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Numerical Solution for a Class of Nonlinear Emden-Fowler Equations by Exponential Collocation Method

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Abstract

In this research, exponential approximation is used to solve a class of nonlinear Emden-Fowler equations. This method is based on the matrix forms of exponential functions and their derivatives using collocation points. To demonstrate the usefulness of the method, we apply it to some different problems. The numerical approximate solutions are compared with available (existing) exact (analytical) solutions to show the accuracy of the proposed method. The method has been checked with several examples to show its validity and reliability. The reported examples illustrate that the method is reasonably efficient and accurate.

Keywords: Exponential approximation; Emden-Fowler equations; Lane-Emden equations; Operational matrix; Collocation method

MSC 2010 No.: 65M06, 65M12

1. Introduction

Differential equations have a remarkable role in several scientific and engineering phenomena that are always of interest in physical and technical applications and appear in various fields such as mathematics, physics and engineering sciences (Siriwardana and Pradhan (2021), Manafian et al. (2016), Shivanian and Aslefallah (2017), Aslefallah et al. (2020), Aslefallah and Shivanian (2015), and Aslefallah et al. (2019)). The Emden-Fowler type of equations are singular initial value problems (IVPs) associated with second-order ordinary differential equations (ODEs) used to model various phenomena in mathematical physics and astrophysics such as thermal explosions, stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and thermionic currents (Bellman (1953); Chandrasekhar (1967)).

In astrophysics, the Lane-Emden equation is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid.

Physically, hydrostatic equilibrium relates potential gradient, density, and pressure gradient, while Poisson's equation relates potential to density. So, if we have another equation that dictates how pressure and density change relative to each other, we can arrive at a solution. The particular choice of a polytropic gas as mentioned above makes the mathematical expression of the problem particularly concise and leads to the Lane-Emden equation. This equation is a useful approximation for self-gravitating spheres of plasma such as stars, but is usually a limiting assumption (Chowdhury and Hashim (2009); Asadpour et al. (2019)).

Recently, Yüzbaşi and Sezer (2013b), Yüzbaşi (2020), Yüzbaşi (2018), and Yüzbaşi and Sezer (2013a) have worked the collocation method based on exponential approximation to solve the linear neutral delay differential, pantograph, singular differential-difference, and Fredholm integro-differential difference equations and Fredholm integro differential equation systems.

In this study, we seek the approximate solution of Emden-Fowler as series of exponential functions. Exponential polynomials or exponential functions have interesting applications in many problems.

The rest of this article is organized as follows. In Section 2 some basic concepts about the main equation and the method are presented. In Section 3, we express briefly required mathematical preliminaries and matrix relations for exponential functions of the method. In Section 4, we present numerical implementation of matrix operation of method. Some numerical examples are reviewed to show the accuracy of the method, and results are reported in Section 5. Finally, in the last section, concluding remarks are given.

2. Basic Concepts

Many problems in applied mathematical physics that occur on semi-infinite interval, are related to Emden-Fowler equation. The equation is of the form:

$$u''(x) + \frac{f'(x)}{f(x)}u'(x) + g\left(x, u(x)\right) = h(x), \qquad x > 0, \tag{1}$$

with initial conditions

$$u(0) = \alpha_0, \qquad u'(0) = \alpha_1, \tag{2}$$

where $g(x, u(x))$ and $h(x)$ are continuous real valued functions $f(x)$ is a continuous and differentiable function with $f(x) \neq 0$.

Equation (1) reduces to the Lane-Emden equation which, with specified $f(x)$, was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents. The solution of the Emden-Fowler equation (1) and (2), as well as other various linear and nonlinear singular IVPs in quantum mechanics and astrophysics, is numerically challenging because of the singularity behavior at the origin.

In this study, we seek the approximate solution of Equation (1) with Equation (2) as series of exponential functions. Exponential polynomials or exponential functions have interesting applications in many problems. These polynomials are based on the exponential base set $\mathcal{B} = \{1, e^{-x}, e^{-2x}, \ldots\}$ and defined by

$$u(x) \simeq u_N(x) = \sum_{n=0}^{N} a_n e^{-nx}, \tag{3}$$

so that a_n are the unknown coefficients ($n = 0, 1, \ldots, N$). The motivation to choose the exponential series is to avoid the singularity at the origin.

3. Matrix Relations for Exponential Functions

In the first step, let us write the approximate solution $u_N(x)$ defined by (3) of Equation (1) in the matrix form as,

$$u(x) = \mathbf{E}(x)\mathbf{A}, \tag{4}$$

where

$$\mathbf{E}(x) = [1 \quad e^{-x} \quad e^{-2x} \ldots \quad e^{-Nx}],$$

and

$$\mathbf{A} = [a_1 \quad a_2 \quad \ldots \quad a_N]^T.$$

Secondly, the relation between $\mathbf{E}(x)$ and its first derivative $\mathbf{E}'(x)$ is given by

$$\mathbf{E}'(x) = \mathbf{E}(x)\mathbf{D},$$

where

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -N \end{bmatrix},$$

and, by repeating the process

$$\mathbf{E}^k(x) = \mathbf{E}(x)\mathbf{D}^k, \quad k = 0, 1, 2, \dots, \tag{5}$$

where $\mathbf{D}^0$ is the unit matrix in dimensional $(N+1) \times (N+1)$. Note that

$$\mathbf{D}^k = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & (-1)^k & 0 & \cdots & 0 \\ 0 & 0 & (-2)^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (-N)^k \end{bmatrix}.$$

From the matrix relations (4) and (5), we have matrix form of the k-th derivative

$$u^{(k)}(x) = \mathbf{E}(x)\mathbf{D}^k\mathbf{A}, \qquad k = 0, 1, \dots, m. \tag{6}$$

By replacing the collocation points $\{x_i\}_{i=0}^N$ we turn (6) into the following system of matrix equations:

$$u^{(k)}(x_i) = \mathbf{E}(x_i)\mathbf{D}^k\mathbf{A}, \qquad i = 0, 1, \dots, N.$$

or, in compact form,

$$\mathbf{U}^{(k)} = \mathbf{E}.\mathbf{D}^k.\mathbf{A},$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}(x_0) \\ \mathbf{E}(x_1) \\ \vdots \\ \mathbf{E}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & e^{-x_0} & e^{-2x_0} & \dots & e^{-Nx_0} \\ 1 & e^{-x_1} & e^{-2x_1} & \dots & e^{-Nx_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-x_N} & e^{-2x_N} & \dots & e^{-Nx_N} \end{bmatrix}, \qquad \mathbf{U}^{(k)} = \begin{bmatrix} u^{(k)}(x_0) \\ u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_N) \end{bmatrix}.$$

4. Implementation of Matrix Operation

To obtain an exponential series solution of (1) under the mixed conditions (2), the following matrix method is used. The foundation of this method is based on calculating the unknown coefficients using the collocation points. Then, by using collocation points on interval $[0, b]$ defined as

$$x_j = \frac{b}{N}j, \qquad j = 0, 1, \dots, N.$$

First, the collocation points are substituted in (1)

$$u''(x_i) + \frac{f'(x_i)}{f(x_i)}u'(x_i) + g\left(x_i, u(x_i)\right) = h(x_i), \qquad i = 0, 1, \dots, N, \tag{7}$$

and then this system is written in the matrix form

$$\mathbf{U}^{(2)} + \mathbf{F}\mathbf{U}^{(1)} + \mathbf{G} = \mathbf{H},$$

where

$$\mathbf{F} = \begin{bmatrix} \frac{f(x_0)}{f'(x_0)} & \frac{f(x_1)}{f'(x_1)} & \dots & \frac{f(x_N)}{f'(x_N)} \end{bmatrix}^T,$$

$$\mathbf{G} = [g(x_0, u(x_0)) \quad g(x_1, u(x_1)) \quad \dots \quad g(x_N, u(x_N))]^T,$$

and

$$\mathbf{H} = [h(x_0) \quad h(x_1) \quad \dots \quad h(x_N)]^T.$$

After the substitution of the above relations, we have the fundamental matrix equation

$$\mathbf{E}\mathbf{D}^2\mathbf{A} + \mathbf{F}\mathbf{E}\mathbf{D}^1\mathbf{A} + \mathbf{G} = \mathbf{H}, \tag{8}$$

Here, (8) corresponds to a system of the nonlinear algebraic equations with the unknown coefficients $a_n, n = 0, 1, \dots, N$.

In a special case, if $g(x, u) = p(x)u^m$, then

$$u''(x_i) + \frac{f'(x_i)}{f(x_i)} u'(x_i) + p(x_i)u^m(x_i) = h(x_i), \qquad i = 0, 1, \dots, N,$$

and this system is written in the matrix form

$$\mathbf{U}^{(2)} + \mathbf{F}\mathbf{U}^{(1)} + \mathbf{P}\mathbf{U}^m = \mathbf{H},$$

where

$$\mathbf{P} = [p(x_0) \quad p(x_1) \quad \dots \quad p(x_N)]^T.$$

After the substitution of the above relations, we have the fundamental matrix equation

$$\mathbf{E}\mathbf{D}^2\mathbf{A} + \mathbf{F}\mathbf{E}\mathbf{D}^1\mathbf{A} + \mathbf{P}(\mathbf{E}\mathbf{A})^m = \mathbf{H},$$

or equivalently,

$$\left(\mathbf{E}\mathbf{D}^2 + \mathbf{F}\mathbf{E}\mathbf{D}^1 + \mathbf{P}(\mathbf{E}\mathbf{A})^{m-1}\mathbf{E}\right)\mathbf{A} = \mathbf{H}. \tag{9}$$

Briefly, (9) can also be written in the form

$$\mathbf{W}\mathbf{A} = \mathbf{H},$$

where

$$\mathbf{W} = \mathbf{E}\mathbf{D}^2 + \mathbf{F}\mathbf{E}\mathbf{D}^1 + \mathbf{P}(\mathbf{E}\mathbf{A})^{m-1}\mathbf{E}.$$

Here, (9) corresponds to a system of the $(N + 1)$ nonlinear algebraic equations with the unknown coefficients $a_n, n = 0, 1, \dots, N$.

According to conditions (2), we have:

$$u(0) = \alpha_0 \qquad \Longrightarrow \qquad a_0 + a_1 + a_2 + \dots + a_N = \alpha_0, \tag{10}$$

and

$$u'(0) = \alpha_1 \qquad \Longrightarrow \qquad -a_1 - 2a_2 - 3a_3 - \ldots - Na_N = \alpha_1. \tag{11}$$

Together with initial conditions, they make a system of non-linear algebraic equations which can be easily solved by using any numerical methods such as Newton's iterative method or just applying mathematical software to solve the nonlinear equations. Hence, $u_N(x)$ can be calculated.

5. Numerical experiments

In this section, we will use the method explained in Section 2 to solve various types of Emden-Fowler equations. All the calculations have been performed using MAPLE. We solve four examples using the method and report the numerical results. The accuracy and efficiency of the method are shown with absolute error. All examples are solved on interval $[0, 1]$, but it is necessary to emphasize that with a proper transformation, examples on interval $[0, \infty)$ can be solved.

Example 5.1.

For first example, consider the Lane-Emden equation as follows

$$u''(x) + \frac{\alpha}{x}u'(x) + g(x, u) = h(x),$$

with initial conditions

$$u(0) = \alpha_0, \qquad u'(0) = \alpha_1.$$

In this special case, for $g(x, u) = u^n(x)$ and $h(x) = 0$, this equation is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics (Yildirim and Ozi (2009); Hosseini and Abbasbandy (2015)).

In this example, let $n = 5, \alpha = 2$ and $h(x) = 0$. Then,

$$u''(x) + \frac{2}{x}u'(x) + u^5(x) = 0,$$

with initial conditions

$$u(0) = 1, \qquad u'(0) = 0.$$

The exact solution to this initial value problem is well known to be $u(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}$.

The mentioned procedure has also been carried out for the values $N = 5$ and $N = 10$. The obtained bounds are shown in Table 1 together with the maximum actual errors corresponding to these N values. Figure 1 shows the graph of absolute error function with $N = 5$ (top left), graph of approximated and exact solution with $N = 5$ (top right), graph of absolute error function with $N = 10$ (bottom left) and graph of approximated and exact solution with $N = 10$ (bottom right).

Table 1. The absolute error obtained by the method for Example 5.1

x_i	N=5	N=10
0.00	0	0
0.10	6.0264×10^{-4}	1.7758×10^{-5}
0.20	1.5729×10^{-3}	2.4552×10^{-5}
0.30	2.3039×10^{-3}	2.5351×10^{-5}
0.40	2.6684×10^{-3}	2.4950×10^{-5}
0.50	2.7316×10^{-3}	2.3806×10^{-5}
0.60	2.6093×10^{-3}	2.2199×10^{-5}
0.70	2.4046×10^{-3}	2.0302×10^{-5}
0.80	2.1854×10^{-3}	1.8219×10^{-5}
0.90	1.9816×10^{-3}	1.6059×10^{-5}
1.00	1.7908×10^{-3}	1.3868×10^{-5}

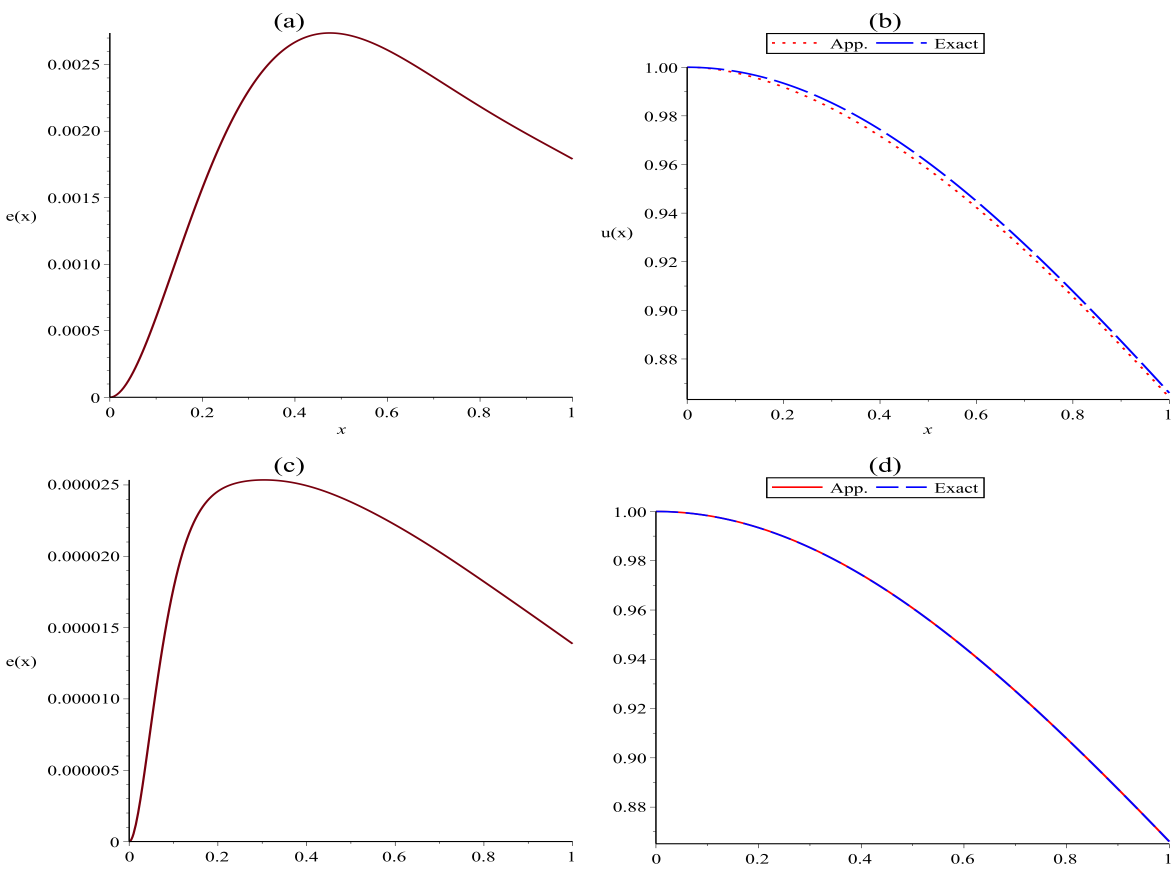

Figure 1. (a) Graph of absolute error function with $N = 5$, (b) graph of approximated and exact solution with $N = 5$, (c) graph of absolute error function with $N = 10$, and (d) graph of approximated and exact solution with $N = 10$ for Example 5.1.

Example 5.2.

Consider the following linear, non-homogeneous singular initial value problem:

$$u''(x) + \frac{2}{x}u'(x) + u = x^4 + x^3 + 21x^2 + 12x + 6,$$

with initial conditions $u(0) = u'(0) = 0$.

The exact solution of this problem is

$$u(x) = x^4 + x^3 + x^2.$$

The behavior of absolute error is reported in Table 2 for the values $N = 10$ and $N = 15$. Figure 2 shows the graph of absolute error function with $N = 10$ (top left), graph of approximated and exact solution with $N = 10$ (top right), graph of absolute error function with $N = 15$ (bottom left) and graph of approximated and exact solution with $N = 15$ (bottom right).

Table 2. The absolute error obtained by the method with $N = 10$ and $N = 15$ for Example 5.2

x	N=10	N=15
0	0	0
0.10	2.9321×10^{-3}	9.3412×10^{-5}
0.20	4.1523×10^{-3}	1.0317×10^{-4}
0.30	4.2314×10^{-3}	1.0671×10^{-4}
0.40	4.3412×10^{-3}	1.0712×10^{-4}
0.50	4.4451×10^{-3}	1.0601×10^{-4}
0.60	4.4561×10^{-3}	1.0452×10^{-4}
0.70	4.4421×10^{-3}	1.0343×10^{-4}
0.80	4.3287×10^{-3}	1.0212×10^{-4}
0.90	4.2367×10^{-3}	1.0076×10^{-4}
1.00	4.1765×10^{-3}	9.7380×10^{-5}

Example 5.3.

As another example, consider the following nonlinear Emden-Fowler type equation

$$u''(x) + \frac{8}{x}u'(x) + u(18 + 4\ln u) = 0,$$

with initial conditions $u(0) = 1$ and $u'(0) = 0$. The exact solution is $u(x) = e^{-x^2}$.

The behavior of absolute error is reported in Table 3 for the values $N = 10$ and $N = 15$. Figure 3 shows graph of absolute error function with $N = 10$ (top left), graph of approximated and exact solution with $N = 10$ (top right), graph of absolute error function with $N = 15$ (bottom left) and graph of approximated and exact solution with $N = 15$ (bottom right).

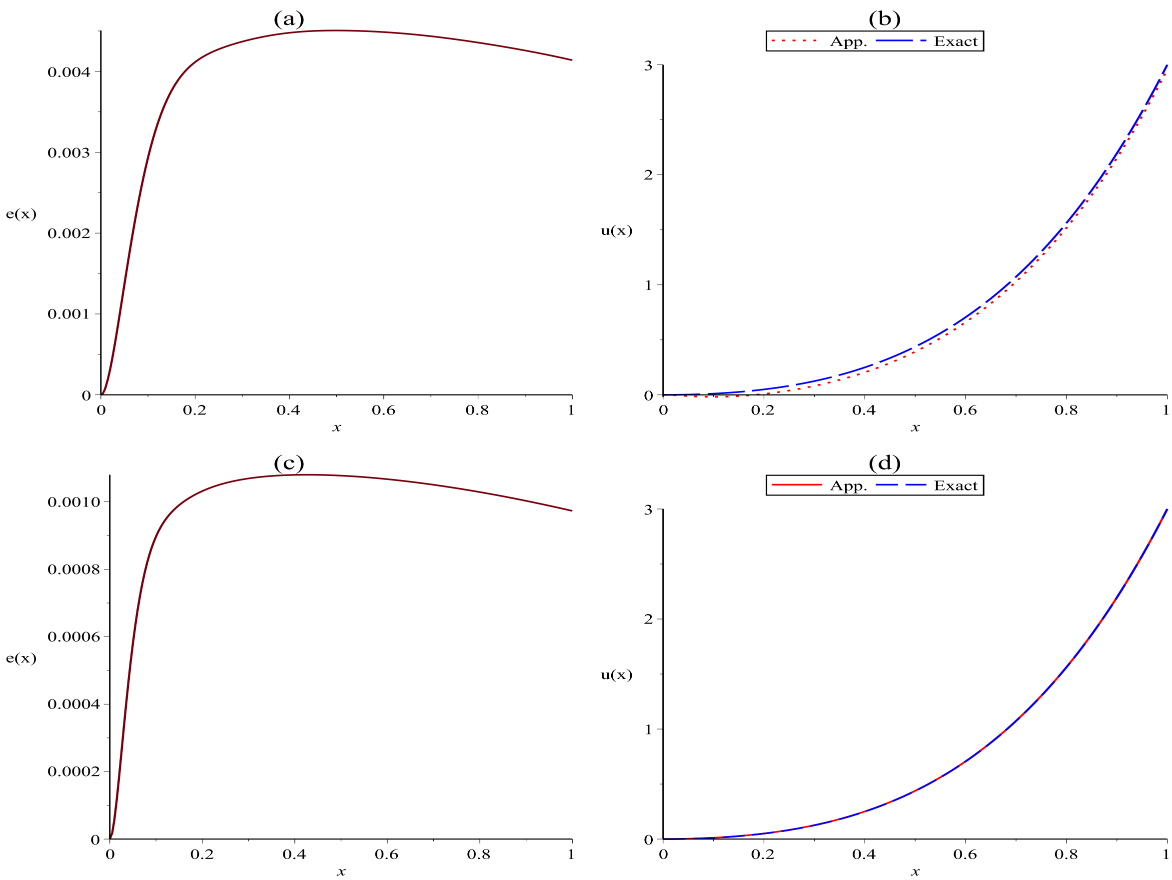

Figure 2. (a) Graph of absolute error function with $N = 10$, (b) graph of approximated and exact solution with $N = 10$, (c) graph of absolute error function with $N = 15$ and (d) graph of approximated and exact solution with $N = 15$ for Example 5.2.

Table 3. The absolute error obtained by the method with $N = 10$ and $N = 15$ for Example 5.3

x	N=10	N=15
0	0	0
0.10	4.1235×10^{-5}	9.2356×10^{-6}
0.20	2.1676×10^{-4}	1.0811×10^{-5}
0.30	4.3299×10^{-4}	1.0918×10^{-5}
0.40	7.7867×10^{-4}	1.1124×10^{-5}
0.50	1.0211×10^{-3}	1.2457×10^{-4}
0.60	1.5432×10^{-3}	1.4321×10^{-4}
0.70	1.7656×10^{-3}	1.7345×10^{-4}
0.80	1.9912×10^{-3}	1.8798×10^{-4}
0.90	2.0985×10^{-3}	2.0011×10^{-4}
1.00	2.2080×10^{-3}	2.2046×10^{-4}

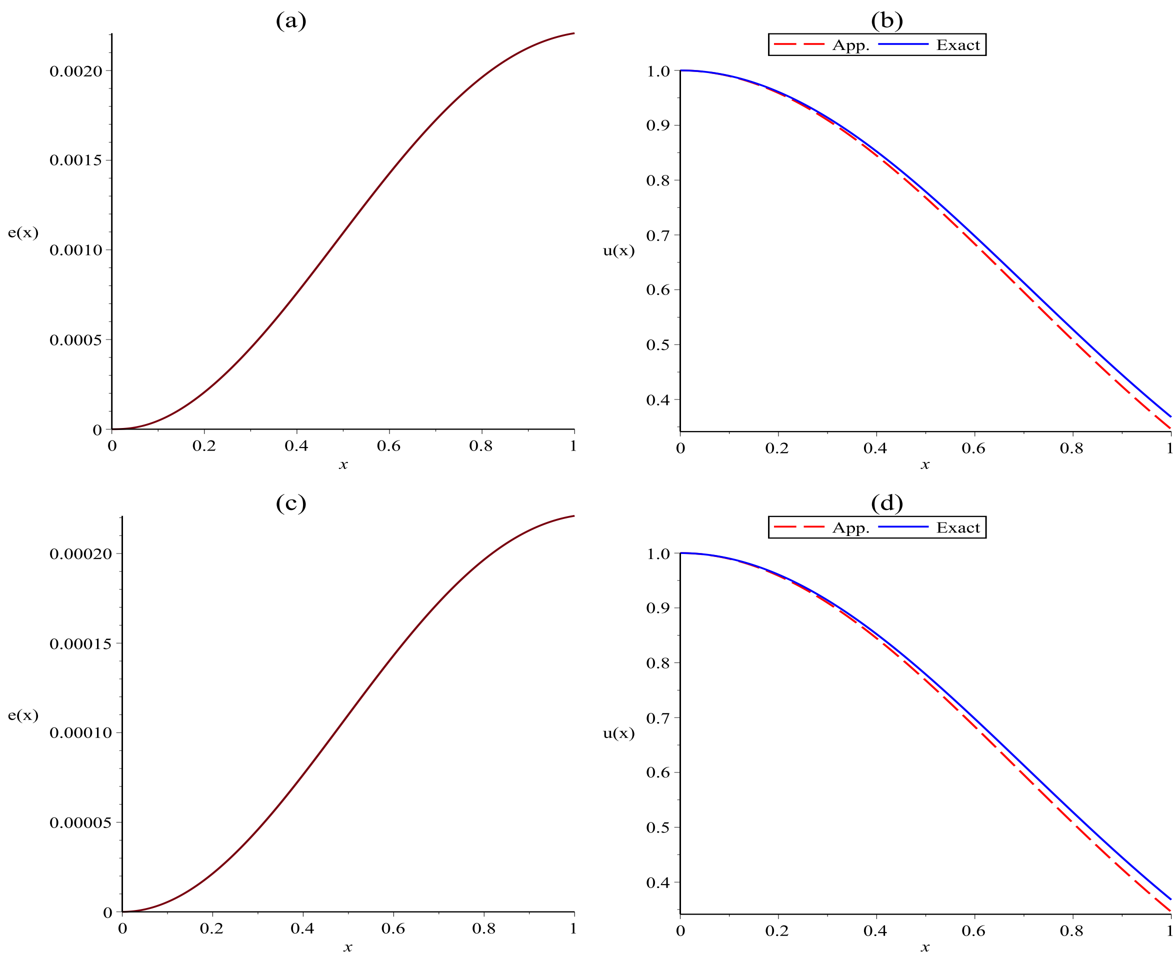

Figure 3. (a) Graph of absolute error function with $N = 10$, (b) graph of approximated and exact solution with $N = 10$, (c) graph of absolute error function with $N = 15$ and (d) graph of approximated and exact solution with $N = 15$ for Example 5.3.

Example 5.4.

This example corresponds to the following nonlinear, homogeneous generalized Emden-Fowler equation:

$$u''(x) + \frac{5}{x}u'(x) + 8(e^u + 2e^{u/2}) = 0,$$

with initial conditions $u(0) = 0$ and $u'(0) = 0$. The exact solution is $u(x) = -2\ln(1 + x^2)$.

The behavior of absolute error is reported in Table 4 for the values $N = 10$ and $N = 15$. Figure 4 shows graph of absolute error function with $N = 10$ (top left), graph of approximated and exact solution with $N = 10$ (top right), graph of absolute error function with $N = 15$ (bottom left) and graph of approximated and exact solution with $N = 15$ (bottom right).

Table 4. The absolute error obtained by the method with $N = 10$ and $N = 15$ for Example 5.4

x	N=10	N=15
0	0	0
0.10	2.2387×10^{-6}	1.3412×10^{-7}
0.20	6.2678×10^{-6}	3.0317×10^{-7}
0.30	4.5565×10^{-5}	2.1129×10^{-6}
0.40	1.4577×10^{-4}	7.1296×10^{-6}
0.50	3.1256×10^{-4}	1.1199×10^{-5}
0.60	6.4490×10^{-4}	3.6532×10^{-5}
0.70	1.2309×10^{-3}	6.5643×10^{-5}
0.80	1.9023×10^{-3}	7.0785×10^{-5}
0.90	2.9766×10^{-3}	1.2297×10^{-4}
1.00	4.2012×10^{-3}	2.1457×10^{-4}

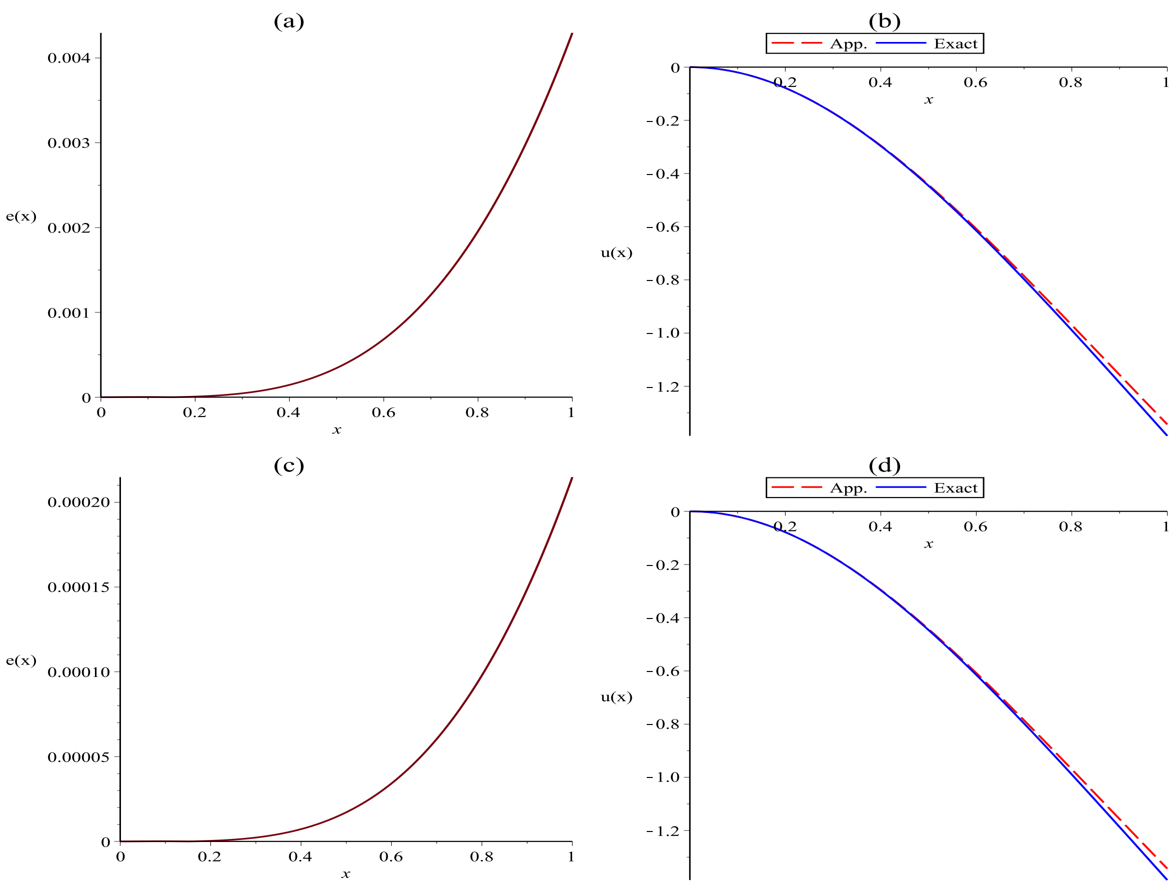

Figure 4. (a) Graph of absolute error function with $N = 10$, (b) graph of approximated and exact solution with $N = 10$, (c) graph of absolute error function with $N = 15$ and (d) graph of approximated and exact solution with $N = 15$ for Example 5.4

6. Conclusions

In this paper exponential approximation has been employed to solve a class of nonlinear Emden-Fowler equations. The method is based on exponential functions and collocation method as operational matrix. Also, to illustrate the accuracy and efficiency of this method, four numerical examples with different order and complexity have been presented. Through numerical experiments, we find that numerical results are in good agreement with the exact analytical solutions. As a result of comparisons with exact solutions, it has been observed that the method presented gives good results. In addition, it is observed that errors decrease as N values increase. As illustrated by the computational results, the implementation of the proposed method is very easy for similar problems.

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