Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/120322

Reliability of a k-out-of-n: G System Subjected to Marshall-Olkin Type Shocks Concerning Magnitude

S. M. Bakry[1,*] *and N. A. Mokhlis*[2]

[1]Basic Science Department, Faculty of Computer and Information Science, Ain Shams University, Cairo, Egypt
[2]Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

Received: 24 Jul. 2022, Revised: 26 Dec. 2022, Accepted: 7 Jan. 2023
Published online: 1 Sep. 2023

Abstract: In this paper the reliability of a k-out-of-n: G system under the effect of shocks having the Marshall-Olkin type shock models, is studied. The magnitudes of the shocks are considered. The system contains n components and only functions when at least k of these components function. The system is subjected to $(n+1)$ shocks coming from $(n+1)$ different sources. The shock coming from the i^{th} source may destroy the i^{th} component, $i = 1,\ldots,n$, while the shock coming from the $(n+1)^{th}$ source may destroy all components simultaneously. A shock is fatal, destroys a component (components), whenever its magnitude exceeds an upper threshold. The system reliability is obtained by considering the arrival time and the magnitude of a shock as a bivariate random variable. It is assumed that the bivariate random variables representing the arrival times and the magnitudes of the shocks are independent with non-identical bivariate distributions. Since the computation of the reliability formula obtained is not easy to handle, an algorithm is introduced for calculating the reliability formula. The reliability of a k-out-of-n: G system subjected to independent and identical shocks is obtained as a special case, as well as the reliabilities of the series and the parallel systems. As an application, the bivariate exponential Gumbel distribution is considered. Also, numerical illustrations are performed to highlight the results obtained.

Keywords: Bivariate exponential Gumbel distribution, k-out-of-n: G system, Order statistics, System reliability, Marshall-Olkin type shocks.

Notations

n:	The number of system components.
k:	The minimum number of working components necessary for the system operation.
T_i:	The arrival time of the shock that affects the i^{th}component, $i = 1,\ 2,\ \ldots,\ n$.
D_i:	The magnitude of the shock that affects the i^{th}component, $i = 1,\ 2,\ \ldots,\ n$.
T_{n+1}:	The arrival time of the shock affecting all components, simultaneously.
D_{n+1}:	The magnitude of the shock affecting all components, simultaneously.
d:	Upper threshold of magnitude. This means that the i^{th} component fails only if $D_i > d,\ i = 1,2,\ldots,n$, and all components fail simultaneously only if $D_{n+1} > d$.
X_i:	The arrival time of the i^{th} shock which has magnitude greater than d (fatal shock), $i = 1,\ 2,\ \ldots,n+1$.
$R_{[\,k\,:\,n\,],\ d}(t)$:	The reliability of a k-out-of-n: G system under the effect of Marshall-Olkin type shock models with the effect of shock magnitude.

1 Introduction

The Marshal-Olkin type shock models were first introduced by Marshall and Olkin [1], where they obtained the joint survival function of a system consisting of two components that are subjected to fatal and non-fatal shocks, and three independent Poisson processes governing the occurrence of those shocks. The first process coincides with the shock affecting the first component, the second process coincides with the shock affecting the second component, and the third process coincides with the shock affecting both components. Due to the practicality of these shocks in everyday life, many researchers have studied the lifetime of systems subjected to these shocks. Ozkut and Bayramoglu [2] obtained the joint

* Corresponding author e-mail: soad.bakry@cis.asu.edu.eg

survival function of the lifetimes of two components under the effect of Marshall-Olkin type shock models, considering the magnitude of the shocks. Bayramoglu and Ozkut [3] investigated the reliability of a k-out-of-n: F system consisting of components under the effect of Marshall-Olkin type shock models, without taking in consideration the magnitudes of the shocks. They obtained the reliability of the system assuming that the times of the shocks are independent, identically distributed, and exchangeable random variables. Bayramoglu and Ozkut [4] obtained the functions of the mean residual life and the mean inactivity time of the system described in Bayramoglu and Ozkut [3]. Ozkut and Eryilmaz [5] obtained the joint survival function of the components of a 2-component system subjected to Marshall-Olkin run shocks type.

However, the reliability of systems subjected to other types of shocks has attracted several authors. Sheu and Liou [6] studied the optimal replacement policy of a k-out-of-$n:F$ system under the effect of shocks arriving according to the nonhomogeneous Poisson process. Skoulakis [7] obtained the reliability of a parallel system subjected to shocks generated by a renewal point process. Huang, Jin, He and He [8] discussed the reliability of a coherent system subjected to internal failures and external random shocks. Eryilmaz and Devrim [9] derived the reliability of a k-out-of-$n:G$ system subjected to shocks destroying a random number of components.

Although, in practical situations, the magnitude of a shock must be taken into consideration, we see that most of the previous studies on Marshall-Olkin type shock models do not consider the magnitude of the shock. In the present article, we obtain the reliability of a k-out-of-n: G system under the effect of shocks having the Marshall-Olkin type shock models, and the magnitude of the shocks has a key role in our work. The k-out-of-n: G system is a system contains n components, and the system functions whenever k or more components function. Series and parallel systems are special cases of such a system. An n-out-of-n: G system is a series system, and 1-out-of-n: G system is a parallel system. A k-out-of-$n:G$ system can be seen in many practical applications such as transportation, redundant networks, manufacturing, production management, transmission, telecommunication systems, and services. The shocks coming at random times will be fatal if their magnitudes are greater than upper threshold d. So, we are interested in the arrival time of the fatal shock, which has a magnitude greater than the upper threshold d. We show that the reliability of the system depends on the joint distribution of the arrival times of the fatal shocks and their magnitudes.

The paper is organized as follows. In Section 2, the reliability formula of the system is obtained assuming the arrival times and the magnitudes of the different shocks to be independent and non-identical bivariate random variables. The reliabilities of the series and the parallel systems are derived as special cases. In Section 3, the reliabilities of the k-out-of-n: G, series, and parallel systems are obtained for the case of identical shocks. In Section 4, an algorithm is introduced for computing the reliability of the systems. As an application, in Section 5 we assume that the times and the magnitudes of the shocks have independent (non-identical and identical) bivariate exponential Gumbel distributions. Under this assumption the exact formulas of $R_{[\,k\,:\,n\,],\,d}(t)$, $R_{[\,n\,:\,n\,],\,d}(t)$, and $R_{[\,1\,:\,n\,],\,d}(t)$ are obtained. Numerical illustrations of the theoretical results are presented in Section 6, showing the effect of the time, and the magnitude of the shocks, as well as the effect of the parameters of the bivariate distributions on the reliability formulas. Finally, in Section 7, a conclusion is presented.

2 Reliability of the System Formula Subjected to Non-Identical Shocks

The following figure displays the shocks affecting the system under consideration

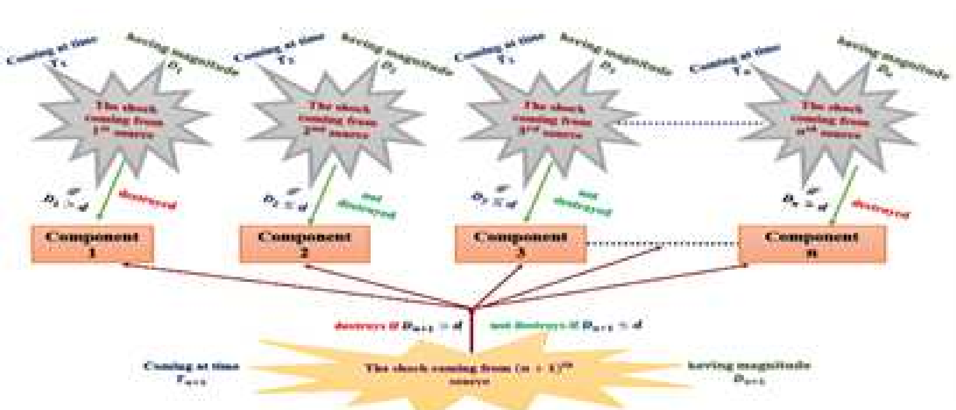

Fig. 1: A system with n components subjected to Marshall-Olkin type shocks concerning magnitude.

Clearly, we have $(n+1)$ independent bivariate random vectors $(T_i, D_i),\ i = 1, \ldots, n+1$. Thus, $R_{[\,k\,:\,n\,],\,d}(t)$ is given by the following theorem.

Theorem 1. *Assume that the bivariate random vectors (T_i, D_i) defining the arrival time and the magnitude of shock i, $i = 1, \ldots, n+1$, are independent but non-identically distributed. Shock i affects component i, $i = 1, \ldots, n$ while shock $(n+1)$ affects all components simultaneously. A shock is fatal if its magnitude exceeds an upper threshold of magnitude d. The reliability of a k-out-of-n: G system under such situation is given by*

$$\begin{aligned} R_{[k:n],d}(t) =& \overline{F}_{T_{n+1}}(t|D_{n+1} > d)\left(1 - \left(\sum_{i=n-k+1}^{n-1} \frac{1}{i!(n-i)!} \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \sum_{j_i=j_{i-1}+1}^{n} \prod_{L=j_1,\ldots,j_i} [1 - \overline{F}_{T_L}(t|D_L > d)]\right.\right. \\ &\left.\left. \times \prod_{\substack{1 \le u \le n \\ u \ne j_1 \ne \cdots \ne j_i}} \overline{F}_{T_u}(t|D_u > d) + \prod_{L=1}^{n} [1 - \overline{F}_{T_L}(t|D_L > d)]\right)\right) \end{aligned} \tag{1}$$

where, $\overline{F}_{T_i}(t|D_i > d)$ is the conditional survival function of T_i given $D_i > d$, $i = 1, \ldots, n+1$.

Proof. Under the conditions of the shocks described above and the structure of the system, the system fails on receiving a fatal shock from source $(n+1)$ or on the failure of the $(n-k+1)^{th}$ component. Thus,

$$R_{[k:n],d}(t) = p\{X_{n+1} > t\}\ p\{X_{n-k+1:n} > t\} \tag{2}$$

where,

$$X_i = (T_i|D_i > d),\ i = 1,\ 2, \ldots, n+1, \tag{3}$$

and $X_{n-k+1:n}$ denotes the $(n-k+1)^{th}$ order statistic structured from the random variables $X_1, \ldots, X_n$. We have

$$p\{X_{n-k+1:n} > t\} = 1 - \sum_{i=n-k+1}^{n} \frac{1}{i!(n-i)!} \sum_{\varphi_{1,2,\ldots,n}} \prod_{L=1}^{i} [1 - p\{X_{j_L} > t\}] \prod_{u=i+1}^{n} p\{X_{j_u} > t\}, \tag{4}$$

where the summation $\varphi_{1,2,\ldots,n}$ extends over all $n!$ permutations $(j_1, \ldots, j_n)$ of $1, \ldots, n$ for which $j_1 < \cdots < j_i$ and $j_{i+1} < \cdots < j_n$, see David and Nagaraja [10]. Substituting (4) in (2) we get

$$R_{[k:n],d}(t) = p\{X_{n+1} > t\}\left[1 - \sum_{i=n-k+1}^{n} \frac{1}{i!(n-i)!} \sum_{\varphi_{1,2,\ldots,n}} \prod_{L=1}^{i} [1 - p\{X_{j_L} > t\}] \prod_{u=i+1}^{n} p\{X_{j_u} > t\}\right] \tag{5}$$

Equation (5) can be rewritten as follows

$$R_{[k:n],d}(t) = p\{X_{n+1} > t\}\left[1 - \sum_{i=n-k+1}^{n} \frac{1}{i!(n-i)!} \sum_{1 \le j_1 < j_2 < \cdots < j_i \le n} \prod_{L=j_1,\ldots,j_i} [1 - p\{X_L > t\}] \prod_{\substack{1 \le u \le n \\ u \ne j_1 \ne \cdots \ne j_i}} p\{X_u > t\}\right]$$

and then we have

$$\begin{aligned} R_{[k:n],d}(t) =& p\{X_{n+1} > t\}\left[1 - \left(\sum_{i=n-k+1}^{n-1} \frac{1}{i!(n-i)!} \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \sum_{j_i=j_{i-1}+1}^{n} \prod_{L=j_1,\ldots,j_i} [1 - p\{X_L > t\}]\right.\right. \\ &\left.\left. \times \prod_{\substack{1 \le u \le n \\ u \ne j_1 \ne \cdots \ne j_i}} p\{X_u > t\} + \prod_{L=1}^{n} [1 - p\{X_L > t\}]\right)\right] \end{aligned} \tag{6}$$

Using (3) in (6), we get

$$R_{[k:n],d}(t) = p\{T_{n+1} > t|D_{n+1} > d\}\left[1 - \left(\sum_{i=n-k+1}^{n-1} \frac{1}{i!(n-i)!} \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \sum_{j_i=j_{i-1}+1}^{n} \prod_{L=j_1,\ldots,j_i} [1 - p\{T_L > t|D_L > d\}] \right.\right.$$
$$\left.\left. \times \prod_{\substack{1 \le u \le n \\ u \ne j_1 \ne \cdots \ne j_i}} p\{T_u > t|D_u > d\} + \prod_{L=1}^{n} [1 - p\{T_L > t|D_L > d\}]\right)\right]$$

Clearly $p\{T_i > t|D_i > d\} = \overline{F}_{T_i}(t|D_i > d),\ i = 1,\ 2,\ \ldots, n+1.$
Thus, the proof is completed.

Special Cases

On substitution with $k = n$ and $k = 1$ in Equation (1), we get the reliabilities of the series and the parallel systems, respectively.

$$R_{[n:n],d}(t) = \overline{F}_{T_{n+1}}(t|D_{n+1} > d)\left[1 - \left(\sum_{i=1}^{n-1} \frac{1}{i!(n-i)!} \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \sum_{j_i=j_{i-1}+1}^{n} \prod_{L=j_1,\ldots,j_i} [1 - \overline{F}_{T_L}(t|D_L > d)] \right.\right.$$
$$\left.\left. \times \prod_{\substack{1 \le u \le n \\ u \ne j_1 \ne \cdots \ne j_i}} \overline{F}_{T_u}(t|D_u > d) + \prod_{L=1}^{n} [1 - \overline{F}_{T_L}(t|D_L > d)]\right)\right], \tag{7}$$

and

$$R_{[1:n],d}(t) = \overline{F}_{T_{n+1}}(t|D_{n+1} > d)\left(1 - \prod_{L=1}^{n} [1 - \overline{F}_{T_L}(t|D_L > d)]\right) \tag{8}$$

3 Reliability of the System Formula Subjected to Identical Shocks

Theorem 2. *Assume that the bivariate random vectors (T_i, D_i) defining the arrival time and the magnitude of shock i, $i = 1, \ldots, n$, are independent and identically distributed. Assume that the bivariate random vector (T_{n+1}, D_{n+1}) defining the arrival time and the magnitude of shock $(n+1)$ is independent of (T_i, D_i), $i = 1, \ldots, n$. Shock i affects component i, $i = 1, \ldots, n$ while shock $(n+1)$ affects all components simultaneously. A shock is fatal if its magnitude exceeds an upper threshold magnitude d. The reliability of a k-out-of-n: G system under such situation is given by*

$$R_{[k:n],d}(t) = \overline{F}_{T_{n+1}}(t|D_{n+1} > d)\left(1 - \sum_{i=n-k+1}^{n} \binom{n}{i} [1 - \overline{F}_T(t|\ D > d)]^i [\overline{F}_T(t|D > d)]^{n-i}\right), \tag{9}$$

where $\overline{F}_T(t|D > d) = p(T > t|D > d) = p(T_i > t|D_i > d)$, $i = 1, \ldots, n$.

Proof. Since (T_i, D_i), $i = 1, \ldots, n$ are assumed to be independent and identical random vectors, then clearly $X_i's$, $i = 1, 2, \ldots, n$ are independent and identical random variables. Using (2), with

$$p\{X_{n-k+1:n} > t\} = 1 - \sum_{i=n-k+1}^{n} \binom{n}{i} [1 - p\{X > t\}]^i [p\{X > t\}]^{n-i},$$

where $X_{n-k+1:n}$ is the $(n-k+1)^{th}$ order statistic structured from the random variables $X_1,\dots,X_n$. We get

$$R_{[k:n],d}(t) = p\{X_{n+1} > t\}\left[1 - \sum_{i=n-k+1}^{n} \binom{n}{i}[1-p\{X>t\}]^i[p\{X>t\}]^{n-i}\right] \tag{10}$$

Using (3) in (10), we have

$$R_{[k:n],d}(t) = p\{T_{n+1} > t|D_{n+1} > d\}\left[1 - \sum_{i=n-k+1}^{n} \binom{n}{i}[1-p\{T>t|D>d\}]^i[p\{T>t|D>d\}]^{n-i}\right].$$

Thus, the proof is completed.

*Remark.*We can get the same result obtained in Equation (9) directly from Equation (1). Clearly the summation in Equation (1) that extends over all $n!$ permutations $(j_1,\dots,j_n)$ of $1,\dots,n$ for which $j_1 < \dots < j_i$ and $j_{i+1} < \dots < j_n$ becomes $n!\prod_{L=1}^{i}[1-\overline{F}_T(t|D>d)]\prod_{u=i+1}^{n}\overline{F}_T(t|D>d)$.

Special Cases

Substitution with $k=n$ and $k=1$ in Equation (9), we get the reliabilities of the series and the parallel systems, respectively, for identical shocks, as

$$R_{[n:n],d}(t) = \overline{F}_{T_{n+1}}(t|D_{n+1} > d)[\overline{F}_T(t|D>d)]^n, \tag{11}$$

and

$$R_{[1:n],d}(t) = \overline{F}_{T_{n+1}}(t|D_{n+1} > d)(1-[1-\overline{F}_T(t|D>d)]^n) \tag{12}$$

4 An Algorithm for Computation of the Reliability of the System in the Case of Non-Identical Shocks

The computation of the formula of $R_{[k:n],d}(t)$ in (1) necessitates computing all the possibilities satisfying $1 \le j_1 < \dots < j_i \le n$ for $i = n-k+1,\dots,n-1$, which is not easy to handle directly even for moderate values of n and k. To solve this problem, we introduce the algorithm presented below. This is applicable to many software's, such as R-Programming, Python, and MATLAB. We have applied the R-Programming. The following notations are used in the construction of the algorithm.

i : Number of failed components, $i = n-k+1,\dots,n-1$.
M : Total number of possible combinations of $1 \le j_1 < \dots < j_i \le n$, $i = n-k+1,\dots,n-1$.
P : A matrix of M rows and i columns. Each row in P represent a possible combination of $1 \le j_1 < \dots < j_i \le n$, $i = n-k+1,\dots,n-1$.

The steps of the algorithm are as follows:

1: Input: "Enter the following values"
$\boldsymbol{n},\boldsymbol{k}$:integers;
$\boldsymbol{t}$, $\boldsymbol{d}$:numeric;

2: Output: The values of reliability formulas.

3: Compute the conditional survival function of $\boldsymbol{T_{n+1}}$ given $\boldsymbol{D_{n+1}} > d$
Conditional survival function of $n+1 \leftarrow$ conditional survival function of T_{n+1} given

$$D_{n+1} > d$$

4: Set productL $\leftarrow$ 1: "Initialize the product of L terms to 1"
Set productu $\leftarrow$ 1: "Initialize the product of u terms to 1"
Set productLu $\leftarrow$ 1: "Initialize the product of L and u terms to 1"
Set sump $\leftarrow$ 0: "Set the sum counter of all possibilities occurs at each i to 0"
Set totalsum $\leftarrow$ 0: "Set the sum counter of all terms to 0"

5:Given a vector of size $\boldsymbol{n}$, generate and print all possible combinations of $\boldsymbol{i}$ elements in matrix.
vector: vector (n): "Create a vector take values from 1 to n"

For $i \leftarrow n-k+1$ **to** $n-1$ **do**
Function combinations (n , r, vec, repeats. Allowed $=$ F)
"We create a matrix 'P' which stores all outputs one by one"
$P \leftarrow$ combinations ($n = length$(vector), $r = i$, $vec =$ vector, repeats. Allowed $= F$);
Return P;
$M \leftarrow num\_rows(P)$;
End function
Moving at each element inside the matrix $\boldsymbol{P}$ **to compute the product terms in Equation** (1)
For $w \leftarrow 1$ **to** M **do**
For $j \leftarrow 1$ **to** i **do**
The failed component "L" is $\leftarrow P[w, j]$;
Compute the conditional survival function of T_L given $D_L > d$:
the value at current $L \leftarrow 1-$ conditional survival function of T_L given $D_L > d$:
productL $\leftarrow$ productL$*$ the value at current L:
End For
Check each row in the matrix P if there are values from 1 to n that do not exist in the current row, then these components are survival
For $u \leftarrow 1$ **to** n **do**
If (u NOT IN $P[w,]$) **then**
Compute the conditional survival function of T_u given $D_u > d$:
the value at current $u \leftarrow$conditional survival function of T_u given $D_u > d$;
productu $\leftarrow$ productu* the value at current u;
End IF
End For
productLu $\leftarrow$ productL $*$ productu :
sump $\leftarrow$ sump $+$ productLu :
Set productL $\leftarrow 1$:
Set productu $\leftarrow 1$:
End For
Function calculate Factorial (integer)
factorial1 $\leftarrow$ Factorial (i);
factorial2 $\leftarrow$ Factorial ($n-i$);
factorial term $\leftarrow 1$ /factorial1 $*$ factorial2;
Return factorial term;
End function
totalsum $\leftarrow$ total sum $+$ (factorialterm*sump):
Set sump $\leftarrow 0$;
End For
6: Compute the possibility of failure of all components (last product in (1)**)**
Set lastproduct $\leftarrow$1: "Initialize the last product term to 1"
For $\boldsymbol{L} \leftarrow 1$ **to** n **do**
the value at current $L \leftarrow 1-$ conditional survival function of T_L given $D_L > d$:
lastproduct $\leftarrow$ lastproduct* the value at current L:
End For
7: Calculate the result of the reliability of the system
System reliability $\leftarrow$ Conditional survival function of $n+1$ * (1- (totalsum + lastproduct)):
Print (System Reliability)
The execution time of the above algorithm depends on n, and k.

5 Exact Reliability Formulas with Bivariate Exponential Gumbel Distribution

The most popular and the most applied lifetime distributions in many areas such as telecommunications, reliability analysis, survival analysis, and life testing are the exponential distributions. In this section, the exact reliability formulas are obtained when the arrival times of the shocks and their magnitudes follow bivariate exponential Gumbel distributions. The bivariate exponential Gumbel distribution was introduced by Gumbel [11]. The survival function of the bivariate exponential Gumbel distribution is given by

$$\overline{F}(x,y) = e^{-\alpha x-\beta y-\theta\alpha\beta xy}, x,y \geq 0, \tag{13}$$

where $\alpha,\ \beta \geq 0$, and $0 < \theta < 1$.

In the case of bivariate exponential Gumbel distribution, the marginal distributions of X and Y are exponential distributions with parameters α and β respectively. Thus,

$$\overline{F}_X(x) = e^{-\alpha x} \tag{14}$$

and

$$\overline{F}_Y(y) = e^{-\beta y} \tag{15}$$

The correlation coefficient between X and Y is given by Gumbel [11] as follows:

$$\rho = -1 - (\frac{1}{\theta})(e^{\frac{1}{\theta}})E\mathrm{i}(-\frac{1}{\theta}), \tag{16}$$

where $E\mathrm{i}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$ gives the exponential integral function.

Also,

$$\lim_{\theta\to 0}(\frac{1}{\theta})(e^{\frac{1}{\theta}})E\mathrm{i}(-\frac{1}{\theta}) \ = -1,$$

and then

$$\lim_{\theta\to 0}\rho \ = 0.$$

It is clear from Equation (16) that as θ increases to 1, the correlation ρ between X and Y decreases until it reaches to -0.40365.

5.1 Exact reliability formula of the system subjected to independent and non-identical shocks

Suppose that (T_i, D_i) are independent random vectors with bivariate exponential Gumbel distributions as in (13) with parameters α_i, β_i, and θ_i, $i = 1,\ldots,\ n+1$. Using (13), (14), and (15), the conditional survival function of T_i given $D_i > d$, $i = 1,\ldots,n+1$ is given by

$$\overline{F}_{T_i}(t|D_i > d) = e^{-\alpha_i t(1+\theta_i\beta_i d)}, t,d \geq 0, \tag{17}$$

where $\alpha_i, \beta_i \geq 0, 0 < \theta_i < 1$, $i = 1,\ldots,n+1$.

On substitution with (17) in (1), (7), and (8), we get

$$\begin{aligned} R_{[k:n],d}(t) =& e^{-\alpha_{n+1}t(1+\theta_{n+1}\beta_{n+1}d)}\left(1-\left(\sum_{i=n-k+1}^{n-1}\frac{1}{i!(n-i)!}\sum_{j_1=1}^{n}\sum_{j_2=j_1+1}^{n}\cdots\sum_{j_i=j_{i-1}+1}^{n}\prod_{L=j_1,\ldots,j_i}[1-e^{-\alpha_L t(1+\theta_L\beta_L d)}]\right.\right.\\ &\left.\left.\times\prod_{\substack{1\leq u\leq n\\ u\neq j_1\neq\cdots\neq j_i}} e^{-\alpha_u t(1+\theta_u\beta_u d)}+\prod_{L=1}^{n}[1-e^{-\alpha_L t(1+\theta_L\beta_L d)}]\right)\right), \end{aligned} \tag{18}$$

$$\begin{aligned} R_{[n:n],d}(t) =& e^{-\alpha_{n+1}t(1+\theta_{n+1}\beta_{n+1}d)}\left(1-\left(\sum_{i=1}^{n-1}\frac{1}{i!(n-i)!}\sum_{j_1=1}^{n}\sum_{j_2=j_1+1}^{n}\cdots\sum_{j_i=j_{i-1}+1}^{n}\prod_{L=j_1,\ldots,j_i}[1-e^{-\alpha_L t(1+\theta_L\beta_L d)}]\right.\right.\\ &\left.\left.\times\prod_{\substack{1\leq u\leq n\\ u\neq j_1\neq\cdots\neq j_i}} e^{-\alpha_u t(1+\theta_u\beta_u d)}+\prod_{L=1}^{n}[1-e^{-\alpha_L t(1+\theta_L\beta_L d)}]\right)\right), \end{aligned} \tag{19}$$

and

$$R_{[1:n],d}(t) = e^{-\alpha_{n+1}t(1+\theta_{n+1}\beta_{n+1}d)}\left(1-\prod_{L=1}^{n}[1-e^{-\alpha_L t(1+\theta_L\beta_L d)}]\right) \tag{20}$$

For any n, and k we can compute $R_{[\,k\,:\,n\,],\,d}(t)$, $R_{[\,n\ \ :n\,],\,d}(t)$, and $R_{[\,1:\,n\,],\,d}(t)$ given by Equations (18), (19), and (20) respectively using the algorithm in Section 4.

5.2 Exact reliability formula of the system subjected to independent and identical shocks

Using (17), the common conditional survival function of T_i given $D_i > d$, $i = 1,\ldots,n$ is given by

$$\overline{F}_T(t|D>d) = e^{-\alpha t(1+\theta\beta d)}, t,\ d \geq 0,\ where\ \alpha,\beta \geq 0, 0<\theta<1, i=1,\ldots,n, \tag{21}$$

while

$$\overline{F}_{T_{n+1}}(t|D_{n+1}>d) = e^{-\alpha_{n+1}t(1+\theta_{n+1}\beta_{n+1}d)}, t,\ d \geq 0,\ where\ \alpha_{n+1},\beta_{n+1} \geq 0,\ 0<\theta_{n+1}<1. \tag{22}$$

Substituting $\overline{F}_T(t|D>d)$, and $\overline{F}_{T_{n+1}}(t|D_{n+1}>d)$, given by (21), and (22) respectively in (9), (11), and (12), we get

$$R_{[k:n],d}(t) = e^{-\alpha_{n+1}t(1+\theta_{n+1}\beta_{n+1}d)}\left(1-\sum_{i=n-k+1}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(-1)^j e^{-\alpha t(n-i+j)(1+\theta\beta d)}\right), \tag{23}$$

$$R_{[n:n],d}(t) = e^{-t[\alpha_{n+1}(1+\theta_{n+1}\beta_{n+1}d)+n\alpha(1+\theta\beta d)]}, \tag{24}$$

and

$$R_{[1:n],d}(t) = \sum_{j=1}^{n}\binom{n}{j}(-1)^{j+1}e^{-t[\alpha_{n+1}(1+\theta_{n+1}\beta_{n+1}d)+j\alpha(1+\theta\beta d)]}, \tag{25}$$

Clearly, we can see that $R_{[\,k\,:\,n\,],\,d}(t)$ is decreasing in t, d, and the parameters α_i, β_i, θ_i, $i = 1,\ldots,n+1$.

6 Numerical Illustration

As a case study, we compute the reliability of $8-\text{out}-\text{of}-10$: G system. We compute $R_{[\,8\,:10\,],\,d}(t)$, $R_{[\,10\,:10\,],\,d}(t)$, and $R_{[\,1\,:10\,],\,d}(t)$ using Equations (23), (24), and (25) for different values of t, d, and the distributions parameters.

Table 1 shows the influence of the rate parameters $(\alpha,\beta,\alpha_{n+1},\ \text{and}\beta_{n+1})$ on the reliability functions, while fixing the association parameters $(\theta,\ \text{and}\theta_{n+1})$. The reliabilities are calculated for fixed values $t=2$, $d=3$, $\theta=0.01$, and $\theta_{n+1}=0.01$. It is clear that the reliability values of all systems increase whenever αor α_{n+1} decreases (the rate of occurrence of the fatal shock that comes from sources $i, i=1,\ldots,n$, or source $n+1$ at time T or T_{n+1}, respectively). We also notice that if $\alpha>\alpha_{n+1}(\beta>\beta_{n+1})$, then $R_{[\,k\,:\,n\,],\,d}(t)$ and $R_{[\,1\,:\,n\,],\,d}(t)$ increase, while the reverse happens for $R_{[\,n\,:\,n\,],\,d}(t)$, which is a logic result for the different construction of the systems.

Table 2 shows the effect of the strength of the association between the arrival times of the shocks and their magnitudes on the reliabilities. The reliabilities are calculated for fixed values $t=2$, $d=3$, $(\alpha=\beta=\alpha_{n+1}=\beta_{n+1}=0.001)$, and different values of the association parameters θ, and θ_{n+1}. We can see that as the $\rho's$ decrease from $\rho=0$ (independence case) to $\rho=-0.40365$, the reliability values of all systems decrease. The reduction in the values of the reliabilities is very slight. This is due to the weak dependence between the shock time and its magnitude for this distribution (bivariate exponential Gumbel distribution).

Table 3 calculates the reliabilities for different values of d $(d=100,\ 500,\ 1000)$, and t $(t=1,\ 3,\ 5)$, while fixing the values of the distribution parameters. Column 1, shows the values of the reliabilities when the system is exposed to independent n shocks, each shock affects one component without considering magnitudes, i. e. , the lifetime of component i is $\alpha e^{-\alpha t}$, $i=1,\ldots,n$. The reliabilities in the first column are calculated by putting $d=\alpha_{n+1}=\ \beta_{n+1}=\theta_{n+1}=0$ in Equations (23), (24), and (25). Column 2, shows the values of the reliabilities, when $d=0$, i. e. , when the magnitude of the shocks is not considered. It is clear that the reliability of systems not exposed to Marshall-Olkin type shocks are greater than those systems exposed to Marshall-Olkin type shocks not considering magnitude. Furthermore, we see that the reliability becomes smaller when the magnitudes of the shocks are considered. We also observe that $R_{[\ k\ :\ n\ ],\ d}(t)$, and $R_{[\ n\ :\ n\ ],\ d}(t)$ decrease dramatically with the increase in d and t, while the decrease in $R_{[\ 1\ :\ n\ ],\ d}(t)$ is not that severe.

The algorithm execution time of calculating $R_{[\ k\ :\ n],\ d}(t)$, $(t=3, d=100, \alpha_{n+1}=0.01, \beta_{n+1}=0.005$, and $\theta_{n+1}=0.2)$, for different values of n and k in the case of non-identical shocks is shown in Table 4. The results in Table 4, are obtained using R-programming. We can see that the execution time of the algorithm for different values of n and k, is very small, it does not exceed few minutes.

Table 1: The effect of the parameters of the distributions on the reliability formulas.

$t=2,\ d=3,\ \theta=0.01,\ \theta_{n+1}=0.01,\ \rho=-0.00981$						
α	β	α_{n+1}	β_{n+1}	$R_{[\ k\ :\ n],\ d}(t)$	$R_{[\ n\ :\ n\ ],\ d}(t)$	$R_{[\ 1\ :\ n\ ],\ d}(t)$
0.01	**0.01**	**0.01**	**0.01**	0.9793694101	0.8024658335	0.9801927921
0.001	0.01	**0.001**	0.01	0.9980004538	0.9782337787	0.9980013999
0.001	0.01	0.01	0.01	0.9801918629	0.9607779097	0.9801927921
0.01	0.01	**0.001**	0.01	0.9971630582	0.8170454135	0.9980013999
0.01	**0.001**	0.01	**0.001**	0.9793753355	0.8025135014	0.9801980852
0.01	**0.001**	0.01	0.01	0.9793700469	0.8025091678	0.9801927921
0.01	0.01	0.01	**0.001**	0.9793746987	0.8024701668	0.9801980852
0. 001	**0. 001**	0. 01	0. 01	0.9801918637	0.960783098	0.9801927921
0. 01	0. 01	**0. 001**	**0. 001**	0.9971635967	0.8170458547	0.9980019388

Table 2: The correlation between the time and the magnitude and its effect on reliability.

$t=2,\ d=3,\ \alpha=0.001,\ \beta=0.001,\ \alpha_{n+1}=0.001,\ \beta_{n+1}=0.001$						
θ	θ_{n+1}	ρ_θ	$\rho_{\theta_{n+1}}$	$R_{[\ k\ :\ n\ ],\ d}(t)$	$R_{[\ n\ :\ n\ ],\ d}(t)$	$R_{[\ 1\ :\ n\ ],\ d}(t)$
0	0	0		0.9980010534	0.9782402351	0.9980019987
0.1	0.1	−0.08436		0.9980004538	0.9782337787	0.9980013999
0.5	0.5	−0.27734		0.9979980552	0.9782079537	0.9979990047
0.7	0.7	−0.33648		0.9979968559	0.9781950414	0.9979978071
1	1	−0.40365		0.9979950569	0.9781756733	0.9979960107

Table 3: Comparison of the reliabilities without and with Marshall-Olkin shocks.

$\alpha=0.01,\ \beta=0.01,\ \alpha_{n+1}=0.001,\ \beta_{n+1}=0.001,\ \theta=1,\ \theta_{n+1}=1,\ \rho=-0.40365$					
$R_{[\ k\ :\ n\ ],\ d}(t)$					
t	**Without Marshall-Olkin shocks**	**With Marshall-Olkin shocks**			
		d			
	$d, \alpha_{n+1}, \beta_{n+1}, \theta_{n+1}=0$	**0**	**100**	**500**	**1000**
1	0.999887815	0.998888427	0.9980622289	0.9811385286	0.9206116912
3	0.9973501456	0.9943625787	0.9793740678	0.7771083225	0.4321854225
5	0.9892536915	0.9843197681	0.9327462501	0.494478807	0.131608664
$R_{[\ n\ :\ n\ ],\ d}(t)$					
1	0.904837418	0.9039330329	0.8178306444	0.5479890357	0.3322060068
3	0.7408182207	0.7385990964	0.5470035427	0.1645567144	0.03666253097
5	0.6065306597	0.6035055754	0.3658616582	0.04941506212	0.004046107383
$R_{[\ 1\ :\ n\ ],\ d}(t)$					
1	1	0.9990004998	0.9989006048	0.9985011244	0.9980019985
3	1	0.9970044955	0.996705439	0.9955100952	0.9940149047
5	1	0.9950124792	0.9945150972	0.9925266972	0.9898680412

Table 4: The execution time of the algorithm for different values of n and k.

n	k	$R_{[\,k\,:\,n\,],\,d}(t)$	Execution time
$\alpha_i = (0.01,\ 1,\ 0.02,\ 0.07,\ 0.2)$, $\beta_i = (0.007,\ 0.0002,\ 0.5,\ 0.03,\ 0.75)$, $\theta_i = (0.1,\ 0.3,\ 0.8,\ 0.6,\ 0.9),\ i = 1,\ldots,5$			
5	1	0. 95571914200	0. 01590896 secs
	3	0. 89765519752	0. 2018201 secs
	5	0. 89195009796	0. 157635 secs
$\alpha_i = (0.01,\ 1,\ 0.02,\ 0.07,\ 0.2,\ 0.25,\ 0.3,\ 0.0005,\ 0.006,\ 0.04)$, $\beta_i = (0.007,\ 0.0002,\ 0.5,\ 0.03,\ 0.75,\ 0.6,\ 0.008,\ 0.001,\ 0.05,\ 0.7)$ $\theta_i = (0.1,\ 0.3, 0.8,\ 0.6,\ 0.9,\ 0.2,\ 0.7,\ 0.5,\ 0.4,\ 0.8), i = 1,\ldots,10$			
10	1	0. 96753782277	0. 02723908 secs
	4	0. 96752765400	0. 732451 secs
	7	0. 96748844780	3. 136231 secs
	10	0. 96748843047	3. 793851 secs
$\alpha_i = (0.01, 1,\ 0.02,\ 0.07,\ 0.2,\ 0.25,\ 0.3, 0.0005,\ 0.006,\ 0.04,\ 0.7,\ 0.05,\ 0.8,\ 0.0001,\ 0.9)$, $\beta_i = (0.007,\ 0.0002,\ 0.5,\ 0.03,\ 0.75,\ 0.6, 0.008,\ 0.001,\ 0.05,\ 0.7,\ 0.0002,\ 0.25,\ 0.3,\ 1,\ 0.0005)$, $\theta_i = (0.1,\ 0.3,\ 0.8,\ 0.6,\ 0.9,\ 0.2,\ 0.7,\ 0.5,\ 0.4,\ 0.8,\ 0.4,\ 0.2,\ 1,\ 0.7,\ 1), i = 1,\ldots,15.$			
15	1	0. 96753855939	0. 01612115 secs
	7	0. 96753855732	54. 15218 secs
	10	0. 96753855712	2. 391665 mins
	15	0. 96753855712	2. 831313 mins

7 Conclusion

In this paper, the reliability formula $R_{[\,k\,:\,n\,],\,d}(t)$ of a system subjected to independent (non-identical and identical) Marshall-Olkin type shocks concerning magnitude is obtained. The reliability formulas of the series and the parallel systems are obtained as special cases. An algorithm is presented to calculate the formula of $R_{[\,k\,:\,n\,],\,d}(t)$ in the case of non-identical shocks, and applied using the R-programming software. The execution time of the algorithm is very small (for $n = 15,\ k = 15$ is 2. 83 mins).

Numerical illustrations are applied to detect the effect of t, d, and the distribution parameters on the reliabilities of the systems. We see that, the reliabilities increase whenever αor α_{n+1} (rate parameters of time) decreases. We also notice that if $\alpha > \alpha_{n+1}(\beta > \beta_{n+1})$, then $R_{[\,k\,:\,n\,],\,d}(t)$ and $R_{[\,1\,:\,n\,],\,d}(t)$ increase, while the reverse happens for $R_{[\,n\,:\,n\,],\,d}(t)$, which is a logic result for the different construction of the systems. The values of the reliabilities decrease very slightly, when $\rho's$ decrease from $\rho = 0$ (independence case) to $\rho = -0.40365$. This is due to the weak dependence between the shock time and its magnitude for this distribution (bivariate exponential Gumbel distribution). The reliability of systems not exposed to Marshall-Olkin type shocks are greater than those systems exposed to Marshall-Olkin type shocks not considering magnitude. Furthermore, we see that the reliability becomes smaller when the magnitudes of the shocks are considered. We also observe that $R_{[\,k\,:\,n\,],\,d}(t)$, and $R_{[\,n\,:\,n\,],\,d}(t)$ decrease dramatically with the increase in d and t, while the decrease in $R_{[\,1\,:\,n\,],\,d}(t)$ is not that severe. For future work, we study the reliability of systems affected by the extension Marshall-Olkin type shock models.

References

[1] A. W. Marshall and I. Olkin, A multivariate exponential distribution, *Journal of the American Statistical Association*, **62**, 30-44 (1967).
[2] M. Ozkut and I. Bayramoglu, On Marshall–Olkin type distribution with effect of shock magnitude, *Journal of Computational and Applied Mathematics*, **271**, 150-162 (2014).
[3] I. Bayramoglu and M. Ozkut, The reliability of coherent systems subjected to Marshall–Olkin type shocks, *IEEE Transactions on Reliability*, **64**, 435-443 (2014).
[4] I. Bayramoglu and M. Ozkut, Mean residual life and inactivity time of a coherent system subjected to Marshall–Olkin type shocks, *Journal of Computational and Applied Mathematics*, **298**, 190-200 (2016).
[5] M. Ozkut and S. Eryilmaz, Reliability analysis under Marshall–Olkin run shock model, *Journal of Computational and Applied Mathematics*, **349**, 52-59 (2019).

[6] S. H. Sheu and C. T. Liou, Optimal replacement of a k-out-of-n system subject to shocks, *Microelectronics Reliability*, **32**, 649-655 (1992).
[7] G. Skoulakis, A general shock model for a reliability system, *Journal of Applied Probability*, **37**, 925-935 (2000).
[8] X. Huang, S. Jin, X. He and D. He, Reliability analysis of coherent systems subject to internal failures and external shocks, *Reliability Engineering & System Safety*, **181**, 75-83 (2019).
[9] S. Eryilmaz and Y. Devrim, Reliability and optimal replacement policy for a k-out-of-n system subject to shocks, *Reliability Engineering & System Safety*, **188**, 393-397 (2019).
[10] H. A. David and H. N. Nagaraja, *Order Statistics*, John Wiley & Sons, New York, (2004).
[11] E. J. Gumbel, Bivariate exponential distributions, *Journal of the American Statistical Association*, **55**, 698-707 (1960).