Applications and Applied Mathematics: An International Journal (AAM)

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Ayyappan, G. and Archana, G. (2023). (R2051) Analysis of MAP/PH1, PH2/2 Queueing Model with Working Breakdown, Repairs, Optional Service, and Balking, Applications and Applied Mathematics: An International Journal (AAM), Vol. 18, Iss. 1, Article 1.
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# Analysis of $M A P / P H_{1}, P_{2} / 2$ Queueing Model with Working Breakdown, Repairs, Optional Service and Balking 

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Received: November 9, 2022; Accepted: June 2, 2023


#### Abstract

In this paper, a classical queueing system with two types of heterogeneous servers has been considered. The Markovian Arrival Process ( $M A P$ ) is used for the customer arrival, while phase type distribution $(P H)$ is applicable for the offering of service to customers as well as the repair time of servers. Optional service are provided by the servers to the unsatisfied customers. The server-2 may get breakdown during the busy period of any type of service. Though the server2 got breakdown, server- 2 has a capacity to provide the service at a slower rate to the current customer who is receiving service when the moment of server-2 struck with breakdown. In the period of vacation/closedown of server-1 and the server- 2 is in working breakdown or under repair process, the arrival of customers may balk the system due to the impatient. Stability conditions has derived for our system and the stationary probability vector was evaluated by using the matrix analytical method. This model also examined at the analysis of busy period, waiting time distribution and system performance measures. The numerical illustrations are provided with the aid of two dimensional and three dimensional graphs.


Keywords: Markovian arrival process; Phase type service; Phase type Repair; Optional service; Working breakdown; Multiple vacation; Closedown; Balking

## 1. Introduction

In the real world scenario, it can be seen that several of the queueing models are associated with server vacations, breakdowns, repairs and impatient customers. System manufacturing, designing of local area communication networks and data communication networks are the most common examples. The customers are bound to be impatient in general. From the real-life experience, we can observe that the customers who require service must form a queue. But because of time constraints, some customers do not join the queue, and some of the customers who joined the queue may become impatient and leave the queue without getting service.

In this theory, Markovian Arrival Process ( $M A P$ ) is the most important tools of point process. Neuts (1979) pioneered the versatile Markovian point process (VMPP). He has used the concept of point process which is Markovian arrival process. Chakravarthy (2011) have analysed the MAP which is represented by $n$-dimensional parameter matrices $\left(D_{0}, D_{1}\right)$ where $D_{0}$ governing the transition for no arrival and $D_{1}$ governing with arrivals.

Krisha Kumar et al. (2008) have studied a two heterogeneous server with Bernoulli vacations. They consider multiple vacations of their model, after the completion of vacation, if system becomes empty, the server takes another vacation. They also derived mean waiting time and waiting time distribution of their model. A multi server queueing model with finite retrial queue was examined by Artalejo et al. (2007).

Lucantoni et al. (1990) investigated a single server queueing model with server vacations and a non-renewal arrival process. They have described the distribution of waiting times. Cao and Xie (2020) investigated the cooperative arrival process of various independent batch Markovian arrival processes. A single server batch arrival queueing model with generally independent and uniformly distributed service times was researched by Lucantoni (2005). For the queue length and waiting time distributions, he has proposed stationary and transient distributions. The three-stage hiring process of tandem queueing with Erlang phase-type selection and bulk arrivals was studied by Haghighi and Mishev (2013).

In a queueing model that incorporates a variable arrival rate, optional service, and repair, Ayyappan and Thilagavathy (2022) have analysed the Markovian arrival process for arrivals and the phase-type distribution for services. They discussed the waiting time distribution and cost analysis for their model. A single server batch queueing approach with a second optional service was investigated by Vijaya Laxmi et al. (2021). Haghighi and Mishev (2014) examined the monograph of models and their applications in a textbook with industrial and business applications. Vijaya Laxmi and George (2020) investigated the transient queueing system of a batch service queue with a second optional service and impatient customers. They carried out the cost analysis for optimisation.

A single server queueing model with two phases along with Bernoulli schedule vacation and multiple vacation policy was done by Choudhury and Paul (2006). They evaluated the analysis of busy period and expected system size of the model. A queue with working breakdown and working vacation was examined by Chakravarthy et al. (2020). They considered a backup server to provide
the service while the main server is in under vacation or breakdown period.
Haghighi and Mishev (2016) examined the external arrival tasks from outside to the system and internal arrival through immediate feedback or through a splitting process and both have with delay concept. At the end of service, any one of the events may occur like it may leave the system completely; or, it seeks feedback or it may go to a unit called splitter. A server's busy period and numerical examples are also given in their model.

A $M A P / P H(1), P H(2) / 1$ queueing model in which the services offered by first come first served basis subject to vacations and optional secondary services was investigated by Chakravarthy (2013). The concept of optional secondary service is that after completion of primary customer service, they may need a secondary service with probability $p$ or the customer leaves the system with probability $q$. An multi server queueing model with impatient customers have been done by Haghighi et al. (1986).

A single server queue with infinite capacity along with single working vacation and reneging during the working vacation period was investigated by Vijaya Laxmi et al. (2019). They considered the arriving customers gets impatient and renege from the system during the period of working vacations. They carried out some numerical analysis to view the effect of the parameters on some performance measures. A queue subject to working breakdown has been investigated by Kalidass and Kasturi (2012).

Swathi et al. (2019) examined a queueing system with balking and reneging. The steady state analysis of the system and several performance measures were also derived by them. A retrial inventory system with multiple server vacations for two heterogeneous servers are considered by Suganya and Sivakumar (2019). They assumed that the customers arrive according to MAP as well as the service follows phase type distribution. The total cost rate and some performance measures are also obtained.

A single server queueing model with N-policy and second optional services have been evaluated by Das et al. (2022). They presented the cost analysis and various performance measures of their model. Chakravarthy and Agarwal (2003) explored a machine repair problem with Unreliable server. In their model, they considered phase type distribution for the service and repair time of server. They also determine the performance measures and some numerical illustrations. $M / M / C$ queueing system with impatient customers was examined by Wang and Zhang (2018). They considered the system capacity as $M$ size and also evaluated some performance measures of their model.

A single server queueing model subject to setup time, Bernoulli vacation, breakdown, repair, Bernoulli feedback and impatient customers was examined by Ayyappan and Gowthami (2021). They evaluated the stationary probability vector and R matrix by the help of matrix-geometric method and also provided some performance measures. Kadi et al. (2020) have investigated a multi-server queueing model with Bernoulli feedback, vacation and impatient customers.

Beena and Jose (2020) have recognized on a retrial production inventory system with heterogeneous servers. The total cost and system performance measures are also evaluated in this model. An
$M / M / 1 / N$ queue with retention of reneging and balking has been investigated by Kumar and Sharma (2012). They evaluated the particular cases and some performance measures of their model. Ayyappan and Thilagavathy (2020) examined a queueing model with a single server, a standby server, impatient customers, setup, and closedown. They have discussed about their model's busy period analysis. A single server queueing model subject to working vacation and two type of server breakdown have been analysed by Agarwal et al. (2021). They considered the server breakdown while server is in working vacation or normal busy period. Numerical illustrations are also examined by them.

The rest of the article is structured as follows. Implementation of our model is provided in Section 2. Section 3 contains the narration for our model. Section 4 describes the process of matrix generation and some notations. In Section 5, the stability condition, the invariant probability vector and the rate matrix $R$ are derived. In Section 6, the busy period analysis is described. Measures of performance are given in Section 7. The cost analysis are presented in Section 8 and the waiting time distribution is discussed in Section 9. Section 10 provides some numerical results and graphical outcomes. The conclusion part is presented in Section 11.

## 2. Implementation of Our Model

In our real life situations, the motivation of this model is pharmaceutical shops. Consider as one the pharmacy shop which has more than one billing sections. Let us consider that there are two servers are working in that shop. The billing process may consists of the following phases:

1. ask the prescription and generate a bill.
2. collect the money through phone pay or cash and enter into the system to make a bill.

3 . organize the items and pack it, in the particular cover.
When the customer arrives, during which they see any server free in the system, then they receive the service without delay; if not, the customers must wait in a lineup to reach the service. Once the customer receives the service from the server, they may leaves the system completely or they have a choice to get the service once again by the server (optional service). At the consummation of service, if there are no customers waiting in the system, server-1 goes on vacation (like receiving the telephone calls, collection of orders, arranging the prescriptions, etc.) or they pursue further service for upcoming customers. Server-1 will put down the system in sleep mode or close the data, before takes the vacation (close down). During the vacation/close down time of server-1 and breakdown/repair time of server- 2 , the incoming customers may not enter into the system due to the lack of patience (balking). While the busy period, the server-2 can attain failure of the service (like hanging the system, lack of network, etc.), then the server- 2 will manage to complete the service in lower rate (working breakdown) and then he goes to repair process. When the repair process is finished, the server begins the service to the customers who have been waiting in the queue. Our model has framed with all these situations.

## 3. Narration of the Model

- In this model, deal with a single queue entry with two heterogeneous servers. Customers arrive in accordance with Markovian arrival process with representation $\left(D_{0}, D_{1}\right)$ of dimension $m$. The matrix $D_{0}$ represents no arrival and $D_{1}$ represents arrival.
- The service offered to the customers in the basis of first come-first served. The duration of normal and optional services of both servers which follows PH -distribution with the notations $\left(\alpha_{1}, T_{1}\right)$, $\left(\alpha_{2}, T_{2}\right),\left(\beta_{1}, S_{1}\right)$ and $\left(\beta_{2}, S_{2}\right)$ of order $n_{1}, n_{2}, n_{3}$ and $n_{4}$, respectively, where $T_{1}^{0}+T_{1} e=0$, $T_{2}^{0}+T_{2} e=0$ and $S_{1}^{0}+S_{1} e=0, S_{2}^{0}+S_{2} e=0$.
- With the process of service completion of server-1 and server-2, the customer either may exit the system with the probability $p_{2}$, or they need second optional service with the probability $p_{1}$, where $p_{1}+p_{2}=1$.
- At the end of the service completion epoch server-1 will closedown the system, if there is no customer present in the system. After completion of vacation by the server-1, if there are no customers in the system, then the server goes on another vacation.
- In the time of busy period, the server-2 may affect the failure at any moment. At that time, the server-2 provides the service continuously to the interrupted customer at a slower rate which also follows PH distribution with representations $\left(\beta_{1}, \theta_{1} S_{1}\right)$ and $\left(\beta_{2}, \theta_{2} S_{2}\right)$ where $\left(0<\theta_{1}, \theta_{2}<1\right)$. Server-2 starts the repair process after completing of service at a slower rate for that interrupted customer only.
- At the end of repair process, either server-2 is ready to give the service if there are customers in the system or he is in idle. The repair time of the server-2 also based on PH-distribution with notation $(\gamma, R)$ of order $s$. During the customers arrival, they may balk the system if server- 1 is on vacation/closedown period and server-2 is on working breakdown/under repair process with the probability $b$.
- The close down, vacation time of server- 1 and breakdown time of server-2 are all based on the exponential distribution with parameters $\varphi, \eta$ and $\tau$, respectively (see Figure 1).


## 4. Matrix Generation under Quasi Birth and Death(QBD) Process

Let us narrate the few notations of this model which followed by generator matrix of the QBD process as follows:

Notations:

- $\otimes$ - Kronecker product of two matrices with various orders.
- $\oplus$ - Kronecker sum of two matrices with various orders.
- $I_{m}$ - Identity matrix of order m.
- $e=e_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right)}$.
- $e_{0}=e_{(1+s) 2 m}$.
- $e_{1}=e_{\left(4 n_{3}+4 n_{4}+2 s+n_{1}+n_{1} s+n_{2}+n_{2} s\right) m}$.
- $e_{2}=e_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right) m}$.


Figure 1. Schematic representation of our model

- The fundamental arrival rate $\lambda$ which is specified as $\lambda=\pi D_{1} e_{m}, \pi$ is the invariant probability vector.
- The rate of normal and optional service of server- 1 is indicated as $\mu_{1}=\left[\alpha_{1}\left(-T_{1}^{-1}\right) e_{t_{1}}\right]^{-1}$ and $\mu_{2}=\left[\alpha_{2}\left(-T_{2}^{-1}\right) e_{t_{2}}\right]^{-1}$.
- The rate of normal and optional service of server-2 is indicated as $\mu_{3}=\left[\beta_{1}\left(-S_{1}^{-1}\right) e_{s_{1}}\right]^{-1}$ and $\mu_{4}=\left[\beta_{2}\left(-S_{2}^{-1}\right) e_{s_{2}}\right]^{-1}$.
- The repair rate for normal/optional service of server-2 as represented by $\sigma=\left[\gamma\left(-R^{-1}\right) e_{r}\right]^{-1}$.
- $N(t)$ represents the total number of customers in the system at time $t$.
- $C_{i}(t)$ stands for server-i's status at time $t$, where $\mathrm{i}=1,2$.

$$
C_{1}(t)= \begin{cases}0, & \text { server-1 is on vacation. } \\ 1, & \text { server-1 is busy in normal serivce } \\ 2, & \text { server-1 is busy in optional service } \\ 3, & \text { server-1 is in closedown. }\end{cases}
$$

$$
C_{2}(t)= \begin{cases}0, & \text { server-2 is idle. } \\ 1, & \text { server-2 is busy in normal Service. } \\ 2, & \text { server-2 is busy in optional Service. } \\ 3, & \text { server-2 is busy with normal service during the breakdown. } \\ 4, & \text { server-2 is busy with optional service during the breakdown. } \\ 5, & \text { server-2 is in PH repair process. }\end{cases}
$$

- $S_{i}(t)$ stands for the service phase of server-i's, where $i=1,2$.
- $K(t)$ stands for the repair phase of server-2.
- $A(t)$ stands for the arrival phase.

Let $\left\{N(t), C_{1}(t), C_{2}(t), S_{1}(t), S_{2}(t), K(t), A(t), \quad t \geq 0\right\}$ be the $C T M C$ with the state space as follows,

$$
\Omega=l(0) \cup l(1) \bigcup_{i=2}^{\infty} l(i),
$$

where

$$
\begin{aligned}
l(0) & =\{(0,0,0, a): 1 \leq a \leq m\} \cup\{(0,0,5, b, a): 1 \leq b \leq s, 1 \leq a \leq m\} \\
& \cup\{(0,3,0, a): 1 \leq a \leq m\} \cup\{(0,3,5, b, a): 1 \leq b \leq s, 1 \leq a \leq m\}, \\
l(1) & =\left\{\left(1,0,1, k_{3}, a\right): 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,0,2, k_{4}, a\right): 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,0,3, k_{3}, a\right): 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,0,4, k_{4}, a\right): 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\{(1,0,5, b, a): 1 \leq b \leq s, 1 \leq a \leq m\} \\
& \cup\left\{\left(1,1,0, k_{1}, a\right): 1 \leq k_{1} \leq n_{1}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,1,5, k_{1}, b, a\right): 1 \leq k_{1} \leq n_{1}, 1 \leq b \leq s, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,2,0, k_{2}, a\right): 1 \leq k_{2} \leq n_{2}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,2,5, k_{2}, b, a\right): 1 \leq k_{2} \leq n_{2}, 1 \leq b \leq s, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,3,1, k_{3}, a\right): 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,3,2, k_{4}, a\right): 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,3,3, k_{3}, a\right): 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(1,3,4, k_{3}, a\right): 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \cup\{(1,3,5, b, a): 1 \leq b \leq s, 1 \leq a \leq m\},
\end{aligned}
$$

and for $i \geq 2$,
$l(i)=\left\{\left(i, 0,1, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\}$
$\cup\left\{\left(i, 0,2, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\}$
$\cup\left\{\left(i, 0,3, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\}$
$\cup\left\{\left(i, 0,4, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\}$
$\cup\left\{(i, 0,5, b, a): i \in \mathbb{Z}^{+}, 1 \leq b \leq s, 1 \leq a \leq m\right\}$
$\cup\left\{\left(i, 1,1, k_{1}, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{1} \leq n_{1}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\}$

$$
\begin{aligned}
& \cup\left\{\left(i, 1,2, k_{1}, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{1} \leq n_{1}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 1,3, k_{1}, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{1} \leq n_{1}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 1,4, k_{1}, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{1} \leq n_{1}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 1,5, k_{1}, b, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{1} \leq n_{1}, 1 \leq b \leq s, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 2,1, k_{2}, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{2} \leq n_{2}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 2,2, k_{2}, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{2} \leq n_{2}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 2,3, k_{2}, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{2} \leq n_{2}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 2,4, k_{2}, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{2} \leq n_{2}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 2,5, k_{2}, b, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{2} \leq n_{2}, 1 \leq b \leq s, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 3,1, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 3,2, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 3,3, k_{3}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{3} \leq n_{3}, 1 \leq a \leq m\right\} \\
& \cup\left\{\left(i, 3,4, k_{4}, a\right): i \in \mathbb{Z}^{+}, 1 \leq k_{4} \leq n_{4}, 1 \leq a \leq m\right\} \\
& \cup\left\{(i, 3,5, b, a): i \in \mathbb{Z}^{+}, 1 \leq b \leq s, 1 \leq a \leq m\right\} .
\end{aligned}
$$

### 4.1. The Infinitesimal Matrix Generation

The quasi birth and death process has the generating matrix $Q$ and is as follows:

$$
Q=\left[\begin{array}{ccccccccc}
B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
B_{10} & B_{11} & B_{12} & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & B_{21} & A_{1} & A_{0} & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & A_{2} & A_{1} & A_{0} & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & A_{2} & A_{1} & A_{0} & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ldots & \ldots
\end{array}\right]
$$

where

$$
\begin{aligned}
& B_{00}=\left[\begin{array}{cc}
B_{11}^{00} & 0 \\
B_{21}^{00} & B_{22}^{00}
\end{array}\right] ; \quad B_{01}=\left[\begin{array}{cccc}
B_{11}^{01} & 0 & 0 & 0 \\
0 & 0 & 0 & B_{24}^{01}
\end{array}\right] ; \quad B_{10}=\left[\begin{array}{cc}
B_{11}^{10} & 0 \\
0 & B_{22}^{10} \\
0 & B_{32}^{10} \\
0 & B_{42}^{10}
\end{array}\right] ; \\
& B_{11}^{00}=\left[\begin{array}{cc}
D_{0} & 0 \\
R^{0} \otimes I_{m}\left(D_{0}+b D_{1}\right) \oplus R
\end{array}\right] ; \quad B_{21}^{00}=\left[\begin{array}{cc}
\varphi I_{m} & 0 \\
0 & \varphi I_{m}
\end{array}\right] ; \\
& B_{22}^{00}=\left[\begin{array}{cc}
D_{0}+\varphi I_{m} & 0 \\
R^{0} \otimes I_{m} & \left(D_{0}+b D_{1}\right) \oplus R-\phi I_{m b}
\end{array}\right] ; \quad B_{11}^{01}=\left[\begin{array}{ccc}
\beta_{1} \otimes D_{1} 000 & 0 \\
0 & 000 & D_{1}(1-b) \otimes I_{s}
\end{array}\right] ; \\
& B_{24}^{01}=\left[\begin{array}{cccc}
\beta_{1} \otimes D_{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} D_{1}(1-b) \otimes I_{s}\right] ; \quad B_{11}^{10}=\left[\begin{array}{cc}
p_{2} S_{1}^{0} \otimes I_{m} & 0 \\
S_{2}^{0} \otimes I_{m} & 0 \\
0 & \gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{m} \\
0 & \gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{m} \\
0 & 0
\end{array}\right] ;
\end{aligned}
$$

$$
\begin{aligned}
& B_{22}^{10}=\left[\begin{array}{cc}
p_{2} T_{1}^{0} \otimes I_{m} & 0 \\
0 & I_{n 1} \otimes p_{2} T_{1}^{0} \otimes I_{m}
\end{array}\right] ; \quad B_{32}^{10}=\left[\begin{array}{cc}
T_{2}^{0} \otimes I_{m} & 0 \\
0 & I_{n 2} \otimes p_{2} T_{2}^{0} \otimes I_{m}
\end{array}\right] ; \\
& B_{42}^{10}=\left[\begin{array}{cc}
p_{2} S_{1}^{0} \otimes I_{m} & 0 \\
S_{2}^{0} \otimes I_{m} & 0 \\
0 & \gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{m} \\
0 & \gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{m} \\
0 & 0
\end{array}\right] ; B_{11}=\left[\begin{array}{cccc}
B_{11}^{11} & B_{12}^{11} & 0 & 0 \\
0 & B_{22}^{11} & B_{23}^{11} & 0 \\
0 & 0 & B_{33}^{11} & 0 \\
B_{41}^{11} & 0 & 0 & B_{44}^{11}
\end{array}\right] ; B_{12}=\left[\begin{array}{cccc}
B_{11}^{12} & 0 & 0 & 0 \\
0 & B_{22}^{12} & 0 & 0 \\
0 & 0 & B_{33}^{12} & 0 \\
0 & 0 & 0 & B_{44}^{12}
\end{array}\right] ; \\
& B_{21}=\left[\begin{array}{cccc}
B_{11}^{21} & 0 & 0 & 0 \\
0 & B_{22}^{21} & 0 & B_{24}^{21} \\
0 & B_{32}^{21} & B_{33}^{21} & B_{34}^{21} \\
0 & 0 & 0 & B_{44}^{21}
\end{array}\right] ; B_{11}^{11}=\left[\begin{array}{cccc}
B_{111} B_{112} & B_{113} & 0 & 0 \\
0 & B_{114} & 0 & B_{115} \\
0 & 0 & B_{116} & 0 \\
0 & 0 & 0 & B_{117} \\
B_{118} & 0 & 0 & 0 \\
B_{119}
\end{array}\right] ;
\end{aligned}
$$

where $B_{111}=S_{1} \oplus D_{0}-\tau I_{n_{3} m}, B_{112}=\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{m}, B_{113}=\tau I_{n_{3} m}$,
$B_{114}=S_{2} \oplus D_{0}-\tau I_{n_{4} m}, B_{115}=\tau \otimes I_{n_{3}}, B_{116}=\theta_{1} S_{1} \oplus\left(D_{0}+b D_{1}\right), B_{117}=\theta_{2} S_{2} \oplus\left(D_{0}+b D_{1}\right)$, and $B_{118}=\beta_{1} \otimes R^{0} \otimes I_{m}, B_{119}=R \oplus\left(D_{0}+b D_{1}\right)-\eta I_{s m}$.

$$
\begin{aligned}
& B_{12}^{11}=\left[\begin{array}{ll}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \alpha_{1} \otimes \eta I_{s m}
\end{array}\right] ; \quad B_{22}^{11}=\left[\begin{array}{c}
T_{1} \oplus D_{0} \\
R^{0} \otimes I_{n_{1} m}\left[\left(T_{1}+R\right) \oplus D_{0}\right] \otimes I_{n_{1}}
\end{array}\right] ; \\
& B_{23}^{11}=\left[\begin{array}{cc}
\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{1} m} & 0 \\
0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{1} m}
\end{array}\right] ; \quad B_{33}^{11}=\left[\begin{array}{cc}
T_{2} \otimes D_{0} & 0 \\
R^{0} \otimes I_{n_{2} m} T_{2} \oplus R \oplus D_{0}
\end{array}\right] ; \\
& B_{41}^{11}=\left[\begin{array}{ccccc}
\varphi I_{n_{3} m} & 0 & 0 & 0 & 0 \\
0 & \varphi I_{n_{4} m} & 0 & 0 & 0 \\
0 & 0 & \varphi I_{n_{3} m} & 0 & 0 \\
0 & 0 & 0 & \varphi I_{n_{4} m} & 0 \\
0 & 0 & 0 & 0 & \varphi I_{s m}
\end{array}\right] ; \quad B_{44}^{11}=\left[\begin{array}{ccccc}
B_{441} & B_{442} & B_{443} & 0 & 0 \\
0 & B_{444} & 0 & B_{445} & 0 \\
0 & 0 & B_{446} & 0 & 0 \\
0 & 0 & 0 & B_{447} & 0 \\
B_{448} & 0 & 0 & 0 & B_{449}
\end{array}\right] ;
\end{aligned}
$$

where $B_{441}=S_{1} \oplus D_{0}-(\varphi+\tau) I_{n_{3} m}, B_{442}=\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{m}, B_{443}=\tau I_{n_{3} m}$, $B_{444}=S_{2} \oplus D_{0}-(\varphi+\tau) I_{n_{4} m}, B_{445}=\tau I_{n_{4} m}, B_{446}=\left(\theta_{1} S_{1} \oplus D_{0}+b D_{1}\right)-\varphi I_{n_{3} m}$, $B_{447}=\left(\theta_{2} S_{2} \oplus D_{0}+b D_{1}\right)-\varphi I_{n_{4} m}, B_{448}=\alpha_{1} \otimes R^{0} \otimes I_{m}, B_{449}=R \oplus\left(D_{0}+b D_{1}\right)-\varphi I_{s m}$.

$$
\begin{aligned}
& B_{11}^{12}=\left[\begin{array}{ccccc}
D_{1} \otimes I_{n_{3}} & 0 & 0 & 0 & 0 \\
0 & D_{1} \otimes I_{n 4} & 0 & 0 & 0 \\
0 & 0 & (1-b) D_{1} \otimes I_{n_{3}} & 0 & 0 \\
0 & 0 & 0 & (1-b) D_{1} \otimes I_{n_{4}} & 0 \\
0 & 0 & 0 & 0 & D_{1} \otimes I_{s}
\end{array}\right] ; \\
& B_{22}^{12}=\left[\begin{array}{cccc}
\beta_{1} \otimes D_{1} \otimes I_{n_{1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
D_{1} \otimes I_{n_{1} s}
\end{array}\right] ; \quad B_{33}^{12}=\left[\begin{array}{c}
\beta_{1} \otimes D_{1} \otimes I_{n_{2}} 000 \\
0 \\
0
\end{array} 000 D_{1} \otimes I_{n_{2} s}\right] ; \\
& B_{44}^{12}=\left[\begin{array}{ccccc}
D_{1} \otimes I_{n_{3}} & 0 & 0 & 0 & 0 \\
0 & D_{1} \otimes I_{n_{4}} & 0 & 0 & 0 \\
0 & 0 & D_{1}(1-b) \otimes I_{n_{3}} & 0 & 0 \\
0 & 0 & 0 & D_{1}(1-b) \otimes I_{n 4} & 0 \\
0 & 0 & 0 & 0 & D_{1}(1-b) \otimes I_{s}
\end{array}\right] ; \\
& B_{11}^{21}=\left[\begin{array}{cccc}
\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{m} & 000 & 0 & \\
\beta_{1} \otimes S_{2}^{0} \otimes I_{m} & 000 & 0 & \\
0 & 000 \beta_{1} \otimes \theta_{1} S_{1}^{0} \otimes I_{m} & \\
0 & 000 & & \beta_{1} \otimes \theta_{2} S_{2}^{0} \otimes I_{m} \\
& 000 & 0 & 0
\end{array}\right] ; \\
& B_{22}^{21}=\left[\begin{array}{cc}
p_{2}\left(T_{1}^{0}+S_{1}^{0}\right) \otimes I_{n_{3} m} & 0 \\
I_{n_{4}} \otimes S_{2}^{0} \otimes I_{m} & 0 \\
0 & \gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{n_{3} m} \\
0 & \gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{n_{4} m} \\
0 & 0
\end{array}\right] ; \\
& B_{24}^{21}=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & I_{n_{1}} \otimes p_{2} T_{1}^{0} \otimes I_{m} & 0 & 0 & 0 \\
0 & 0 & I_{n_{1}} \otimes p_{2} T_{1}^{0} \otimes I_{m} & 0 & 0 \\
0 & 0 & 0 & I_{n_{1}} \otimes p_{2} T_{1}^{0} \otimes I_{m} & 0 \\
0 & 0 & 0 & 0 & I_{n_{1}} \otimes p_{2} T_{1}^{0} \otimes I_{m}
\end{array}\right] ; \\
& B_{32}^{21}=\left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{2} m}
\end{array}\right] ; \quad B_{33}^{21}=\left[\begin{array}{cc}
I_{n_{3}} \otimes p_{2} S_{1}^{0} \otimes I_{n} & 0 \\
e \otimes \gamma \otimes S_{2}^{0} \otimes I_{m} & 0 \\
0 & \gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{n_{3} m} \\
0 & \gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{n_{4} m} \\
0 & 0
\end{array}\right] ;
\end{aligned}
$$

$B_{34}^{21}=\left[\begin{array}{ccccc}I_{n_{2}} \otimes T_{2}^{0} \otimes I_{m} & 0 & 0 & 0 & 0 \\ 0 & I_{n_{2}} \otimes T_{2}^{0} \otimes I_{m} & 0 & 0 & 0 \\ 0 & 0 I_{n_{2}} \otimes T_{2}^{0} \otimes I_{m} & 0 & 0 & \\ 0 & 0 & 0 & I_{n_{2}} \otimes T_{2}^{0} \otimes I_{m} 0 \\ 0 & 0 & 0 & 0 & 0\end{array}\right] ;$
$B_{44}^{21}=\left[\begin{array}{cccc}\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{m} & 0 & 0 & 0 \\ \beta_{1} \otimes S_{2}^{0} \otimes I_{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \gamma \otimes \theta S_{1}^{0} \otimes I_{m} \\ 0 & 0 & 00 \gamma \otimes \theta S_{2}^{0} \otimes I_{m} \\ 0 & 0 & 0 & 0\end{array}\right] ; \quad A_{1}=\left[\begin{array}{cccc}A_{11}^{1} & A_{12}^{1} & 0 & 0 \\ 0 & A_{22}^{1} & A_{23}^{1} & 0 \\ 0 & 0 & A_{33}^{1} & 0 \\ A_{41}^{1} & 0 & 0 & A_{44}^{1}\end{array}\right] ;$
$A_{0}=\left[\begin{array}{cccc}A_{11}^{0} & 0 & 0 & 0 \\ 0 & A_{22}^{0} & 0 & 0 \\ 0 & 0 & A_{33}^{0} & 0 \\ 0 & 0 & 0 & A_{44}^{0}\end{array}\right] ; A_{2}=\left[\begin{array}{cccc}A_{11}^{2} & 0 & 0 & 0 \\ 0 & A_{22}^{2} & 0 & 0 \\ 0 & A_{32}^{2} & A_{33}^{2} & 0 \\ 0 & 0 & 0 & A_{44}^{2}\end{array}\right] ; A_{11}^{1}=\left[\begin{array}{ccccc}A_{111} & A_{112} & A_{113} & 0 & 0 \\ 0 & A_{114} & 0 & A_{115} & 0 \\ 0 & 0 & A_{116} & 0 & 0 \\ 0 & 0 & 0 & A_{117} & 0 \\ A_{118} & 0 & 0 & 0 & A_{119}\end{array}\right] ;$
where $A_{111}=\left(S_{1} \oplus D_{0}\right)-(\tau+\eta) I_{n_{3} m}, A_{112}=\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{m}, A_{113}=\tau I_{n_{3} m}$, $A_{114}=\left(S_{2} \oplus D_{0}\right)-(\tau+\eta) I_{n_{4} m}, A_{115}=\tau I_{n_{4} m}, A_{116}=\left(\theta_{1} S_{1} \oplus D_{0}+\lambda D_{1}\right)-\eta I_{n_{3} m}$, and $A_{117}=\left(\theta_{2} S_{2} \oplus D_{0}+\lambda D_{1}\right)-\eta I_{n_{4} m}, A_{118}=\beta_{1} \otimes R^{0} \otimes I_{m}, A_{119}=R \oplus\left(D_{0}+b D_{1}\right)-\eta I_{s m}$.
$A_{12}^{1}=\left[\begin{array}{ccccc}\alpha_{1} \otimes \eta I_{n_{3} m} & 0 & 0 & 0 & 0 \\ 0 & \alpha_{1} \otimes \eta I_{n_{4} m} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{1} \otimes \eta I_{n_{3} m} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{1} \otimes \eta I_{n_{4} m} & 0 \\ 0 & 0 & 0 & 0 & \alpha_{1} \otimes \eta I_{s m}\end{array}\right] ;$
$A_{22}^{1}=\left[\begin{array}{ccccc}A_{221} & A_{222} & A_{223} & 0 & 0 \\ 0 & A_{224} & 0 & A_{225} & 0 \\ 0 & 0 & A_{226} & 0 & 0 \\ 0 & 0 & 0 & A_{227} & 0 \\ A_{228} & 0 & 0 & 0 & A_{229}\end{array}\right] ;$
where $A_{221}=\left(T_{1} \oplus S_{1} \oplus D_{0}\right)-\tau I_{n_{1} n_{3} m}, A_{222}=\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{n_{4} m}, A_{223}=\tau I_{n_{1} n_{3} m}$, $A_{224}=\left(T_{1} \oplus S_{2} \oplus D_{0}\right)-\tau I_{n_{1} n_{4} m}, A_{225}=\tau I_{n_{1} n_{4} m}, A_{225}=\tau I_{n_{1} n_{4} m}, A_{226}=T_{1} \oplus \theta_{1} S_{1} \oplus D_{0}$, and $A_{227}=T_{1} \oplus \theta_{2} S_{2} \oplus D_{0}, A_{228}=\beta_{1} \otimes R^{0} \otimes I_{n_{3} m}, A_{229}=T_{1} \oplus R \oplus D_{0}$.
$A_{23}^{1}=$
$\left[\begin{array}{ccccc}\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{3} m} & 0 & 0 & 0 & 0 \\ 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{4} m} & 0 & 0 & 0 \\ 0 & 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{3} m} & 0 & 0 \\ 0 & 0 & 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{4} m} & 0 \\ 0 & 0 & 0 & 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{s m} .\end{array}\right] ;$
$A_{33}^{1}=\left[\begin{array}{ccccc}A_{331} & A_{332} & A_{333} & 0 & 0 \\ 0 & A_{334} & 0 & A_{335} & 0 \\ 0 & 0 & A_{336} & 0 & 0 \\ 0 & 0 & 0 & A_{337} & 0 \\ A_{338} & 0 & 0 & 0 & A_{339}\end{array}\right] ;$
where $A_{331}=\left(T_{2} \oplus S_{1} \oplus D_{0}\right)-\tau I_{n_{2} n_{3} m}, A_{332}=\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{n_{4} m}, A_{333}=\tau I_{n_{2} n_{3} m}$, $A_{334}=\left(T_{2} \oplus S_{2} \oplus D_{0}\right)-\tau I_{n_{2} n_{4} m}, A_{335}=\tau I_{n_{2} n_{4} m}, A_{336}=T_{2} \oplus \theta_{1} S_{1} \oplus D_{0}$, and $A_{337}=T_{2} \oplus \theta_{2} S_{2} \oplus D_{0}, A_{338}=\beta_{1} \otimes R^{0} \otimes I_{n_{3} m}, A_{339}=T_{2} \oplus R \oplus D_{0}$.
$A_{41}^{1}=\left[\begin{array}{ccccc}\varphi I_{n_{3} m} & 0 & 0 & 0 & 0 \\ 0 & \varphi I_{n_{4} m} & 0 & 0 & 0 \\ 0 & 0 & \varphi I_{n_{3} m} & 0 & 0 \\ 0 & 0 & 0 & \varphi I_{n_{4} m} & 0 \\ 0 & 0 & 0 & 0 & \varphi I_{s m}\end{array}\right] ; \quad A_{44}^{1}=\left[\begin{array}{ccccc}A_{441} & A_{442} & A_{443} & 0 & 0 \\ 0 & A_{444} & 0 & A_{445} & 0 \\ 0 & 0 & A_{446} & 0 & 0 \\ 0 & 0 & 0 & A_{447} & 0 \\ A_{448} & 0 & 0 & 0 & A_{449}\end{array}\right] ;$
where $A_{441}=\left(S_{1} \oplus D_{0}\right)-(\varphi+\tau) I_{n_{3} m}, A_{442}=\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{m}, A_{443}=\tau I_{n_{3} m}$, $A_{444}=\left(S_{2} \oplus D_{0}\right)-(\varphi+\tau) I_{n_{4} m}, A_{445}=\tau I_{n_{4} m}, A_{446}=\left(\theta_{1} S_{1} \oplus D_{0}+b D_{1}\right)-\varphi I_{n_{3} m}$, and $\left.A_{447}=\left(\theta_{2} S_{2} \oplus D_{0}+b D_{1}\right)-\varphi I_{n_{4} m}, A_{448}=\beta_{1} \otimes R^{0} \otimes I m, A_{449}=R \oplus D_{0}+b D_{1}\right)-\varphi I_{s m}$.
$A_{11}^{0}=\left[\begin{array}{ccccc}D_{1} \otimes I_{n_{3}} & 0 & 0 & 0 & 0 \\ 0 & D_{1} \otimes I_{n_{4}} & 0 & 0 & 0 \\ 0 & 0 & D_{1}(1-b) \otimes I_{n_{3}} & 0 & 0 \\ 0 & 0 & 0 & D_{1}(1-b) \otimes I_{n_{4}} & 0 \\ 0 & 0 & 0 & 0 & D_{1}(1-b) \otimes I_{s}\end{array}\right] ;$
where $A_{11}^{0}=A_{44}^{0}, A_{22}^{0}=\left[\begin{array}{ccccc}D_{1} \otimes I_{n_{1} n_{3}} & 0 & 0 & 0 & 0 \\ 0 & D_{1} \otimes I_{n_{1} n 4} & 0 & 0 & 0 \\ 0 & 0 & D_{1} \otimes I_{n_{1} n_{3}} & 0 & 0 \\ 0 & 0 & 0 & D_{1} \otimes I_{n_{1} n_{4}} & 0 \\ 0 & 0 & 0 & 0 & D_{1} \otimes I_{n_{1} s}\end{array}\right]$;
$A_{33}^{0}=\left[\begin{array}{ccccc}D_{1} \otimes I_{n_{2} n_{3}} & 0 & 0 & 0 & 0 \\ 0 & D_{1} \otimes I_{n_{2} n 4} & 0 & 0 & 0 \\ 0 & 0 & D_{1} \otimes I_{n_{2} n_{3}} & 0 & 0 \\ 0 & 0 & 0 & D_{1} \otimes I_{n_{2} n_{4}} & 0 \\ 0 & 0 & 0 & 0 & D_{1} \otimes I_{n_{2} s}\end{array}\right] ;$
$A_{11}^{2}=\left[\begin{array}{ccc}\beta_{1} \otimes p_{1} S_{1}^{0} \otimes I_{m} & 000 & 0 \\ \beta_{1} \otimes S_{2}^{0} \otimes I_{m} & 000 & 0 \\ 0 & 000 \beta_{1} \otimes \theta S_{1}^{0} \otimes I_{m} \\ 0 & 000 \beta_{1} \otimes \theta S_{2}^{0} \otimes I_{m} \\ 0 & 000 & 0\end{array}\right] ; A_{22}^{2}=\left[\begin{array}{ccccc}A_{221} & 0 & 0 & 0 & 0 \\ A_{222} & A_{223} & 0 & 0 & 0 \\ 0 & 0 & A_{224} & 0 & A_{225} \\ 0 & 0 & 0 & A_{226} & A_{227} \\ 0 & 0 & 0 & 0 & A_{228}\end{array}\right] ;$
where $A_{221}=p_{2}\left(T_{1}^{0} \alpha_{1}+S_{1}^{0} \beta_{1}\right) \otimes I_{n_{1} m}, A_{222}=\beta_{1} \otimes S_{2}^{0} \otimes I_{n_{3} m}, A_{223}=\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{4} m}$, $A_{224}=\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{3} m}, A_{225}=\gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{s m}, A_{226}=\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{4} m}$, and $A_{227}=\gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{s m}, A_{228}=\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{s m}$,

$$
\begin{aligned}
& A_{32}^{2}=\left[\begin{array}{cccc}
\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{3} m} & 0 & 0 & 0 \\
0 & \alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{4} m} & 0 & 0 \\
0 & 0 & \alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{3} m} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{4} m} \\
0 & 0 & 0 \\
\alpha_{1} \otimes T_{2}^{0} \otimes I_{s m}
\end{array}\right] ; \\
& A_{33}^{2}=\left[\begin{array}{ccc}
\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{n_{3} m} & 000 & 0 \\
\beta_{1} \otimes S_{2}^{0} \otimes I_{n_{3} m} & 000 & 0 \\
0 & 000 \gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{s m} \\
0 & 000 \gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{s m} \\
0 & 000 & 0 \\
0 & 0
\end{array}\right] ; \\
& A_{44}^{2}=\left[\begin{array}{ccc}
\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{m} & 000 & 0 \\
\beta_{1} \otimes S_{2}^{0} \otimes I_{m} & 000 & 0 \\
0 & 000 \gamma \otimes \theta S_{1}^{0} \otimes I_{m} \\
0 & 000 \gamma \otimes \theta S_{2}^{0} \otimes I_{m} \\
0 & 000 & 0
\end{array}\right] .
\end{aligned}
$$

## 5. System Analysis

We evaluate this model, beneath the certain conditions, to ensure that the system is stable.

### 5.1. Analysis of System Stability condition

Let $A$ be the matrix , where $A=A_{0}+A_{1}+A_{2}$. The invariant probability vector $\varsigma$, which is referred to as a generator matrix, satisfies

$$
\varsigma A=0, \quad \varsigma e=1
$$

The vector $\varsigma$, which represents the states which partitioned by
$\varsigma=\left(\varsigma_{0}, \varsigma_{1}, \varsigma_{2}, \varsigma_{3}, \varsigma_{4}, \varsigma_{5}, \varsigma_{6}, \varsigma_{7}, \varsigma_{8}, \varsigma_{9}, \varsigma_{10}, \varsigma_{11}, \varsigma_{12}, \varsigma_{13}, \varsigma_{14}, \varsigma_{15}, \varsigma_{16}, \varsigma_{17}, \varsigma_{18}, \varsigma_{19}\right)$
is evaluated by the aid of the subsequent equation:

$$
\begin{aligned}
& \varsigma_{0}\left[\left(D_{1} \otimes I_{n_{3}}+\left(\left(S_{1} \oplus D_{0}\right)-(\tau+\eta) I_{n_{3} m}\right)+\left(\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{m}\right)\right]\right. \\
& \quad+\varsigma_{1}\left[\beta_{1} \otimes S_{2}^{0} \otimes I_{m}\right]+\varsigma_{4}\left[\beta_{1} \otimes R^{0} \otimes I_{m}\right]+\varsigma_{15}\left[\varphi I_{n_{3} m}\right]=0, \\
& \varsigma_{0}\left[\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{m}\right]+\varsigma_{1}\left[\left(D_{1} \otimes I_{n_{4}}\right)+\left(S_{2} \oplus-(\tau+\eta) I_{n_{4} m}\right)\right]+\varsigma_{16}\left[\varphi I_{n_{4} m}\right]=0, \\
& \varsigma_{0}\left[\tau I_{n_{3} m}\right]+\varsigma_{2}\left[\left(D_{1}(1-b) \otimes I_{n_{3}}\right)+\left(\left(\theta S_{1} \oplus D_{0}+\lambda D_{1}\right)-\eta I_{n_{3} m}\right)\right]+\varsigma_{17}\left[\varphi I_{n_{3} m}\right]=0, \\
& \varsigma_{1}\left[\tau I_{n_{4} m}\right]+\varsigma_{3}\left[\left(D_{1}(1-b) \otimes I_{n_{4}}\right)+\left(\left(\theta S_{2} \oplus D_{0}+\lambda D_{1}\right)-\eta I_{n_{4} m}\right)\right]+\varsigma_{18}\left[\varphi I_{n_{4} m}\right]=0, \\
& \varsigma_{2}\left[\beta_{1} \otimes \theta_{1} S_{1}^{0} \otimes I_{m}\right]+\varsigma_{3}\left[\beta_{1} \otimes \theta_{2} S_{2}^{0} \otimes I_{m}\right] \\
& \quad+\varsigma_{4}\left[\left(D_{1}(1-b) \otimes I_{s}\right)+\left(R \oplus\left(D_{0}+b D_{1}\right)-\eta I_{s m}\right)\right]+\varsigma_{19}\left[\varphi I_{s m}\right]=0, \\
& \varsigma_{0}\left[\alpha_{1} \otimes \eta I_{n_{3} m}\right]+\varsigma_{5}\left[\left(D_{1} \otimes I_{n_{1} n_{3}}\right)+\left(\left(T_{1} \oplus S_{1} \oplus D_{0}\right)-\tau I_{n_{1} n_{3} m}\right)+\left(p_{2}\left(T_{1}^{0} \alpha_{1}+S_{1}^{0} \beta_{1}\right) \otimes I_{n_{1} m}\right)\right] \\
& \quad+\varsigma_{6}\left[\beta_{1} \otimes \theta_{2} S_{2}^{0} \otimes I_{m n_{3}}\right]+\varsigma_{9}\left[\beta_{1} \otimes R^{0} \otimes I_{n_{3} m}\right]+\varsigma_{10}\left[\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{3} m}\right]=0, \\
& \varsigma_{1}\left[\alpha_{1} \otimes \eta I_{n_{4} m}\right]+\varsigma_{5}\left[\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{n_{4} m}\right] \\
& \quad+\varsigma_{6}\left[\left(D_{1} \otimes I_{n_{1} n_{4}}\right)+\left(\left(T_{1} \oplus S_{2} \oplus D_{0}\right)-\tau I_{n_{1} n_{4} m}\right)+\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{4} m}\right)\right] \\
& \quad+\varsigma_{11}\left[\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{4} m}\right]=0, \\
& \varsigma_{2}\left[\alpha_{1} \otimes \eta I_{n_{3} m}\right]+\varsigma_{5}\left[\tau I_{n_{1} n_{3} m}\right]+\varsigma_{7}\left[\left(D_{1} \otimes I_{n_{1} n_{3}}\right)+\left(T_{1} \oplus \theta_{1} S_{1} \oplus D_{0}\right)+\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{3} m}\right)\right] \\
& \quad+\varsigma_{12}\left[\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{3} m}\right]=0, \\
& \varsigma_{3}\left[\alpha_{1} \otimes \eta I_{n_{4} m}\right]+\varsigma_{6}\left[\tau I_{n_{1} n_{4} m}\right]+\varsigma_{8}\left[\left(D_{1} \otimes I_{n_{1} n_{4}}\right)+\left(T_{1} \oplus \theta_{2} S_{2} \oplus D_{0}\right)+\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{4} m}\right)\right] \\
& \quad+\varsigma_{13}\left[\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{4} m}\right]=0, \\
& \varsigma_{4}\left[\alpha_{1} \otimes \eta I_{s m}\right]+\varsigma_{7}\left[\gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{s m}\right]+\varsigma_{8}\left[\gamma \otimes \theta S_{2}^{0} \otimes I_{s m}\right] \\
& \quad+\varsigma_{9}\left[\left(D_{1} \otimes I_{n_{1} s}\right)+\left(T_{1} \oplus R \oplus D_{0}\right)+\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{s m}\right)\right]+\varsigma_{14}\left[\alpha_{1} \otimes T_{2}^{0} \otimes I_{s m}\right]=0,
\end{aligned}
$$

$$
\varsigma_{5}\left[\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{3} m}\right]+\varsigma_{10}\left[\left(D_{1} \otimes I_{n_{2} n_{3}}\right)+\left(\left(T_{2} \oplus S_{1} \oplus D_{0}\right)-\operatorname{tau}_{n_{2} n_{3} m}\right)+\left(\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{n_{3} m}\right]\right.
$$

$$
+\varsigma_{11}\left[\beta_{1} \otimes S_{2}^{0} \otimes I_{n_{3}}\right]+\varsigma_{14}\left[\beta_{1} \otimes R^{0} \otimes I_{n_{3} m}\right]=0
$$

$$
\varsigma_{6}\left[\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{4} m}\right]+\varsigma_{10}\left[\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{n_{4} m}\right]
$$

$$
+\varsigma_{11}\left[\left(D_{1} \otimes I_{n_{2} n_{4}}\right)+\left(\left(T_{2} \oplus S_{2} \oplus D_{0}\right)-\tau I_{n_{2} n_{4} m}\right)\right]=0
$$

$$
\varsigma_{7}\left[\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{3} m}\right]+\varsigma_{10}\left[\tau I_{n_{2} n_{3} m}\right]+\varsigma_{12}\left[\left(D_{1} \otimes I_{n_{2} n_{3}}\right)+\left(T_{2} \oplus \theta_{1} S_{1} \oplus D_{0}\right)\right]=0
$$

$$
\varsigma_{8}\left[\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{4} m}\right]+\varsigma_{11}\left[\tau I_{n_{2} n_{4} m}\right]+\varsigma_{13}\left[\left(D_{1} \otimes I_{n_{2} n_{4}}\right)+\left(T_{2} \oplus \theta_{2} S_{2} \oplus D_{0}\right)\right]=0
$$

$$
\varsigma_{9}\left[\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{s m}\right]+\varsigma_{12}\left[\gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{s m}\right]+\varsigma_{13}\left[\gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{s m}\right]
$$

$$
+\varsigma_{14}\left[\left(D_{1} \otimes I_{n_{2} s}\right)+\left(T_{2} \oplus R \oplus D_{0}\right)\right]=0
$$

$$
\varsigma_{15}\left[\left(D_{1} \otimes I_{n_{3}}\right)+\left(\left(S_{1} \oplus D_{0}\right)-(\varphi+\tau) I_{n_{3} m}\right)+\left(\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{m}\right)\right]+\varsigma_{16}\left[\beta_{1} \otimes S_{2}^{0} \otimes I_{m}\right]
$$

$$
+\varsigma_{19}\left[\beta_{1} \otimes R^{0} \otimes I_{m}\right]=0,
$$

$$
\varsigma_{15}\left[\beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{m}\right]+\varsigma_{16}\left[\left(D_{1} \otimes I_{n_{4}}\right)+\left(\left(S_{2} \oplus D_{0}\right)-(\phi+\tau) I_{n_{4} m}\right)\right]=0
$$

$$
\varsigma_{15}\left[\tau I_{n_{3} m}\right]+\varsigma_{17}\left[\left(D_{1}(1-b) \otimes I_{n_{3}}\right)+\left(\left(\theta_{1} S_{1} \oplus D_{0}+b D_{1}\right)-\varphi I_{n_{3} m}\right)\right]=0,
$$

$$
\varsigma_{16}\left[\tau I_{n_{4} m}\right]+\varsigma_{18}\left[\left(D_{1}(1-b) \otimes I_{n_{4}}\right)+\left(\left(\theta_{2} S_{2} \oplus D_{0}+b D_{1}\right)-\varphi I_{n_{4} m}\right)\right]=0,
$$

$$
\varsigma_{17}\left[\gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{m}\right]+\varsigma_{18}\left[\gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{m}\right]
$$

$$
+\varsigma_{19}\left[\left(D_{1}(1-b) \otimes I_{s}\right)+\left(\left(R \oplus\left(D_{0}+b D_{1}\right)\right)-\varphi I_{s m}\right)\right]=0
$$

subject to

$$
\begin{aligned}
& \varsigma_{0} e_{n_{3} m}+\varsigma_{1} e_{n_{4} m}+\varsigma_{2} e_{n_{3} m}+\varsigma_{3} e_{n_{4} m}+\varsigma_{4} e_{s m}+\varsigma_{5} e_{n_{1} n_{3} m}+\varsigma_{6} e_{n_{1} n_{4} m}+\varsigma_{7} e_{n_{1} n_{3} m}+\varsigma_{8} e_{n_{1} n_{4} m} \\
& \quad+\varsigma_{9} e_{n_{1} s m}+\varsigma_{10} e_{n_{2} n_{3} m}+\varsigma_{11} e_{n_{2} n_{4} m}+\varsigma_{12} e_{n_{2} n_{3} m}+\varsigma_{13} e_{n_{2} n_{4} m}+\varsigma_{14} e_{n_{2} s m}+\varsigma_{15} e_{n_{3} m}+\varsigma_{16} e_{n_{4} m} \\
& \quad+\varsigma_{17} e_{n_{3} m}+\varsigma_{18} e_{n_{4} m}+\varsigma_{19} e_{s m}=1
\end{aligned}
$$

The necessary and sufficient condition of a QBD process which satisfy the condition $\varsigma A_{0} e<\varsigma A_{2} e$ allow that system to stay in stable condition.

Therefore,

$$
\begin{aligned}
& {\left[\left(\varsigma_{0}+\varsigma_{15}\right)\left(D_{1} \otimes I_{n_{3}}\right)+\left(\varsigma_{1}+\varsigma_{16}\right)\left(D_{1} \otimes I_{n_{4}}\right)+\left(\varsigma_{2}+\varsigma_{17}\right)\left(D_{1}(1-b) \otimes I_{n_{3}}\right)\right.} \\
& \quad+\left(\varsigma_{3}+\varsigma_{18}\right)\left(D_{1}(1-b) \otimes I_{n_{4}}\right)+\left(\varsigma_{4}+\varsigma_{19}\right)\left(D_{1}(1-b) \otimes I_{s}\right)+\left(\varsigma_{5}+\varsigma_{7}\right)\left(D_{1} \otimes I_{n_{1} n_{3}}\right) \\
& \quad+\left(\varsigma_{6}+\varsigma_{8}\right)\left(D_{1} \otimes I_{n_{1} n_{4}}\right)+\varsigma_{9}\left(D_{1} \otimes I_{n_{1} s}\right)+\left(\varsigma_{10}+\varsigma_{12}\right)\left(D_{1} \otimes I_{n_{2} n_{3}}\right)+\left(\varsigma_{11}+\varsigma_{13}\right)\left(D_{1} \otimes I_{n_{2} n_{4}}\right) \\
& \left.\quad+\varsigma_{14}\left(D_{1} \otimes I_{n_{2} s}\right)\right] \\
& < \\
& {\left[\left(\varsigma_{0}+\varsigma_{15}\right)\left(\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{m}\right)+\left(\varsigma_{1}+\varsigma_{16}\right)\left(\beta_{1} \otimes S_{2}^{0} \otimes I_{m}\right)+\varsigma_{2}\left(\beta_{1} \otimes \theta_{1} S_{1}^{0} \otimes I_{m}\right)\right.} \\
& \quad+\varsigma_{3}\left(\beta_{1} \otimes \theta_{2} S_{2}^{0} \otimes I_{m}\right)+\varsigma_{5}\left(p_{2}\left(T_{1}^{0} \alpha_{1}+S_{1}^{0} \beta_{1}\right) \otimes I_{n_{1} m}\right)+\varsigma_{6}\left(\beta_{1} \otimes S_{2}^{0} \otimes I_{n_{3} m}\right) \\
& \quad+\left(\varsigma_{10}+\varsigma_{12}\right)\left(\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{3} m}\right)+\left(\varsigma_{6}+\varsigma_{8}\right)\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{4} m}\right)+\left(\varsigma_{11}+\varsigma_{13}\right)\left(\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{4} m}\right) \\
& \quad+\varsigma_{7}\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{n_{3} m}\right)+\left(\varsigma_{7}+\varsigma_{12}\right)\left(\gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{s m}\right)+\left(\varsigma_{8}+\varsigma_{13}\right)\left(\gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{s m}\right) \\
& \quad+\varsigma_{9}\left(\alpha_{1} \otimes p_{2} T_{1}^{0} \otimes I_{s m}\right)+\varsigma_{14}\left(\alpha_{1} \otimes T_{2}^{0} \otimes I_{s m}\right)+\varsigma_{10}\left(\beta_{1} \otimes p_{2} S_{1}^{0} \otimes I_{n_{3} m}\right) \\
& \left.\quad+\varsigma_{11}\left(\beta_{1} \otimes S_{2}^{0} \otimes I_{n_{3} m}\right)+\varsigma_{17}\left(\gamma \otimes \theta_{1} S_{1}^{0} \otimes I_{m}\right)+\varsigma_{18}\left(\gamma \otimes \theta_{2} S_{2}^{0} \otimes I_{m}\right)\right] .
\end{aligned}
$$

### 5.2. Analysis of Invariant Probability Vector

Let $X$ represents the infinitesimal generator matrix $Q$ and which is split by $X=\left(X_{0}, X_{1}, X_{2}, \ldots\right)$.
For $X_{0}$ is of dimension $2 m(1+s)$, $X_{1}$ is of dimension $\left(4 n_{3}+4 n_{4}+2 s+n_{1}+n_{1} s+n_{2}+n_{2} s\right) m$ and $X_{i}^{\prime} s$ are of dimension $\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right) m$, $i \geq 2$. The vector $X$ of $Q$ satisfies

$$
X Q=0, \quad X e=1
$$

After satisfying the stability criterion, use the below equation to find the invariant probability vector $X$,

$$
X_{i+1}=X_{2} R^{i-1}, \quad i=2,3, \ldots
$$

where $R$ is the solution of the matrix quadratic equation and this matrix $R$ is called rate matrix,

$$
R^{2} A_{2}+R A_{1}+A_{0}=0
$$

With the aid of succeeding equation we can find the vectors, namely, $X_{0}, X_{1}$ and $X_{2}$,

$$
X_{0} B_{00}+X_{1} B_{10}=0
$$

$$
\begin{gathered}
X_{0} B_{01}+X_{1} B_{11}+X_{2} B_{21}=0, \\
X_{1} B_{12}+X_{2}\left[A_{1}+R A_{2}\right]=0
\end{gathered}
$$

subject to normalizing condition

$$
X_{0} e_{0}+X_{1} e_{1}+X_{2}[I-R]^{-1} e_{2}=1
$$

Therefore, the rate matrix $R$ can be found with the help of logarithmic reduction algorithm, as described by Latouche and Ramaswami (1999).

## Logarithmic Reduction Algorithm

Step 1: $H \leftarrow\left(-A_{1}\right)^{-1} A_{0}, L \leftarrow\left(-A_{1}\right)^{-1} A_{2}, G=L$ and $T=H$.
Step 2: $U=H L+L H$
$M=H^{2}$
$H \leftarrow(I-U)^{-1} M$
$M \leftarrow L^{2}$
$L \leftarrow(I-U)^{-1} M$
$G \leftarrow G+T L$
$T \leftarrow T H$
Continue Step 1 until $\|e-G e\|_{\infty}<\epsilon$
Step 3: $R=-A_{0}\left(A_{1}+A_{0} G\right)^{-1}$.

## 6. Analysis of Active Period

- Under the busy period of $M A P / P H_{1}, P H_{2} / 2$ queuing model, we will understand the epoch of the time interval starts from a new arrival which find the empty system and ends when the system becomes empty again at the completion of service.
- A busy cycle which is defined by the initial passage time of the level between 1 and 0 and the time return to 0 level with minimum one visit to any other level.
- From level $i$ to $i-1, i=3,4, \ldots$ which is the initial passage time under the consideration of the Quasi birth and death process. The boundary states, namely, $i=0,1$ and 2 are dealt with separately.
- For all the levels $i$, where $i=2,3, \ldots$, we seen that there are $\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+\right.$ $\left.n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right) m$ states.


## Notations:

- Let $G_{j, j^{\prime}}(k, x)$ represent the conditional probability that the QBD process, starts in the state $(i, j)$ at time $t=0$ and ends up in the state $\left(i, j^{\prime}\right)$ by making meticulously k left jumps and obtaining both stages at the same period.
- The joint transform matrix is

$$
\tilde{G}_{j, j^{\prime}}(z, s)=\sum_{k=1}^{\infty} z^{k} \int_{0}^{\infty} e^{-s x} d G_{j, j^{\prime}}(k, x) ; \quad|z| \leq 1, \quad \operatorname{Re}(s) \geq 0 .
$$

- The matrix $\tilde{G}(z, s)=\tilde{G}_{j, j^{\prime}}(z, s)$. (Neuts (1981)
- Except for the boundary states, the matrix $G=G_{j j^{\prime}}=\tilde{G}(1,0)$ concerns the initial passage times.
- At time $t=0$, when returning from stage 2 to stage 1 , the conditional probability is described in the first return time and it is denoted as $G_{j j^{\prime}}^{(2,1)}(k, x)$.
- At time $t=0$, when returning from stage 1 to stage 0 , the conditional probability is described in the first return time and it is denoted as $G_{j j^{\prime}}^{(1,0)}(k, x)$.
- At time $t=0$, when returning to stage 0 , the conditional probability is described and it is denoted as $G_{j j^{\prime}}^{(0,0)}(k, x)$.
- At time $t=0$, let $S_{1 j}$ be the average initial passage time between the stages $i$ and $i-1$ and the process in the state $(i, j)$.
- At time $t=0$, let $S_{2 j}$ be the average number of customers who received service during the first passage process between levels $i$ and $i-1$, which begins in the state $(i, j)$.
- $\tilde{S}_{1}, \tilde{S}_{2}$ be the column vectors along with $S_{1 j}$ and $S_{2 j}$ as their entries, respectively.
- The average first return time between stage 2 and stage 1 is represented by $\tilde{S}_{1}^{(2,1)}$.
- In the first return period from stage 2 to the stage 1 , the expected number of services completed is represented by $\tilde{S}_{2}^{(2,1)}$.
- The average first return time between stage 1 and stage 0 is represented by $\tilde{S}_{1}^{(1,0)}$.
- In the first return period from stage 1 to stage 0 , the expected number of services completed is represented by $\tilde{S}_{2}^{(1,0)}$.
- The average first return time to stage 0 is represented by $\tilde{S}_{1}^{(0,0)}$.
- In the first return period to stage 0 , the expected number of services completed is represented by $\tilde{S}_{2}^{(0,0)}$.

We evaluate $\tilde{G}(z, s)$ matrix which satisfies the equation

$$
\tilde{G}(z, s)=z\left(s I-A_{1}\right)^{-1} A_{2}+\left(s I-A_{1}\right)^{-1} A_{0} \tilde{G}^{2}(z, s) .
$$

Once the rate matrix $R$ found, we can evaluate $G$ matrix by using logarithmic reduction algorithm method which is given by Latouche and Ramaswami (1999)

$$
G=-\left(A_{1}+R A_{2}\right)^{-1} A_{2}
$$

In the boundary states, namely 2,1 and 0 , it is represented by the equations $\tilde{G}^{(2,1)}(z, s), \tilde{G}^{(1,0)}(z, s)$ and $\tilde{G}^{(0,0)}(z, s)$.

$$
\begin{aligned}
& \tilde{G}^{(2,1)}(z, s)=z\left(s I-A_{1}\right)^{-1} B_{21}+\left(s I-A_{1}\right)^{-1} A_{0} \tilde{G}(z, s) \tilde{G}^{(2,1)}(z, s), \\
& \tilde{G}^{(1,0)}(z, s)=z\left(s I-B_{11}\right)^{-1} B_{10}+\left(s I-B_{11}\right)^{-1} B_{12} \tilde{G}^{(2,1)}(z, s) \tilde{G}^{(1,0)}(z, s), \\
& \tilde{G}^{(0,0)}(z, s)=\left(s I-B_{00}\right)^{-1} B_{01} \tilde{G}^{(1,0)}(z, s) .
\end{aligned}
$$

since $G, \tilde{G}^{2,1}(1,0), \tilde{G}^{(1,0)}(1,0)$ and $\tilde{G}^{(0,0)}(1,0)$ are all stochastic matrices.
The instants can be calculated as follows:

$$
\tilde{S}_{1}=-\left.\frac{\partial \tilde{G}(z, s)}{\partial s}\right|_{s=0, z=1} e=-\left[A_{0}(G+I)+A_{1}\right]^{-1} e
$$

$$
\begin{aligned}
& \left.\tilde{S}_{2}=\left.\frac{\partial \tilde{G}(z, s)}{\partial z}\right|_{s=0, z=1} e=-\left[A_{0}(G+I)+A_{1}\right]^{-1}\right] A_{2} e, \\
& \tilde{S}_{1}^{(2,1)}=-\left.\frac{\partial \tilde{G}^{(2,1)}(z, s)}{\partial s}\right|_{s=0, z=1} e=-\left[A_{1}+A_{0} G\right]^{-1}\left[e+A_{0} \tilde{S}_{1}\right], \\
& \tilde{S}_{2}^{(2,1)}=\left.\frac{\partial \tilde{G}^{(2,1)}(z, s)}{\partial z}\right|_{s=0, z=1} e=-\left[A_{1}+A_{0} G\right]^{-1}\left[B_{21} e+A_{0} \tilde{S}_{2}\right], \\
& \tilde{S}_{1}^{(1,0)}=-\left.\frac{\partial \tilde{G}^{(1,0)}(z, s)}{\partial s}\right|_{s=0, z=1}=-\left[B_{11}+B_{12} \tilde{G}^{(2,1)}(1,0)\right]^{-1}\left[e+B_{12} \tilde{S}_{1}^{(2,1)}\right], \\
& \tilde{S}_{2}^{(1,0)}=\left.\frac{\partial \tilde{G}^{(1,0)}(z, s)}{\partial z}\right|_{s=0, z=1} e=-\left[B_{12} \tilde{G}^{(2,1)}(1,0)+B_{11}\right]^{-1}\left[B_{12} \tilde{S}_{2}^{(2,1)}+B_{10} e\right], \\
& \tilde{S}_{1}^{(0,0)}=-\left.\frac{\partial \tilde{G}^{(0,0)}(z, s)}{\partial s}\right|_{s=0, z=1} e=-B_{00}^{-1}\left[B_{01} \tilde{S}_{1}^{(1,0)}+e\right], \\
& \tilde{S}_{2}^{(0,0)}=\left.\frac{\partial \tilde{G}^{(0,0)}(z, s)}{\partial z}\right|_{s=0, z=1} e=-B_{00}^{-1}\left[B_{01} \tilde{S}_{2}^{(1,0)}\right] .
\end{aligned}
$$

## 7. System Performance Measures

- Probability of server- 1 is busy with normal service:

$$
P_{B N S 1}=X_{1}\left(e_{1}(6)+e_{1}(7)\right)+X_{2}(I-R)^{-1}\left(e_{2}(6)+e_{2}(7)+e_{2}(8)+e_{2}(9)+e_{2}(10)\right) .
$$

- Probability of server- 1 is with in optional service:

$$
P_{B O S 1}=X_{1}\left(e_{1}(8)+e_{1}(9)\right)+X_{2}(I-R)^{-1}\left(e_{2}(11)+e_{2}(12)+e_{2}(13)+e_{2}(14)+e_{2}(15)\right) .
$$

- Probability of server-2 is busy with normal service:

$$
P_{B N S 2}=X_{1}\left(e_{1}(1)+e_{1}(10)\right)+X_{2}(I-R)^{-1}\left(e_{2}(1)+e_{2}(6)+e_{2}(11)+e_{2}(16)\right) .
$$

- Probability of server-2 is busy with optional service:

$$
P_{B O S 2}=X_{1}\left(e_{1}(2)+e_{1}(11)\right)+X_{2}(I-R)^{-1}\left(e_{2}(2)+e_{2}(7)+e_{2}(12)+e_{2}(17)\right) .
$$

- Probability of server-1 is in vacation:

$$
\begin{aligned}
P_{V 1}= & X_{0}\left(e_{0}(1)+e_{0}(2)\right)+X_{1}\left(e_{1}(6)+e_{1}(7)\right)+X_{2}(I-R)^{-1}\left(e_{2}(6)+e_{2}(7)\right. \\
& \left.+e_{2}(8)+e_{2}(9)+e_{2}(10)\right) .
\end{aligned}
$$

- Probability of server-1 is in closedown:

$$
\begin{array}{r}
P_{C D 1}=X_{0}\left(e_{0}(3)+e_{0}(4)\right)+X_{1}\left(e_{1}(10)+e_{1}(11)+e_{1}(12)+e_{1}(13)+e_{1}(14)\right) \\
+X_{2}(I-R)^{-1}\left(e_{2}(16)+e_{2}(17)+e_{2}(18)+e_{2}(19)+e_{2}(20)\right) .
\end{array}
$$

- Probability of server-2 to be idle:

$$
P_{\text {Idle } 2}=X_{0}\left(e_{0}(1)+e_{0}(3)\right)+X_{1}\left(e_{1}(6)+e_{1}(8)\right) .
$$

- Probability of server-2 is busy in working breakdown with normal service:

$$
P_{W B N S 2}=X_{1}\left(e_{1}(3)+e_{1}(12)\right)+X_{2}(I-R)^{-1}\left(e_{2}(8)+e_{2}(7)+e_{2}(13)+e_{2}(18)\right) .
$$

- Probability of server-2 is busy in working breakdown with optional service:

$$
P_{W B O S 2}=X_{1}\left(e_{1}(4)+e_{1}(13)\right)+X_{2}(I-R)^{-1}\left(e_{2}(4)+e_{2}(9)+e_{2}(14)+e_{2}(19)\right) .
$$

- Probability of server-2 is under repair state:

$$
\begin{aligned}
P_{R 2}= & X_{0}\left(e_{0}(2)+e_{0}(4)\right)+X_{1}\left(e_{1}(5)+e_{1}(7)+e_{1}(9)+e_{1}(14)\right) \\
& +X_{2}(I-R)^{-1}\left(e_{2}(5)+e_{2}(10)+e_{2}(15)+e_{2}(20)\right) .
\end{aligned}
$$

- Expected size of system

$$
E_{S y s}=\sum_{z=1}^{\infty} z x_{z} e=X_{1} e_{1}+X_{2}\left(2(I-R)^{-1}+R(I-R)^{-2}\right) e_{2}
$$

## 8. Analysis of Cost Model

In this section, we introduce a cost function TAC (fixed cost) with the following assumption:

- TAC - Total cost per unit time.
- $C_{H}$ - Holding cost of each customer in the system.
- $C_{B N S 1}$ - Unit time cost of service that server-1 is busy with normal service.
- $C_{B O S 1}$ - Unit time cost of service that server-1 is busy with optional service.
- $C_{B N S 2}$ - Unit time cost of service that server-2 is busy with normal service.
- $C_{B O S 2}$ - Unit time cost of service when server-2 is busy with optional service.
- $C_{I d l e 2}$ - Unit time cost that occurred server-2 is in idle.
- $C_{W B N S 2}$ - Unit time cost that the server-2 is busy during working breakdown in normal service.
- $C_{W B O S 2}$ - Unit time cost that the server-2 is busy during working breakdown in optional service.
- $C_{C D 1}$ - Unit time cost that the server-1 is in close down.
- $C_{V 1}$ - Unit time cost that the server- 1 is on vacation.
- $C_{R 2}$ - Unit time cost that the server-2 is under PH repair due to server caused by breakdown.
- $C_{1}$ - Cost obtained by the server-1 for providing normal service.
- $C_{2}$ - Cost obtained by the server-1 for providing optional service.
- $C_{3}$ - Cost obtained by the server-2 for providing normal service.
- $C_{4}$ - Cost obtained by the server-2 for providing optional service.
- $C_{5}$ - Cost obtained by the server-2 is busy with working breakdown in normal service.
- $C_{6}$ - Cost obtained by the server- 2 is busy with working breakdown in optional service.
- $C_{7}$ - Cost obtained in carrying out breakdown process by server- 2 .
- $C_{8}$ - Cost obtained in carrying out the closedown process by server-1.
- $C_{9}$ - Cost obtained by the server carrying out the repair process.

The total average cost per unit time is given by

$$
\begin{aligned}
& T A C=C_{H} E_{S Y S}+C_{B N S 1} P_{B N S 1}+C_{B O S 1} P_{B O S 1}+C_{B N S 2} P_{B N S 2}+C_{B O S 2} P_{B O S 2} \\
& +C_{I d l e 2} P_{I d l e 2}+C_{W B N S 2} P_{W B N S 2}+C_{W B O S 2} P_{W B O S 2}+C_{C D 1} P_{C D 1}+C_{V 1} P_{V 1} \\
& +C_{R 2} P_{R 2}+\mu_{1} C_{1}+\mu_{2} C_{2}+\mu_{3} C_{3}+\mu_{4} C_{4}+\theta \mu_{3} C_{5}+\theta \mu_{4} C_{6}+\tau C_{7}+\phi C_{8}+\sigma C_{9}
\end{aligned}
$$

## 9. Investigation of Waiting Time Distribution

We explore the waiting time distribution is described as the customers enters into the empty system. Let $\mathrm{W}(\mathrm{t})$ be the waiting time distribution function, which takes into account new customers joining the queue. If the server is idle when a customer arrives, they will get service immediately(absorption time); otherwise, if the server is busy or on vacation, they will have to wait in a queue to receive service from the server. While the busy time, repair time, close down time or in vacation time of server, the customers expect the service from server after waiting some amount of time in the queue.

In a CTMC, with the state space

$$
\tilde{\Omega}=\{\bar{*}\} \cup\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \ldots\},
$$

provides the absorption time where

$$
\begin{gathered}
\overline{0}=\left\{\left(0, j_{1}, 5, b\right): j_{1}=0,3,1 \leq b \leq s\right\}, \\
\overline{1}=\left\{\left(1,0, j_{2}, k_{3}\right): j_{2}=1,3,1 \leq k_{3} \leq n_{3}\right\} \cup\left\{\left(1,0, j_{2}, k_{4}\right): j_{2}=2,4,1 \leq k_{4} \leq n_{4}\right\} \\
\cup\{(1,0,5, b): 1 \leq b \leq s\} \cup\left\{\left(1,1,5, k_{1}, b\right): 1 \leq k_{1} \leq n_{1}, 1 \leq b \leq s\right\} \\
\cup\left\{\left(1,2,5, k_{2}, b\right): 1 \leq k_{2} \leq n_{2}, 1 \leq b \leq s\right\} \cup\left\{\left(1,3, j_{2}, k_{3}\right): j_{2}=1,3,1 \leq k_{3} \leq n_{3}\right\} \\
\cup\left\{\left(1,3, j_{2}, k_{4}\right): j_{2}=2,4,1 \leq k_{4} \leq n_{4}\right\} \cup\{(1,3,5, b): 1 \leq b \leq s\},
\end{gathered}
$$

and for $\bar{i} \geq 2$,

$$
\begin{aligned}
\bar{i} & =\left\{\left(i, 0, j_{2}, k_{3}\right): i \in \mathbb{Z}^{+}, j_{2}=1,3,1 \leq k_{3} \leq n_{3}\right\} \\
& \cup\left\{\left(i, 0, j_{2}, k_{4}\right): i \in \mathbb{Z}^{+}, j_{2}=2,4,1 \leq k_{4} \leq n_{4}\right\} \\
& \cup\left\{(i, 0,5, b): i \in \mathbb{Z}^{+}, 1 \leq b \leq s\right\} \\
& \cup\left\{\left(i, 0, j_{2}, k_{3}\right): i \in \mathbb{Z}^{+}, j_{2}=1,3,1 \leq k_{3} \leq n_{3}\right\} \\
& \cup\left\{\left(i, 1, j_{2}, k_{1}, k_{3}\right): i \in \mathbb{Z}^{+}, j_{2} 1,3,1 \leq k_{1} \leq n_{1}, 1 \leq k_{3} \leq n_{3}\right\} \\
& \cup\left\{\left(i, 1, j_{2}, k_{1}, k_{4}\right): i \epsilon \mathbb{Z}^{+}, j_{2}=2,4,1 \leq k_{1} \leq n_{1}, 1 \leq k_{4} \leq n_{4}\right\} \\
& \cup\left\{\left(i, 1,5, k_{1}, b\right): i \in \mathbb{Z}^{+}, 1 \leq k_{1}, \leq n_{1}, 1 \leq k_{3} \leq n_{3}\right\} \\
& \cup\left\{\left(i, 2, j_{2}, k_{2}, k_{3}\right): i \epsilon \mathbb{Z}^{+}, j_{2}=1,3,1 \leq k_{2} \leq n_{2}, 1 \leq k_{3} \leq n_{3}\right\} \\
& \cup\left\{\left(i, 2, j_{2}, k_{2}, k_{4}\right): i \epsilon \mathbb{Z}^{+}, j_{2}=2,4,1 \leq k_{2} \leq n_{2}, 1 \leq k_{4} \leq n_{4}\right\} \\
& \cup\left\{\left(i, 2,5, k_{2}, b\right): i \in \mathbb{Z}^{+}, 1 \leq k_{2} \leq n_{2}, 1 \leq b \leq s\right\} \\
& \cup\left\{\left(i, 3, j_{2}, k_{3}\right): i \in \mathbb{Z}^{+}, j_{2}=1,3,1 \leq k_{3} \leq n_{3}\right\} \\
& \cup\left\{\left(i, 3, j_{2}, k_{4}\right): i \in \mathbb{Z}^{+}, j_{2}=2,4,1 \leq k_{4} \leq n_{4}\right\} \\
& \cup\left\{(i, 3,5, b): i \in \mathbb{Z}^{+}, 1 \leq b \leq s\right\} .
\end{aligned}
$$

Let $(*)$ represent the state space of an absorbing continuous time Markov chain and the arriving tagged customers will start to take the service before waiting in the queue.

The state space $(*)$ obtained from the states which have at least one idle server at the time of arrival is as follows:

$$
\bar{*}=\left\{\left(0, j_{1}, 0\right): j_{1}=0,3\right\} \cup\left\{\left(1,1,0, k_{1}\right): 1 \leq k_{1} \leq n_{1}\right\} \cup\left\{\left(1,2,0, k_{2}\right): 1 \leq k_{2} \leq n_{2}\right\} .
$$

Let $\tilde{Q}$ be the transition matrix of the absorbing Markov chain as follows:

$$
\tilde{Q}=\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
M_{0} & P_{0} & 0 & 0 & 0 & 0 & \ldots \\
M_{1} & P_{10} & P_{1} & 0 & 0 & 0 & \ldots \\
M_{2} & 0 & P_{21} & P & 0 & 0 & \ldots \\
0 & 0 & 0 & P_{2} & P & 0 & \ldots \\
0 & 0 & 0 & 0 & P_{2} & P & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ldots
\end{array}\right] .
$$

The $\tilde{Q}$ entries are as follows:
$M_{0}=\left[\begin{array}{l}R^{0} \\ R^{0}\end{array}\right] ; P_{0}=\left[\begin{array}{cc}R & 0 \\ \varphi I_{s} R-\varphi I_{s}\end{array}\right] ; M_{1}=\left[\begin{array}{c}p_{2} S_{1}^{0} \\ S_{2}^{0} \\ 0 \\ 0 \\ 0 \\ R^{0} \otimes e_{n_{1}} \\ R^{0} \otimes e_{n_{2}} \\ p_{2} S_{1}^{0} \\ S_{2}^{0} \\ 0 \\ 0 \\ 0\end{array}\right] ; P_{10}=\left[\begin{array}{cc}0 & 0 \\ 0 & 0 \\ \gamma \otimes \theta_{1} S_{1}^{0} & 0 \\ \gamma \otimes \theta_{2} S_{2}^{0} & 0 \\ 0 & 0 \\ 0 & I_{n_{1}} \otimes p_{2} T_{1}^{0} \\ 0 & I_{n_{2}} \otimes T_{2}^{0} \\ 0 & 0 \\ 0 & 0 \\ 0 & \gamma \otimes \theta_{1} S_{1}^{0} \\ 0 & \gamma \otimes \theta_{2} S_{2}^{0} \\ 0 & 0\end{array}\right] ;$
$P_{1}=\left[\begin{array}{cccc}P_{11}^{1} & P_{12}^{1} & 0 & 0 \\ 0 & P_{22}^{1} & P_{23}^{1} & 0 \\ 0 & 0 & P_{33}^{1} & 0 \\ P_{41}^{1} & 0 & 0 & P_{44}^{1}\end{array}\right] ;$ where $P_{22}^{1}=T_{1} \oplus R, P_{23}^{1}=\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{1}}, P_{33}^{1}=T_{2} \oplus R$.
$P_{11}^{1}=\left[\begin{array}{ccccc}S_{1}-\tau I_{n_{3}} & \beta_{2} \otimes p_{1} S_{1}^{0} & \tau I_{n_{3}} & 0 & 0 \\ 0 & S_{2}-\tau I_{n_{4}} & 0 \tau I_{n_{3}} & 0 & \\ 0 & 0 & \theta_{1} S_{1} & 0 & 0 \\ 0 & 0 & 0 & \theta_{2} S_{2} & 0 \\ \beta_{1} \otimes R^{0} & 0 & 0 & 0 & R-\eta I_{s}\end{array}\right] ; P_{12}^{1}=\left[\begin{array}{c}0 \\ 0 \\ 0 \\ 0 \\ \alpha_{1} \otimes \eta I_{s}\end{array}\right] ;$
$P_{41}^{1}=\operatorname{diag}\left[p_{i j}\right]$, where $p_{11}=p_{33}=\phi I_{n_{3}}, p_{22}=p_{44}=\varphi I_{n_{4}}, p_{55}=\varphi I_{s}$.
$M_{2}=\left[m_{i 1}\right]$, of order $(20 \times 1)$ where $m_{61}=q\left(T_{1}^{0}+S_{1}^{0}\right) \otimes e_{n_{3}}, m_{71}=e_{n_{4}} \otimes S_{2}^{0}$, $m_{111}=e_{n_{3}} \otimes p_{2} S_{1}^{0}, m_{121}=e_{n_{4}} \otimes S_{2}^{0}$ and the remaining elements are zero.

$$
\begin{aligned}
& P_{24}^{21}=\left[\begin{array}{lcccc}
0 & 0 & 0 & 0 & 0 \\
0 & I_{n_{4}} \otimes p_{2} T_{1}^{0} & 0 & 0 & 0 \\
0 & 0 & I_{n_{3}} \otimes p_{2} T_{1}^{0} & 0 & 0 \\
0 & 0 & 0 & I_{n_{4}} \otimes p_{2} T_{1}^{0} & 0 \\
0 & 0 & 0 & 0 & I_{s} \otimes p_{2} T_{1}^{0}
\end{array}\right] ;
\end{aligned}
$$

$$
\begin{aligned}
& P_{32}^{21}=\left[\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\alpha_{1} \otimes T_{2}^{0} \otimes I_{n_{2}}
\end{array}\right] ; P_{34}^{21}=\left[\begin{array}{cccccc}
I_{n_{3}} \otimes T_{2}^{0} & 0 & 0 & 0 & 0 \\
0 & I_{n_{4}} \otimes T_{2}^{0} & 0 & 0 & 0 \\
0 & 0 & I_{n_{3}} \otimes T_{2}^{0} & 0 & 0 \\
0 & 0 & 0 & I_{n_{4}} \otimes T_{2}^{0} & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right] ; \\
& P_{11}^{21}=P_{44}^{21}, P_{22}^{21}=P_{33}^{21}, P=\left[\begin{array}{cccc}
P_{11} & P_{12} & 0 & 0 \\
0 & P_{22} & P_{23} & 0 \\
0 & 0 & P_{33} & 0 \\
P_{41} & 0 & 0 & P_{44}
\end{array}\right] ; \\
& P_{11}=\left[\begin{array}{ccccc}
S_{1}-(\tau+\eta) I_{n_{3}} & \beta_{2} \otimes p_{1} S_{1}^{0} & \tau I_{n_{3}} & 0 & 0 \\
0 & S_{2}-(\tau+\eta) I_{n_{4}} & 0 & \tau I_{n_{4}} & 0 \\
0 & 0 & \theta_{1} S_{1}-\eta I_{n_{3}} & 0 & 0 \\
0 & 0 & 0 & \theta_{2} S_{2}-\eta I_{n_{4}} & 0 \\
\beta_{1} \otimes R^{0} & 0 & 0 & 0 & R-\eta I_{s}
\end{array}\right] ; \\
& P_{12}=\left[\begin{array}{ccccc}
\alpha_{1} \otimes \eta I_{n_{3}} & 0 & 0 & 0 & 0 \\
0 & \alpha_{1} \otimes \eta I_{n_{4}} & 0 & 0 & 0 \\
0 & 0 & \alpha_{1} \otimes \eta I_{n_{3}} & 0 & 0 \\
0 & 0 & 0 & \alpha_{1} \otimes \eta I_{n_{4}} & 0 \\
0 & 0 & 0 & 0 & \alpha_{1} \otimes \eta I_{s m}
\end{array}\right] ; \\
& P_{22}=\left[\begin{array}{ccccc}
\left(T_{1}+S_{1}\right)-\tau I_{n_{1} n_{3}} & \beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{n_{4}} & \tau I_{n_{1} n_{3}} & 0 & 0 \\
0 & \left(T_{1}+S_{2}\right)-\tau I_{n_{1} n_{4}} & 0 & \tau I_{n_{1} n_{4}} & 0 \\
0 & 0 & T_{1}+\theta_{1} S_{1} & 0 & 0 \\
0 & 0 & 0 & T_{1}+\theta_{2} S_{2} & 0 \\
\beta_{1} \otimes R^{0} \otimes I_{n_{3}} & 0 & 0 & 0 & T_{1}+R
\end{array}\right] ; \\
& P_{23}=\left[\begin{array}{ccccc}
\alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{3}} & 0 & 0 & 0 & 0 \\
0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{4}} & 0 & 0 & 0 \\
0 & 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{3}} & 0 & 0 \\
0 & 0 & 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{n_{4}} & 0 \\
0 & 0 & 0 & 0 & \alpha_{2} \otimes p_{1} T_{1}^{0} \otimes I_{s}
\end{array}\right] ; \\
& P_{33}=\left[\begin{array}{ccccc}
\left(T_{2}+S_{1}\right)-\tau I_{n_{2} n_{3}} & \beta_{2} \otimes p_{1} S_{1}^{0} \otimes I_{n_{4}} & \tau I_{n_{2} n_{3}} & 0 & 0 \\
0 & \left(T_{2}+S_{2}\right)-\tau I_{n_{2} n_{4}} & 0 & \tau I_{n_{2} n_{4}} & 0 \\
0 & 0 & T_{2}+\theta_{1} S_{1} & 0 & 0 \\
0 & 0 & 0 & T_{2}+\theta_{2} S_{2} & 0 \\
\beta_{1} \otimes R^{0} \otimes I_{n_{3}} & 0 & 0 & 0 & T_{2}+R
\end{array}\right] ;
\end{aligned}
$$



Let us define $Z(0)=\left(Z_{0}(0), Z_{1}(0), Z_{2}(0), \ldots\right)$ to the conditional probability distribution for the system state defined on the arrival of the tagged clients and probability vector of $Z_{0}(0), Z_{1}(0)$. The vector $Z_{0}(0)$ and $Z_{1}(0)$ may further partitioned as $Z_{0}(0)=\left(Z_{02}, Z_{04}\right)$ and
$Z_{1}(0)=\left(Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{17}, Z_{19}, Z_{110}, Z_{111}, Z_{112}, Z_{113}, Z_{114}\right)$. Then $Z_{i}(0)$ as follows

$$
\begin{aligned}
& Z_{0}(0)=X_{0}\left[I_{2 s} \otimes \frac{D_{1} e_{m}}{\lambda}\right] \\
& Z_{1} 0=X_{1}\left[I_{4 n_{3}+4 n_{4}+2 s+n_{1} s+n_{2} s} \otimes \frac{D_{1} e_{m}}{\lambda}\right], \\
& Z_{i}(0)=X_{i}\left[I_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right.} \otimes \frac{D_{1} e_{m}}{\lambda}\right], \quad \text { for } i \geq 2
\end{aligned}
$$

where $\lambda$ as the basic arrival rate of the $M A P$.
Define

$$
Z(t)=\left(Z_{*}(t), Z_{0}(t), Z_{1}(t), \ldots\right)
$$

where
$Z_{0}(t)$ - vector of order $(1 \times 2 s)$,
$Z_{1}(t)-$ vector of order $\left(1 \times\left(4 n_{3}+4 n_{4}+2 s+n_{1} s+n_{2} s\right)\right)$,
$Z_{i}(t), i \geq 2-$ vector of order $\left(1 \times\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right)\right)$.
The entries of $Z_{i}(t)$ indicates the probabilities of the continuous time Markov chain in which the respective states of level $i$ with the generator matrix $\tilde{Q}$ at epoch $t$. The probability of tagged customer is in the absorbing state at epoch $t$ is represented by $Z_{*}(t)$.

Clearly, $W(t)=Z_{*}(t)$, for $t \geq 0$.
The differential equation $Z^{\prime}(t)=Z(t) \tilde{Q}$, where $t \geq 0$, becomes

$$
\begin{aligned}
& Z_{*}^{\prime}(t)=Z_{0}(t) M_{0}+Z_{1}(t) M_{1}+Z_{2}(t) M_{2} \\
& Z_{0}^{\prime}(t)=Z_{0}(t) P_{0}+z_{1}(t) P_{10} \\
& Z_{1}^{\prime}(t)=Z_{1}(t) P_{1}+z_{2}(t) P_{21} \\
& Z_{i}^{\prime}(t)=Z_{i}(t) P+Z_{i+1}(t) P_{2}, \text { for } i \geq 2
\end{aligned}
$$

the derivative with respect to $t$ is represented by ${ }^{\prime}$.
Laplace Stieltjes Transform (LST) for $W(t)$ is found with the technique was provided by Neuts and Lucantoni (1979). In commencing the process at the state $i$ as initial probability vector with $Z_{i}(0)$, $i \geq 2$. Let $w(s)$ as the row vector which specifies the LST of the first passage process to stage 1 .

As indicated by Neuts (1979), we get

$$
\begin{equation*}
w(s)=\sum_{i=2}^{\infty} Z_{i}(0)\left[(s I-P)^{-1} P_{2}\right]^{i-2} . \tag{1}
\end{equation*}
$$

Let the LST of absorbing time to the state $\left(^{*}\right)$ correspond to the process starting at state level
$i=0,1,2$ as indicated by $\phi(i, s)$. As a result in Neuts (1979), we have

$$
\begin{align*}
\phi(0, s) & =\left[s I-P_{0}\right]^{-1} M_{0}  \tag{2}\\
\phi(1, s) & =\left[s I-P_{1}\right]^{-1} P_{10} \phi(0, s)+\left[s I-P_{1}\right]^{-1} M_{1}  \tag{3}\\
\phi(2, s) & =[s I-P]^{-1} P_{21} \phi(1, s)+[s I-P]^{-1} M_{2} \tag{4}
\end{align*}
$$

Thus, we evaluated the LST of the distribution of its waiting time $\tilde{W}(s)$, provided by

$$
\begin{equation*}
\tilde{W}(s)=Z_{0}(0) \phi(0, s)+Z_{1}(0) \phi(1, s)+w(s) \phi(2, s) . \tag{5}
\end{equation*}
$$

### 9.1. Expected Waiting Time

The expected waiting time is denoted by

$$
\begin{equation*}
E(W)=-Z_{0}(0) \phi^{\prime}(0,0)-Z_{1}(0) \phi^{\prime}(1,0)-w^{\prime}(0) e-w(0) \phi^{\prime}(2,0) \tag{6}
\end{equation*}
$$

On the point that the system has the state levels $i=0,1$, then the mean time to enter into the absorbing state $(*)$ is represented by the foremost terms of the following equation.

Besides, if the system has the state level $i \geq 2$, the average amount of time to enter into the absorbing state $(*)$ is represented by the end two terms of the preceding equation.

On differentiating (2), (3) and (4), we substitute $s=0$ and will get

$$
\begin{align*}
& \phi^{\prime}(0,0)=(-1)\left[-P_{0}\right]^{-2} M_{0},  \tag{7}\\
& \phi^{\prime}(1,0)=(-1)\left[-P_{1}\right]^{-2} P_{10} \phi(0,0)+\left[-P_{1}\right]^{-1} P_{10} \phi^{\prime}(0,0)-\left[-P_{1}\right]^{-2} M_{1},  \tag{8}\\
& \phi^{\prime}(2,0)=(-1)[-P]^{-2} P_{21} \phi(1,0)+[-P]^{-1} P_{21} \phi^{\prime}(1,0)-[-P]^{-2} M_{2} . \tag{9}
\end{align*}
$$

By use the expression (7) and (8) together with the primary condition

$$
Z(0)=\left(Z_{0}(0), Z_{1}(0), Z_{2}(0), \ldots\right),
$$

we can determine the first two terms of (6). From (1), we have

$$
\begin{equation*}
w(0)=\sum_{i=2}^{\infty} Z_{i}(0) V^{i-2} \tag{10}
\end{equation*}
$$

where $V=[-P]^{-1} P_{2}$. Since V indicate a stochastic matrix, we get

$$
\begin{equation*}
w(0) e=1-Z_{0}(0) e_{0}-Z_{1}(0) e_{1} . \tag{11}
\end{equation*}
$$

With the help of (9) and (10) together with the primary condition $Z(0)=\left(Z_{0}(0), Z_{1}(0), Z_{2}(0), \ldots\right)$, we can compute the final term of (6).

On differentiating (1) with respect to $s$ and substituting $s=0$, we get

$$
\begin{equation*}
w^{\prime}(0)=(-1) \sum_{i=1}^{\infty} Z_{2+i}(0) \sum_{j=0}^{i-1} V^{j}[-P]^{-1} V^{i-j} \tag{12}
\end{equation*}
$$

Since $V$ is a stochastic in nature, we get

$$
\begin{equation*}
(-1) w^{\prime}(0) e=(-1) \sum_{i=1}^{\infty} Z_{2+i}(0) \sum_{j=0}^{i-1} V^{j}[-P]^{-1} e \tag{13}
\end{equation*}
$$

The value of $(-1) w^{\prime}(0) e$ can be evaluated with the help of the method mentioned by Neuts (1979). Let us take $V_{2}$, a stochastic matrix which is satisfying two conditions, namely, $I-V+V_{2}$ is non-singular and the generalized inverse is of the form $I-V$. Then, the matrix $V_{2}$ may be chosen as $V_{2}=v_{0} e$, where $v_{0}$ is the invariant probability vector of $V$ such that

$$
v_{0} V=v_{0} \text { and } v_{0} e=1
$$

in view of the property that

$$
V V_{2}=V_{2} V=V_{2} .
$$

Therefore, we get,

$$
\begin{equation*}
\sum_{j=0}^{i-1} V^{j}\left(I-V+V_{2}\right)=I-V^{i}+i V_{2}, \quad \text { for } i \geq 1 \tag{14}
\end{equation*}
$$

Substituting (14) in (13) and carrying out some simplifications, we get

$$
\begin{align*}
& (-1) w^{\prime}(0) e_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right)}= \\
& \quad\left[x_{2}[I-R]^{-1}\left[I_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right)} \otimes \frac{D_{1} e_{m}}{\lambda}\right]\right.  \tag{15}\\
& \quad-w(0)+x_{2} R[I-R]^{-2}\left[I_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right)} \otimes \frac{D_{1} e_{m}}{\lambda} V_{2}\right] \\
& \quad \times\left[I-V+V_{2}\right]^{-1}[-P]^{-1} e_{\left(4 n_{3}+4 n_{4}+2 s+2 n_{1} n_{3}+2 n_{1} n_{4}+n_{1} s+2 n_{2} n_{3}+2 n_{2} n_{4}+n_{2} s\right)}
\end{align*}
$$

Thus, we have formed all the terms of (6). The expected waiting time can be assess easily by using (6).

## 10. Numerical Results

From this part, we determine the results of our system by representing numerically and graphically. The representations of $M A P$ are distinct with the following variance and correlation structures and their mean values are 1 . The arrival process like $E R L-A, E X P-A$ and $H Y P-E X P-A$ accord with renewal process and their correlation is zero. This values are taken from Chakravarthy (2011).

## Erlang of order 2 (ERL-A):

$$
D_{0}=\left[\begin{array}{cc}
-2 & 2 \\
0 & -2
\end{array}\right] ; \quad D_{1}=\left[\begin{array}{cc}
0 & 0 \\
2 & 0
\end{array}\right] ;
$$

## Exponential (EXP-A):

$$
D_{0}=[-1] ; \quad D_{1}=[1] ;
$$

## Hyper exponential (HYP-EXP-A):

$$
D_{0}=\left[\begin{array}{cc}
-1.90 & 0 \\
0 & -0.19
\end{array}\right] ; \quad D_{1}=\left[\begin{array}{cc}
1.710 & 0.190 \\
0.171 & 0.019
\end{array}\right]
$$

Let us consider the service and repair process as PH-distributions and these values are incurred from Chakravarthy (2011) which are as follows:

ERL-S (Erlang of order 2):

$$
\alpha_{1}=\alpha_{2}=(1,0) ; \quad T_{1}=T_{2}=\left[\begin{array}{cc}
-2 & 2 \\
0 & -2
\end{array}\right] ;
$$

ERL-R (Erlang of order 2):

$$
\beta_{1}=\beta_{2}=(1,0) ; \quad S_{1}=S_{2}=\left[\begin{array}{cc}
-2 & 2 \\
0 & -2
\end{array}\right]
$$

EXP-S (Service in Exponential):

$$
\alpha_{1}=\alpha_{2}=(1) ; \quad T_{1}=T_{2}=[-1] ;
$$

EXP-R (Repair in Exponential):

$$
\beta_{1}=\beta_{2}=(1) ; \quad S_{1}=S_{2}=[-1] ;
$$

HYP-EXP-S (Service in Hyper exponential):

$$
\alpha_{1}=\alpha_{2}=(0.8,0.2) ; \quad T_{1}=T_{2}=\left[\begin{array}{cc}
-2.8 & 0 \\
0 & -0.28
\end{array}\right] ;
$$

HYP-EXP-R (Repair in Hyper exponential):

$$
\beta_{1}=\beta_{2}=(0.8,0.2) ; \quad S_{1}=S_{2}=\left[\begin{array}{cc}
-2.8 & 0 \\
0 & -0.28
\end{array}\right]
$$

## Illustrative Example 10.1.

In Tables 1, 2 and 3, we determine the outcome of the normal service rate of server-1 $\left(\mu_{1}\right)$ on the expected system size $(E S)$.

Fix $\mu_{2}=44, \mu_{3}=50, \mu_{4}=44, \theta_{1}=0.5, \theta_{2}=0.4, \sigma=\eta=3, \tau=1, \phi=2, p=0.8, q=0.2$, $b=0.001$.

We observe that from the following Tables 1, 2, and 3:

- As increasing the normal service rate of server-1 $\left(\mu_{1}\right)$, ES values are reduced for various combinations of arrival and service times.
- With estimating the values of different service times, the expected system size decreases speedier in hyper exponential service time while slow in Erlang service time.

Table 1. Normal Service rate ( $\mu_{1}$ ) vs $E S$ - EXP-S

| $\mu_{1}$ | EXP-A | ERL-A | HYP-A |
| :---: | :---: | :---: | :---: |
| 50 | 0.04581451 | 0.04404593 | 0.04697277 |
| 51.5 | 0.04581117 | 0.04404435 | 0.04696835 |
| 52 | 0.04580789 | 0.04404280 | 0.04696402 |
| 52.5 | 0.04580467 | 0.04404128 | 0.04695976 |
| 53 | 0.04580150 | 0.04403979 | 0.04695558 |
| 53.5 | 0.04579839 | 0.04403833 | 0.04695148 |
| 54 | 0.04579533 | 0.04403689 | 0.04694745 |
| 54.5 | 0.04579233 | 0.04403549 | 0.04694349 |
| 55 | 0.04578938 | 0.04403410 | 0.04693960 |
| 55.5 | 0.04578647 | 0.04403274 | 0.04693578 |

Table 2. Normal Service rate ( $\mu_{1}$ ) vs $E S$ - ERL-S

| $\mu_{1}$ | EXP-A | ERL-A | HYP-A |
| :---: | :---: | :---: | :---: |
| 50 | 0.04314870 | 0.05036041 | 0.05144265 |
| 50.5 | 0.04314627 | 0.05035753 | 0.05143934 |
| 51 | 0.04314389 | 0.05035470 | 0.05143609 |
| 51.5 | 0.04314155 | 0.05035193 | 0.05143290 |
| 52 | 0.04313925 | 0.05034920 | 0.05142978 |
| 52.5 | 0.04313699 | 0.05034653 | 0.05142671 |
| 53 | 0.04313476 | 0.05034391 | 0.05142370 |
| 53.5 | 0.04313258 | 0.05034133 | 0.05142075 |
| 54 | 0.04313043 | 0.05033880 | 0.05141785 |
| 54.5 | 0.04312832 | 0.05033632 | 0.05141501 |

## Illustrative Example 10.2.

From Figure 2 as well as the figures in Appendices 6 through 13 representing the two dimensional graphs, we determine the outcome of vacation rate $(\eta)$ on the probability of busy with normal service and optional service server-1 $\left(P_{B N S 1}\right.$ and $\left.P_{B O S_{1}}\right)$.

Fix $\lambda=1, \mu_{1}=\mu_{3}=20, \mu_{2}=\mu_{4}=10, \theta_{1}=0.5, \theta_{2}=0.4, \sigma=3, \tau=1, \phi=2, \psi=10$, $p=0.8, q=0.2, b=0.001$.

Observing from Figures 2 and 6 through 13, when lifting the vacation rate $\eta$, the $P_{B N S 1}$ and $P_{B O S 1}$ are increased and which is high in case of $H Y P . E X P-A$ and slow in $E R L-A$. Likewise it

Table 3. Normal Service rate $\left(\mu_{1}\right)$ vs $E S$ - HYP-EXP-S

| $\mu_{1}$ | EXP-A | ERL-A | HYP-A |
| :---: | :---: | :---: | :---: |
| 50 | 0.06531732 | 0.02158777 | 0.06616394 |
| 50.5 | 0.06530943 | 0.02158677 | 0.06614837 |
| 51 | 0.06530170 | 0.02158579 | 0.06613309 |
| 51.5 | 0.06529415 | 0.02158486 | 0.06611809 |
| 52 | 0.06528675 | 0.02158395 | 0.06610337 |
| 52.5 | 0.06527951 | 0.02158308 | 0.06608890 |
| 53 | 0.06527241 | 0.02158223 | 0.06607470 |
| 53.5 | 0.06526546 | 0.02158142 | 0.06606074 |
| 54 | 0.06525865 | 0.02158063 | 0.06604703 |
| 54.5 | 0.06525198 | 0.02157987 | 0.06603356 |

is high in $E R L-S$ and slow in $H Y P . E X P-S$. We examined the following view point of the graphs, $E S$ decreases faster in $E R L-S$ rather than $E X P-S$ and $H Y P . E X P-S$.


Figure 2. Vacation rate ( $\eta$ ) vs. $P_{B N S 1}$ and $P_{B O S 1}-M / M / 2$

## Illustrative Example 10.3.

In Figure 3 as well as the figures in Appendices 14 through 21 that represents the three dimensional graphs, we evaluate the outcome of the normal service rate of server-1 $\left(\mu_{1}\right)$ and the rate of vacation $(\eta)$ on the expected system size $(E S)$.

Fix $\mu_{2}=44, \mu_{3}=50, \mu_{4}=44, \theta_{1}=0.5, \theta_{2}=0.4, \sigma=3, \tau=1, \phi=2, p=0.8, q=0.2$, $b=0.001$.

We see from the Figures 3 and 14 through 21, when elevating the vacation rate ( $\eta$ ) and normal service rate of server-1 $\left(\mu_{1}\right)$ then $E S$ decreases for both arrival and service patterns. It maximizes quickly in $M A P-N C-A$ and slowly in $H Y P . E X P-A$. Similarly, it quickly reduces in $E R L-S$ and increases in $H Y P . E X P-S$.

## Hk/Hk/2



Figure 3. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate ( $\eta$ )vs. ES - $H k / H k / 2$

## Illustrative Example 10.4.

In the Figure 4 as well as the figures in Appendices 22 through 29 that represents the three dimensional graphs, we determine the outcome of the breakdown rate $(\tau)$ and the rate of vacation $(\eta)$ on the total cost of the system (TAC).

Fix $\lambda=1, \mu_{1}=\mu_{3}=20, \mu_{2}=\mu_{4}=10, \theta_{1}=0.5, \theta_{2}=0.4, \sigma=3, \phi=2, p=0.8$, $q=0.2, b=0.001, C H=15, C_{B N S 1}=C_{B N S 2}=2, C_{B O S 1}=C_{B O S 2}=1, C_{I d l e 2}=1$, $C_{W B N S 2}=C_{W B O S 2}=1, C_{C} D 1=1, C_{V 1}=1, C_{R 2}=1, C_{1}=C_{2}=C_{3}=C_{4}=C_{5}=C_{6}=$ $C_{7}=C_{8}=C_{9}=1$.

We see from the Figures 4 and 22 through 29, when elevating the vacation rate $(\eta)$ and breakdown rate $(\tau)$, then the total cost decreases for both arrival and service patterns. It maximizes quickly in $M A P-N C-A$ and slowly in $H Y P . E X P-A$. Similarly, it quickly reduces in $E R L-S$ and increases in $H Y P . E X P-S$.

## Illustrative Example 10.5.

From the Figure 5 as well as the figures in Appendices 30 through 37 that represents the two dimensional graphs, we determine the outcome of normal service rate of server- $1\left(\mu_{1}\right)$ on the expected waiting time (EW).

Fix $\lambda=1, \mu_{2}=10, \mu_{3}=20, \mu_{4}=10, \theta_{1}=0.5, \theta_{2}=0.4, \sigma=3, \eta=3, \tau=1, \phi=2, p=0.8$, $q=0.2, b=0.001$.

Observing from the Figures 5 and 30 through 37, when lifting the normal service rate of server$1\left(\mu_{1}\right)$, then EW values are reduces for both arrival and service patterns. With estimating the values of different service times, the expected waiting time decreases more fast in hyper exponential service and slow in Erlang service time.


Figure 4. Breakdown $\operatorname{rate}(\tau)$ and Vacation rate $(\eta)$ vs. TC $-E k / H k / 2$


Figure 5. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-M / E k / 2$

## 11. Conclusion

In this paper, we have developed the queueing model with working breakdown, vacation, closedown, optional service and repair. A queue with two server in which arrival occurs MAP and service times of both server follows to be phase type distribution. By using matrix analytic method, we found the stationary probability since the queueing systems are Quasi Birth-Death process. The stability condition for the $M A P / P H_{1}, P H_{2} / 2$ queuing system has analyzed and some performance measures for queueing system was selected and implemented in numerical illustrations by using two dimensional and three dimensional graphs. For further work, the model can be investigate with batch arrival in according to Markovian arrival process and various service rates with N -policy.

## Acknowledgment:

The authors are gratefully acknowledge the anonymous referees for their valuable remarks and recommendations for the improvement of the manuscript.

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## Appendix

The following Figures 6 through 13 belong to Illustrative Example 2:


Figure 6. Vacation rate ( $\eta$ ) vs. $P_{B N S 1}$ and $P_{B O S 1}-E k / M / 2$


Figure 7. Vacation rate ( $\eta$ ) vs. $P_{B N S 1}$ and $P_{B O S 1}-H k / M / 2$


Figure 8. Vacation rate $(\eta)$ vs. $P_{B N S 1}$ and $P_{B O S 1}-M / E k / 2$


Figure 9. Vacation rate ( $\eta$ ) vs. $P_{B N S 1}$ and $P_{B O S 1}-E k / E k / 2$


Figure 10. Vacation rate $(\eta)$ vs. $P_{B N S 1}$ and $P_{B O S 1}-H k / E k / 2$


Figure 11. Vacation rate $(\eta)$ vs. $P_{B N S 1}$ and $P_{B O S 1}-M / H k / 2$


Figure 12. Vacation rate $(\eta)$ vs. $P_{B N S 1}$ and $P_{B O S 1}-E k / H k / 2$


Figure 13. Vacation rate ( $\eta$ ) vs. $P_{B N S 1}$ and $P_{B O S 1}-H k / H k / 2$

The following Figures $14-21$ belong to Illustrative Example 3:


Figure 14. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate $(\eta)$ vs. ES - $M / M / 2$

Ek/M/2


Figure 15. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate $(\eta)$ vs. ES - $E k / M / 2$

Hk/M/2


Figure 16. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate ( $\eta$ ) vs. ES $-H k / M / 2$


Figure 17. Normal service rate ( $\mu_{1}$ ) and Vacation rate ( $\eta$ ) vs. ES $-M / E k / 2$

## Ek/Ek/2



Figure 18. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate ( $\eta$ ) vs. ES $-E k / E k / 2$

## Hk/Ek/2



Figure 19. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate ( $\eta$ ) vs. ES $-H k / E k / 2$


Figure 20. Normal service rate $\left(\mu_{1}\right)$ and Vacation rate $(\eta)$ vs. ES $-M / H k / 2$
Ek/Hk/2


Figure 21. Normal service rate ( $\mu_{1}$ ) and Vacation rate ( $\eta$ )vs. ES - $E k / H k / 2$

The following Figures $22-29$ belong to Illustrative Example 4:


Figure 22. Breakdown rate $(\tau)$ and Vacation rate $(\eta)$ vs. TC $-M / M / 2$


Figure 23. Breakdown rate $(\tau)$ and Vacation rate $(\eta)$ vs. TC $-E k / M / 2$


Figure 24. Breakdown $\operatorname{rate}(\tau)$ and Vacation rate $(\eta)$ vs. TC $-H k / M / 2$


Figure 25. Breakdown $\operatorname{rate}(\tau)$ and Vacation rate $(\eta)$ vs. TC $-M / E k / 2$


Figure 26. Breakdown $\operatorname{rate}(\tau)$ and Vacation $\operatorname{rate}(\eta)$ vs. TC $-E k / E k / 2$


Figure 27. Breakdown rate $(\tau)$ and Vacation rate $(\eta)$ vs. TC $-H k / E k / 2$


Figure 28. Breakdown rate $(\tau)$ and Vacation rate $(\eta)$ vs. TC $-M / H k / 2$


Figure 29. Breakdown rate $(\tau)$ and Vacation rate $(\eta)$ vs. TC $-H k / H k / 2$

The following Figures $30-37$ belong to Illustrative Example 5:


Figure 30. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-M / M / 2$


Figure 31. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-M / H k / 2$


Figure 32. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW - $E k / M / 2$


Figure 33. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. $\mathrm{EW}-E k / E k / 2$


Figure 34. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-E k / H k / 2$


Figure 35. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-H k / M / 2$


Figure 36. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-H k / E k / 2$


Figure 37. Normal service rate of server-1 $\left(\mu_{1}\right)$ vs. EW $-H k / H k / 2$

