A formulation of optimal control problem with quasi-geometric discounting

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Abstract

In the previous paper, I characterized some dynamic aspects on dynastic utility incorporating two-sided altruism with discrete-time OLG settings. In its extension, hereby I formulate the continuous-time version of the optimal control problem with quasi-geometric discounting.

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1. Introduction

In my previous paper (Aoki (2015)), I described a dynamics of one-sector growth model with an OLG setting under two-sided altruism, where we a modified Euler equation for dynamic inconsistency was derived. To be natural, the model with dynamic inconsistency proves to be equivalent to that with quasi-geometric discounting (as in Krussel et al. (2003)). There I focused on self-consistency, differentiability, and fragility against recursive perturbation of policy/value functions, in a local area around any arbitrary fixed point. The indeterminacy of self-consistent policy functions, as proved in Krussel et al. (2003), suggests that a slight functional perturbation might not necessarily result in the contraction in the sup norm to the original one.

In the present note, I extend the model to a continuous-time case (as in Ekeland et al. (2006)), but with a finite, nonzero lifetime for each generation, as the first step for investigating its dynamic properties, especially the fragility/robustness against perturbation from some of self-consistent value functions in some functional spaces.

1. Discrete-time model with quasi-geometric discounting: Review

At first, we review a discrete-time model with quasi-geometric discounting, as discussed in Krussel et al. (2003) and Aoki (2015). Assume that time is now finite and discrete as period t=0,1,⋯,T, each generation (t=0,1,⋯,T−1.) lives only for each single period t, and that discount factors are $\beta\mu$ for the second period, but β's afterwards. Then

The *objective function* of generation t is:

$$V_{t,T}(k)=u(c_t)+\beta\mu u(c_{t+1})+\beta^2\mu u(c_{t+2})\cdots+\beta^{T-t-1}\mu u(c_{T-1})+\beta^{T-t}\mu\Phi(k_T). \qquad (1)$$

Here c_t and k_t are the consumption and capital stock at period t, respectively, and $u(\cdot)$ is a utility function. $\Phi(\cdot)$ is the value function at terminal period T.

We also define a following modified objective function:

$$\tilde{V}_{t,T}(k)=u(c_t)+\beta u(c_{t+1})+\beta^2 u(c_{t+2})\cdots+\beta^{T-t-1}u(c_{T-1})+\beta^{T-t}\Phi(k_T). \qquad (2)$$

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The corresponding value functions to the above value functions, $V_{t,T}(k_t)$ and $\tilde{V}_{t,T}(k_t)$, are also defined as $W_t(k_t)$ and $\tilde{W}_t\ (k_t)$, respectively.

Then generation t solves:

$$W_t(k_t)=\max_{c_t}\ (u(c_t)+\beta\mu\widetilde{W}_{t+1}(k_{t+1})), \quad (3)$$
$$s.t.\ k_{t+1}=f(k_t)+(1-\delta)k_t-c_t.$$

where

$$\widetilde{W}_t(k_t)=(1/\mu)W_t(k_t)+(1-1/\mu)u(\hat{c}_t). \quad (4)$$

Here $\hat{c}_t$ is the solution in c_t of the above problem. $u(\cdot)$ is a utility function, and δ is a depreciation rate. Then *the modified Euler equation* is derived as:

$$-u'\ ((f-g_t)(k))+\beta u'\ ((f-g_{t+1})\circ g_t(k))[(\mu f'-(\mu-1)g_{t+1}')\circ g_t(k)]=0. \quad (5)$$

2. Continuous-time model with quasi-geometric discounting

Now we move to the continuous-time case. The assumptions of the model are as following.

Assumptions:

- The world starts at $t=0$, and ends at $t=T(=n\Delta T)$, during which there exist n periods with interval ΔT, and n corresponding generations.
- Each of n generations ($i=0,1,\cdots,n-1.$) lives only for one period. The life of generation i starts at time $t=i\Delta T$ and ends at time $t=(i+1)\Delta T$.

Fig. 1 Time course in continuous-time model

- Discount *factor* function of each generation $\beta(s)$ as a function of s, time from its birth. (So for generation i, $s=0$ when $t=i\Delta T$.)
- *Objective function* of generation $n-l$ ($l=1,2,\cdots,n.$):

$$V_{n-l}(k_{n-l}^0)=\int_0^{l\Delta T}e^{-\int_0^s\beta(s')ds'}u(c(s))ds+e^{-\int_0^{l\Delta T}\beta(s')ds'}\Phi(k_n^0), \quad (6)$$

where

k_{n-l}^0 : Initial capital of generation $n-l$,

$\Phi(k_n^0)$: Value of capital (k_n^0) at the terminal period $t=T(=n\Delta T)$.

Then the corresponding value function for generation $n-l$ (W_{n-l}) is calculated as:

$$W_{n-l}(k_{n-l}^0)=\max_{\{c(t)\}_0^{\Delta T},\{k(t)\}_0^{\Delta T}}\left(\int_0^{\Delta T}e^{-\int_0^s\beta(s')ds'}u(c(s))ds+e^{-\int_0^{\Delta T}\beta(s')ds'}\widetilde{W}_{n-l+1}(k_{n-l+1}^0)\right). \quad (7)$$
$$s.t.\ \dot{k}(s)=f(k(s))-c(s)(-\delta k(s)),k(0)=k_{n-l}^0,\ k_{n-l+1}^0=k(\Delta T)\ (k_n^0\ is\ not\ fixed).$$

Here let $\left(\{\hat{c}_{n-l}(k_{n-l}^0,s)\}_{s=0}^{\Delta T},\{\hat{k}_{n-l}(k_{n-l}^0,s)\}_{s=0}^{\Delta T}\right)$ be the solutions of $\{c(s)\}_0^{\Delta T},\{k(s)\}_0^{\Delta T}$ in the above problem. Then define the subsequently realized paths of consumption and capital until the terminal period

$t = T(= n\Delta T)$, given $k^0_{n\text{-}l}$, the initial capital of generation $n-l$, as $\{\tilde{c}_{n-l}(k^0_{n-l}, s)\}_{s=0}^{l\Delta T}$ and $\{\tilde{k}_{n-l}(k^0_{n-l}, s)\}_{s=0}^{l\Delta T}$.

Then the modified value of $V_{n\text{-}l}$, transformed according to the discount structure of generation $n-l-1$ (the preceding generation), is represented as:

$$\widetilde{W}_{n-l}(k^0_{n-l}) = \int_0^{l\Delta T} e^{-\int_0^s \beta(s'+\Delta T)ds'} u\left(\tilde{c}_{n-l}(k^0_{n-l}, s)\right) ds + e^{-\int_0^{l\Delta T}\beta(s'+\Delta T)ds'} \Phi\left(\tilde{k}_{n-l}(k^0_{n-l}, l\Delta T)\right). \quad (8)$$

Now we assume, for simplicity, that discount factor is β_0 during the own life of the current generation $n-l$, but β_1 afterwards. That is, $\beta(s')=\beta_0$ *for* $0 \le s' \le \Delta T$, and $\beta(s')=\beta_1$ *for* $\Delta T \le s' \le l\Delta T$, where $\beta_0/\beta_1 \equiv \mu$. Note that $\beta(s'+\Delta T)=\beta(s'+2\Delta T)$ *for* $0 \le t'$. (The discount structure after one period is the same as that after two periods.) Then now the Bellman equations can be constructed as:

$$W_{n-l}(k) = \max_{\{c(s)\}_0^{\Delta T}, \{k(s)\}_0^{\Delta T}} \left(\int_0^{\Delta T} e^{-\beta_0 s} u(c(s))dt + e^{-\beta_0 \Delta T} \widetilde{W}_{n-l+1}(k(\Delta T))\right), \quad (9)$$

$$s.t.\ \dot{k}(s) = f(k(s)) - c(s)(-\delta k(s)), k(0) = k,$$

and

$$\widetilde{W}_{n-l}(k) = e^{(\beta_0-\beta_1)\Delta T} W_{n-l}(k) + \int_0^{\Delta T}\left[e^{-\beta_1 s} - e^{-\beta_0 s + (\beta_0 - \beta_1)\Delta T}\right] u(\hat{c}_{n-l}(k,s))ds$$
$$= e^{\beta_1(\mu-1)\Delta T} W_{n-l}(k) + \int_0^{\Delta T}\left[e^{-\beta_1 s} - e^{\beta_1(\mu-1)\Delta T - \beta_1 \mu s}\right] u(\hat{c}_{n-l}(k,s))ds. \quad (10)$$

In *time consistency*, if it occurred, we could drop the subscripts $n-l$ and $n-l+1$ from the above equation. Then the corresponding Hamilton–Jacobi–Bellman equation is derived just as in the ordinary case, but only for $0 \le t \le \Delta T$.

$$W(k) = \max_{\{c(t)\}_0^{\Delta T}, \{k(t)\}_0^{\Delta T}} \left(\int_0^{\Delta T} e^{-\beta_0 t} u(c(t))dt + e^{-\beta_0 \Delta T} \widetilde{W}(k(\Delta T))\right), \quad (11)$$

$$s.t.\ \dot{k}(t) = f(k(t)) - c(t)(-\delta k(t)), k(0) = k.$$

$$\left(\{\hat{c}(k,t)\}_{t=0}^{\Delta T}, \{\hat{k}(k,t)\}_{t=0}^{\Delta T}\right) = \underset{\{c(t)\}_0^{\Delta T}, \{k(t)\}_0^{\Delta T}}{argmax} \left(\int_0^{\Delta T} e^{-\beta_0 t} u(c(t))dt + e^{-\beta_0 \Delta T} \widetilde{W}(k(\Delta T))\right) \quad (12)$$

$$\widetilde{W}(k) = e^{(\beta_0-\beta_1)\Delta T} W(k) + \int_0^{\Delta T}\left[e^{-\beta_1 t} - e^{(\beta_0-\beta_1)\Delta T - \beta_0 t}\right] u(\hat{c}(k,t))dt$$
$$= e^{\beta_1(\mu-1)\Delta T} W(k) + \int_0^{\Delta T}\left[e^{-\beta_1 t} - e^{\beta_1(\mu-1)\Delta T - \beta_1 \mu t}\right] u(\hat{c}(k,t))dt \quad (13)$$

5. Final remarks

This note formulates the continuous-time optimal control problem with quasi-geometric discounting. The results shown here will help analytical investigations including the existence or robustness of time-consistency, which are still left for future work.

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