(R2026) Special Smarandache Ruled Surfaces According to Flc Frame in E^3

Süleyman Şenyurt
Ordu University

Kebire Hilal Ayvacı
Ordu University

Davut Canlı
Ordu University

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[1]**Süleyman Şenyurt,** [2*] **Kebire Hilal Ayvacı, and** [3] **Davut Canlı**

[1,2*,3] Department of Mathematics
Faculty of Arts and Science
Ordu University
52200, Ordu,Turkey
[1]senyurtsuleyman52@gmail.com; [2*]kebirehilalayvaci@odu.edu.tr;
[3] davutcanli@odu.edu.tr

*Corresponding Author

Abstract

In this study, we introduce some special ruled surfaces according to the Flc frame of a given polynomial curve. We name these ruled surfaces as TD_2, TD_1 ve D_2D_1 Smarandache ruled surfaces and provide their characteristics such as Gauss and mean curvatures in order to specify their developability and minimality conditions. Moreover, we examine the conditions if the parametric curves of the surfaces are asymptotic, geodesic or curvature line. Such conditions are also argued in terms of the developability and minimality conditions. Finally, we give an example and picture the corresponding graphs of ruled surfaces by using Maple 17.

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1. Introduction

The theory of curves and surfaces with its vast application and field of study is an important subject of differential geometry. Surfaces, especially, have potential to provide solutions to the real world problems as the technological developments are continuously advancing and letting them to be more integrated in. As a special kind, the ruled surfaces introduced first by G. Monge are very popular since they are easy to be handled in computational sense. These surfaces are widely referred in the fields such as kinematics, architectural designs, computer aided geometric designs (CAGDs), etc. For example, the kinematic characteristic of a spatial motion particle can be explained by ruled surfaces and their corresponding orthogonal frames (Karger and Novak (1978)). The examples of ruled surfaces are also found in important architectural structures like the Ciechanow water tower, Kobe Port Tower, Shuckhov tower, and so on. In the geometric point of view, a ruled surface is defined to be family of lines as it is formed by a moving line along a given curve. The most famous examples of those are cylinders and cones. The extensive use of ruled surfaces lead researchers to question some of their characterizations. Topics such as the developability and minimality of ruled surfaces, and the characterization of the curves on a ruled surface are the most researched subjects.

For instance, the cylindrical helix and Bertrand curves on ruled surfaces were discussed by Izumiya and Takeuchi (2003) and they provided the relationships between the cylindrical helix curve (respectively, Bertrand curve) and Gaussian (respectively, mean) curvature of the ruled surface in Izumiya and Takeuchi (2003). Yu et al. (2014) defined the structural functions of ruled surfaces and investigated the invariants, kinematic properties and geometric properties of non-developable ruled surfaces. Ruled surfaces accepting the focal curve as a direction curve were examined by Alegre et al. (2010) and their various characterizations were discussed. Hu et al. (2020) constructed new ruled surfaces by referring Bezier curves. Özsoy (2019) introduced normal and binormal ruled surfaces where the base curve was assigned to be W-direction curve, and by providing the characteristics of these surfaces, they worked the cases of the base curve as asymptotic, geodesic and curvature line. More recently, Ouarab (2021) put forth a new concept on ruled surfaces by utilizing the definition of Smarandache curve and introduced new ruled surfaces. Smarandache curves are another interested subject of differential geometry. A Smarandache curve is defined to be the linear combinations of the vector elements of Frenet frame of a regular curve (Ashbacher (1997)).

The Smarandache curve was first introduced by Turgut and Yılmaz (2008) in Minkowski space and they studied the Serret-Frenet apparatus of it. Ali A.T. (2010) considered these curves in the Euclidean space E^3. Following these, Çetin et al. (2011) defined new Smarandache curves according to Bishop frame while Bektaş and Yüce (2013) worked on these curves according to Darboux frame. Taşköprü and Tosun (2014) studied Smarandache curves on unit sphere S^2. Şenyurt and Çalışkan (2016) defined the N^*C^* Smarandache curves of Mannheim and Bertrand curve pairs according to the Frenet frame respectively. Demircan (2015) presented new Smarandache curves according to Type 2-Bishop frame and gave some geometric characterizations of those. Mandal (2019) exploited the q-frame to construct Smarandache curves in both Euclidean and space. Şenyurt et al. (2020) Smarandache curves constructed by the Frenet vectors of spacelike anti-Salkowski curve with a spacelike principal normal defined. As already pointed out, Ouarab

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(2021) exploited the idea of the construction of Smarandache curves and extended the concept to the ruled surfaces. She introduced some special Smarandache ruled surfaces according to Frenet frame, alternative frame and Darboux frame.

Motivated by these, in this paper, we have studied some special ruled surfaces according to the Flc frame of a given polynomial curve. We name these ruled surfaces as TD_2, TD_1 ve D_2D_1 Smarandache ruled surfaces and provide their characteristics such as Gauss and mean curvatures in order to specify their developability and minimality conditions. Moreover, we examine the conditions if the parametric curves of the surfaces are asymptotic, geodesic or curvature line. Such conditions are also argued in terms of the developability and minimality conditions. Finally, we give an example and picture the corresponding graphs of ruled surfaces by using Maple 17.

2. Some Special Smarandache Ruled Surfaces According to Flc Frame in E^3

In this section, we introduce some of new special ruled surfaces according to the vectors of Flc frame.

Definition 2.1.

Let $\alpha = \alpha(s)$ be a polynomial curve in E^3 and denote $\{T, D_2, D_1\}$ as the Flc frame of α. By taking the base curve as TD_2 Smarandache curve and the generator vector as D_1, we define TD_2 Smarandache ruled surface as following:

$$\varphi_1(s, v) = \frac{1}{\sqrt{2}}(T + D_2) + vD_1. \tag{1}$$

Theorem 2.1.

The Gauss and mean curvatures of TD_2 Smarandache ruled surface denoted by $\varphi_1(s, v)$ are given by

$$K = -\frac{d_1{}^2(d_2 + d_3)^2}{2\left((d_1)^2 - d_1\sqrt{2}vd_3 + (vd_3)^2 + d_1\sqrt{2}vd_2 + (vd_2)^2\right)^2},$$

$$H = \frac{\sqrt{2}d_1\eta^{-2}(X_1 + Y_1) + 2d_1(d_2 + d_3)^2 - 2v\eta^{-2}(d_3X_1 - d_2Y_1)}{4\left((d_1)^2 - d_1\sqrt{2}vd_3 + (vd_3)^2 + d_1\sqrt{2}vd_2 + (vd_2)^2\right)^{3/2}},$$

respectively, where the coefficients X_1, Y_1, Z_1 are

$$X_1 = -\frac{\sqrt{2}}{2}(\eta d_2)^2 - \frac{\sqrt{2}}{2}(\eta d_1)^2 - v(\eta d_2)' - \frac{\sqrt{2}}{2}\eta^2 d_2 d_3 - \frac{\sqrt{2}}{2}(\eta d_1)' + v\eta^2 d_1 d_3,$$

$$Y_1 = \frac{\sqrt{2}}{2}(\eta d_1)' - \frac{\sqrt{2}}{2}(\eta d_3)^2 - \frac{\sqrt{2}}{2}\eta^2 d_2 d_3 - v\eta^2 d_1 d_2 - v(\eta d_3)' - \frac{\sqrt{2}}{2}(\eta d_1)^2,$$

$$Z_1 = \frac{\sqrt{2}}{2}(\eta d_3)' + \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 - v(\eta d_3)^2 + \frac{\sqrt{2}}{2}(\eta d_2)' - v(\eta d_2)^2 - \frac{\sqrt{2}}{2}\eta^2 d_1 d_2.$$

Proof:

The first and second order partial derivatives of $\varphi_1(s,v)$ with respect to s and v are

$$\varphi_{1s}(s,v) = -\eta\left(\frac{\sqrt{2}}{2}d_1 + v d_2\right)T + \eta\left(\frac{\sqrt{2}}{2}d_1 - v d_3\right)D_2 + \frac{\sqrt{2}}{2}\eta(d_2 + d_3)D_1,$$

$$\varphi_{1ss}(s,v) = -\left(\frac{\sqrt{2}}{2}(\eta d_2)^2 + \frac{\sqrt{2}}{2}(\eta d_1)^2 + v(\eta d_2)' + \frac{\sqrt{2}}{2}\eta^2 d_2 d_3 + \frac{\sqrt{2}}{2}(\eta d_1)' - v\eta^2 d_1 d_3\right)T$$

$$+ \left(\frac{\sqrt{2}}{2}(\eta d_1)' - \frac{\sqrt{2}}{2}(\eta d_3)^2 - \frac{\sqrt{2}}{2}\eta^2 d_2 d_3 - v\eta^2 d_1 d_2 - v(\eta d_3)' - \frac{\sqrt{2}}{2}(\eta d_1)^2\right)D_2$$

$$+ \left(\frac{\sqrt{2}}{2}(\eta d_3)' + \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 - v(\eta d_3)^2 + \frac{\sqrt{2}}{2}(\eta d_2)' - v(\eta d_2)^2 - \frac{\sqrt{2}}{2}\eta^2 d_1 d_2\right)D_1,$$

$$\varphi_{1v}(s,v) = D_1, \qquad \varphi_{1sv}(s,v) = -\eta d_2 T - \eta d_3 D_2, \quad \varphi_{1vv}(s,v) = 0.$$

The unit normal vector field of $\varphi_1(s,v)$ defined by $\dfrac{\varphi_{1s} \times \varphi_{1v}}{\|\varphi_{1s} \times \varphi_{1v}\|}$ is given in the following:

$$N_{\varphi_1} = \frac{\left(d_1\sqrt{2} - 2v d_3\right)T + \left(d_1\sqrt{2} + 2v d_2\right)D_2}{2\sqrt{(d_1)^2 - d_1\sqrt{2}v d_3 + (v d_3)^2 + d_1\sqrt{2}v d_2 + (v d_2)^2}}.$$

For the sake of simplicity, we restate the expression of φ_{1ss} as

$$\varphi_{1ss}(s,v) = X_1 T + Y_1 D_2 + Z_1 D_1,$$

and then, we calculate the coefficients of first and second fundamental forms as

$$E_{\varphi_1} = \langle \varphi_{1s}, \varphi_{1s} \rangle = (\eta d_1)^2 + \eta^2 d_1\sqrt{2}v d_2 + (\eta v d_2)^2 - \eta^2 d_1\sqrt{2}v d_3$$

$$+ (\eta v d_3)^2 + \frac{1}{2}(\eta d_2)^2 + \eta^2 d_2 d_3 + \frac{1}{2}(\eta d_3)^2,$$

$$F_{\varphi_1} = \langle \varphi_{1s}, \varphi_{1v} \rangle = \frac{\sqrt{2}}{2}\eta(d_2 + d_3), \quad G_{\varphi_1} = \langle \varphi_{1v}, \varphi_{1v} \rangle = 1,$$

$$e_{\varphi_1} = \langle \varphi_{1ss}, N_{\varphi_1} \rangle = \frac{d_1\sqrt{2}X_1 + d_1\sqrt{2}Y_1 + 2d_2 Y_1 v - 2d_3 X_1 v}{2\sqrt{(d_1)^2 - d_1\sqrt{2}v d_3 + (v d_3)^2 + d_1\sqrt{2}v d_2 + (v d_2)^2}},$$

$$f_{\varphi_1} = \langle \varphi_{1sv}, N_{\varphi_1} \rangle = -\frac{\eta d_1(d_2 + d_3)\sqrt{2}}{2\sqrt{(d_1)^2 - d_1\sqrt{2}v d_3 + (v d_3)^2 + d_1\sqrt{2}v d_2 + (v d_2)^2}},$$

$$g_{\varphi_1} = \langle \varphi_{1vv}, N_{\varphi_1} \rangle = 0.$$

Finally, when substituted these relations in Gaussian and Mean curvatures of the ruled surface φ_1, the proof is complete. ∎

Theorem 2.2.

The necessary and sufficient condition for TD_2 Smarandache ruled surface, $\varphi_1(s,v)$ to have a

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singularity on the point (s_0, v_0) is that

$$\sqrt{2}d_1 = v_0 (d_3 - d_2).$$

Proof:

From the definition (see Do-Carmo (1976)), $\varphi_1 (s, v)$ has a singular point, (s_0, v_0), if and only if $\|\varphi_{1s} \times \varphi_{1v}\| (s_0, v_0) = 0$. Therefore,

$$\|\varphi_{1s} \times \varphi_{1v}\| (s_0, v_0) = \eta \sqrt{\left(\frac{1}{2}d_1 \sqrt{2} - v_0 d_3\right)^2 + \left(\frac{1}{2}d_1 \sqrt{2} + v_0 d_2\right)^2} = 0 \quad (\eta \neq 0)$$

$$\implies \sqrt{2}d_1 = v_0 (d_3 - d_2),$$

which completes the proof. ∎

Theorem 2.3.

Any parametric curve of $\varphi_1 (s, v)$ Smarandache ruled surface is a curvature line if and only if $d_2 = -d_3$.

Proof:

By definition (Do-Carmo (1976)), a parametric curve of a ruled surface is a curvature line if and only if $F = f = 0$. Thus, we have

$$F_{\varphi_1} = \frac{\sqrt{2}}{2}\eta (d_2 + d_3) = 0,$$

$$f_{\varphi_1} = -\frac{\eta d_1 (d_2 + d_3) \sqrt{2}}{2\sqrt{(d_1)^2 - d_1 \sqrt{2}v d_3 + (v d_3)^2 + d_1 \sqrt{2}v d_2 + (v d_2)^2}} = 0.$$

The common solution to the above two relations is that $d_2 = -d_3$. ∎

Theorem 2.4.

For $\varphi_1 (s, v)$ Smarandache ruled surface:

(i.) s parametric curves are asymptotic if and only if

$$\frac{X_1 + Y_1}{d_3 X_1 - d_2 Y_1} = \frac{\sqrt{2}v}{d_1}.$$

(ii.) v parametric curves are always asymptotic.

Proof:

(i.) s parametric curves are asymptotic on a surface if and only if $e = 0$, (Do-Carmo (1976)).

Therefore,

$$e_{\varphi_1} = \frac{d_1\sqrt{2}X_1 + d_1\sqrt{2}Y_1 + 2d_2 Y_1 v - 2d_3 X_1 v}{2\sqrt{(d_1)^2 - d_1\sqrt{2}v d_3 + (v d_3)^2 + d_1\sqrt{2}v d_2 + (v d_2)^2}} = 0$$
$$\Rightarrow d_1\sqrt{2}X_1 + d_1\sqrt{2}Y_1 + 2d_2\,(s)\,Y_1 v - 2d_3 X_1 v = 0$$
$$\Rightarrow \frac{X_1 + Y_1}{d_3 X_1 - d_2 Y_1} = \frac{\sqrt{2}v}{d_1}.$$

(ii.) On the other hand, v parametric curves are asymptotic on a surface if and only if $g = 0$ (Do-Carmo (1976)). Since it is the case in $g_{\varphi_1} = 0$, v parametric curves on $\varphi_1(s, v)$ are always asymptotic. ∎

Theorem 2.5.

For $\varphi_1(s, v)$ Smarandache ruled surface:

(i.) s parametric curves are non-geodesic.

(ii.) v parametric curves are always geodesic.

Proof:

(i.) In order for s parametric curves to be geodesic on $\varphi_1(s, v)$ Smarandache ruled surface, the following relations should hold:

$$N_{\varphi_1} \times \varphi_{1ss} = \frac{\begin{aligned}-Z_1\left(d_1\sqrt{2} + 2v d_2\right)T + Z_1\left(d_1\sqrt{2} - 2v d_3\right)D_2\\ + \left(\sqrt{2}d_1 X_1 - \sqrt{2}d_1 Y_1 + 2d_2 X_1 v + 2d_3 Y_1 v\right)D_1\end{aligned}}{2\sqrt{(d_1)^2 - d_1\sqrt{2}v d_3 + (v d_3)^2 + d_1\sqrt{2}v d_2 + (v d_2)^2}} = 0.$$

With some manipulations on this, we get

$$\Longrightarrow d_1\sqrt{2} + 2v d_2 = 0, \quad d_1\sqrt{2} - 2v d_3 = 0$$
$$\Longrightarrow \sqrt{2}d_1 = v\left(d_3 - d_2\right)$$

However, the points satisfying the above relations are the singular ones. Therefore, s parametric curves cannot be geodesic on $\varphi_1(s, v)$.

(ii.) On the other hand, since $N_{\varphi_1} \times \varphi_{1vv} = 0$, v parametric curves are always geodesic on $\varphi_1(s, v)$. ∎

Corollary 2.1.

If any parametric curve of $\varphi_1(s, v)$ is a line of curvature, then the surface is developable.

Proof:

For the φ_1 surface to be devolapable, $K = 0$. This is achieved if the parameter curves on the surface are lines of curvature. ∎

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Corollary 2.2.

If any parametric curve of $\varphi_1(s, v)$ is asymptotic and $d_1 = 0$, then the surface is minimal.

Proof:

For the φ_1 surface to be minimal, $H = 0$. This is achieved if the parameter curves on the surface are asymptotic and $d_1 = 0$. ∎

Definition 2.2.

Let $\alpha = \alpha(s)$ be a polynomial curve in E^3 and denote $\{T, D_2, D_1\}$ as the Flc frame of α. By taking the base curve as TD_1 Smarandache curve and the generator vector as D_2, we define TD_1 Smarandache ruled surface as following:

$$\varphi_2(s, v) = \frac{1}{\sqrt{2}}(T + D_1) + vD_2. \tag{2}$$

Theorem 2.6.

The Gauss and mean curvatures of TD_1 Smarandache ruled surface denoted by $\varphi_2(s, v)$ are given

$$K = -\frac{(d_2)^2(d_1 - d_3)^2}{2\left((d_2)^2 + d_2\sqrt{2}vd_3(s) + (vd_3)^2 + d_2\sqrt{2}vd_1 + (vd_1)^2\right)^2},$$

$$H = -\frac{2d_2(d_1 - d_3)^2 + \eta^{-2}d_2\sqrt{2}(X_2 - Z_2) + 2v\eta^{-2}(d_3X_2 + d_1Z_2)}{4\left((d_2)^2 + d_2\sqrt{2}vd_3 + (vd_3)^2 + d_2\sqrt{2}vd_1 + (vd_1)^2\right)^{3/2}},$$

where the coefficients X_2, Y_2, Z_2 are

$$X_2 = -\frac{\sqrt{2}}{2}(\eta d_2)' - \frac{\sqrt{2}}{2}(\eta d_1)^2 - \frac{\sqrt{2}}{2}(\eta d_2)^2 + \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 - v(\eta d_1)' - v\eta^2 d_2 d_3,$$

$$Y_2 = -\frac{\sqrt{2}}{2}\eta^2 d_2 d_3 - \frac{\sqrt{2}}{2}(\eta d_3)' - \frac{\sqrt{2}}{2}\eta^2 d_1 d_2 - v(\eta d_1)^2 + \frac{\sqrt{2}}{2}(\eta d_1)' - v(\eta d_3)^2,$$

$$Z_2 = v(\eta d_3)' + \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 - \frac{\sqrt{2}}{2}(\eta d_2)^2 + \frac{\sqrt{2}}{2}(\eta d_2)' - \frac{\sqrt{2}}{2}(\eta d_3)^2 - v\eta^2 d_1 d_2.$$

Proof:

The first and second order partial derivatives of $\varphi_2(s, v)$ are

$$\varphi_{2s}(s, v) = -\eta\left(\frac{\sqrt{2}}{2}d_2 + vd_1\right)T + \frac{\sqrt{2}}{2}\eta(d_1 - d_3)D_2 + \eta\left(\frac{\sqrt{2}}{2}d_2 + vd_3\right)D_1,$$

$$\varphi_{2ss}(s, v) = -\left(\frac{\sqrt{2}}{2}(\eta d_2)' + \frac{\sqrt{2}}{2}(\eta d_1)^2 + \frac{\sqrt{2}}{2}(\eta d_2)^2 - \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 + v(\eta d_1)' + v\eta^2 d_2 d_3\right) T$$

$$-\left(\frac{\sqrt{2}}{2}\eta^2 d_2 d_3 + \frac{\sqrt{2}}{2}(\eta d_3)' + \frac{\sqrt{2}}{2}\eta^2 d_1 d_2 + v(\eta d_1)^2 - \frac{\sqrt{2}}{2}(\eta d_1)' + v(\eta d_3)^2\right) D_2$$

$$+\left(v(\eta d_3)' + \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 - \frac{\sqrt{2}}{2}(\eta d_2)^2 + \frac{\sqrt{2}}{2}(\eta d_2)' - \frac{\sqrt{2}}{2}(\eta d_3)^2 - v\eta^2 d_1 d_2\right) D_1,$$

$$\varphi_{2sv}(s, v) = -\eta(d_1 T + d_3 D_1), \quad \varphi_{2v}(s, v) = D_2, \quad \varphi_{2vv}(s, v) = 0.$$

Next, the unit normal vector field of $\varphi_2(s, v)$ defined as $\dfrac{\varphi_{2s} \times \varphi_{2v}}{\|\varphi_{2s} \times \varphi_{2v}\|}$ is calculated by following:

$$N_{\varphi_2} = -\frac{\left(d_2\sqrt{2} + 2vd_3\right) T + \left(d_2\sqrt{2} + 2vd_1\right) D_1}{2\sqrt{(d_2)^2 + d_2\sqrt{2}vd_3 + (vd_3)^2 + d_2\sqrt{2}vd_1 + (vd_1)^2}}.$$

For simplicity, we restate the expression this time for $\varphi_{2ss}(s, v)$ as

$$\varphi_{2ss}(s, v) = X_2 T + Y_2 D_2 + Z_2 D_1.$$

Then, we compute the coefficients of first and second fundamental forms such that

$$E_{\varphi_2} = \langle \varphi_{2s}, \varphi_{2s} \rangle = \eta^2 \left(\frac{\sqrt{2}}{2}d_2 + vd_1\right)^2 + \frac{\eta^2}{2}(d_1 - d_3)^2 + \eta^2 \left(\frac{\sqrt{2}}{2}d_2 + vd_3\right)^2,$$

$$F_{\varphi_2} = \langle \varphi_{2s}, \varphi_{2v} \rangle = \frac{\sqrt{2}}{2}\eta(d_1 - d_3), \quad G_{\varphi_2} = \langle \varphi_{2v}, \varphi_{2v} \rangle = 1,$$

$$e_{\varphi_2} = \langle \varphi_{2ss}, N_{\varphi_2} \rangle = -\frac{2d_1 Z_2 v + d_2\sqrt{2}X_2 + d_2\sqrt{2}Z_2 + 2d_3 X_2 v}{2\sqrt{(d_2)^2 + d_2\sqrt{2}vd_3 + v^2(d_3)^2 + d_2\sqrt{2}vd_1 + v^2(d_1)^2}},$$

$$f_{\varphi_2} = \langle \varphi_{2sv}, N_{\varphi_2} \rangle = \frac{\eta d_2(d_1 - d_3)\sqrt{2}}{2\sqrt{(d_2)^2 + d_2\sqrt{2}vd_3 + (vd_3)^2 + d_2\sqrt{2}vd_1 + (vd_1)^2}},$$

$$g_{\varphi_2} = \langle \varphi_{2vv}, N_{\varphi_2} \rangle = 0.$$

Finally, when substituted these relations in Gaussian and Mean curvatures of the ruled surface φ_2, the proof is complete. ∎

Theorem 2.7.

The necessary and sufficient condition for TD_2 Smarandache ruled surface, $\varphi_2(s, v)$ to have a singularity on the point (s_0, v_0) is that

$$\sqrt{2}d_2 = -v_0(d_1 + d_3).$$

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Proof:

$\varphi_2\left(s, v\right)$ has a singular point, $\left(s_0, v_0\right)$, if and only if $\left\|\varphi_{2s} \times \varphi_{2v}\right\|\left(s_0, v_0\right) = 0$ (Do-Carmo (1976)). Therefore,

$$\left\|\varphi_{2s} \times \varphi_{2v}\right\|\left(s_0, v_0\right) = \eta\sqrt{\left(\frac{\sqrt{2}}{2}d_2 + v_0 d_3\right)^2 + \left(\frac{\sqrt{2}}{2}d_2 + v_0 d_1\right)^2} = 0 \quad (\eta \neq 0)$$

$$\implies \sqrt{2}d_2 = -v_0\left(d_1 + d_3\right),$$

which completes the proof. ∎

Theorem 2.8.

Any parametric curve of $\varphi_2\left(s, v\right)$ Smarandache ruled surface is a curvature line if and only if $d_1 = d_3$.

Proof:

A parametric curve of a ruled surface is a curvature line if and only if $F = f = 0$, (Do-Carmo (1976)). Thus, we have

$$F_{\varphi_2} = \frac{\sqrt{2}}{2}\eta\left(d_1 - d_3\right) = 0,$$

$$f_{\varphi_2} = \frac{\eta d_2\left(d_1 - d_3\right)\sqrt{2}}{2\sqrt{\left(d_2\right)^2 + d_2\sqrt{2}v d_3 + \left(v d_3\right)^2 + d_2\sqrt{2}v d_1 + \left(v d_1\right)^2}} = 0.$$

The common solution to the above two relations is that $d_1 = d_3$, completing the proof. ∎

Theorem 2.9.

For $\varphi_2\left(s, v\right)$ Smarandache ruled surface:

(i.) s parametric curves are asymptotic if and only if

$$\frac{X_2 + Z_2}{d_3 X_2 + d_1 Z_2} = -\frac{\sqrt{2}v}{d_2}.$$

(ii.) v parametric curves are always asymptotic.

Proof:

(i.) s parametric curves are asymptotic on a surface if and only if $e = 0$ (Do-Carmo (1976)). Therefore,

$$e_{\varphi_2} = -\frac{2d_1 Z_2 v + d_2\sqrt{2}X_2 + d_2\sqrt{2}Z_2 + 2d_3 X_2 v}{2\sqrt{\left(d_2\right)^2 + d_2\sqrt{2}v d_3 + \left(v d_3\right)^2 + d_2\sqrt{2}v d_1 + \left(v d_1\right)^2}} = 0$$

$$\Rightarrow 2d_1 Z_2 v + d_2\sqrt{2}X_2 + d_2\sqrt{2}Z_2 + 2d_3 X_2 v = 0$$

$$\Rightarrow \frac{X_2 + Z_2}{d_3 X_2 + d_1 Z_2} = -\frac{\sqrt{2}v}{d_2}.$$

(ii.) On the other hand, v parametric curves are asymptotic on a surface if and only if $g = 0$ (Do-Carmo (1976)). Since it is the case in $g_{\varphi_2} = 0$, v parametric curves on $\varphi_2(s, v)$ are always asymptotic.   ∎

Theorem 2.10.

For $\varphi_2(s, v)$ Smarandache ruled surface:

(i.) s parametric curves are non-geodesic.

(ii.) v parametric curves are always geodesic.

Proof:

(i.) In order for s parametric curves to be geodesic on $\varphi_2(s, v)$ Smarandache ruled surface, the following relation should hold:

$$N_{\varphi_2} \times \varphi_{2ss} = \frac{\begin{array}{c} -Y_2\left(d_2\sqrt{2} + 2vd_1\right)T + \left(\sqrt{2}d_2X_2 - \sqrt{2}d_2Z_2 + 2d_1X_2v - 2d_3Z_2v\right)D_2 \\ + Y_2\left(d_2\sqrt{2} + 2vd_3\right)D_1 \end{array}}{2\sqrt{\left(d_2\right)^2 + d_2\sqrt{2}vd_3 + \left(vd_3\right)^2 + d_2\sqrt{2}vd_1 + \left(vd_1\right)^2}} = 0.$$

With some elementary manipulations on this, we have

$$\Longrightarrow d_2\sqrt{2} + 2vd_1 = 0, \quad d_2\sqrt{2} + 2vd_3 = 0$$
$$\Longrightarrow \sqrt{2}d_2 = -v\left(d_1 + d_3\right).$$

However, the points satisfying the above relations are the singular ones. Therefore, s parametric curves cannot be geodesic on $\varphi_2(s, v)$.

(ii.) On the other hand, since $N_{\varphi_2} \times \varphi_{2vv} = 0$, v parametric curves are always geodesic on $\varphi_2(s, v)$.   ∎

Corollary 2.3.

If any parametric curve of $\varphi_2(s, v)$ is a line of curvature, then the surface is developable.

Proof:

For the φ_2 surface to be developable, $K = 0$. This is achieved if the parameter curves on the surface are lines of curvature.   ∎

Corollary 2.4.

If any parametric curve of $\varphi_2(s, v)$ is asymptotic and $d_2 = 0$, then the surface is minimal.

Proof:

For the φ_2 surface to be minimal, $H = 0$. This is achieved if the parameter curves on the surface are asymptotic and $d_2 = 0$.   ∎

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Definition 2.3.

Let $\alpha = \alpha(s)$ be a polynomial curve in E^3 and denote $\{T, D_2, D_1\}$ as the Flc frame of α. By taking the base curve as $D_2 D_1$ Smarandache curve and the generator vector as T, we define $D_2 D_1$ Smarandache ruled surface as following:

$$\varphi_3(s, v) = \frac{1}{\sqrt{2}}(D_2 + D_1) + vT. \tag{3}$$

Theorem 2.11.

The Gauss and mean curvatures of $D_2 D_1$ Smarandache ruled surface denoted by $\varphi_3(s, v)$ are given

$$K = -\frac{(d_3)^2 (d_1 + d_2)^2}{2\left((d_3)^2 + d_3\sqrt{2}vd_2 + (vd_2)^2 - d_3\sqrt{2}vd_1 + (vd_1)^2\right)^2},$$

$$H = \frac{2d_3(d_1 + d_2)^2 + 2v\eta^{-2}(d_2 Y_3 - d_1 Z_3) + \eta^{-2}d_3\sqrt{2}(Y_3 + Z_3)}{4\left((d_3)^2 + d_3\sqrt{2}vd_2 + (vd_2)^2 - d_3\sqrt{2}vd_1 + (vd_1)^2\right)^{3/2}},$$

where the coefficients X_3, Y_3, Z_3 are

$$X_3 = -\frac{\sqrt{2}}{2}(\eta d_2)' + \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 - v(\eta d_2)^2 - \frac{\sqrt{2}}{2}\eta^2 d_2 d_3 - \frac{\sqrt{2}}{2}(\eta d_1)' - v(\eta d_1)^2,$$

$$Y_3 = -\frac{\sqrt{2}}{2}(\eta d_3)' - \frac{\sqrt{2}}{2}(\eta d_1)^2 - \frac{\sqrt{2}}{2}\eta^2 d_1 d_2 + v(\eta d_1)' - \frac{\sqrt{2}}{2}(\eta d_3)^2 - v\eta^2 d_2 d_3,$$

$$Z_3 = -\frac{\sqrt{2}}{2}(\eta d_2)^2 - \frac{\sqrt{2}}{2}\eta^2 d_1 d_2 - \frac{\sqrt{2}}{2}(\eta d_3)^2 + v\eta^2 d_1 d_3 + \frac{\sqrt{2}}{2}(\eta d_3)' + v(\eta d_2)'.$$

Proof:

The first and second order partial derivatives of $\varphi_3(s, v)$ are

$$\varphi_{3s}(s, v) = -\eta\frac{\sqrt{2}}{2}(d_2 + d_1)T + \eta\left(-\frac{\sqrt{2}}{2}d_3 + vd_1\right)D_2 + \eta\left(\frac{\sqrt{2}}{2}d_3 + vd_2\right)D_1,$$

$$\varphi_{3ss}(s, v) = -\left(\frac{\sqrt{2}}{2}(\eta d_2)' - \frac{\sqrt{2}}{2}\eta^2 d_1 d_3 + v(\eta d_2)^2 + \frac{\sqrt{2}}{2}\eta^2 d_2 d_3 + \frac{\sqrt{2}}{2}(\eta d_1)' + v(\eta d_1)^2\right)T$$

$$- \left(\frac{\sqrt{2}}{2}(\eta d_3)' + \frac{\sqrt{2}}{2}(\eta d_1)^2 + \frac{\sqrt{2}}{2}\eta^2 d_1 d_2 - v(\eta d_1)' + \frac{\sqrt{2}}{2}(\eta d_3)^2 + v\eta^2 d_2 d_3\right)D_2$$

$$- \left(\frac{\sqrt{2}}{2}(\eta d_2)^2 + \frac{\sqrt{2}}{2}\eta^2 d_1 d_2 + \frac{\sqrt{2}}{2}(\eta d_3)^2 - v\eta^2 d_1 d_3 - \frac{\sqrt{2}}{2}(\eta d_3)' - v(\eta d_2)'\right)D_1,$$

$$\varphi_{3sv}(s, v) = \eta(d_1 D_2 + d_2 D_1), \quad \varphi_{3v}(s, v) = T, \quad \varphi_{3vv}(s, v) = 0.$$

Next, the unit normal vector field of $\varphi_3 \left(s, v\right)$ defined as $\dfrac{\varphi_{3s} \times \varphi_{3v}}{\left\|\varphi_{3s} \times \varphi_{3v}\right\|}$ is calculated by following:

$$N_{\varphi_3} = \frac{\left(d_3 \sqrt{2} + 2v d_2\right) D_2 + \left(d_3 \sqrt{2} - 2v d_1\right) D_1}{2\sqrt{\left(d_3\right)^2 + d_2 \sqrt{2} v d_3 + \left(v d_2\right)^2 - d_3 \sqrt{2} v d_1 + \left(v d_1\right)^2}}.$$

For simplicity, we restate the expression this time for $\varphi_{3ss} \left(s, v\right)$ as

$$\varphi_{3ss} \left(s, v\right) = X_3 T + Y_3 D_2 + Z_3 D_1.$$

Then, we compute the coefficients of first and second fundamental forms such that

$$E_{\varphi_3} = \left\langle \varphi_{3s}, \varphi_{3s} \right\rangle = \frac{\eta^2}{2} \left(d_2 + d_1\right)^2 + \eta^2 \left(-\frac{\sqrt{2}}{2} d_3 + v d_1\right)^2 + \eta^2 \left(\frac{\sqrt{2}}{2} d_3 + v d_2\right)^2,$$

$$F_{\varphi_3} = \left\langle \varphi_{3s}, \varphi_{3v} \right\rangle = -\frac{\sqrt{2}}{2} \eta \left(d_2 + d_1\right), \quad G_{\varphi_3} = \left\langle \varphi_{3v}, \varphi_{3v} \right\rangle = 1,$$

$$e_{\varphi_3} = \left\langle \varphi_{3ss}, N_{\varphi_3} \right\rangle = \frac{2 d_2 Y_3 v + d_3 \sqrt{2} Y_3 + d_3 \sqrt{2} Z_3 - 2 d_1 Z_3 v}{2\sqrt{\left(d_3\right)^2 + d_3 \sqrt{2} v d_2 + v^2 \left(d_2\right)^2 - d_3 \sqrt{2} v d_1 + v^2 \left(d_1\right)^2}},$$

$$f_{\varphi_3} = \left\langle \varphi_{3sv}, N_{\varphi_3} \right\rangle = \frac{\eta d_3 \left(d_1 + d_2\right) \sqrt{2}}{2\sqrt{\left(d_3\right)^2 + d_3 \sqrt{2} v d_2 + v^2 \left(d_2\right)^2 - d_3 \sqrt{2} v d_1 + v^2 \left(d_1\right)^2}},$$

$$g_{\varphi_3} = \left\langle \varphi_{3vv}, N_{\varphi_3} \right\rangle = 0.$$

Finally, when substituted these relations in Gaussian and Mean curvatures of the ruled surface φ_3, the proof is complete. ∎

Theorem 2.12.

The necessary and sufficient condition for $D_2 D_1$ Smarandache ruled surface, $\varphi_3 \left(s, v\right)$ to have a singularity on the point $\left(s_0, v_0\right)$ is that

$$\sqrt{2} d_3 = v_0 \left(d_1 - d_2\right).$$

Proof:

$\varphi_3 \left(s, v\right)$ has a singular point $\left(s_0, v_0\right)$ if and only if $\left\|\varphi_{3s} \times \varphi_{3v}\right\| \left(s_0, v_0\right) = 0$ (Do-Carmo (1976)). Thus,

$$\left\|\varphi_{3s} \times \varphi_{3v}\right\| \left(s_0, v_0\right) = \eta \sqrt{\left(\frac{\sqrt{2}}{2} d_3 + v d_2\right)^2 + \left(\frac{\sqrt{2}}{2} d_3 - v d_1\right)^2} = 0 \quad \left(\eta \neq 0\right)$$

$$\implies \sqrt{2} d_3 = v_0 \left(d_1 - d_2\right),$$

which completes the proof. ∎

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Theorem 2.13.

Any parametric curve of $\varphi_3(s, v)$ Smarandache ruled surface is a curvature line if and only if $d_1 = -d_2$.

Proof:

A parametric curve of a ruled surface is a curvature line if and only if $F = f = 0$ (Do-Carmo (1976)). Thus, we have

$$F_{\varphi_3} = -\frac{\sqrt{2}}{2}\eta(d_1 + d_2) = 0,$$

$$f_{\varphi_3} = \frac{\eta d_3(d_1 + d_2)\sqrt{2}}{2\sqrt{(d_3)^2 + d_3\sqrt{2}v d_2 + v^2(d_2)^2 - d_3\sqrt{2}v d_1 + v^2(d_1)^2}} = 0.$$

The common solution to the above two relations is that $d_1 = -d_2$, which completes the proof. ∎

Theorem 2.14.

For $\varphi_3(s, v)$ Smarandache ruled surface:

(i.) s parametric curves are asymptotic if and only if

$$\frac{Y_3 + Z_3}{d_1 Z_3 - d_2 Y_3} = \frac{\sqrt{2}v}{d_3},$$

(ii.) v parametric curves are always asymptotic.

Proof:

(i.) s parametric curves are asymptotic on a surface if and only if $e = 0$ (Do-Carmo (1976)). Therefore,

$$e_{\varphi_3} = \frac{2d_2 Y_3 v + d_3\sqrt{2}Y_3 + d_3\sqrt{2}Z_3 - 2d_1 Z_3 v}{2\sqrt{(d_3)^2 + d_3\sqrt{2}v d_2 + v^2(d_2)^2 - d_3\sqrt{2}v d_1 + v^2(d_1)^2}} = 0$$

$$\Rightarrow 2d_2 Y_3 v + d_3\sqrt{2}Y_3 + d_3\sqrt{2}Z_3 - 2d_1 Z_3 v = 0$$

$$\Rightarrow \frac{Y_3 + Z_3}{d_1 Z_3 - d_2 Y_3} = \frac{\sqrt{2}v}{d_3}.$$

(ii.) v parametric curves are asymptotic on a surface if and only if $g = 0$ (Do-Carmo (1976)). Since it is the case in $g_{\varphi_3} = 0$, v parametric curves on $\varphi_3(s, v)$ are always asymptotic. ∎

Theorem 2.15.

For $\varphi_3(s, v)$ Smarandache ruled surface:

(i.) s parametric curves are non-geodesic.

(ii.) v parametric curves are always geodesic.

Proof:

(i.) In order for s parametric curves to be geodesic on $\varphi_3\left(s,v\right)$ Smarandache ruled surface, the following relation should hold

$$N_{\varphi_3} \times \varphi_{3ss} = \frac{\left(\sqrt{2}d_3\,Y_3 - \sqrt{2}d_3\,Z_3 - 2d_2\,Z_3v - 2d_1\,Y_3v\right)T + X_3\left(2vd_1 - d_3\sqrt{2}\right)D_2 + \left(2vd_2 + d_3\sqrt{2}\right)D_1}{2\sqrt{\left(d_3\right)^2 + d_3\sqrt{2}vd_2 + v^2\left(d_2\right)^2 - d_3\sqrt{2}vd_1 + v^2\left(d_1\right)^2}} = 0.$$

With some elementary manipulations on this, we have

$$\Longrightarrow 2vd_1 - d_3\sqrt{2} = 0, \quad 2vd_2 + d_3\sqrt{2} = 0,$$
$$\Longrightarrow v(d_1 - d_2) = \sqrt{2}d_3.$$

However, the points satisfying the above relations are the singular ones. Therefore, s parametric curves cannot be geodesic on $\varphi_3\left(s,v\right)$.

(ii.) On the other hand, since $N_{\varphi_3} \times \varphi_{3vv} = 0$, then v parametric curves are always geodesic on $\varphi_3\left(s,v\right)$. ∎

Corollary 2.5.

If any parametric curve of $\varphi_3\left(s,v\right)$ is a line of curvature, then the surface is developable.

Proof:

For the φ_3 surface to be developable, $K = 0$. This is achieved if the parameter curves on the surface are lines of curvature. ∎

Corollary 2.6.

If any parametric curve of $\varphi_3\left(s,v\right)$ is asymptotic and $d_3 = 0$, then the surface is minimal.

Proof:

For the φ_3 surface to be minimal, $H = 0$. This is achieved if the parameter curves on the surface are asymptotic and $d_3 = 0$. ∎

3. Examples

Let us consider a helical polynomial curve parameterized as $\alpha(s) = \left(6s, 3s^2, s^3\right)$. Then the Flc frame elements of α are given by

$$T(s) = \left(\frac{2}{s^2+2}, \frac{2s}{s^2+2}, \frac{s^2}{s^2+2}\right),$$
$$D_2(s) = \left(\frac{s}{\sqrt{s^2+1}}, \frac{-1}{\sqrt{s^2+1}}, 0\right),$$

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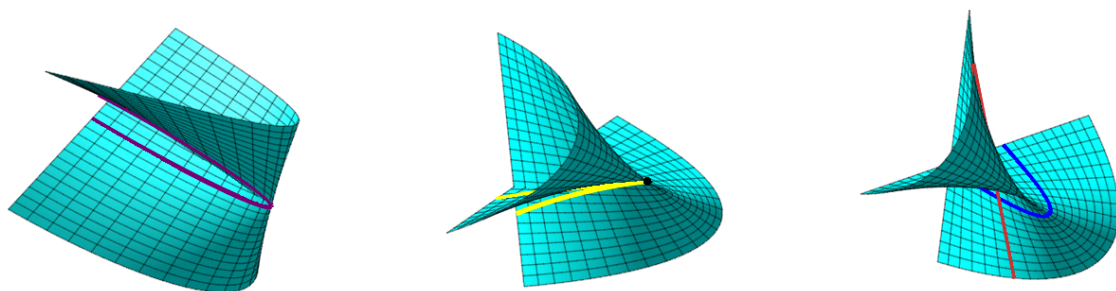AAM: Intern. J., Vol. 18, Issue 1 (June 2023) 15

$$D_1(s) = \left(-\frac{s^2}{\sqrt{s^2+1}(s^2+2)}, -\frac{s^3}{\sqrt{s^2+1}(s^2+2)}, \frac{2\sqrt{s^2+1}}{s^2+2} \right),$$

and the corresponding curvatures according to Flc frame are as following:

$$d_1(s) = \frac{2s}{3\sqrt{s^2+1}(s^2+2)^2}, \quad d_2(s) = -\frac{2}{3\sqrt{s^2+1}(s^2+2)^2}, \quad d_3(s) = \frac{s^2}{3(s^2+1)(s^2+2)^2}.$$

(i) Thus, we have the parametric form for TD_2 Smarandache ruled surface as following:

$$\varphi_1(s,v) = \left(\frac{\sqrt{2}}{2}\left(\frac{2}{s^2+2} - \frac{s^2}{\sqrt{s^2+1}(s^2+2)} \right) + \frac{vs}{\sqrt{s^2+1}}, \right.$$
$$\left. \frac{\sqrt{2}}{2}\left(\frac{2s}{s^2+2} - \frac{s^3}{\sqrt{s^2+1}(s^2+2)} \right) - \frac{v}{\sqrt{s^2+1}}, \quad \frac{\sqrt{2}}{2}\left(\frac{s^2+2\sqrt{s^2+1}}{s^2+2} \right) \right).$$

(a) with TD_2 Smarandache curve   (b) with striction curve and singular point   (c) with parametric curves

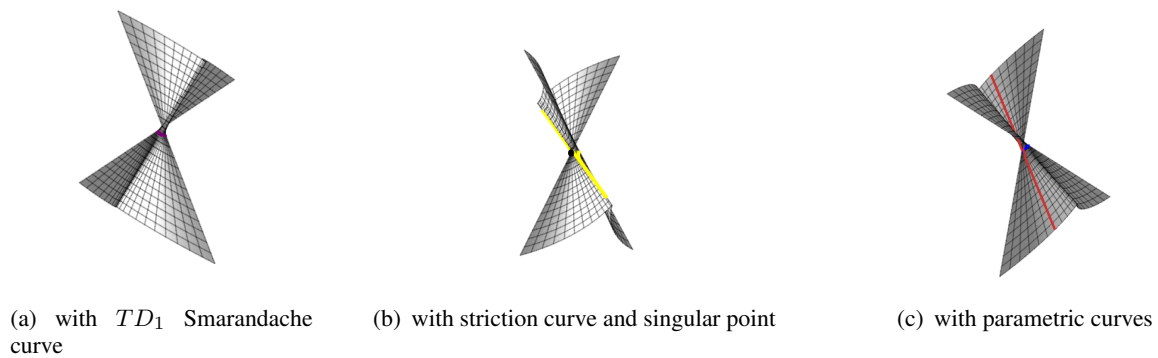**Figure 1.** TD_2 Smarandache ruled surfaces from different angles

(ii) Next, the parametric form for TD_1 Smarandache ruled surface is as:

$$\varphi_2(s,v) = \left(\frac{\sqrt{2}}{2}\left(\frac{2}{s^2+2} + \frac{s}{\sqrt{s^2+1}} \right) - \frac{vs^2}{\sqrt{s^2+1}(s^2+2)}, \right.$$
$$\left. \frac{\sqrt{2}}{2}\left(\frac{2s}{\sqrt{s^2+2}} - \frac{1}{\sqrt{s^2+1}} \right) - \frac{vs^3}{\sqrt{s^2+1}(s^2+2)}, \quad \frac{\sqrt{2}}{2}\left(\frac{s^2}{s^2+2} \right) + \frac{2v\sqrt{s^2+1}}{s^2+2} \right).$$

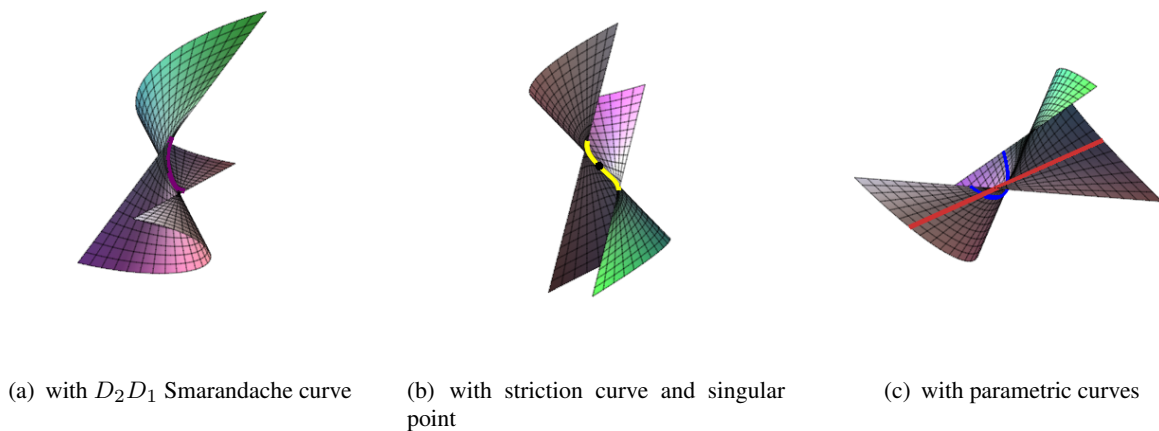(iii) Last, we parameterize the D_2D_1 Smarandache ruled surface as

$$\varphi_3(s,v) = \left(\frac{\sqrt{2}}{2}\left(\frac{s}{\sqrt{s^2+1}} - \frac{s^2}{\sqrt{s^2+1}(s^2+2)} \right) + \frac{2v}{s^2+2}, \right.$$
$$\left. -\frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{s^2+1}} + \frac{s^3}{\sqrt{s^2+1}(s^2+2)} \right) + \frac{2vs}{s^2+2}, \quad \frac{\sqrt{2s^2+2}}{s^2+2} + \frac{vs^2}{s^2+2} \right).$$

(a) with TD_1 Smarandache curve

(b) with striction curve and singular point

(c) with parametric curves

Figure 2. TD_1 Smarandache ruled surfaces from different angles

(a) with D_2D_1 Smarandache curve

(b) with striction curve and singular point

(c) with parametric curves

Figure 3. D_2D_1 Smarandache ruled surfaces from different angles

4. Conclusion

In this paper, we have studied some special ruled surfaces according to the Flc frame of a given polynomial curve. We examine the conditions if the parametric curves of the surfaces are asymptotic, geodesic or curvature line. Such conditions are also argued in terms of the developability and minimality conditions. Finally, we give an example and picture the corresponding graphs of ruled surfaces.

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