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McDonald Generalized Power Weibull Distribution: Properties, and Applications

Shei B. Sayibu[1,*], *A. Luguterah*[2] *and S. Nasiru*[2]

[1]Department of Statistics, Faculty of Physical Sciences, University for Development Studies, Box 1350, West Africa, Tamale, Ghana
[2]Department of Statistics and Actuarial Science, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, Box 24, West Africa, Navrongo, Ghana

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Abstract: This research introduces a novel six-parameter model called the McDonald Generalized Power Weibull distribution. The model contains several sub-models that prove highly valuable in modeling real-life scenarios, including the McDonald Weibull, McDonald exponential, McDonald Nadarajah-Haghighi, beta generalized power Weibull distribution, and Kumaraswamy generalized power distributions, among others. The proposed model demonstrates suitability in modeling survival/reliability data, accommodating various hazard failure rates such as increasing, decreasing, unimodal (upside-down bathtub), modified bathtub, and reversed J-shape. Various properties of the new model are investigated, including moments, quantiles, incomplete moments, moment-generating functions, and order statistics. The maximum likelihood estimation method is employed to estimate the model parameters. The study concludes by illustrating the flexibility of the proposed model through the use of lifetime data to demonstrate its applicability.
Keywords: Bathtub, flexibility, generalized power Weibull, quantile, and maximum likelihood

1 Introduction

Probability distributions are integral part of modeling real data. The capacity of a distribution to describe different data sets relies on the flexibility of the model. Real data comes with different characteristics that are non-monotonic in nature. The complex nature of lifetime data renders the classical models unfit to model such data sets [1]. Modifications are therefore carried out on these classical distributions to improve upon their flexibility and performance. For researchers to achieve this desired flexibility, existing distributions are modified to produce new distributions with higher flexibility and performance. This can be achieved using different techniques. One such method is the introduction of additional shape parameters to the baseline model aimed at enhancing its skewness and varying its tail weight [2]. An important characteristic of making extensions to a distribution is to inject some amount of flexibility into the baseline distribution. This will increase its capacity in fitting data that suitably cannot be fitted by the baseline model [3]. Developing new extended distributions to model both non-monotonic and monotonic failure rates is therefore necessary.

Extended distributions are more appropriate for modeling lifetime data in various fields such as finance, economics, engineering, biology, medicine, and manufacturing compared to non-extended distributions. Recent studies have introduced several generalized distributions that fall into this category. Examples include the extended cosine generalized family of distributions [4], the odd inverse exponential class of distributions [3], the exponentiated power generalized Weibull power series family of distributions [5], the generalized power generalized Weibull distribution [6], the McDonald modified Weibull distribution [7], the McDonald extended Weibull distribution [8], and the McDonald normal distribution [9], among others.

The proposed generalized power Weibull distribution by [10] demonstrated various shapes of failure rates, including increasing, decreasing, and bathtub-shaped patterns. Therefore, the generalized power Weibull distribution is not suitable for accurately modeling datasets that exhibit modified bathtub shapes, reversed J-shapes, or right-skewed failure rates. Consider a random variable represented by X which follows a generalized power Weibull (GPW) distribution, characterized by its probability density function (PDF) and cumulative distribution function (CDF). The PDF and CDF of the GPW model are respectively given:

$$t(x) = \varepsilon\phi\varphi x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1} e^{(1-(1+\varphi x^{\phi})^{\varepsilon})} \tag{1}$$

and

*Corresponding author e-mail: b.shei@yahoo.com

$$T(x)=1-e^{\left(1-\left(1+\varphi x^{\phi}\right)^{\varepsilon}\right)}, \varepsilon>0, \phi>0, \varphi>0, x>0. \tag{2}$$

Where ε and ϕ are shape parameters and φ is a scale parameter.

This study focuses on the examination and development of the McDonald GPW distribution (MGPWD). The purpose behind developing this distribution is to enhance the capability of the GPW to model datasets that exhibit both non-monotonic and monotonic failure rates. Additionally, the aim is to derise a distribution that can effectively handle datasets with varying levels of skewness and kurtosis. The remaining sections of the paper are organized as follows: Section 2 presents the development of the PDF and CDF for the MGPWD and the sub-models associated with the MGPWD. Section 3 addresses the mixture representation of the PDF. Statistical properties of the MGPWD are explored in Section 4. Parameter estimation and Monte Carlo simulation techniques are discussed in Section 5. Section 6 focuses on the application of the MGPWD to real-world data. Finally, concluding remarks are provided in Section 7.

2. The New Model

In this section, the PDF and CDF of the MGPWD are developed using the GPW model as the baseline distribution. If $t(x)$ and $T(x)$ are the baseline PDF and CDF, then the PDF of McDonald's generalized family of distribution as defined by [9] and [7] is presented as:

$$f(x,u,v,z)=\frac{z}{B(u,v)}t(x)T^{uz-1}(x)\left\{1-T^{z}(x)\right\}^{v-1}, \tag{3}$$

where $t(x)=\frac{dT(x)}{dx}$, $B(u,v)=\int_0^1 x^{u-1}(1-x)^{v-1}$ is the beta distribution and $u>0, v>0$ and $z>0$ represents the shape parameters of the generalized family of distributions. The parameters u, v and z are additional shape parameters that regulate the skewness as well as the lightness of the tail weight. It is essential to note that when $z=1$ the beta family is realized [11]. Also, when $u=1$, the Kumaraswamy family is obtained [12]. A significant attribute of the McDonald generalized family of distributions is the lightness of their tails' weight due to the increase in the number of shape parameters.

For X being a random variable having a density equation (3) and denoted as $X\sim MG(u,v,z,\omega)$ in which ω represents a vector of parameters. The CDF corresponding to equation (3) is:

$$F(x,u,v,z)=I_{T(x)^z}(u,v)=\frac{1}{B(u,v)}\int_0^{T(x)^z} x^{1-u}(1-x)^{v-1}dx,$$

where $I_{T(x)^z}(u,v)=\frac{1}{B(u,v)}\int_0^{T(x)^z} x^{1-u}(1-x)^{v-1}dx$, represents the incomplete beta distribution ratio [13]. The CDF in hypergeometric function form as stated by [13] is defined as:

$$F(x)=\frac{zT(x)^z}{uB(u,v)}\,{}_2F_1\left(u,1-v;u+1;T(x)^z\right), \tag{4}$$

where

${}_2F_1(u,v;z;x)=\frac{\Gamma(z)}{\Gamma(u)\Gamma(v)}\sum_{j=0}^{\infty}\frac{\Gamma(u+j)\Gamma(v+j)}{\Gamma(z+j)}\frac{x^j}{j!}$, and $\Gamma(u)=\int_0^{\infty}x^{u-1}e^{-x}dx$ is the gamma function. The PDF of the MGPWD can be obtained by substituting equations (1) and (2) into equation (3) to obtain:

$$f(x,u,v,z,\varepsilon,\phi,\varphi)=\frac{z\varepsilon\phi\varphi}{B(u,v)}x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{uz-1}\left[1-\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right]^{v-1}, \quad (5)$$

where $x>0, u>0, v>0, z>0, \varepsilon>0, \phi>0$ .and $\varphi>0$.

Figure 1 illustrates several desired shapes manifested by the PDF of the MGPWD. These shapes encompass a reversed J-shape, right skewed, nearly symmetric, increasing monotonically, and platykurtic shapes of different kurtosis.

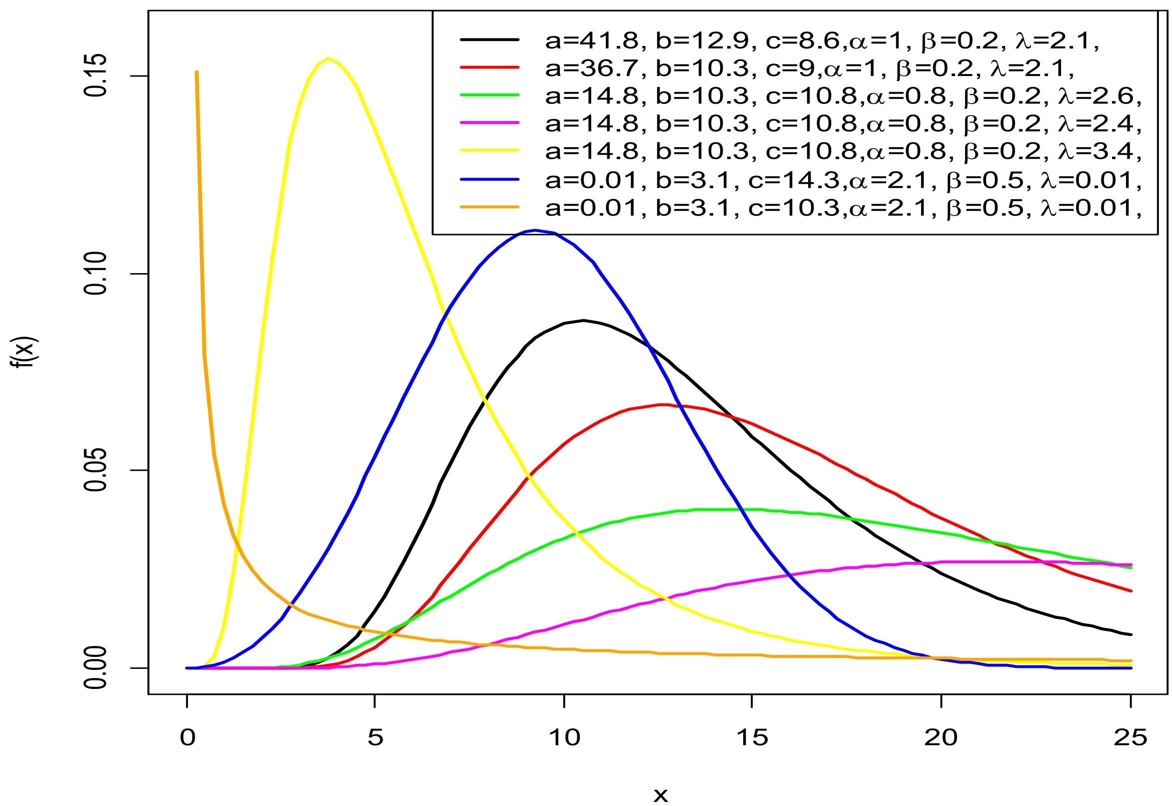

Fig.1: PDF shapes of the MGPWD

By substituting equation (2) into equation (4), the CDF of the MGPWD is obtained as follows:

$$F\left(x,u,v,z,\varepsilon,\phi,\varphi\right)=\frac{\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{uz}}{uB\left(u,v\right)}\,{}_2F_1\left(u,1-v;u+1;\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right), \quad (6)$$

where $x>0, u>0, v>0, z>0, \varepsilon>0, \phi>0$ .and $\varphi>0$..

The CDF can also be defined as,

$$F\left(x\right)=I_{\left(1-e^{\left(1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right)}\right)^{c}}\left(a,b\right). \quad (7)$$

The CDF defined in equation (7) is very useful in the calculation of the quantile function of the distribution and subsequent generation of random numbers. The survival function of the MGPWD is given as,

$$S(x)=1-\frac{\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{uz}}{uB(u,v)}\,{}_2F_1\left(u,1-v;u+1;\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right). \tag{8}$$

Figure 2 displays the hazard failure rate associated with the MGPWD. The hazard function exhibits various appropriate shapes such as monotonically increasing and decreasing patterns, bathtub-shaped patterns, upside-down bathtubs, and modified bathtubs. The hazard rate function of the MGPWD is mathematically represented as follows:

$$h(x)=\frac{z\varepsilon\phi\varphi x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\left[1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right]^{uz-1}\left[1-\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right]^{v-1}}{\frac{uB(u,v)-\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{uz}}{u}\,{}_2F_1\left(u,1-v;u+1;\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right)} \tag{9}$$

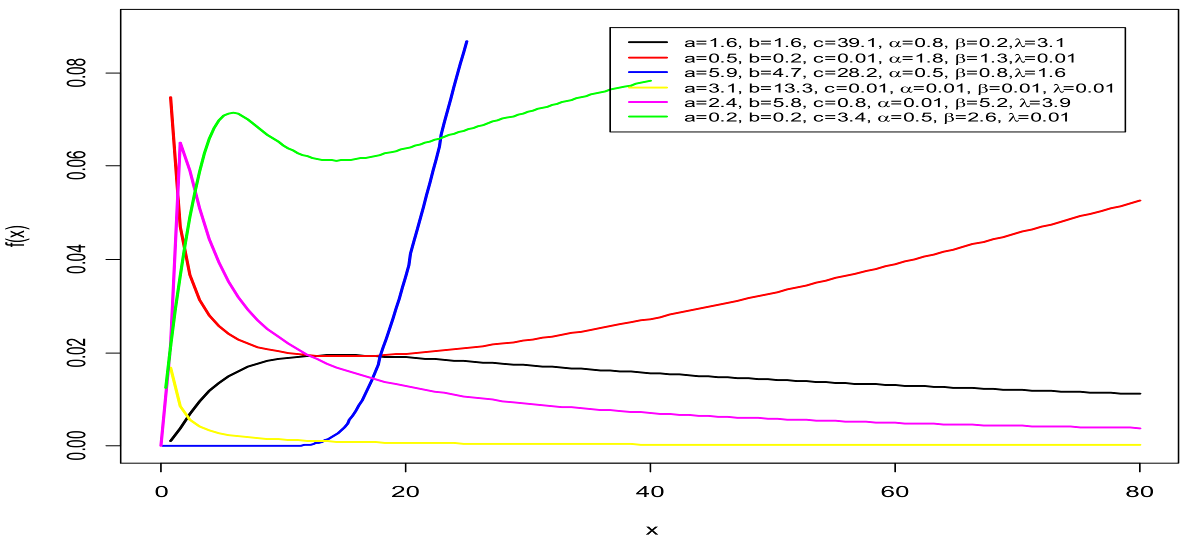

Fig. 2: Hazard function failure rates of the MGPWD

The reverse hazard function refers to the division of the probability density by the cumulative density. It represents the conditional density of an immediate failure, given that the failure occurred. The reverse hazard function is given as,

$$r(x)=\frac{z\varepsilon\phi\varphi x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\left[1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right]^{uz-1}\left[1-\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right]^{v-1}}{\frac{\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{uz}}{u}\,{}_2F_1\left(u,1-v;u+1;\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right)} \tag{10}$$

2.1 Sub-divided models of the MGPWD

The MGPWD converges to different important distributions given that its parameters vary. The MGPWD has some known special sub-models. For $X\sim MGPWD(u,v,z,\varepsilon,\phi,\varphi)$. The following sub-models are obtained.

1. For $\varepsilon=1$, the PDF reduces to the MacDonald Weibull distribution (MW)

2. For $\varepsilon=1$ and $\varphi=1$, the PDF becomes MacDonald exponential distribution (ME)
3. For $\varphi=1$, the PDF reduces to MacDonald Nadarajah-Haghighi distribution (MNH)
4. For $\varepsilon=1$ and $\varphi=2$, the PDF reduces to MacDonald Rayleigh distribution (MR)
5. For $z=1$, the MGPWD becomes the Beta Generalized power Weibull distribution (BGPW)
6. For $\varepsilon=1$ and $z=1$, the MGPWD becomes the Beta Weibull distribution (BW)
7. For $\varepsilon=1$, $\varphi=1$ and $z=1$, the PDF reduces to Beta exponential distribution (BE)
8. For $z=1$ and $\varphi=1$, the MGPWD becomes the Beta Nadarajah-Haghighi distribution (BNH)
9. For $\varepsilon=1$, $z=1$ and $\varphi=2$, the PDF reduces to Beta Rayleigh distribution (BR)
10. For $u=1$, the MGPWD reduces to the Kumaraswamy generalized power Weibull distribution (KGPW)
11. For $u=1$ and $\varepsilon=1$, the MGPWD reduces to Kumaraswamy Weibull distribution (KW)
12. For $u=1$, $\varepsilon=1$ and $\varphi=1$, the PDF of the MGPWD becomes the Kumaraswamy exponential distribution (KE)
13. $u=1$ and $\varphi=1$, the MGPWD reduces to the Kumaraswamy Nadarajah-Haghighi distribution (KNH)
14. $u=1$, $\varepsilon=1$ and $\varphi=2$, the PDF reduces to the Kumaraswamy Rayleigh distribution (KR)

3. Representation of the PDF in Mixture form

The mixture form of the PDF of the MGPWD is presented in this section. The mixture form of representing the PDF is essential in the development of the statistical properties of the new distribution.

Lemma 1. The density function of the MGPWD in a mixture form is given as,

$$f(x,u,v,z)=\frac{z\varepsilon\phi\varphi}{B(u,v)}x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{(i+j)(1-(1+\varphi x^{\phi})^{\varepsilon})}, \quad (11)$$

where $\omega_{ij}=(-1)^{i+j}\binom{v-1}{i}\binom{z(u+i)-1}{j}$.

Proof. Using binomial series expansion $(1-x)^n=\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}x^i$ where $|x|\le 1$, then

$$\left[1-\left(1-e^{(1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z}\right]^{v-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{v-1}{i}\left(\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{zi}\right).$$

substituting the expanded form back into equation (5) gives,

$$f(x,u,v,z,\varepsilon,\phi,\varphi)=\frac{z\varepsilon\phi\varphi}{B(u,v)}x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{uz-1}\sum_{i=0}^{\infty}(-1)^{i}\binom{v-1}{i}\left(\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{zi}\right).$$

This is simplified as,

$$f(x,u,v,z,\varepsilon,\phi,\varphi)=\frac{z\varepsilon\phi\varphi x^{\phi-1}}{B(u,v)}(1+\varphi x^{\phi})^{\varepsilon-1}e^{(1-(1+\varphi x^{\phi})^{\varepsilon})}\sum_{i=0}^{\infty}(-1)^{i}\binom{v-1}{i}\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z(u+i)-1}.$$

Expanding further using the last term gives,

$$\left(1-e^{1-(1+\varphi x^{\phi})^{\varepsilon}}\right)^{z(u+i)-1}=\sum_{j=0}^{\infty}(-1)^{i}\binom{z(u+i)-1}{j}e^{j\left(1-\left(1+\varphi x^{\phi}\right)^{\varepsilon}\right)}.$$

Substituting and simplifying gives,

$$f(x,u,v,z,\varepsilon,\phi,\varphi)=\frac{z\varepsilon\phi\varphi}{B(u,v)}x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{v-1}{i}\binom{z(u+i)-1}{j}e^{(i+j)(1-(1+\varphi x^{\phi})^{\varepsilon})}.$$

and the PDF is finally represented as,

$$f(x,u,v,z,\varepsilon,\phi,\varphi)=\frac{z\varepsilon\phi\varphi}{B(u,v)}x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{(i+j)(1-(1+\varphi x^{\phi})^{\varepsilon})}.$$

4. Statistical properties of the MGPWD

This section provides an overview of several statistical properties associated with the MGPWD. The properties encompass the quantile, the moment, moment generating function, incomplete moment, and order statistics.

4.1 Quantile function

The quantile function, denoted as Q, is a mapping that takes a probability value p and returns the corresponding threshold value x such that the random draws from the CDF fall below probability p. The quantile function is applied in the generation of random numbers.

Lemma 2. The quantile function of the MGPWD is

$$x_p=\left[\frac{1}{\varphi}\left[\left\{1-\log\left(1-\left(Q_{(u,v)}(p)\right)^{1/z}\right)\right\}^{1/\varepsilon}-1\right]\right]^{1/\phi},\ p\in[0,1]. \tag{12}$$

Proof. Given a strictly monotonic function, the quantile function Q, returns a threshold value x below which random draws from the given CDF could fall p percent of the time. $Q(p)\le x$ if and only if $p\le F(x)$. Therefore if $F(x)$ is continuous and strictly monotonically increasing, the inequalities are replaced by equality. Hence, the quantile function can be written in terms of p as;

$$Q(p)=F^{-1}(p).$$

Thus,

$$F(x)=I_{\left(1-e^{\left(1-\left(1+\varphi x^{\phi}\right)^{\varepsilon}\right)}\right)^{z}}(u,v)=p.$$

This implies;

$$\left(1-e^{1-\left(1+\varphi x^{\phi}\right)^{\varepsilon}}\right)^{z}=Q_{(u,v)}(p)$$

This reduces by multiplying both sides by the z^{th} root to give,

$$1-e^{1-\left(1+\varphi x^{\phi}\right)^{\varepsilon}}=\left(Q_{(u,v)}\left(p\right)\right)^{1/z}.$$

Rearranging and making x the subject results into,

$$x=\left[\frac{1}{\varphi}\left[\left\{1-\log\left(1-\left(Q_{(u,v)}\left(p\right)\right)^{1/z}\right)\right\}^{1/\varepsilon}-1\right]\right]^{1/\varphi};$$

which completes the proof.

4.2 Moment

The moments in statistical analysis are very useful when it comes to the calculation and determination of central tendencies and measures of dispersion.

Proposition 1. If $X\sim MGPWD$, the r^{th} moment of the random variable X is

$$\mu_r^{'}=\frac{z}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\omega_{ijk}e^{i+j}\left(\frac{1}{\varphi}\right)^{r/\phi}\Gamma\left(\frac{r+\phi\left(\varepsilon-k\right)}{\varepsilon\phi},j+1\right),r=1,2,... \tag{13}$$

where $\omega_{ijk}=\left(-1\right)^{i+j+k}\binom{v-1}{i}\binom{z\left(\varepsilon+i\right)-1}{j}\binom{r/\phi}{k}\left(\frac{1}{j+1}\right)^{\frac{r+\phi\left(\varepsilon-k\right)}{\varepsilon\phi}}$.

Proof. The non-central r^{th} moment is defined as,

$$\mu_r^{'}=E\left[X^r\right]=\int_0^{\infty}x^r f\left(x\right)dx. \tag{14}$$

Using the mixture form for the PDF in equation (11) and substituting it into equation (14) gives,

$$\mu_r^{'}=\int_0^{\infty}x^r\frac{z\varepsilon\phi\varphi}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{\left(i+j\right)(1-(1+\varphi x^{\phi})^{\varepsilon})}dx. \tag{15}$$

Factorizing the constants,

$$\mu_r^{'}=\frac{z\varepsilon\phi\varphi}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}\int_0^{\infty}x^r x^{\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{\left(i+j\right)(1-(1+\varphi x^{\phi})^{\varepsilon})}dx.$$

Simplifying gives,

$$\mu_r^{'}=\frac{z\varepsilon\phi\varphi}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{i+j}\int_0^{\infty}x^{r+\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{-\left(i+j\right)(1-(1+\varphi x^{\phi})^{\varepsilon})}dx.$$

Considering the integral part and let,

$$I=\int_0^{\infty}x^{r+\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{-\left(i+j\right)(1-(1+\varphi x^{\phi})^{\varepsilon})}dx.$$

Also, let

$$u=(i+j)\left(1+\varphi y^{\phi}\right)^{\varepsilon}.$$

Then

$$x=\left\{\frac{1}{\varphi}\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right]\right\}^{1/\phi}. \tag{16}$$

Differentiating u with respect to x,

$$\frac{du}{dx}=\varepsilon\phi\varphi x^{\phi-1}(j+1)\left(1+\varphi x^{\phi}\right)^{\varepsilon-1}.$$

This implies,

$$\frac{du}{\varepsilon\phi\varphi x^{\phi-1}(j+1)\left(1+\varphi x^{\phi}\right)^{\varepsilon-1}}=dx.$$

Substituting dx into the integral yields,

$$\mu_r^{'}=\frac{z}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{i+j}\int_0^{\infty}x^r e^{-u}\frac{du}{(j+1)\left(1+\varphi x^{\phi}\right)^{\varepsilon-1}},$$

Putting the expression for x in equation (13) gives,

$$\mu_r^{'}=\frac{z}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{i+j}\int_0^{\infty}\left\{\frac{1}{\varphi}\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right]\right\}^{r/\phi}e^{-u}\frac{du}{(j+1)}.$$

This reduces to,

$$\mu_r^{'}=\frac{z}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{i+j}\left(\frac{1}{\varphi}\right)^{r/\phi}\int_0^{\infty}\left\{\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right]\right\}^{r/\phi}e^{-u}\frac{du}{(j+1)}.$$

Using the binomial theorem identity. This power series converges for $n\geq 0$ and $\left|\frac{x}{u}\right|<1$.

Thus, $\left|\left(\frac{u}{j+1}\right)^{1/\varepsilon}\right|<1$. It follows that

$$\mu_r^{'}=\frac{z}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{i+j}\left(\frac{1}{\varphi}\right)^{r/\phi}\int_{j+1}^{\infty}\sum_{k=0}^{\infty}(-1)^k\binom{r/\phi}{k}\left\{\left(\frac{u}{j+1}\right)^{1/\varepsilon}\right\}^{r/\phi-k}e^{-u}\frac{du}{(j+1)}.$$

Further simplification gives,

$$\mu_r^{'} = \frac{z}{B(u,v)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \omega_{ij} e^{i+j} \left(\frac{1}{\varphi}\right)^{r/\phi} \int_{j+1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{r/\phi}{k} \left\{\left(\frac{u}{j+1}\right)\right\}^{\frac{r-\phi k}{\varepsilon\phi}} e^{-u} \frac{du}{(j+1)}.$$

This yields,

$$\mu_r^{'} = \frac{z}{B(u,v)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{r/\phi}{k} \left(\frac{1}{j+1}\right)^{\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}} \omega_{ij} e^{i+j} \left(\frac{1}{\phi}\right)^{r/\phi} \int_{j+1}^{\infty} \{(u)\}^{\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}-1} e^{-u} du.$$

Thus;

$$\mu_r^{'} = \frac{z}{B(u,v)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \omega_{ijk} e^{i+j} \left(\frac{1}{\varphi}\right)^{r/\phi} \Gamma\left(\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}, j+1\right).$$

4.3 Moment Generating Function

Proposition 2. If $X \sim MGPWD$ for any integer value, the moment generating function $M_X(t)$ is

$$M_X(t) = \frac{z}{B(u,v)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \omega_{ijk} e^{i+j} \left(\frac{1}{\varphi}\right)^{r/\phi} \Gamma\left(\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}, j+1\right). \quad (17)$$

Proof. The moment-generating function is defined as follows;

$$M_X(t) = E\left[e^{tX}\right] = \int_0^{\infty} e^{tx} f(x) dx. \quad (18)$$

Using the Taylor series:

$$M_X(t) = E\left[\sum_{r=0}^{\infty} \frac{t^r X^r}{r!}\right].$$

This implies,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left[X^r\right].$$

$\mu_r^{'} = E(X^r)$ as given in equation (13) and substituting it into the expression gives,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left\{ \frac{z}{B(u,v)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{r/\phi}{k} \left(\frac{1}{j+1}\right)^{\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}} \omega_{ij} e^{i+j} \left(\frac{1}{\phi}\right)^{r/\phi} \Gamma\left(\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}, j+1\right) \right\}.$$

Simplifying gives,

$$M_X(t) = \frac{z}{B(u,v)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \omega_{ijk} e^{i+j} \left(\frac{1}{\varphi}\right)^{r/\phi} \Gamma\left(\frac{r+\phi(\varepsilon-k)}{\varepsilon\phi}, j+1\right).$$

4.4 Incomplete Moment

Proposition 3. If $X \sim MGPWD$ then the incomplete moment is

$$M_r(y)=\frac{z}{B(u,v)}\varphi^{-r/\phi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}w_{ijk}e^{j+1}\left(\frac{1}{j+1}\right)^{r-\phi k/\varepsilon\phi}\Gamma\left(\frac{r-\phi k+\varepsilon\phi}{\varepsilon\phi},(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}\right). \quad (19)$$

Proof. The r^{th} incomplete moment of the random variable X is presented as follows,

$$M_r(y)=E\left[X^r / X>y\right]. \quad (20)$$

This is further simplified as,

$$M_r(y)=\int_y^{\infty}x^r f(x)\,dx. \quad (21)$$

Substituting equation (13) into equation (21) yields,

$$M_r(y)=\frac{z\varepsilon\phi\varphi}{B(u,b)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}\int_y^{\infty}x^{r+\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{(j+1)(1-(1+\varphi x^{\phi})^{\varepsilon})}dx.$$

Factorizing gives,

$$M_r(y)=\frac{z\varepsilon\phi\varphi}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{j+1}\int_y^{\infty}x^{r+\phi-1}(1+\varphi x^{\phi})^{\varepsilon-1}e^{-(j+1)(1+\varphi x^{\phi})^{\varepsilon}}dx. \quad (22)$$

Setting $u=(j+1)\left(1+\varphi x^{\phi}\right)^{\varepsilon}$, then $x=\left(\varphi^{-1}\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right]\right)^{1/\phi}$.

Differentiating u with respect to x gives,

$$dx=\frac{du}{\varepsilon\phi\varphi x^{\phi-1}(j+1)\left(1+\varphi x^{\phi}\right)^{\varepsilon-1}}$$

Substituting the expressions for x and dx into equation (22), $x\to\infty, u\to\infty$ and $x\to y$, $u\to(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}$.

$$M_r(y)=\frac{z\varepsilon\phi\varphi}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}e^{j+1}\int_{(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}}^{\infty}\left(\varphi^{-1}\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right]\right)^{r/\phi}x^{\phi-1}\left(1+\varphi x^{\phi}\right)^{\varepsilon-1}e^{-u}\frac{du}{\varepsilon\phi\varphi x^{\phi-1}(j+1)\left(1+\varphi x^{\phi}\right)^{\varepsilon-1}}.$$

Simplifying gives,

$$M_r(y)=\frac{z}{B(u,v)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}\varphi^{-r/\phi}e^{j+1}\left(\frac{1}{j+1}\right)\int_{(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}}^{\infty}\left(\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right)^{r/\phi}e^{-u}du. \quad (23)$$

Dealing with the integral expression,

$$I=\int_{(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}}^{\infty}\left(\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right)^{r/\phi}e^{-u}du.$$

Using the Binomial expansion for the expression in the integral,

$$\left(\left(\frac{u}{j+1}\right)^{1/\varepsilon}-1\right)^{r/\phi}=\sum_{k=0}^{\infty}(-1)^{k}\binom{r/\varepsilon\phi}{k}\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}\right]^{r/\phi-k}.$$

Substituting the expansion into equation (22) gives,

$$M_{r}\left(y\right)=\frac{z}{B(u,v)}\varphi^{-r/\phi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}w_{ij}\left(-1\right)^{k}\binom{r/\varepsilon\phi}{k}e^{j+1}\int_{(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}}^{\infty}\left[\left(\frac{u}{j+1}\right)^{1/\varepsilon}\right]^{r/\phi-k}e^{-u}du.$$

Further simplification gives;

$$M_{r}\left(y\right)=\frac{z}{B(u,v)}\varphi^{-r/\phi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}w_{ijk}e^{j+1}\left(\frac{1}{j+1}\right)^{r-\phi k/\varepsilon\phi}\int_{(j+1)\left(1+\varphi y^{\phi}\right)^{\varepsilon}}^{\infty}u^{\frac{r-\phi k+\alpha\phi}{\varepsilon\phi}-1}e^{-u}du.$$

Thus,

$$M_{r}\left(y\right)=\frac{z}{B(u,v)}\varphi^{-r/\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}w_{ijk}e^{j+1}\left(\frac{1}{j+1}\right)^{r-\phi k/\varepsilon\phi}\Gamma\left(\frac{r-\phi k+\varepsilon\phi}{\varepsilon\phi},\left(j+1\right)\left(1+\varphi y^{\phi}\right)^{\varepsilon}\right).$$

4.5 Order statistics

In most cases, certain sample values, such as the smallest, largest, or middle observations from a random sample, carry significant information about the entire population. For instance, knowing the highest recorded floodwater or the lowest recorded income can be valuable when making development plans. Similarly, the median price of houses sold in the past year can aid in estimating the cost of living. The statistical estimation of these extreme or central values is referred to as order statistics.

Proposition 4. If X has the MGPWD, then the r^{th} order statistic of the MGPWD is given by:

$$f_{X(r)}(x)=\frac{c\alpha\beta\lambda n!}{B\left(a,b\right)\left(n-r\right)!\left(r-1\right)!}\sum_{i=1}^{n-r}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(-1\right)^{i+j+k}\binom{n-r}{i}\binom{b-1}{j}\binom{c\left(j+a\right)-1}{k}$$
$$\left[x^{\beta-1}(1+\lambda x^{\beta})^{\alpha-1}e^{\left((1-(1+\lambda x^{\beta})^{\alpha}\right)k}\right]\left[\frac{\left(1-e^{(1-(1+\lambda x^{\beta})^{\alpha}}\right)^{ac}}{aB\left(a,b\right)}{}_{2}F_{1}\left(a,1-b;a+1;\left(1-e^{(1-(1+\lambda x^{\beta})^{\alpha}}\right)^{c}\right)\right]^{r+i-1}. \quad (24)$$

The r^{th} order statistic of the random sample $X_1,...,X_n$ is the random variable $X_{(r)}$, where $X_{(1)}\leq X_{(2)}\leq...\leq X_{(n)}$ is the ordered sample. Accordingly, the PDF of X_r is,

$$f_{X(r)}(x)=\frac{n!}{(n-r)!(r-1)!}\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r} f_x(x) \quad (25)$$

Using the Binomial expansion for the term,

$$\left[1-F_X(x)\right]^{n-r}=\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}\left[F_X(x)\right]^i.$$

Then substituting it into equation (25) gives,

$$f_{X(r)}(x)=\frac{n!}{(n-r)!(r-1)!}\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}\left[F_X(x)\right]^i\left[F_X(x)\right]^{r-1} f_x(x).$$

It is then simplified as,

$$f_{X(r)}(x)=\frac{n!}{(n-r)!(r-1)!}\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}\left[F_X(x)\right]^{r+i-1} f_x(x) \quad (26)$$

Substituting equations (5) and (6) into equation (26) produces,

$$f_{X(r)}(x)=\frac{n!}{(n-r)!(r-1)!}\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}\left[\frac{\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^{ac}}{aB(a,b)}\,{}_2F_1\left(a,1-b;a+1;\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^c\right)\right]^{r+i-1}$$

$$\left[\frac{c\alpha\beta\lambda}{B(a,b)}x^{\beta-1}(1+\lambda x^\beta)^{\alpha-1}e^{(1-(1+\lambda x^\beta)^\alpha}\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^{ac-1}\left(1-\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^c\right)^{b-1}\right].$$

Using Binomial expansion, for the last expression,

$$\left(1-\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^c\right)^{b-1}=\sum_{j=1}^{\infty}(-1)^j\binom{b-1}{j}\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^{cj}.$$

Substituting it back gives,

$$f_{X(r)}(x)=\frac{n!}{(n-r)!(r-1)!}\sum_{i=1}^{n-r}\sum_{j=1}^{\infty}(-1)^{i+j}\binom{n-r}{i}\binom{b-1}{j}\times$$

$$\left[\frac{\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^{ac}}{aB(a,b)}\,{}_2F_1\left(a,1-b;a+1;\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^c\right)\right]^{r+i-1}\times$$

$$\left[\frac{c\alpha\beta\lambda}{B(a,b)}x^{\beta-1}(1+\lambda x^\beta)^{\alpha-1}e^{(1-(1+\lambda x^\beta)^\alpha}\left(1-e^{(1-(1+\lambda x^\beta)^\alpha}\right)^{c(j+a)-1}\right].$$

Again, using Binomial expansion for the last expression,

$$\left(1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right)^{c(j+a)-1}=\sum_{k=1}^{\infty}(-1)^{k}\binom{c(j+a)-1}{k}\left(e^{\left(1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right)^{k}}\right).$$

Substituting it back gives,

$$f_{X(r)}(x)=\frac{n!}{(n-r)!(r-1)!}\sum_{i=1}^{n-r}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{i+j+k}\binom{n-r}{i}\binom{b-1}{j}\times$$
$$\binom{c(j+a)-1}{k}\left[\frac{c\alpha\beta\lambda}{B(a,b)}x^{\beta-1}(1+\lambda x^{\beta})^{\alpha-1}e^{\left((1-(1+\lambda x^{\beta})^{\alpha}\right)k}\right]\times$$
$$\left[\frac{\left(1-e^{(1-(1+\lambda x^{\beta})^{\alpha}}\right)^{ac}}{aB(a,b)}{}_2F_1\left(a,1-b;a+1;\left(1-e^{(1-(1+\lambda x^{\beta})^{\alpha}}\right)^{c}\right)\right]^{r+i-1}.$$

It is then simplified as,

$$f_{X(r)}(x)=\frac{c\alpha\beta\lambda n!}{B(a,b)(n-r)!(r-1)!}\sum_{i=1}^{n-r}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{i+j+k}\binom{n-r}{i}\times$$
$$\binom{b-1}{j}\binom{c(j+a)-1}{k}\left[x^{\beta-1}(1+\lambda x^{\beta})^{\alpha-1}e^{\left((1-(1+\lambda x^{\beta})^{\alpha}\right)k}\right]\times$$
$$\left[\frac{\left(1-e^{(1-(1+\lambda x^{\beta})^{\alpha}}\right)^{ac}}{aB(a,b)}{}_2F_1\left(a,1-b;a+1;\left(1-e^{(1-(1+\lambda x^{\beta})^{\alpha}}\right)^{c}\right)\right]^{r+i-1}$$

5. Estimation of the Parameters

This section considers the method of parameters estimation and the Monte Carlo simulation procedure.

5.1 Maximum Likelihood estimation

In this section, the Maximum likelihood estimates (MLE) of the unknown parameters of the MGPWD are derived. Let the likelihood function of the MGPWD of the random sample $X_1, X_2, ..., X_n$ be:

$$L(\theta)=\prod_{i=1}^{n}f(x_i,\theta).$$

The logarithm of the likelihood function is obtained as,

$$\ell=n\log\varepsilon+n\log\phi+n\log\varphi+n\log z+n\log\left[\Gamma(u+v)\right]-n\log\Gamma(u)-$$
$$n\log\Gamma(v)+(v-1)\sum_{i=1}^{n}x_i+(\varepsilon-1)\sum_{i=1}^{n}\left(1+\varphi x_i^{\phi}\right)+\left(1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}\right)+$$
$$(uz-1)\sum_{i=1}^{n}\log\left(1-e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)+(v-1)\sum_{i=1}^{n}\left(1-\left(1-e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}\right). \quad (27)$$

Taking the partial derivative of equation (27) with respect to $u, v, z, \varepsilon, \phi$ and φ gives the score functions as:

$$\frac{\partial \ell}{\partial u} = n\psi(u+v) - n\psi(u) + z\sum_{i=1}^{n} \log\left[1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right]. \quad (28)$$

$$\frac{\partial \ell}{\partial v} = n\psi(u+v) + n\psi(v) + \sum_{i=1}^{n} \log\left[1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}\right] + \sum_{i=1}^{n} \log x_i. \quad (29)$$

$$\frac{\partial \ell}{\partial z} = \frac{n}{z} + u\sum_{i=1}^{n} \log\left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right) + (1-u)\sum_{i=1}^{n} \frac{\left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z} \log\left[1 - \left(1+\varphi x_i^{\phi}\right)^{\varepsilon}\right]}{1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}}. \quad (30)$$

$$\frac{\partial \ell}{\partial \varepsilon} = \frac{n}{\varepsilon} + \sum_{i=1}^{n} \log\left(1+\varphi x_i^{\phi}\right) + (uz-1)\sum_{i=1}^{n} \frac{e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}} \log\left[\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}\right]\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}{1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}} -$$

$$(v-1)\sum_{i=1}^{n} \frac{z e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}} \left(1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z-1}\right) \log\left[\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}\right]\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}{1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}}. \quad (31)$$

$$\frac{\partial \ell}{\partial \phi} = \frac{n}{\phi} - \varphi \log[x_i] x_i^{\phi} + (\varepsilon - 1)\sum_{i=1}^{n} \frac{\varphi \log[x_i] x_i^{\phi}}{1+\varphi x_i^{\phi}} +$$

$$(uz-1)\sum_{i=1}^{n} \frac{\left(1+\varphi x_i^{\phi}\right)^{\varepsilon} \varepsilon\varphi \log[x_i] x_i^{\phi} \left(1+\varphi x_i^{\phi}\right)^{\varepsilon-1}}{1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}} +$$

$$(v-1)\sum_{i=1}^{n} \frac{e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}} \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z-1} \varepsilon\varphi \log[x_i] x_i^{\phi} \left(1+\varphi x_i^{\phi}\right)^{\varepsilon-1}}{1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}}. \quad (32)$$

$$\frac{\partial \ell}{\partial \varphi} = \frac{n}{\varphi} - x_i^{\phi} + (\varepsilon - 1)\sum_{i=1}^{n} \frac{x_i^{\phi}}{1 + \varphi x_i^{\phi}} +$$

$$(uz - 1)\sum_{i=1}^{n} \frac{e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}} \varepsilon x_i^{\phi} \left(1 + \varphi x_i^{\phi}\right)^{\varepsilon - 1}}{1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}} - (v - 1) \times$$

$$\sum_{i=1}^{n} \frac{z e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}} \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z-1} \varepsilon x_i^{\phi} \left(1 + \varphi x_i^{\phi}\right)^{\varepsilon - 1}}{1 - \left(1 - e^{1-\left(1+\varphi x_i^{\phi}\right)^{\varepsilon}}\right)^{z}}, \quad (33)$$

where ψ represents the digamma function.

By setting each of these equations to zero and then solving them concurrently produces the maximum likelihood estimates of the parameters. With regards to the interval estimation of the model parameters, the observed information matrix, $J(\theta)$ is obtained. Thus,

$$J^{-1}(\theta) = \begin{pmatrix} J_{uu} & J_{uv} & J_{uz} & J_{u\varepsilon} & J_{u\phi} & J_{u\varphi} \\ . & J_{vv} & J_{vz} & J_{v\varepsilon} & J_{v\phi} & J_{v\varphi} \\ . & . & J_{zz} & J_{z\varepsilon} & J_{z\phi} & J_{z\varphi} \\ . & . & . & J_{\varepsilon\varepsilon} & J_{\varepsilon\phi} & J_{\varepsilon\varphi} \\ . & . & . & . & J_{\phi\phi} & J_{\phi\varphi} \\ . & . & . & . & . & J_{\varphi\varphi} \end{pmatrix}.$$

5.2 Monte Carlo Simulation

In this section, Monte Carlo simulation studies is carried out to analyze the performance of the maximum likelihood estimates of the parameters. The average bias (AB) and root mean square error (RMSE) of the estimates were computed and examined. The simulation was done by using different parameters values as well as varying the sample size. The random numbers of the model were obtained using the quantile function given in equation (12) with the aid of R software. The simulation process was repeated for N = 1000 times each containing sample sizes n = 30, 50, 80, 120, 150, 200 and 250. The values of first and second sets of parameter values are $u = 0.4, v = 0.7, z = 0.5, \varepsilon = 0.2, \phi = 0.8, \varphi = 0.1$ and $u = 0.6, v = 0.9, z = 0.7, \varepsilon = 0.4, \phi = 0.3, \varphi = 1.0$ respectively. The AB and the RMSE of the parameters are computed using the relation:

$$AB = \frac{1}{N}\sum_{i=1}^{n}\left(\hat{\theta}_i - \theta\right)$$

and

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{n}\left(\hat{\theta}_i - \theta\right)^2}.$$

where $\theta = u, v, z, \varepsilon, \phi$ and φ

The values of AB and RMSE for the parameters' estimates are presented in Table 1. It shows the values of the AB and RMSE of the parameters $u, v, z, \varepsilon, \phi, \varphi$ for different sample sizes. It is observed that these estimates decrease and approach zero as the sample size increases. This indicates that the maximum likelihood estimates possess asymptotic properties. Consequently, they can be utilized for constructing confidence intervals.

Table 1: Monte Carlo simulation results for the two sets of parameter values

		I		II	
Parameter	n	AB	RMSE	AB	RMSE
	30	3.0419	5.2633	5.2858	6.7411
	50	1.2420	3.0746	3.0315	5.1567
	80	0.4387	0.7158	1.2092	2.9947
$\hat{u}$	120	0.3674	0.5064	0.3759	0.4845
	150	0.3145	0.4123	0.3418	0.3975
	200	0.2625	0.3426	0.3033	0.3800
	250	0.2312	0.2627	0.2116	0.2394
	30	2.3189	4.0663	3.9776	5.6581
	50	1.2149	2.9505	3.3257	5.0885
	80	1.1194	2.1033	1.9118	2.6794
$\hat{v}$	120	1.0997	1.9592	1.2251	1.7848
	150	0.6668	1.2358	1.2047	1.6612
	200	0.3684	0.4262	1.1030	1.6134
	250	0.2603	0.3798	0.6019	0.6390
	30	4.9871	6.7317	3.8349	5.3319
	50	4.5047	6.1565	2.8025	4.3763
	80	2.9093	4.5538	2.4214	4.2944
$\hat{z}$	120	2.2189	4.2923	1.5473	3.1494
	150	1.6670	3.2069	1.3353	3.0258
	200	1.4356	3.1316	1.3165	2.9934
	250	0.8665	1.2805	0.3995	0.4781
	30	0.2639	0.3260	0.4355	0.7069
	50	0.1661	0.2439	0.3627	0.6258
	80	0.1503	0.2205	0.2693	0.4221
$\hat{\varepsilon}$	120	0.1478	0.2180	0.2207	0.3361
	150	0.1397	0.1936	0.1806	0.2477
	200	0.1086	0.1342	0.1387	0.1607
	250	0.0744	0.1282	0.1105	0.1309
	30	0.0452	0.0531	0.1981	0.2473
	50	0.0433	0.0458	0.1619	0.2233
	80	0.0338	0.0402	0.1417	0.1683
$\hat{\phi}$	120	0.0333	0.0397	0.0996	0.1140
	150	0.0326	0.0375	0.0848	0.0928
	200	0.0242	0.0292	0.0788	0.1140
	250	0.02007	0.0266	0.0668	0.0851

$\hat{\varphi}$	30	7.2214	7.9138	8.1848	8.5424
	50	5.8551	7.1599	4.1605	5.8196
	80	5.0445	6.5723	4.1325	5.7640
	120	2.9026	4.7361	3.5166	5.2485
	150	2.0983	4.1371	3.1063	4.9733
	200	1.0959	3.1342	3.0019	4.5498
	250	0.7368	0.9918	2.3424	4.1036

6. Applications of the MGPWD

This section investigates the applications of the MGPWD in two sets of reliability data. The goodness of fit statistics of the MGPWD is compared to its sub-models, and other existing models obtained from [14]. These are the exponentiated generalized exponential Dagum (EGED), the McDonald Dagum (McD) distribution, and the exponentiated Kumaraswamy Dagum (EKD) using Kolmogorov-Smirnov (K-S), Crame′r-von (W^*) Mises distance value, as well as AIC, BIC, and AICc statistics. To test the goodness-of-fit of the distributions, the maximized values of the log-likelihood $\left(\ell_n\right)$, the K-S statistics with their respective p-values, the AIC, AICc and the BIC. These criteria have the following forms: $AIC = -2\ln \ell_n\left(k\right) + 2k$, $AICc = -2AIC + \frac{2k\left(k+1\right)}{n-k-1}$ and $BIC = -2\ln \ell_n\left(k\right) + k\ln(n)$, where k is the number of parameters and n is the sample size. The estimates of the MGPWD parameters were obtained with the aid of the subroutine mle2 using the bbmle package in [15].

6.1 Data on yarn

The first data set represents the failure times of 100 cm polyester/viscose yarn which were subjected to 2.3% strain level in textile experiment with the aim of assessing the tensile resistance ability of the yarn. The data is obtained from [16] and Pal and [17]. The observations are as follows: 86, 175, 157, 282, 38, 211, 497, 246, 393, 198, 146, 178, 220, 224, 337, 180, 182, 185, 396, 264, 251, 76, 42, 149, 65, 93, 423, 188, 203, 105, 653, 264, 321, 180, 151, 315, 185, 568, 829, 203, 98, 15, 180, 325, 341, 353, 229, 55, 239, 124, 249, 364, 198, 250, 40, 571, 400, 55, 236, 137, 400, 195, 38, 196, 40, 124, 338, 61, 286, 135, 292, 262, 20, 90, 135, 279, 290, 244, 294, 350, 131, 88, 61, 229, 597, 81, 398, 20, 277, 193, 169, 264, 121, 166, 246, 186, 71, 289, 143, and 188.

The descriptive statistics of the yarn data are shown in Table 3. The mean failure time of the data is observed as 223.1 with median and standard deviation values as 197 and 144.8 respectively. The data is right skewed and moderately peaked as compared to the normal distribution.

Table 1: Descriptive statistics of yarn data

Mean	Median	Standard deviation	Skewness	Kurtosis
223.1000	197.0000	144.8000	1.3600	3.0000

The TTT transform graph for the yarn data is presented in Figure 5. The graph indicates that hazard rate function for the yarn data has an increasing hazard failure rate.

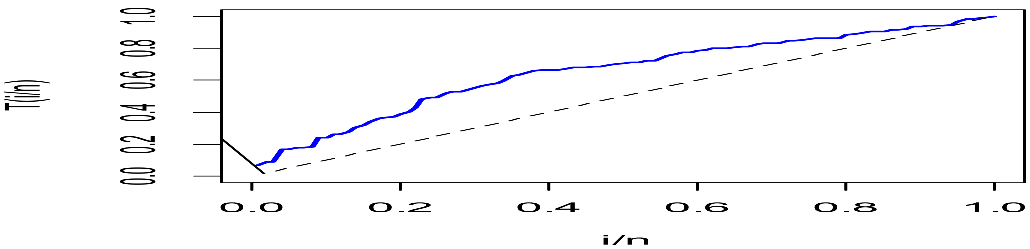

Fig. 1: TTT transform plot of the yarn data set

Table 4 shows the maximum likelihood estimates of the parameters of the MGPWD, MWD, BGPWD, BWD, KGPWD, KWD, EGED, McD and EKD for the yarn data set. The results in Table 4 indicates that two parameters, (α and β) of the parameters MCGPWD are significant at the 5% level of significance.

Table 2: Maximum likelihood estimate for the yarn data

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
MGPW	11.4820	0.6791	2.5278	2.7017	0.6948	0.0133
	(0.0202)	(0.7369)	(0.1557)	(1.8911)	(0.2411)	(0.0126)
MW	12.2145	1.3827	0.2343		0.9192	0.0096
	(0.0633)	(1.4482)	(0.1968)		(0.3943)	(0.0204)
BGPW	3.2697	0.8918		2.7413	0.6734	0.0131
	(1.0992)	(1.1326)		(1.5230)	(0.0687)	(0.0132
BW	2.4967	0.9363			0.9795	0.0088
	(1.5607)	0.8164)			(0.3326)	(0.0156)
KGPW		5.5733	3.5504	1.1308	0.6406	0.0226
		(4.1756)	(3.2422)	(0.9573)	(0.4067)	(0.0572)
KW		5.9366	3.2971		0.6852	0.0196
		(2.6769)	(2.6889)		(0.3165)	(0.0523)
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$

EGED	0.0260	75.3100	0.0170	3.5130	45.6920	0.0900
	(0.0070)	(0.0070)	(0.0050)	(0.6310)	(0.0360)	(0.0110)
	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
McD	0.0270	39.4130	98.7800	0.3330	25.0420	46.2760
	(1.848×10^{-2})	(9.647×10^{-2})	(2.180×10^{-5})	(1.504×10^{-1})	(4.507×10^{-4})	(4.654×10^{-5})
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\theta}$	
EKD	46.1090	39.4130	5.1880	0.2030	31.1690	
	(1.2950)	(5.0060)	(0.961)	(0.0400)	(11.023)	

The log-likelihood, goodness of fit statistics and information criteria of the fitted distributions have been examined and the results are presented in Table 5. The model that has the highest value of the log-likelihood (ℓ) and the least values of the K-S, W*, AICc, BIC, and AIC gives a better fit to the yarn data. The MGPWD is said to have performed better in fitting the yarn data than the other models as shown in Table 5.

Table 3: Log-likelihood and goodness of fit statistics of the yarn data set

Model	]	AIC	AICc	BIC	K-S	W^*
MGWP	**- 625.6300**	**1258.2550**	**1258.1580**	**1265.8860**	**0.0590**	**0.1099**
MW	- 626.4500	1262.9090	1263.5470	1275.9350	0.1019	0.1534
BGPW	- 625.6600	1261.3140	1261.9520	1274.3400	0.0878	0.1125
BW	- 626.42	1260.844	1261.265	1271.264	0.1026	0.1512
KGPW	- 625.6400	1261.2890	1261.9270	1274.3150	0.0884	0.1139
KW	- 625.68	1259.368	1259.789	1269.789	0.0901	0.1166
EGED	-628.1700	1268.336	1269.5530	1283.9670	0.1240	0.2490
MCD	-628.2000	1268.3990	12696160	1284.0300	0.1280	0.2850
EKD	-653.9600	1317.9130	1318.8160	1330.9380	0.1780	0.9850

*Bolded means best based on the selection criteria.

Figure 6 displays the histograms depicting the densities of the fitted models and the empirical cumulative distribution functions. Additionally, the CDFs of the fitted models for the yarn data are also presented in the plot.

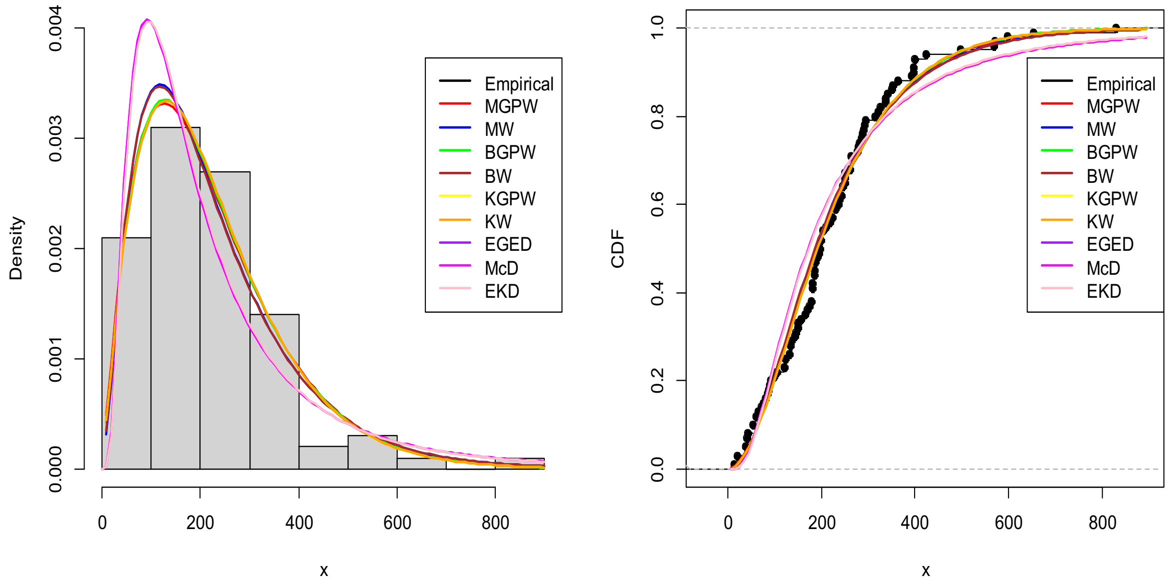

Fig.2: PDFs and CDFs plots of the yarn data set

Also, the variance-covariance matrix for the parameter estimates of the MGPWD for the yarn data is given as follows;

$$J^{-1}=\begin{pmatrix} 4.089\times10^{-4} & 3.778\times10^{-3} & 2.010\times10^{-3} & 3.614\times10^{-2} & -4.204\times10^{-3} & 7.361\times10^{-5} \\ 3.778\times10^{-3} & 5.431\times10^{-1} & 6.307\times10^{-2} & -5.096\times10^{-2} & -5.854\times10^{-2} & -5.835\times10^{-5} \\ 2.010\times10^{-3} & 6.307\times10^{-2} & 2.426\times10^{-2} & 1.130\times10^{-1} & -3.335\times10^{-2} & 1.455\times10^{-3} \\ 3.614\times10^{-2} & -5.096\times10^{-2} & 1.130\times10^{-1} & 3.576 & -3.236\times10^{-1} & 3.598\times10^{-3} \\ -4.204\times10^{-3} & -5.854\times10^{-2} & -3.335\times10^{-2} & -3.236\times10^{-1} & 5.813\times10^{-2} & -2.144\times10^{-3} \\ 7.361\times10^{-5} & -5.835\times10^{-5} & 1.455\times10^{-3} & 3.598\times10^{-3} & -2.144\times10^{-3} & 1.583\times10^{-5} \end{pmatrix}$$

The variances of the maximum likelihood estimates of the parameters of the MGPWD are: $\operatorname{var}(\hat{a})=4.089\times10^{-4}$, $\operatorname{var}(\hat{b})=5.431\times10^{-1}$, $\operatorname{var}(\hat{c})=2.426\times10^{-2}$, $\operatorname{var}(\hat{\alpha})=3.576$, $\operatorname{var}(\hat{\beta})=5.813\times10^{-2}$, and $\operatorname{var}(\hat{\lambda})=1.583\times10^{-5}$. The 95% confidence intervals for the parameters a,b,c,α,β and γ of the MGPWD are estimated and presented respectively as follows: (11.4820, 11.5216), (0.7652, 2.1234), (2.226, 2.8330), (0, 6.4082), (0.2222, 1.1674) and (0, 0.0380).

The P-P plot of the fitted models is shown in Figure 7. The plots show that the MGPWD has performed better in fitting the yarn data better veras compared to other models.

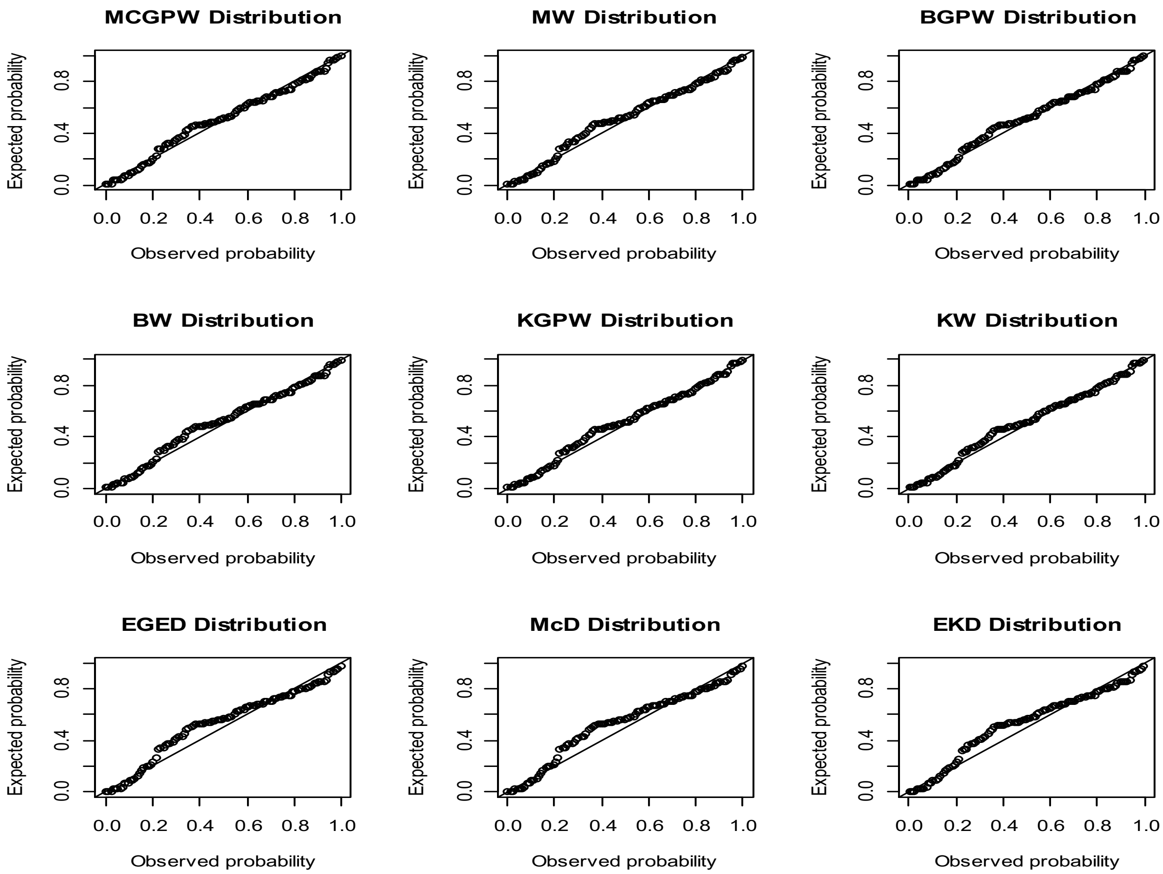

Fig.3: P-P plots of the yarn data set

6.2 Data set on failure times of appliances

The second data set applied to the MGPWD is obtained from [18]. The data consist of failure times for 36 appliances subjected to an automatic life test. The observations are as follows: 11, 1990, 2831, 35, 2223, 3034, 49, 2327, 3059, 170, 2400, 3112, 329, 2451, 3214, 381, 2471, 3478, 708, 2551, 3504, 958, 2565, 4329, 1062, 2568, 6367, 1167, 2694, 6976, 1594, 2702, 78,46, 1925, 2761 and 13403.

The descriptive statistics are shown in Table 6. The mean failure rate was observed to be 2757 with median and standard deviation values of 2511 and 2569 respectively. The results from Table 6 further show the appliance data set is positively skewed and highly peaked than the normal distribution.

Table 4: Descriptive statistics of the appliance data

Mean	Median	Standard deviation	Skewness	Kurtosis
2757.0000	2511.0000	2569.0000	2.3800	7.8800

Figure 8 illustrates the TTT transform curve of the appliance dataset. The curve showcases a convex shape initially, followed by a concave shape, and then another convex shape. This pattern indicates that the failure rate of the dataset exhibits a modified bathtub.

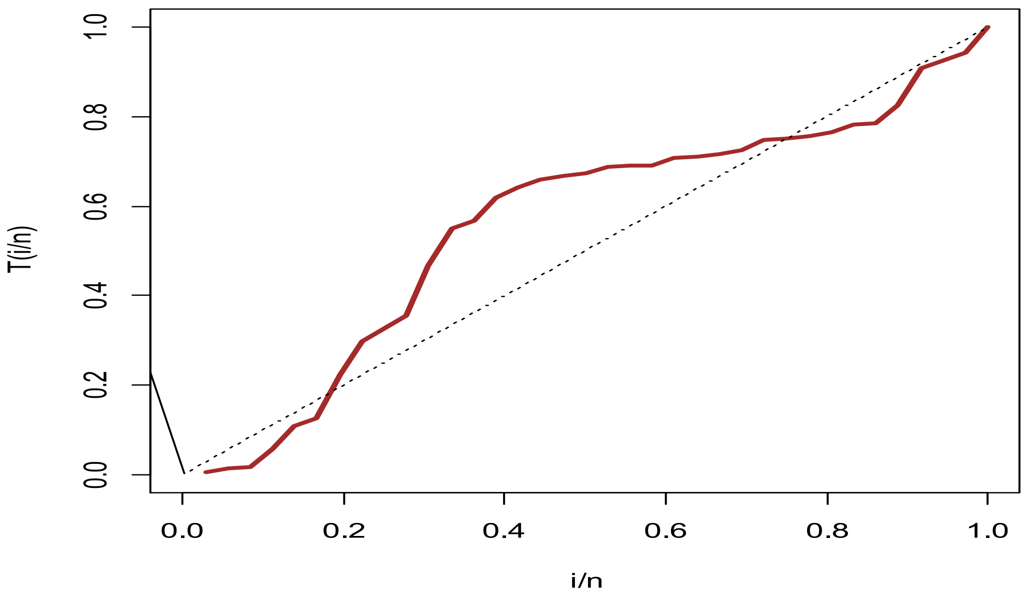

Fig.4: TTT transform plots of the appliance data

Table 7 presents the maximum likelihood estimates for the parameters of the MGPWD and other fitted models for the appliance dataset. The corresponding standard errors are provided in brackets. for the MGPWD and the other models fitted to the appliance data set. Upon examining Table 7, it can be observed that the majority of the parameters are statistically significant at a 5% level of significance considering the standard errors.

Table 5: Maximum likelihood parameter estimates for the appliance data

Model	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
MGPW	0.3663	7.8374	5.2830	3.7109	0.4531	0.0044
	(0.1894)	(0.0031)	(0.0042)	(0.0032)	(0.1540)	(0.0063)
MW	0.3137	11.0593	5.5835		0.5478	0.0086
	(0.1798)	(0.0040)	(0.0122)		(0.2042)	(0.0163)
BGPW	1.4915	2.0429		2.0235	0.6512	0.0018
	(0.8839)	(0.2355)		(1.3350)	(0.2742)	(0.0045)

BW	1.6156	0.9638			0.7216	0.0049
	(0.4852)	(0.7613)			(0.1498)	(0.0095)
KGPW		27.7274	7.5513	1.2248	0.1955	0.1652
		(15.2798)	(4.3676)	(0.4375)	(0.0589)	(0.0855)
KW		5.7371	2.2307		0.5340	0.0085
		(0.2850)	(1.6303)		(0.2778)	(0.0258)
		(2.6769)				
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{c}$	$\hat{d}$
EGED	0.001	27.1980	4.5600	2.8380	20.8660	0.0700
	(1.00×10^{-4})	(0.001)	(0.8470)	(0.1230	(0.0100)	(0.0030)
	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{c}$
McD	1.4270	3.4550	1.2750	10.5050	0.0640	500.5560
	(0.0920)	(0.2120)	(6.8750)	(56.9060)	(0.0120)	(6.7960)
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{\theta}$	
EKD	5.5620	12.6830	3.7160	0.1280	11.6090	
	(1.5170)	(2.1580)	(0.7550)	(0.029)	(3.9220)	

Table 8 reveals that the MGPWD provided a superior fit to the appliance data compared to the other models. This can be deduced from its highest log-likelihood value and smallest values of AIC, BIC, AICc, W*, and K-S.

Table 6: Log-likelihood, the goodness of fit statistics, and information criteria of the appliance data

Model	ℓ	AIC	AICc	BIC	K-S	W^*
MGWPD	**- 320.5400**	**653.0836**	**655.9801**	**662.5847**	**0.1831**	**0.2704**
MWD	- 320.7700	653.5427	656.4392	663.0438	0.1937	0.2826
BGPWD	- 321.6100	655.2193	658.1158	664.7204	0.2186	0.3184
BWD	- 322.5100	657.0155	659.9120	666.5166	0.2174	0.3571
KGPWD	- 322.9200	657.8478	660.7444	667.3490	0.1963	0.3720

KWD	- 322.2300	656.4619	659.3585	665.9630	0.2081	0.3456
EGED	-328.8700	669.7400	670.9570	679.2410	0.2530	0.5690
McD	-356.4800	724.9550	728.9500	734.4560	0.3470	0.9860
EKD	-341.6500	693.2950	694.1980	701.2130	0.2690	0.9250

*Bolded means best based on the selection criteria.

Figure 9 displays histograms overlaid with the densities of the fitted models, as well as the empirical CDFs of the fitted models of the appliance data. It is noticeable that the MGPWD distribution closely resembles the empirical density and CDF of the appliance dataset.

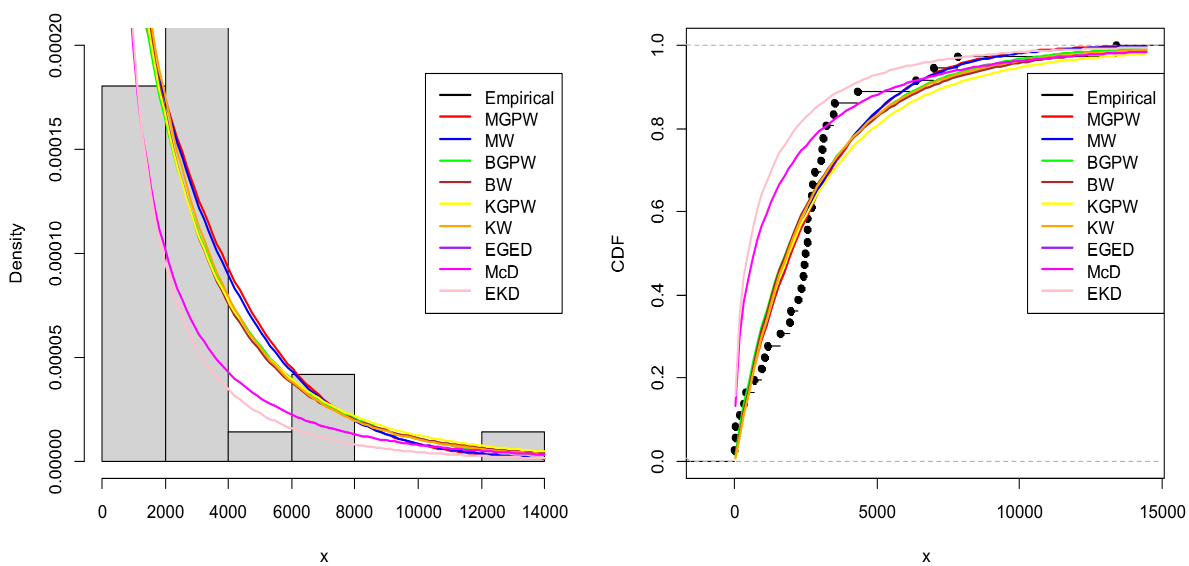

Fig. 5: PDFs and CDFs of the appliance data

The variance-covariance matrix for the parameter estimates of the MGPWD for the appliance data is represented by;

$$J^{-1}=\begin{pmatrix} 3.5876\times10^{-2} & -5.6731\times10^{-4} & -5.7509\times10^{-4} & -5.9090\times10^{-4} & -2.6819\times10^{-2} & 1.1125\times10^{-3} \\ -5.6731\times10^{-4} & 9.3446\times10^{-6} & 1.0902\times10^{-5} & 9.8306\times10^{-6} & 4.6105\times10^{-4} & -1.8959\times10^{-5} \\ -5.7509\times10^{-4} & 1.0902\times10^{-5} & 1.7970\times10^{-5} & 1.1827\times10^{-5} & 6.0876\times10^{-4} & -2.4448\times10^{-5} \\ -5.9090\times10^{-4} & 9.8306\times10^{-6} & 1.1827\times10^{-5} & 1.0366\times10^{-5} & 4.8986\times10^{-4} & -2.0104\times10^{-5} \\ -2.6819\times10^{-2} & 4.6105\times10^{-4} & 6.0876\times10^{-4} & 4.8986\times10^{-4} & 2.3704\times10^{-2} & -9.6685\times10^{-4} \\ 1.1125\times10^{-3} & -1.8959\times10^{-5} & -2.4448\times10^{-5} & -2.0104\times10^{-5} & -9.6685\times10^{-4} & 3.9591\times10^{-5} \end{pmatrix}$$

The variances of the maximum likelihood estimates of the appliance data of the MGPWD are as follows: $\mathrm{var}(\hat{a})=3.5876\times10^{-2}$, $\mathrm{var}(\hat{b})=9.3446\times10^{-6}$, $\mathrm{var}(\hat{c})=1.7970\times10^{-5}$, $\mathrm{var}(\hat{\alpha})=1.0366\times10^{-5}$, $\mathrm{var}(\hat{\beta})=2.3704\times10^{-2}$ and $\mathrm{var}(\hat{\lambda})=3.9591\times10^{-5}$. The 95% confidence intervals for the parameters a, b, c, α, β and λ were estimated and restively given as (0, 0.7375), (7.8313, 7.8435), (5.2748, 5.2912), (3.7046, 3.7153), (0.1513, 0.7549) and (0, 0.0168).

Figure 10 shows the P-P plots for the fitted models. It can be observed that MGPWD gives a better fit to the data as it most part of its point on the diagonal line.

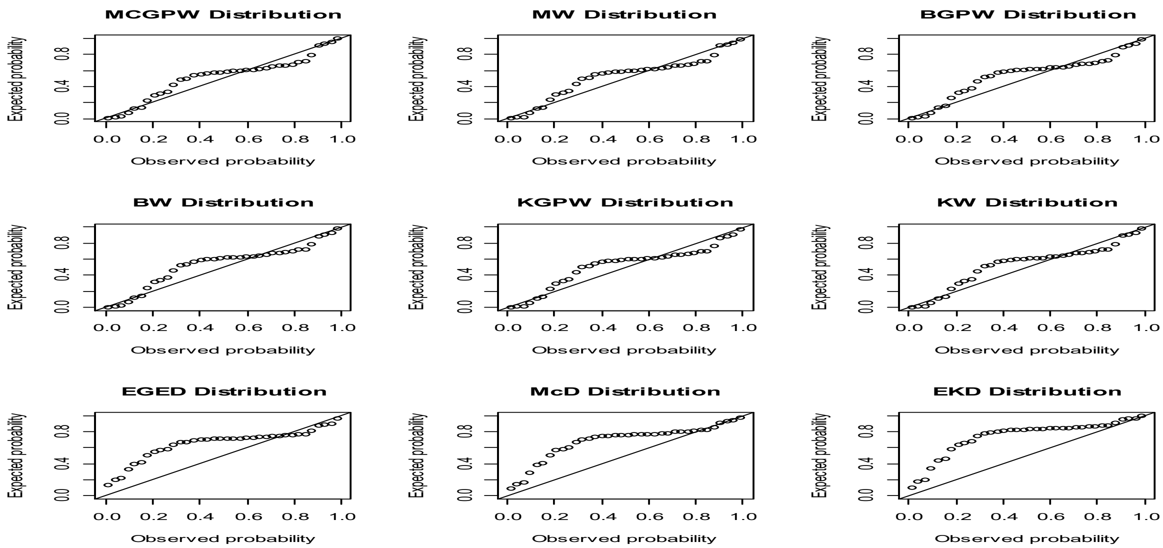

Fig. 6: P-P plots of the appliance data

7. Conclusion

The research introduced and examined the statistical properties of the MGPWD, a distribution with several sub-models that find applications in survival and reliability data analysis. Various statistical properties such as moments, incomplete moments, quantiles, moment generating functions, and order statistics were derived for the MGPWD. The parameters of the model were estimated using maximum likelihood estimation. The plots of the probability density function and hazard functions indicated that the proposed model is well-suited for modeling survival and reliability data sets exhibiting both monotonic and non-monotonic failure rates. The application of the MGPWD was demonstrated using real-world data, highlighting its practicability. Further research is recommended to explore alternative estimation methods for the proposed model.

References

[1] S. Nasiru, P N. Mwita, and O. Ngesa, Discussion on generalized modified inverse Rayleigh. *An International Journal of Applied Mathematics and Information Sciences*. **12**(1), 113-124, 2018.

[2] S. Nasiru, P. N. Mwita, and O. Ngesa, Exponentiated generalized exponential Dagum distribution. *Journal of King Saud University*, **31**(3), 2017.

[3] Y. Amadu, A. Luguterah and S. Nasiru, Odd inverse exponential class of distributions: properties and cure regression. *Journal of Statistics and Management Systems.* https:/doii.org /10.1080/09720510.2021.1923944, 2022.

[4] Z. Mahmood, T. M. Jawa, N. Sayed-Ahmed, E. M. Khalid, A. H. Muse and A. H. Tolba, An extended cosine generalized family of distributions for reliability modeling: characteristics and applications with a simulation study. *Hindawi, Mathematical problems in Engineering*, 2022,

[5] M. A. Aldahlan, F. Jamal, C. Chesneau, I. Elbatal, and M. Elgarhy, Exponentiated power generalized Weibull power series family of distributions: properties, estimation, and applications. *PLoS One*, **15**(3), 2020.

[6] M. A. Salim, The generalized power generalized Weibull distribution: properties and application. *arXiv preprint arXiv*: 1807.10763, 2020.

[7] F. Merovci and I. Elbatal, McDonald Modified Weibull Distribution: properties and applications. *arXiv*: 1309.2961v1 [stat.ME], 2013.

[8] M. Elizabeth, E. Hashimoto, E. M. Ortega, G. M. Cordeiro, and A. R. P. Marcelino, McDonald extended Weibull distribution. *Journal of Statistical Theory and Practice*. **9**(3), 608–632, 2014.

[9] G. M. Cordeiro, R. J. Cintra, L. C. Rego and E. M. M. Ortega, The McDonald Normal Distribution. *Pakistan Journal of Operation Research*. **8**(3), 301-329, 2012.

[10] M. Nikulin and F. Haghighi, A Chi-squared test for the generalized power Weibull family for the head-and-neck cancer censored data. *Journal of Mathematical Science*. **133**(3), 1333-1341, 2006.

[11] N. Eugene, C. Lee and F. Famoye, Beta-normal distribution, and its applications. Communications in Statistics: *Theory and Methods*. **31**(4), 497-512, 2002.

[12] G. M. Cordeiro and M. D. Castro, A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, **81**, 883-898, 2011.

[13] I. S. Gradshtyn and I. M. Ryzhik,. *Table of Integral*s, Series, and Products. 6th edition New York: Academic, 2000.

[14] Q. Liao, Z. Ahmad, E. Mahamoudi and G. G. Hamedani, A new flexible bathtub-shaped modification of the Weibull model: properties and applications. *Mathematical Problems in Engineering*, 2020. Article ID3206257, https//:doi.org/10.1155/3206257.

[15] B. Bolker, *Tools for general maximum likelihood estimation.* R development core team, 2014.

[16] C. P. Quesenberry and J. Kent, Selecting among probability distributions used in reliability. *Technometrics*. **24**(1), 59-65, 1982.

[17] M. Pal and M. Tiensuwan, Beta transmuted exponentiated Weibull geometric distribution. *Austrian Journal of Statistics*. **43**(2), 133-149, 2014.

[18] J. F. Lawless, *Statistical models and methods for lifetime data*. Wiley, New York, 1982.