, and its solution of stresses in terms of temperature and its variations under assumed initial and boundary conditions matches Nowacki (1975).

Case 2: For $\alpha = 1, \beta = 1$, and $\beta = 1.5$, the graphical representation of Equation (1) shows the variation of stresses in terms of temperature respectively which interpolates the classical heat conduction equations.

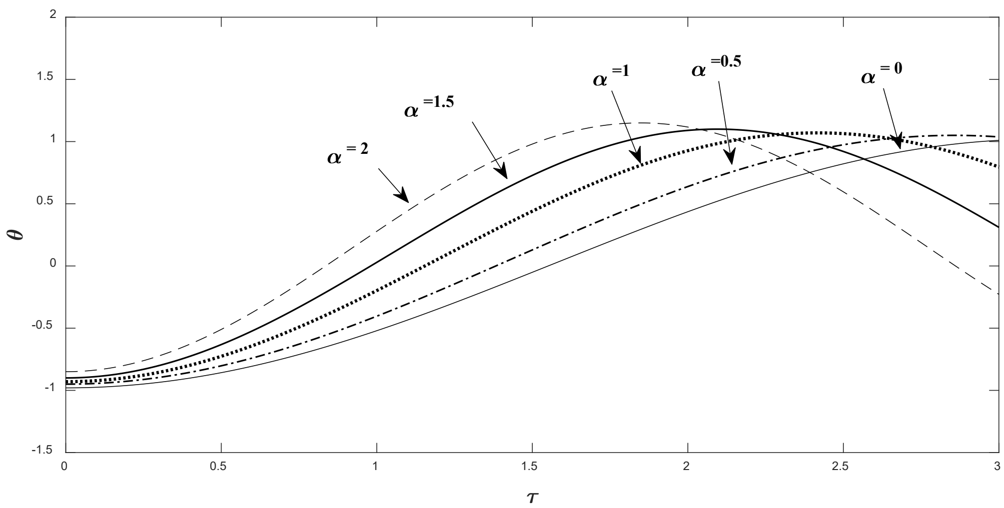

Figure 2. Non-dimensional temperature (stress) flow with respect to non-dimensional time at $\xi = 1$ for different $\alpha(\beta = 2)$

Figure 2 represents the non-dimensional temperature (stress) distributions in time at a point $\xi = 1$ for different values $\alpha(\beta = 2)$. From the plotting, it is observed that the variation of the temperature (stress) distribution is directly proportional to the fractional order parameter α . Also, it is observed that temperature equilibrium appears faster for small values α as compared to large values. Further, the following cases are observed:

Case 1: For $\beta = 2$ and $\alpha = 0$, Equation (1) is a Laplace equation. In this case, the obtained solution of stresses in terms of temperature distribution and its variation under assumed conditions found a good match with the standard plot.

Case 2: For $\beta = 2$ and $\alpha = 0.5$, $\alpha = 1.5$, $\alpha = 2$, Equation (1) and its solution of stresses in terms of temperature distribution and its variation show the fractional order behaviour under assumed initial and boundary conditions. For $\alpha = 0.5$ and $\alpha = 1.5$, which interpolates the wave equation and shows the reflection of the memory effect, it is observed that the obtained result follows the numerical analysis done by Povstenko (2009).

Case 3: For $\beta = 2$ and $\alpha = 1$, $\alpha = 2$, Equation (1) is a diffusion and wave equation, respectively, and its corresponding solutions in terms of temperature (stress) with variation along time under assumed initial and boundary conditions match standard plotting.

Case 4: Also, it is observed that for $\beta = 2$ and L tends to ∞ the solution of the problem in such a situation is presented for $\alpha = 0.5$, $\alpha = 1$ $\alpha = 1.5$ $\alpha = 2$ and for a fixed relaxation parameter, these results are in good agreement with Fil'Shtinskii et al. (2013) results.

5. Conclusion

In this research article, we assumed a one-dimensional quasi-static time-space fractional order thermoelastic problem in a finite domain with Caputo and Riesz fractional order derivatives. Temperature distribution and corresponding thermal stresses are obtained by the integral transform method. The solution satisfies the appropriate boundary conditions. Limiting cases coincide with the solutions of standard equations like the wave equation, Laplace equation, and diffusion equation.

From the above numerical analysis, the following observations have been noted:

- The space-time fractional-order derivative heat equation predicts the global tendency. The local diffusion equation and the wave equation interpolate the classical equation of heat conduction for the fractional value of α and β .
- Due to the fractional order theory, the non-local operator predicts the retarded response to physical stimuli found in nature, which opposes the instantaneous response to the non-localized generalized theory of heat conduction.
- Also, because of thermal conductivity, we observe a retarded response and long-range interactions for weak, moderate, and superconductivity.

Hence, the assumed modelling can describe the anomalous behaviour of many complex processes including the multi-scaling of the time and space variables. Such thermoelastic problems apply to investigating the dynamics of viscoelastic materials such as polymers, the diffusion of pollution in the atmosphere, diffusion processes involving cells, signals theory, control theory, electromagnetic theory, etc.

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Nomenclature

ν - Poisson's ratio,

λ, μ -Lames constants,

δ_{ij} - Kronecker symbol,

σ_{ij} -Stress components

ε_{ij} -Strain components

α - Order of non-local derivative of Caputo type

β - Order of finite Riesz space fractional derivative

$T(x,t)$ - Temperature function,

a - Thermal diffusivity coefficient,

$P(t)$ - Prescribed jump function

u_i -Displacement components

$E_{\alpha,\beta}$ - generalized Mittag-Leffler function

x -Space variable

t -Time variable

$I_{0+}^{2-\beta}$, $I_{-}^{2-\beta}T(x)$ - Riemann-Liouville fractional integrals

s -Laplace transforms parameter

α' -Elastic constant

$q(t)$ -Heat flux

$E_{\alpha,\beta}(x)$ - Generalized Mittag-Leffler function

τ -Non-dimensional time parameters

ξ -Non-dimensional space parameters

θ -Non-dimensional temperature parameters

$\sum_{22}$ -Non-dimensional stress parameters

L -Denotes the scale of length

T_0 - Initially distributed temperature