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Proportional Hazard Bivariate Kumaraswamy Model Applied on Fish Mercury Concentration

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Abstract: International advisory bodies have developed guidelines for testing mercury and aquatic items to protect human health and international trade. The mercury absorption in fish has a great effect on human health. For modeling this problem, a new bivariate distribution using the proportional hazard rate (PHR) model with Kumaraswamy marginal called BKPH is derived and studied via statistical properties and reliability measures. Moreover, several methods of parameter estimation are discussed, including maximum likelihood estimation (MLE), method of moments estimation (MME), and inference function for margins estimation (IFM). In the simulation study, the performance of estimators depending on their estimation methodologies is compared. Finally, a comparative study of the proposed BKPH with several bivariate Kumaraswamy distributions via goodness of fit criteria was introduced. The results of the study proved the potentiality of the BKPH model and has a best fitting on mercury fish absorption data.

Keywords: Proportional hazard rate model; Kumaraswamy distribution; Copula; Estimation methods; Simulation.

1. Introduction

When working with bivariate lifetime data, it is critical to consider several distributions that might be used to describe the data. In recent years, statisticians have been increasingly interested in the construction of continuous bivariate distributions. Essentially, the method is based on the statistical interpretation of marginal distributions and the copula of dependence. According to [14], there are several methods for generating bivariate distributions. The following characteristics have been identified by [10] for constructing bivariate distributions:

a. A stochastic representation, such as a mixture, should exist.

b. Margin should, at the very least, be within the same parametric family.

c. A parameter should be utilized to describe the bivariate dependence between the margins, and it should cover a wide range of dependence.

d. While a closed-form representation is ideal, a numerical assessment of the joint distribution and density should be possible.

All these desirable features cannot be achieved together. While bivariate normal distributions may be the closest, there is no known bivariate family that encompasses all the characteristics; see [10], Section 5.

The proportional reversed hazard rates (PRHR) and proportional hazard rates (PHR) models are two flexible families of distributions that have been used to model failure time data in reliability and survival analysis see, by [19]. Later, many researchers continued the study and found many results for the PHR model. In 1953, Lehmann first introduced the PHR model in the two-sample hypothesis testing context, which was named Lemann alternatives. The PHR model covers some commonly used statistical lifetime distributions which apply to model component lifetimes. According to the PHR model,

$$H(x) = 1-\left(1-F(x)\right)^{\lambda} \quad ,x \geq 0, \lambda > 0$$

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where $F(x)$ represents the baseline cumulative distribution and $H(x)$ represents the generated cumulative distribution using the PHR model.

Motivated by Joe's assertion, the purpose of this study is to propose a bivariate exponential distribution that satisfies all the properties (a), (b), and (d) and has a usable range of dependence.

In bivariate cases, several models have been constructed for PHR and PRHR ; for example; [6, 18, 22, 23] and the references cited therein. Based on the baseline univariate cumulative distributions $F_1(x)$ and $F_2(y)$, a bivariate PHR model was proposed as follows:

$$H_\lambda(x,y) = 1 - \left(1 - F_1(x)F_2(y)\right)^\lambda, \;\; x, y \in \mathbb{R} \; and \; 0 < \lambda \leq 1. \tag{1}$$

with the univariate marginal distributions

$$H_i(x) = 1 - \left(1 - F_i(x)\right)^\lambda, \;\; i = 1,2. \quad \text{where } \lambda \text{ is the dependence parameter.}$$

Some bivariate distribution using this concept are introduced in the literature for example, [22, 23] where the baseline CDF are Exponential and Rayleigh distributions. There has been considerable interest in copula structure for studying dependence structures in bivariate distributions. The dependence properties of bivariate distributions in PHR and PRHR models have been studied by [3, 6]. The double-bounded probability distribution proposed in [17] has been extensively studied in hydrology and related fields. The CDF of Kumaraswamy (KW) with two shape parameters $\alpha > 0$ and $\beta > 0$ is defined by

$$F(x) = 1 - (1 - x^\alpha)^\beta \qquad , x \in (0,1), \quad \alpha, \beta > 0$$

Due to [12], although Beta and KW distributions have many similarities, they also have some significant differences. Continuous distributions come in a broad range of forms in the statistical literature, including univariate and bivariate continuous distributions, see [1,5,12,16]. According to [23], a new structure of bivariate Rayleigh distribution is presented using the PHR model and applied to the COVID-19 data set.

Two pelagic fish species were studied by [13] to determine whether there were any differences in mercury concentration that could affect the measurement of mercury exposure for users. They observed that there are correlations between fish size and mercury concentrations in fish, as well as intake concentrations, suggesting that this pollutant is important in users' decisions to eat fish. Only if people are aware of the dangers of eating large fish can they change their eating habits.

The organization of this paper is: the materials and methods are shown in Section 2 which illustrates the derivation of a new bivariate Kumaraswamy (BK) distribution using the PHR model (BKPH) and its parameter estimation via 3 techniques. In Section 3, results and discussion of the paper is introduced through; several statistical properties of BKPH distribution, measures of reliability and the copula structure of the proposed distribution are presented and discussed. Moreover, a simulation study to see how the MLE, MME, and IFM estimators perform. Finally, the potentiality of the proposed distribution in comparison to other bivariate Kumaraswamy distributions is demonstrated with biological application to fish mercury concentrations.

2. Materials and Methods

This section discusses the bivariate Kumaraswamy distribution using the PHR model, several statistical properties of BKPH, including the Joint PDF and the CDF, as well as several estimation methods.

2.1 Bivariate Kumaraswamy Proportional Hazard (BKPH) Distribution

Using (1) with the baseline Kumaraswamy distribution

$$F_i(x) = 1 - (1 - x^{\alpha_i})^{\beta_i}, i = 1,2.$$

The CDF **of BKPH distribution** for parameter vector $\Theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ is defined as

$$H_\lambda(x,y;\lambda,\Theta) = 1 - \left\{1 - \left\{\{1 - (1 - x^{\alpha_1})^{\beta_1}\}\{1 - (1 - y^{\alpha_2})^{\beta_2}\}\right\}\right\}^\lambda \tag{2}$$

where $0 \leq x, y \leq 1, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0, 0 < \lambda \leq 1$

whose univariate CDF are ordinary Kumaraswamy distributions

$$H_i(x) = 1 - (1 - x^{\alpha_i})^{\beta_i \lambda}, \qquad i = 1,2.$$

Equation (1) can be rewritten using the binomial expansion, see [23],

$$(1-z)^t = \sum_{j=0}^{\infty} \binom{t}{j} (-1)^j z^j, \ |z| < 1, t \in \mathcal{R} \tag{3}$$

as

$$H_\lambda(x,y) = 1 - \sum_{j=0}^{\infty} \binom{\lambda}{j} (-1)^j \big(F_1(x)F_2(y)\big)^j \tag{4}$$

where $0 < F_1(x)F_2(y) < 1$.

According to [3], the **BKPH joint pdf** is defined using

$$h_\lambda(x,y) = \sum_{j=1}^{\infty} \binom{\lambda}{j} (-1)^{j+1} j^2 \, f_1(x) \, \big(F_1(x)\big)^{j-1} f_2(y)\big(F_2(y)\big)^{j-1}$$

as,

$$h_\lambda(x,y;\lambda,\Theta) = \ \alpha_1\alpha_2\beta_1\beta_2 \sum_{j=1}^{\infty} \binom{\lambda}{j} (-1)^{j+1} j^2 \left\{x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{j-1}\right\}\left\{y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}\right\} \tag{5}$$

As shown in Figures 1 and 2, the joint pdf (5) can deal with bivariate skewed data with different parameter values.

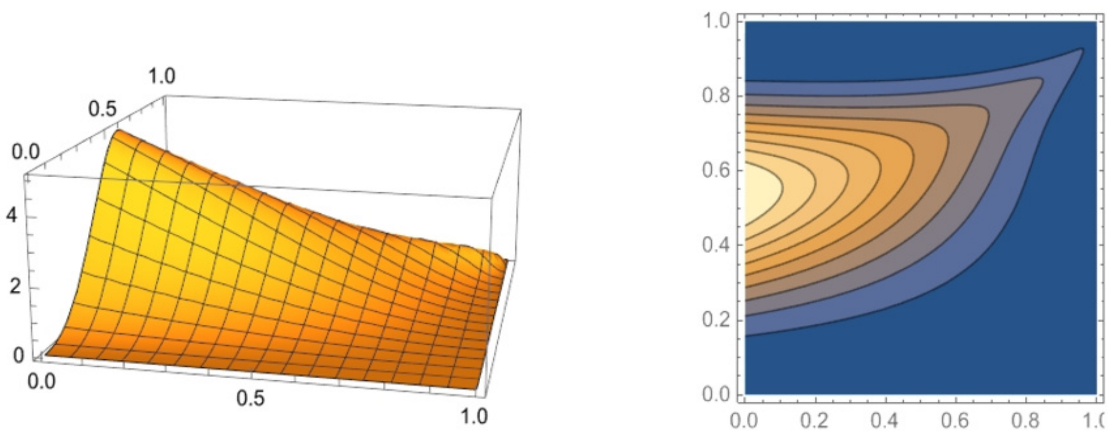

Fig. 1. Bivariate Density Plots and Contours of BKPH for parameters $\alpha_1 = 1, \alpha_2 = 3.4, \ \beta_1 = 2.8, \ \beta_2 = 6, \lambda = 0.75$

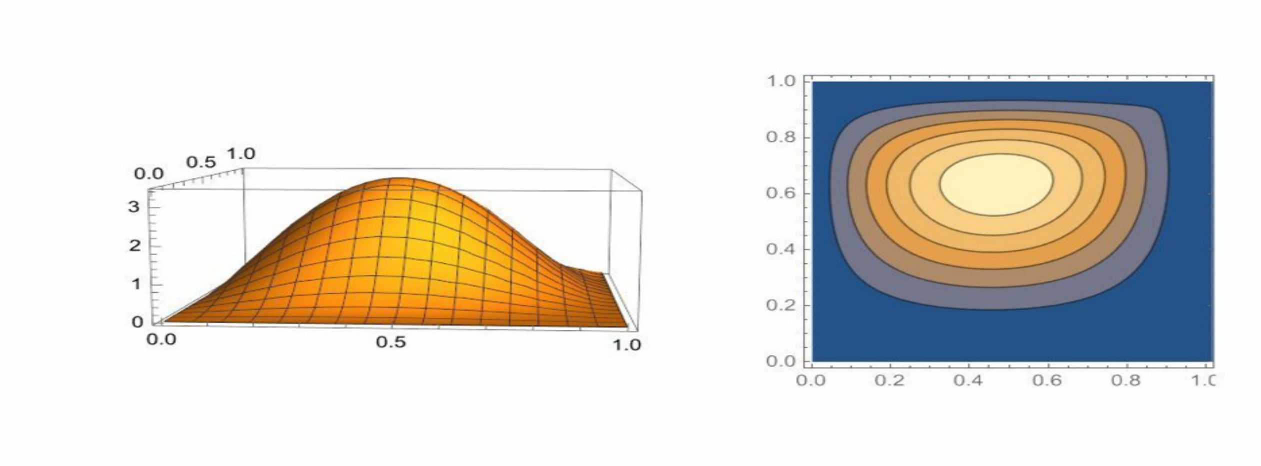

Fig 2. Bivariate Density Plots and Contours of BKPH for parameters $\alpha_1 = 2, \alpha_2 = 3,\ \beta_1 = 3,\ \beta_2 = 3, \lambda = 0.95$

2.2 Parameter Estimation

In this section, several estimation methods will be discussed based on random samples $(x_i, y_i), i = 1,2, \dots, n$,to estimate the unknown parameters of the BKPH distribution.

2.2.1 Maximum Likelihood Estimation (MLE)

Here, we discussed the estimation of the unknown parameters of the BKPH model using the maximum likelihood method. Suppose we have n observations from the bivariate density in (5). Therefore, the log-likelihood is expressed as follows:

$$\ell(\lambda, \Theta) = Ln\, L(\lambda, \Theta) = Ln \prod_{i=1}^{n} h_\lambda(x_i, y_i; \lambda, \Theta)$$

$$\ell(\lambda, \Theta) = n\, Ln[\lambda \alpha_1 \beta_1 \alpha_2 \beta_2] + (\alpha_1 - 1) \sum_{i=1}^{n} Ln[x_i] + (\beta_1 - 1) \sum_{i=1}^{n} Ln[1 - {x_i}^{\alpha_1}] + (\alpha_2 - 1) \sum_{i=1}^{n} Ln[y_i] + (\beta_2 - 1) \sum_{i=1}^{n} Ln[1 - {y_i}^{\alpha_2}] + (\lambda - 2) \sum_{i=1}^{n} Ln\left[1 - \left\{\left(1 - (1 - {x_i}^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - {y_i}^{\alpha_2})^{\beta_2}\right)\right\}\right] \sum_{i=1}^{n} Ln\left[1 - \lambda\left\{\left(1 - (1 - {x_i}^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - {y_i}^{\alpha_2})^{\beta_2}\right)\right\}\right] \quad (6)$$

The maximum likelihood estimates can be obtained by maximizing (6) for the unknown parameters. By solving the following normal Equations numerically, the MLE estimates are obtained

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} Ln(x_i) + (1 - \beta_1) \sum_{i=1}^{n} \frac{Ln(x_i)\, x_i^{\alpha_1}}{1 - x_i^{\alpha_1}} + (2 - \lambda) \sum_{i=1}^{n} \frac{Ln[x_i] x_i^{\alpha_1} \left((1 - x_i^{\alpha_1})\right)^{\beta_1 - 1} \left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right) \beta_1}{1 - \left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)} - \lambda \sum_{i=1}^{n} \frac{Ln[x_i] x_i^{\alpha_1} (1 - x_i^{\alpha_1})^{\beta_1 - 1} \left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right) \beta_1}{1 - \lambda\left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)} = 0 \quad (7)$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n} Ln\left[1 - x_i^{\alpha_1}\right] + (\lambda - 2) \sum_{i=1}^{n} \frac{Ln[1 - x_i^{\alpha_1}] (1 - x_i^{\alpha_1})^{\beta_1} \left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)}{1 - \left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)} + \lambda \sum_{i=1}^{n} \frac{Ln[1 - x_i^{\alpha_1}] (1 - x_i^{\alpha_1})^{\beta_1} \left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)}{1 - \lambda\left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)} = 0 \quad (8)$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{i=1}^{n} \mathrm{Ln}[y_i] + (1 - \beta_2) \sum_{i=1}^{n} \frac{\mathrm{Ln}[y_i] y_i^{\alpha_2}}{1 - y_i^{\alpha_2}} + (2 - \lambda) \sum_{i=1}^{n} \frac{\mathrm{Ln}[y_i]\left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right) y_i^{\alpha_2} (1 - y_i^{\alpha_2})^{\beta_2 - 1} \beta_2}{1 - \left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)} - \lambda \sum_{i=1}^{n} \frac{\mathrm{Ln}[y_i]\left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right) y_i^{\alpha_2} (1 - y_i^{\alpha_2})^{\beta_2 - 1} \beta_2}{1 - \lambda\left(1 - (1 - x_i^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y_i^{\alpha_2})^{\beta_2}\right)} = 0 \quad (9)$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n}{\beta_2} + \sum_{i=1}^{n} \mathrm{Ln}[1 - y_i^{\alpha_2}] + (\lambda - 2) \sum_{i=1}^{n} \frac{\mathrm{Ln}[1-y_i^{\alpha_2}]\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)(1-y_i^{\alpha_2})^{\beta_2}}{1-\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)\left(1-(1-y_i^{\alpha_2})^{\beta_2}\right)} + \lambda \sum_{i=1}^{n} \frac{\mathrm{Ln}[1-y_i^{\alpha_2}]\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)(1-y_i^{\alpha_2})^{\beta_2}}{1-\lambda\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)\left(1-(1-y_i^{\alpha_2})^{\beta_2}\right)} = 0 \quad (10)$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \mathrm{Ln}[1 - (1 - (1 - x_i^{\alpha_1})^{\beta_1})(1 - (1 - y_i^{\alpha_2})^{\beta_2})] \sum_{i=1}^{n} \frac{\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)\left(1-(1-y_i^{\alpha_2})^{\beta_2}\right)}{1 - \lambda\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)\left(1-(1-y_i^{\alpha_2})^{\beta_2}\right)} = 0 \quad (11)$$

It is not possible to obtain them explicitly, since the previous equations are non-linear for the parameters, so the initial assumptions can be used to solve them numerically.

2.2.2 Method of Moments (MME)

Consider the population moment for j th variate is $\mu_j = E(X_j), \mu_j^2 = E(X_j^2), j = 1,2$, the sample and the joint moment are $m_j = \frac{\sum_{i=1}^{n} X_{ji}}{n}$, $m_j^2 = \frac{\sum_{i=1}^{n} X_{ji}^2}{n}$, $\overline{XY} = \sum_{i=1}^{n} \frac{X_i Y_i}{n}$.

According to this method of estimating parameters $\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2$ and $\hat{\lambda}$ for $BKPH$ distribution can be computed by solving the following equations:

$$m_j = E(X_j) = \beta_j \, \mathrm{B}\left(\beta_j, \frac{\alpha_j + 1}{\alpha_j}\right), \qquad m_j^2 = E(X_j^2) = \beta_j \, \mathrm{B}\left(\beta_j, \frac{\alpha_j + 2}{\alpha_j}\right), \qquad j = 1,2$$

$$\overline{XY} = E(XY) = \sum_{J=1}^{\infty} \binom{\lambda}{J} (-1)^{J+1} J^2 \prod_{k=1}^{2} \left(\left(\beta_k \left\{ \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, \mathrm{B}\left(\beta_k (1+i), \frac{\alpha_k + 1}{\alpha_k}\right) \right\}\right) \quad (12)$$

2.2.3 Inference Function for Margins (IFM)

The IFM method has been introduced by [9] for the estimation of parameters for multivariate models using marginal distributions. This method is computed by estimating model parameters from separately maximizing marginal likelihood function, and then estimating dependence parameters from separate joint likelihood function. Using log-likelihood equations, we can obtain estimates $\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2$ and $\hat{\lambda}$. Assume that $\ell_j, j = 1,2$ is the log-likelihood function of X and Y.

$$\ell_1 = nLn[\alpha_1] + nLn[\beta_1] + (\alpha_1 - 1) \sum_{i=1}^{n} Ln[x_i] + (\beta_1 - 1) \sum_{i=1}^{n} Ln[1 - x_i{}^{\alpha_1}]$$

$$\frac{\partial \ell_1}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} \mathrm{Ln}[x_i] + (-1 + \beta_1) \sum_{i=1}^{n} - \frac{\mathrm{Ln}[x_i] x_i^{\alpha_1}}{1 - x_i^{\alpha_1}} = 0 \quad (13)$$

$$\frac{\partial \ell_1}{\partial \beta_1} = \frac{n}{\beta_1} + \sum_{i=1}^{n} \mathrm{Ln}[1 - x_i^{\alpha_1}] = 0 \quad (14)$$

$$\ell_2 = n\mathrm{Ln}[\alpha_2] + n\mathrm{Ln}[\beta_2] + (\alpha_2 - 1) \sum_{i=1}^{n} \mathrm{Ln}[y_i] + (\beta_2 - 1) \sum_{i=1}^{n} \mathrm{Ln}[1 - y_i{}^{\alpha_2}]$$

$$\frac{\partial \ell_2}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{i=1}^{n} \mathrm{Ln}[y_i] + (-1 + \beta_2) \sum_{i=1}^{n} - \frac{\mathrm{Ln}[y_i] y_i^{\alpha_2}}{1 - y_i^{\alpha_2}} = 0 \quad (15)$$

$$\frac{\partial \ell_2}{\partial \beta_2} = \frac{n}{\beta_2} + \sum_{i=1}^{n} \mathrm{Ln}[1 - y_i^{\alpha_2}] = 0 \quad (16)$$

By differentiating the above log-likelihood function (6) for λ and equating it to zero we get the likelihood equation as given below

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \mathrm{Ln}[1 - (1 - (1 - x_i^{\alpha_1})^{\beta_1})(1 - (1 - y_i^{\alpha_2})^{\beta_2})] \sum_{i=1}^{n} \frac{\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)\left(1-(1-y_i^{\alpha_2})^{\beta_2}\right)}{1 - \lambda\left(1-(1-x_i^{\alpha_1})^{\beta_1}\right)\left(1-(1-y_i^{\alpha_2})^{\beta_2}\right)} = 0$$

MLE of λ has no closed-form expression and cannot be solved analytically. Hence, we can numerically obtain the MLE of λ by employing the iterative procedure of the Newton-Raphson technique or any suitable iterative methods.

3. Results and Discussion

Statistical and reliability characteristics of BKPH distribution, copula structure, simulation, and application of real data are discussed in this section.

3.1 Statistical characteristics

3.1.1 Product Moments and Pearson's Correlation Coefficient

For the random vector (X,Y), the product moments $E(X^{r_1}Y^{r_2})$, is obtained by

$$E(X^{r_1}Y^{r_2}) = \int_0^1\int_0^1 x^{r_1}y^{r_2}h_\lambda(x,y;\lambda,\Theta)dx\,dy$$

Theorem 1. If $(X,Y)\sim BKPH(\lambda,\Theta)$, for $r_1,r_2 \geq 1$, then the r^{th} product moment of $X and\ Y$, denoted by $E(X^{r_1}Y^{r_2})$, is given by

$$E(X^{r_1}Y^{r_2}) = \sum_{J=1}^{\infty}\binom{\lambda}{J}(-1)^{J+1}J^2\prod_{k=1}^{2}\left((\beta_k\left\{\sum_{i=0}^{J-1}\binom{J-1}{i}(-1)^i\,\mathrm{B}\left(\beta_k(1+i),\frac{\alpha_k+r_k}{\alpha_k}\right)\right\}\right) \quad (17)$$

where $\mathrm{B}(.,.)$ denotes the beta function.

proof. From (5) we have

$$E(X^{r_1}Y^{r_2}) = \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\int_0^1\int_0^1 x^{r_1}y^{r_2}\alpha_1\alpha_2\beta_1\beta_2\left\{x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{j-1}\right\}\left\{y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}\right\}dx\,dy$$

$$=\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\int_0^1\alpha_1\,\beta_1 x^{r_1}\left\{x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{j-1}\right\}dx\int_0^1 y^{r_2}\alpha_2\beta_2\left\{y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}\right\}dy$$

briefly,

$$E(X^{r_1}Y^{r_2}) = \sum_{J=1}^{\infty}\binom{\lambda}{J}(-1)^{J+1}J^2\prod_{k=1}^{2}I_k \qquad for\ k=1,2$$

where for $k=1$,

$$I_1 = \int_0^1 \alpha_1\,\beta_1 x^{r_1}x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{j-1}dx$$

$$=\alpha_1\beta_1\int_0^1 x^{r_1+\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{j-1}dx$$

By using the expansion (3), then

$$I_1 = \beta_1\sum_{i=0}^{J-1}\binom{J-1}{i}(-1)^i\int_0^1(1-\mathrm{u})^{\frac{r_1}{\alpha_1}}u^{\beta_1(\mathrm{i}+1)-1}du$$

Then,

$$I_1 = \beta_1\sum_{i=0}^{J-1}\binom{J-1}{i}(-1)^i\mathrm{B}\left(\beta_1(1+i),\frac{\alpha_1+r_1}{\alpha_1}\right)$$

Similarly,

$$I_2 = \int_0^1 \alpha_2\,\beta_2\,y^{r_2}\left\{y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}\right\}dy$$

$$= \beta_2 \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \mathrm{B}\left(\beta_2(1+i), \frac{\alpha_2 + r_2}{\alpha_2}\right)$$

Hence,

$$I_k = \beta_k \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, \mathrm{B}\left(\beta_k(1+i), \frac{\alpha_k + r_k}{\alpha_k}\right), \qquad for\ k = 1,2$$

As a result, the expression for r^{th} product moment is demonstrated.

Using (17) the product moment is obtained as

$$E(XY) = \sum_{J=1}^{\infty} \binom{\lambda}{J} (-1)^{J+1} J^2 \prod_{k=1}^{2} \left((\beta_k \left\{ \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, B\left(\beta_k(1+i), \frac{\alpha_k + 1}{\alpha_k}\right) \right\} \right)$$

Pearson Coefficient of correlation (ρ) for the BKPH distribution is given by

$$\rho = \frac{\sum_{J=1}^{\infty} \binom{\lambda}{J} (-1)^{J+1} J^2 \prod_{k=1}^{2} \left((\beta_k \left\{ \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, \mathrm{B}\left(\beta_k(1+i), \frac{\alpha_k+1}{\alpha_k}\right) \right\} \right) - \prod_{k=1}^{2} \left((\beta_k \, \mathrm{B}\left(\beta_k, \frac{\alpha_k+1}{\alpha_k}\right) \right)}{\prod_{k=1}^{2} \left\{ \sqrt{\beta_k \mathrm{B}\left[\beta_k, \frac{2+\alpha_k}{\alpha_k}\right] - \left(\beta_k \mathrm{B}\left[\beta_k, \frac{\alpha_k+1}{\alpha_k}\right]\right)^2} \right\}} \tag{18}$$

3.1.2 Moment Generating Function

The joint moment generating function of the bivariate distributions is defined as

$$M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y})$$

Theorem 2: The joint moment-generating function of $BKPH(\lambda, \Theta)$, is given by

$$M_{X,Y}(t_1, t_2) = \sum_{J=1}^{\infty} \binom{\lambda}{J} (-1)^{J+1} J^2 \prod_{k=1}^{2} \left(\beta_k \sum_{r=0}^{\infty} \frac{t_k^r}{r!} \left\{ \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, \mathrm{B}\left(\beta_k(1+i), \frac{\alpha_k + r_k}{\alpha_k}\right) \right\} \right) \tag{19}$$

Proof. From (5) we have

$$M_{X,Y}(t_1, t_2) = \int_0^1 \int_0^1 e^{t_1 x + t_2 y} h_\lambda(x, y; \lambda, \Theta) dx\, dy$$

$$= \sum_{J=1}^{\infty} \binom{\lambda}{J} (-1)^{J+1} J^2 \prod_{k=1}^{2} M_k(t_k; \alpha_k, \beta_k)$$

where $M_k(t_k; \alpha_k, \beta_k)$ is the moment generating function of Kumaraswamy distribution, for $k = 1,2$

$$M_k(t_k; \alpha_k, \beta_k) = E(e^{t_k X}) = E\left(\sum_{r=0}^{\infty} \frac{x^r {t_k}^r}{r!}\right) = \sum_{r=0}^{\infty} \frac{{t_k}^r}{r!} E(x^r) = \sum_{r=0}^{\infty} \frac{{t_k}^r}{r!} I_k$$

and,

$$I_k = \beta_k \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, \mathrm{B}\left(\beta_k(1+i), \frac{\alpha_k + r_k}{\alpha_k}\right)$$

Then,

$$M_k(t_k; \alpha_k, \beta_k) = \beta_k \sum_{r=0}^{\infty} \frac{t_k^r}{r!} \left\{ \sum_{i=0}^{J-1} \binom{J-1}{i} (-1)^i \, \mathrm{B}\left(\beta_k(1+i), \frac{\alpha_k + r_k}{\alpha_k}\right) \right\}$$

which gives the required result.

Table 1 provides a few numerical values for the (ρ). Note that if the values of α are less than 1, the values of β and the dependence parameter λ values increase, then the values of ρ are very small, approaching zero. If the values of α are more than 1 and the values of β increase, the ρ is a strong negative, then it changes to be positive and tends to be a strong positive correlation if λ approaches 0.6, then it declines to zero when the value λ approaches *one* where the independence is fulfilled.

Table 1. Correlation Coefficient for some values of $\alpha_1, \beta_1, \alpha_2, \beta_2$ *and* λ.

Correlation Coefficient (ρ)									
		λ							
		0.6	**0.65**	**0.7**	**0.75**	**0.8**	**0.85**	**0.9**	**0.95**
β_1	β_2	$\alpha_1 = 0.05, \alpha_2 = 0.05$							
0.03	**0.02**	-0.7878	-0.5135	-0.3039	-0.1509	-0.0469	0.0146	0.0393	0.0329
0.07	**0.08**	0.2624	0.3001	0.3113	0.2997	0.2684	0.2204	0.1582	0.0840
0.3	**0.4**	0.6215	0.5640	0.4971	0.4229	0.3433	0.2598	0.1740	0.0871
0.7	**0.8**	0.4484	0.3983	0.3441	0.2871	0.2288	0.1700	0.1118	0.0549
2	**5**	0.0125	0.0109	0.0094	0.0077	0.0061	0.0045	0.0029	0.0014
10	**15**	1.7×10^{-5}	1.5×10^{-5}	1.3×10^{-5}	1.1×10^{-5}	8.7×10^{-6}	6.4×10^{-6}	4.1×10^{-6}	2×10^{-6}
		$\alpha_1 = 0.5, \alpha_2 = 0.5$							
0.2	**0.02**	-0.4379	-0.1408	0.0659	0.1946	0.2555	0.2584	0.2118	0.1234
0.3	**0.05**	0.2186	0.3303	0.3897	0.4039	0.3793	0.3218	0.2365	0.1278
0.4	**0.1**	0.4629	0.4999	0.5006	0.4704	0.4138	0.3349	0.2376	0.1249
0.7	**0.8**	0.7454	0.6909	0.6201	0.5359	0.4412	0.3382	0.2292	0.1159
2	**5**	0.6205	0.5584	0.4883	0.4123	0.3322	0.2497	0.1660	0.0825
10	**15**	0.4591	0.4101	0.3560	0.2986	0.2391	0.1785	0.1179	0.0583
		$\alpha_1 = 2, \alpha_2 = 3$							
0.3	**0.4**	-0.6978	-0.3138	-0.0404	0.1369	0.2316	0.2552	0.2183	0.1305
0.7	**0.8**	0.2871	0.4429	0.5258	0.5461	0.5132	0.4351	0.3194	0.1724
2	**5**	0.7911	0.8116	0.7851	0.7189	0.6198	0.4934	0.3450	0.1792
10	**15**	0.8875	0.8751	0.8232	0.7382	0.6260	0.4919	0.3402	0.1750
20	**25**	0.8967	0.8800	0.8249	0.7378	0.6243	0.4896	0.3382	0.1738
50	**100**	0.9021	0.8823	0.8249	0.7364	0.6222	0.4873	0.3362	0.1726

3.1.3 Conditional Distribution

The conditional density function of Y given $X = t$ is given by

$$h_{Y/X}(y/x = t) = \frac{\alpha_2\beta_2}{\lambda}(1 - t^{\alpha_1})^{\beta_1(1-\lambda)} y^{\alpha_2 - 1}(1 - y^{\alpha_2})^{\beta_2 - 1} \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1} j^2 \left\{\left(1 - (1 - t^{\alpha_1})^{\beta_1}\right)\left(1 - (1 - y^{\alpha_2})^{\beta_2}\right)\right\}^{j-1} \quad (20)$$

Theorem 3: If $(X, Y) \sim BKPH(\lambda, \Theta)$, then the conditional moment of Y^r on $X = t$ is given by

$$E(Y^r/X = t) = \frac{\beta_2}{\lambda}(1 - t^{\alpha_1})^{\beta_1(1-\lambda)} \sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1} j^2 \left(1 - (1 - t^{\alpha_1})^{\beta_1}\right)^{j-1} \sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i \,\mathrm{B}\left(\beta_2(1 + i), \frac{\alpha_2 + r}{\alpha_2}\right) \quad (21)$$

Proof. From (9) we have

$$E(Y^r/X=t)=\int_0^1 y^r\, h_{Y/X}(y/t)dy$$

$$=\frac{1}{\lambda}(1-t^{\alpha_1})^{\beta_1(1-\lambda)}\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-(1-t^{\alpha_1})^{\beta_1}\right)^{j-1}\left(\alpha_2\beta_2\int_0^1 y^r\, y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}dy\right)$$

By using the expansion (3), then

$$E(Y^r/X=t)=\frac{\beta_2}{\lambda}(1-t^{\alpha_1})^{\beta_1(1-\lambda)}\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\left(1-(1-t^{\alpha_1})^{\beta_1}\right)^{j-1}\left(\sum_{i=0}^{J-1}\binom{J-1}{i}(-1)^i\int_0^1 u^{\beta_2(i+1)-1}(1-u)^{\frac{r}{\alpha_2}}du\right)$$

and

$$\int_0^1 u^{\beta_2(i+1)-1}(1-u)^{\frac{r}{\alpha_2}}du=\mathrm{B}\left(\beta_2(1+i),\frac{\alpha_2+r}{\alpha_2}\right)$$

Hence, the proof is done.

3.2 Reliability characteristics

3.2.1 Survival Function

Following, the survival function of the bivariate distribution which is in the form

$$S(x,y)=1-H_1(x)-H_2(x)+H_\lambda(x,y)$$

The joint survival function is given by

$$S(x,y;\lambda,\Theta)=\left((1-x^{\alpha_1})^{\beta_1}\right)^{\lambda}+\left((1-y^{\alpha_2})^{\beta_2}\right)^{\lambda}-\left(1-\left(1-(1-x^{\alpha_1})^{\beta_1}\right)\left(1-(1-y^{\alpha_2})^{\beta_2}\right)\right)^{\lambda} \quad (22)$$

3.2.2 Stress-Strength Parameter

According to reliability theory, a component's life is described by the stress-strength model, which includes a random strength (X) subjected to a random stress (Y). Components fail instantly when the level of stress applied exceeds the level of strength. Therefore, component reliability is measured by $R=P(X<Y)$.

Theorem 4: The stress-strength parameter of $BKPH(\lambda,\Theta)$ is given by

$$R=P(X<Y)=\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}\mathrm{j}\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\sum_{k=0}^{j}\binom{j}{k}(-1)^k\sum_{l=0}^{\beta_1 k}\binom{\beta_1 k}{l}(-1)^l\beta_2\,\mathrm{B}\left(\beta_2(1+i),\frac{\alpha_2+\alpha_1 l}{\alpha_2}\right) \quad (23)$$

Proof.

$$R=P(X<Y)=\int_0^1\int_0^y h_\lambda(x,y;\lambda,\Theta)\,dx\,dy$$

$$=\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}j^2\int_0^1\alpha_2\,\beta_2 y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}dy\int_0^y\alpha_1\,\beta_1\,x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{j-1}dx$$

$$=\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}\,\mathrm{j}\int_0^1\alpha_2\,\beta_2 y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}\left(1-(1-y^{\alpha_1})^{\beta_1}\right)^{j}dy$$

By using the expansion (3), then

$$\left(1-(1-y^{\alpha_2})^{\beta_2}\right)^{j-1}=\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i(1-y^{\alpha_2})^{\beta_2 i}$$

and

$$\left(1-(1-y^{\alpha_1})^{\beta_1}\right)^j=\sum_{k=0}^{j}\binom{j}{k}(-1)^k(1-y^{\alpha_1})^{\beta_1 k}$$

Hence,

$$R=\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}\,\mathrm{j}\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\sum_{k=0}^{j}\binom{j}{k}(-1)^k\alpha_2\beta_2\int_0^1 y^{\alpha_2-1}\,(1-y^{\alpha_2})^{\beta_2(\mathrm{i}+1)-1}(1-y^{\alpha_1})^{\beta_1 k}dy$$

$$=\beta_2\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}\,\mathrm{j}\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\sum_{k=0}^{j}\binom{j}{k}(-1)^k\int_0^1\mathrm{u}^{\beta_2(\mathrm{i}+1)-1}\left(1-(1-u)^{\frac{\alpha_1}{\alpha_2}}\right)^{\beta_1 k}du$$

By using the expansion (3),

$$\left(1-(1-u)^{\frac{\alpha_1}{\alpha_2}}\right)^{\beta_1 k}=\sum_{l=0}^{\beta_1 k}\binom{\beta_1 k}{l}(-1)^l(1-u)^{\frac{\alpha_1}{\alpha_2}l}$$

hence,

$$R=\beta_2\sum_{j=1}^{\infty}\binom{\lambda}{j}(-1)^{j+1}\,\mathrm{j}\sum_{i=0}^{j-1}\binom{j-1}{i}(-1)^i\sum_{k=0}^{j}\binom{j}{k}(-1)^k\sum_{l=0}^{\beta_1 k}\binom{\beta_1 k}{l}(-1)^l\int_0^1\mathrm{u}^{\beta_2(\mathrm{i}+1)-1}\,(1-u)^{\frac{\alpha_1}{\alpha_2}l}du$$

where,

$$\int_0^1\mathrm{u}^{\beta_2(\mathrm{i}+1)-1}\,(1-u)^{\frac{\alpha_1}{\alpha_2}l}du=\mathrm{B}\left(\beta_2(1+i),\frac{\alpha_2+\alpha_1 l}{\alpha_2}\right)$$

which gives the required result.

3.2.3 The Joint Hazard Rate Function

There are several ways in which the bivariate failure rate is defined in the literature. According to [2], it is defined as follows:

$$r(x,y;\lambda,\Theta)=\frac{h_\lambda(x,y;\lambda,\Theta)}{S(x,y;\lambda,\Theta)}$$

According to this definition, the joint hazard rate function for BKPH distribution is

$$r(x,y;\lambda,\Theta)=$$

$$\frac{\lambda\alpha_1\alpha_2\beta_1\beta_2\;x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left(1-(-1+(1-x^{\alpha_1})^{\beta_1})(-1+(1-y^{\alpha_2})^{\beta_2})\right)^{\lambda-2}\left(1-(-1+(1-x^{\alpha_1})^{\beta_1})(-1+(1-y^{\alpha_2})^{\beta_2})\lambda\right)}{\left((1-x^{\alpha_1})^{\beta_1}\right)^{\lambda}+\left((1-y^{\alpha_2})^{\beta_2}\right)^{\lambda}-\left(1-(-1+(1-x^{\alpha_1})^{\beta_1})(-1+(1-y^{\alpha_2})^{\beta_2})\right)^{\lambda}}\tag{24}$$

According to [15], it is defined hazard gradient of a bivariate random vector as follows:

$$\eta(x,y;\lambda,\Theta)=\left(\frac{-\partial\ln S(x,y;\lambda,\Theta)}{\partial x},\frac{-\partial\ln S(x,y;\lambda,\Theta)}{\partial y}\right)$$

Therefore, the hazard gradient for BKPH distribution is

$$\frac{-\partial\ln S(x,y;\lambda,\Theta)}{\partial x}=-\frac{\alpha_1\beta_1 x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}\left((1-(1-y^{\alpha_2})^{\beta_2})-\lambda\left((1-x^{\alpha_1})^{\beta_1}\right)^{\lambda-1}\right)}{\left((1-x^{\alpha_1})^{\beta_1}\right)^{\lambda}+\left((1-y^{\alpha_2})^{\beta_2}\right)^{\lambda}-\left(1-(1-(1-x^{\alpha_1})^{\beta_1})(1-(1-y^{\alpha_2})^{\beta_2})\right)}$$

and,
$$\frac{-\partial\ln S(x,y;\lambda,\Theta)}{\partial y}=-\frac{\alpha_2\beta_2 y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\left((1-(1-x^{\alpha_1})^{\beta_1})-\lambda\left((1-y^{\alpha_2})^{\beta_2}\right)^{\lambda-1}\right)}{\left((1-x^{\alpha_1})^{\beta_1}\right)^{\lambda}+\left((1-y^{\alpha_2})^{\beta_2}\right)^{\lambda}-\left(1-(1-(1-x^{\alpha_1})^{\beta_1})(1-(1-y^{\alpha_2})^{\beta_2})\right)}$$

3.3 Copula structure and dependence properties

The study of stochastic dependence relies heavily on bivariate distribution functions with uniform marginals. They have been rediscovered many times and used in various contexts under different names, such as “copulas”.

The PHR bivariate distributions has an Archimedean copula see, [22] which defined as:

The Archimedean copula $C: [0,1]^2 \to [0,1]$ has a form

$$C_{\phi}(u,v) = \phi^{[-1]}\{\phi(u) + \phi(v)\}, \ \ 0 \le u, v \le 1$$

for some convex decreasing function $\phi: [0,1] \to [0, \infty[$.

We discuss the dependence properties of the BKPH distribution through Archimedean copula via According to Sklar’s theorem [4], solving the equation

$$C_{\lambda}\big(H_1(x), H_2(y)\big) = H(x,y) \tag{25}$$

for the function $C_{\lambda}: [0,1]^2 \to [0,1]$ yields the underlying copula associated with (2), as

$$C_{\lambda}(u,v) = 1 - \left\{1 - \left(1 - (1-u)^{\frac{1}{\lambda}}\right)\left(1 - (1-v)^{\frac{1}{\lambda}}\right)\right\}^{\lambda}, \tag{26}$$

for all $u, v \in (0,1)$ and $0 < \lambda \le 1$. This copula belongs to the Archimedean family of copulas with the strict generator $\phi(t) = -\ln\left[1 - (1-t)^{\frac{1}{\lambda}}\right]$ (see, [20] for detail). Note that the density function of the BKPH distribution defined by (5) could be rewritten as

$$h_{\lambda}(x,y) = h_1(x) h_2(y) c_{\lambda}\big(H_1(x), H_2(y)\big)$$

where $H_i(x) = 1 - (1 - x^{\alpha_i})^{\beta_i \lambda} and\ h_i(x) = \alpha_i \beta_i \lambda\, x^{\alpha_i - 1}(1 - x^{\alpha_i})^{\beta_i \lambda - 1}$, $i = 1,2$, are the marginal distribution functions and the marginal density functions, respectively and $c_{\lambda}(u,v) = \frac{\partial^2}{\partial u \partial v} C_{\lambda}(u,v)$, is the density function for copula (14) given by

$$c_{\lambda}(u,v) = \lambda\left\{1 - \lambda\left(1 - (1-u)^{\frac{1}{\lambda}}\right)\left(1 - (1-v)^{\frac{1}{\lambda}}\right)\right\}\left\{1 - \left(1 - (1-u)^{\frac{1}{\lambda}}\right)\left(-(1-v)^{\frac{1}{\lambda}}\right)\right\}^{\lambda - 2} \tag{27}$$

In [22,23], copula properties are discussed as measures of association, concordance ordering, tail monotonicity, and symmetry.

- **Measures of Association**

  Dependence properties such as Kendall's tau (τ) and Spearman's rho (ρ_s) which depend only on the copula C and are given by

$$\tau = 4 \int_0^1 \int_0^1 C(u,v)\, dC(u,v) - 1 \tag{28}$$

$$\rho_s = 12 \int_0^1 \int_0^1 C(u,v)\, du\, dv - 3, \tag{29}$$

Due to [3], the following result provides expressions for these measures associated.

Proposition 1. Suppose that $C_{\lambda}(u,v)$ be a copula defined in (14) then for every $0 < \lambda \le 1$

$$\tau = 1 + 4\lambda B(2, 2\lambda - 1)(\Psi(2) - \Psi(2\lambda + 1)),$$

$$\rho = 9 - 12\lambda^2 \sum_{j=0}^{\infty} (-1)^j \binom{\lambda}{j} [B(j+1, \lambda)]^2,$$

where B denotes the beta function and Ψ is the digamma function.

3.4 Simulation

We present Monte Carlo simulation studies in this section to investigate the performance of the proposed estimators in this paper. The bias and mean square error (MSE) criteria are used to compare maximum likelihood (MLE), Method of Moments (MME), and Inference Function for Margins (IFM) estimators by considering different sample sizes and different parameter values. For each sample size (n) and the specified values of the parameters, 1000 data sets are generated from the BKPH distribution with three sets of the parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda) = (2,1,2,1,0.5)$, $(\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda) = (1.8,1,1.7,1,0.6)$ and $(\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda) = (1.7,1.1,1.6,1,0.5)$. The sample sizes (n) are 30, 50, 70, and 90 observations. The results of the bias and mean square error (MSE) over 1000 replications are reported in Table 2, Table 3, and Table 4.

Table 2. Estimation results based on MLE, MME, and IFM approaches for parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda) = (2,1,2,1,0.5)$ of BKK distribution for $n = 30,50,70\ and\ 90$.

	n	MSE			Bias		
		MLE	MME	IFM	MLE	MME	IFM
α_1	30	0.347	1.076	0.152	0.206	-0.992	0.067
	50	0.17	1.017	0.095	0.105	-0.985	0.044
	70	0.131	1.026	0.061	0.081	-0.998	0.019
	90	0.087	0.992	0.049	0.042	-0.982	0.043
β_1	30	0.15	1.295	0.037	0.024	1.056	0.033
	50	0.084	1.149	0.023	0.001	1.018	0.019
	70	0.049	1.137	0.017	-0.001	1.029	0.024
	90	0.041	1.042	0.011	-0.007	0.992	0.003
α_2	30	0.379	1.126	0.155	0.223	-1.008	0.046
	50	0.184	1.051	0.092	0.115	-0.995	0.045
	70	0.118	1.042	0.063	0.073	-1.002	0.025
	90	0.088	1.026	0.052	0.053	-0.997	0.023
β_2	30	0.184	1.321	0.038	0.049	1.060	0.040
	50	0.081	1.163	0.022	-0.0002	1.024	0.013
	70	0.054	1.151	0.016	-0.0006	1.033	0.019
	90	0.04	1.099	0.013	-0.002	1.016	0.012
λ	30	1.789	0.039	0.342	0.849	-0.178	0.562
	50	0.678	0.037	0.300	0.669	-0.179	0.536
	70	0.457	0.038	0.299	0.601	-0.188	0.539
	90	0.416	0.036	0.277	0.582	-0.183	0.521

Table 3. Estimation results based on MLE, MME, and IFM approaches for parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda) = (1.8,1,1.7,1,0.6)$ of BKK distribution for $n = 30,50,70\ and\ 90$.

	n	MSE			Bias		
		MLE	MME	IFM	MLE	MME	IFM
α_1	30	0.298	0.534	0.123	0.174	-0.684	0.061
	50	0.145	0.508	0.077	0.099	-0.685	0.039
	70	0.102	0.505	0.049	0.085	-0.692	0.018
	90	0.071	0.500	0.040	0.046	-0.691	0.039
β_1	30	0.146	0.635	0.037	0.031	0.716	0.033
	50	0.075	0.539	0.023	-0.016	0.680	0.019
	70	0.053	0.518	0.017	-0.007	0.681	0.024
	90	0.037	0.495	0.011	-0.009	0.672	0.003
α_2	30	0.264	0.496	0.112	0.159	-0.659	0.039
	50	0.132	0.468	0.066	0.106	-0.658	0.038
	70	0.081	0.484	0.046	0.065	-0.679	0.021
	90	0.062	0.454	0.038	0.048	-0.661	0.019
β_2	30	0.151	0.642	0.038	0.033	0.716	0.040
	50	0.089	0.549	0.022	0.002	0.688	0.013
	70	0.055	0.553	0.016	-0.02	0.706	0.019
	90	0.036	0.486	0.013	-0.005	0.668	0.012
λ	30	1.745	0.195	0.239	0.741	0.403	0.462
	50	0.713	0.176	0.202	0.597	0.394	0.436
	70	0.498	0.175	0.197	0.539	0.403	0.435
	90	0.268	0.164	0.180	0.474	0.392	0.418

Table 4. Estimation results based on MLE, MME, and IFM approaches for parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda) = (1.7,1.1,1.6,1,0.5)$ of BKK distribution for $n = 30,50,70\ and\ 90$.

	n	MSE			Bias		
		MLE	MME	IFM	MLE	MME	IFM
α_1	30	0.219	0.793	0.099	0.119	-0.856	0.052
	50	0.104	0.759	0.063	0.05	-0.852	0.034
	70	0.090	0.763	0.040	0.024	-0.857	0.015
	90	0.069	0.753	0.033	0.001	-0.858	0.035
β_1	30	0.278	1.559	0.045	0.316	1.163	0.033
	50	0.113	1.384	0.028	0.187	1.123	0.019
	70	0.077	1.339	0.020	0.156	1.118	0.024
	90	0.047	1.328	0.013	0.116	1.119	0.003

α_2	30	0.211	0.817	0.099	0.129	-0.855	0.037
	50	0.117	0.742	0.059	0.061	-0.834	0.036
	70	0.077	0.759	0.041	0.033	-0.853	0.020
	90	0.057	0.702	0.034	0.004	-0.825	0.019
β_2	30	0.246	1.305	0.038	0.297	1.059	0.040
	50	0.100	1.155	0.022	0.182	1.02	0.013
	70	0.059	1.172	0.016	0.132	1.046	0.019
	90	0.04	1.036	0.013	0.111	0.991	0.012
λ	30	0.137	0.058	0.342	0.358	-0.229	0.562
	50	0.162	0.056	0.300	0.395	-0.230	0.536
	70	0.171	0.057	0.299	0.408	-0.236	0.539
	90	0.178	0.055	0.277	0.416	-0.233	0.521

- In all the cases the performances of all methods of estimation are quite satisfactory. The biases and MSE of the MLE, MME, and IFM approaches for each $\alpha_1, \beta_1, \alpha_2, \beta_2, \lambda$ decay towards zero as the sample size increases, as expected.
- The biases and MSE of the IFM approach for each $\alpha_1, \beta_1, \alpha_2, \beta_2$ are better than the MLE and MME. The biases and MSE of the MME for λ are better than the MLE and IFM.
- The performance of the IFM estimates behaves in a very similar manner to the corresponding MLE exception for λ .

3.5 Application

A study was conducted to determine the total mercury concentrations in various size classes of two pelagic fish species of high commercial significance, horse mackerel (Trachurus trachurus) and Mediterranean horse mackerel (Trachurus mediterraneus), in order to determine whether total mercury concentrations and fish sizes are related and if any differences may affect the quantitative assessment of mercury exposure to consumers., see [13].

Table 5. Mean total of mercury concentrations and length of two pelagic fish species

Fish Species	**Horse Mackerela**		**Mediterranean Horse**	
Group no.	Fish length (m)	Mean total of mercury $(mg\ kg^{-1})$	Fish length (m)	Mean total of mercury $(mg\ kg^{-1})$
1	0.195	0.00017	0.185	0.00009
2	0.214	0.0002	0.199	0.00009
3	0.244	0.00016	0.229	0.00021
4	0.25	0.00026	0.234	0.00016
5	0.29	0.00043	0.243	0.00021
6	0.231	0.00029	0.246	0.00026
7	0.271	0.00042	0.265	0.00028
8	0.285	0.00044	0.291	0.00041

9	0.321	0.00059	0.296	0.00054
10	0.326	0.00139	0.341	0.0007
11	0.344	0.00142	0.369	0.00155
12	0.372	0.00241	0.397	0.00162

According to Table 6, the following are the most important descriptive statistics

Table 6. Summary statistics for the data set

Fish Species	**Horse Mackerela**		**Mediterranean Horse**	
Statistics	Fish length (m)	Mean total of mercury Hg	Fish length (m)	Mean total of mercury Hg
Minimum	0.195	0.00016	0.185	0.00009
1st Quartile	0.23425	0.0002150	0.23025	0.0001725
Median	0.27800	0.0004250	0.25550	0.0002700
Mean	0.27858	0.0006817	0.27458	0.0005100
3rd Quartile	.32475	.0011900	.32975	.0006600
Maximum	0.372	0.00241	0.397	0.00162
Standard Error	0.015743	0.00020094	0.019150	0.00015402
Pearson's Correlation (ρ)	0.861195		0.929182	
τ	0.848485		0.953959	
ρ_s	0.937063		0.985971	
ρ_s/τ	1.104395		1.03356	

By first performing the Kolmogorov-Smirnov test and Anderson-Darling test (AD), Critical Value (Cv) for the marginal Kumaraswamy distribution, we determine the initial estimates. The results of Table 7and Table 8 indicate a good fit for the Kumaraswamy distributions of two pelagic fish species.

Table 7. Goodness of fit measures for the marginal to Kumaraswamy distribution of Horse Mackerela

	MLE Estimates	**KS**	**p-value**	**AD**	**Critical value at 0.05**
Fish length (m)	$\hat{\alpha}_1 = 2.1$ $\hat{\beta}_1 = 9.7$	0.2728	0.3339	1.6593	2.5018
Mean total of mercury Hg	$\hat{\alpha}_2 = 0.5$ $\hat{\beta}_2 = 26.2$	0.2836	0.2893	1.9116	2.5018

Table 8. Goodness of fit measures for the marginal to Kumaraswamy distribution of Mediterranean Horse

	MLE Estimates	**KS**	**p-value**	**AD**	**Critical value at 0.05**
Fish length (m)	$\hat{\alpha}_1 = 2.1$ $\hat{\beta}_1 = 10$	0.2543	0.4198	1.3111	2.5018
Mean total of mercury Hg	$\hat{\alpha}_2 = 0.5$ $\hat{\beta}_2 = 32$	0.2686	0.3522	1.6343	2.5018

The BKPH model is compared with different bivariate distribution models. The following bivariate distributions models are:

- ***Bivariate Kumaraswamy type Exponential distribution (BKE),*** *see [22]*

$$g_{BKE}(x,y;\ \alpha_1,\alpha_2,\lambda) = \alpha_1\alpha_2\lambda e^{-(\alpha_1 x+\alpha_2 y)}[1-\alpha(1-e^{-\alpha_1 x})(1-e^{-\alpha_2 y})][1-(1-e^{-\alpha_1 x})(1-e^{-\alpha_2 y})]^{\lambda-2} \quad (30)$$

where, $x, y > 0, \alpha_1, \alpha_2 > 0,\ 0 \le \lambda \le 1$.

- ***Bivariate Cubic Kumaraswamy Distribution (BCK)***

According to [8], a cubic copula is defined as follows:

$$C(u,v) = uv\left(1+\theta(u-1)(v-1)(2u-1)(2v-1)\right) \qquad -1 \le \theta \le 2$$

Using it, we defined Bivariate Cubic Kumaraswamy distribution as follows:

$$g_{CK}(x,y;\ \alpha_1,\beta_1,\alpha_2,\beta_2,\theta) = \alpha_1\alpha_2\beta_1\beta_2\left(x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}(1+(1-6(1-x^{\alpha_1})^{\beta_1}+6(1-x^{\alpha_1})^{2\beta_1})\ (1-6(1-y^{\alpha_2})^{\beta_2}+6(1-y^{\alpha_2})^{2\beta_2})\theta)\right) \quad (31)$$

- ***Bivariate Farlie-Gumbel Morgenstern Kumaraswamy Distribution (BFGMK)***

In [11], Farlie-Gumbel Morgenstern Kumaraswamy type copulas are defined as follows:

$$C\big(F(x),F(y)\big) = \Big(\big(1-(1-x^{\alpha_1})^{\beta_1}\big)\big(1-(1-y^{\alpha_2})^{\beta_2}\big)\Big)\{1+\ \theta((1-x^{\alpha_1})^{\beta_1}(1-y^{\alpha_2})^{\beta_2})\}$$

where, $-1 \le\ \theta \le 1$.

We used it to define Bivariate Farlie-Gumbel Morgenstern Kumaraswamy distribution as

$$g_{FGMK}(x,y;\ \alpha_1,\beta_1,\alpha_2,\beta_2,\theta) = \alpha_1\alpha_2\beta_1\beta_2\left(\left(x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\right)\left\{1+\theta\left(\left(2(1-x^{\alpha_1})^{\beta_1}-1\right)\left(2(1-y^{\alpha_2})^{\beta_2}-1\right)\right)\right\}\right) \quad (32)$$

- ***Bivariate Nelsen–Ten Kumaraswamy Distribution (BNTK)***

In [11], bivariate Nelsen–Ten Kumaraswamy type copulas are defined as follows:

$$C\big(F(x),F(y)\big) = \frac{(1-(1-x^{\alpha 1})^{\beta 1})(1-(1-y^{\alpha 2})^{\beta 2})}{[1+(1-(1-(1-x^{\alpha 1})^{\beta 1})^{\theta})(1-(1-(1-y^{\alpha 2})^{\beta 2})^{\theta})]^{1/\theta}}\ ,\quad 0 < \theta \le 1$$

We used it to define Bivariate Nelsen–Ten Kumaraswamy distribution as

$$g_{NTK}(x,y;\ \alpha_1,\beta_1,\alpha_2,\beta_2,\theta) = \alpha_1\alpha_2\beta_1\beta_2\left(\left(x^{\alpha_1-1}(1-x^{\alpha_1})^{\beta_1-1}y^{\alpha_2-1}(1-y^{\alpha_2})^{\beta_2-1}\right)\left((2-\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{\theta}+(-1\ +(1-(1-x^{\alpha_1})^{\beta_1})^{\theta})(1-(1-y^{\alpha_2})^{\beta_2}\right)^{\theta})^{-(2+\frac{1}{\theta})}\right)\left(4-2\left(1-(1-x^{\alpha_1})^{\beta_1}\right)^{\theta}+\ (1-(1-y^{\alpha_2})^{\beta_2})^{\theta}(-2-(1-(1-x^{\alpha 1})^{\beta 1})^{\theta}(-1+\theta)))\right)\Bigg) \quad (33)$$

The MLEs are derived from the parameters using the initial estimates. Based on the results of these tests, we will decide whether the proposed model fits better than the comparable bivariate Kumaraswamy models based on Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), and Consistent AIC (CAIC). According to the results, the BKPH model appears to provide a better fit to describe the relationship between the fish length and mercury concentration which is based on a highly positive correlation between them for both kinds of the studied types of fish. According to the goodness of fit criteria, the BKPH model modeled the dependency relationship with the best fit compared with other bivariate models being studied where the above criteria are minimal for BKPH model. In Tables 9 and 10, the results are summarized.

Table 9. The analytical measures of the fitted models for mercury concentration of Horse Mackere

Distribution	MLEs	Ln	AIC	BIC	CAIC	HQIC
BKPH	$\hat{\alpha}_1 = 2.35179$ $\hat{\alpha}_2 = 0.425821$ $\hat{\beta}_1 = 23.96369$ $\hat{\beta}_2 = 31.58676$ $\hat{\lambda} = 0.524483$	83.62	-157.23	-154.81	-147.23	-158.13
BKE	$\hat{\alpha}_1 = 4.68993$ $\hat{\alpha}_2 = 1578.289$ $\hat{\lambda} = 0.75854$	79.196	-152.39	-150.94	-149.39	-152.93
BFGMK	$\hat{\alpha}_1 = 1.858018$ $\hat{\alpha}_2 = 0.360625$ $\hat{\beta}_1 = 6.635428$ $\hat{\beta}_2 = 9.9983894$ $\hat{\theta} = 0.880875$	77.43	-144.866	-142.441	-134.866	-145.763
BCK	$\hat{\alpha}_1 = 1.985136$ $\hat{\alpha}_2 = 0.264347$ $\hat{\beta}_1 = 1.40101$ $\hat{\beta}_2 = 2.274994$ $\hat{\theta} = 0.92661$	55.45	-100.896	-98.47	-90.895	-101.793
BNTK	$\hat{\alpha}_1 = 1.463249$ $\hat{\alpha}_2 = 0.320039$ $\hat{\beta}_1 = 1.689775$ $\hat{\beta}_2 = 2.916852$ $\hat{\theta} = 0.8570699$	83.396	-156.793	-154.386	-146.793	-157.69

Table 10. The analytical measures of the fitted models for mercury concentration of Mediterranean Horse Mackere

Distribution	MLEs	Ln	AIC	BIC	CAIC	HQIC
BKPH	$\hat{\alpha}_1 = 3.37113$ $\hat{\alpha}_2 = 0.379294$ $\hat{\beta}_1 = 108.672$ $\hat{\beta}_2 = 33.9846$ $\hat{\lambda} = 0.48612$	90.22	-170.45	-168.02	-160.45	-171.35

BKE	$\hat{\alpha}_1 = 7.52558$ $\hat{\alpha}_2 = 3407.524$ $\hat{\lambda} = 0.50203$	83.73	−161.46	-160	-158.46	-166.99
BFGMK	$\hat{\alpha}_1 = 5.02965$ $\hat{\alpha}_2 = 0.32009$ $\hat{\beta}_1 = 15.57205$ $\hat{\beta}_2 = 11.39262$ $\hat{\theta} = 0.765614$	54.94	−99.87	−97.45	−89.87	−100.77
BCK	$\hat{\alpha}_1 = 1.656216$ $\hat{\alpha}_2 = 0.31686$ $\hat{\beta}_1 = 5.15974$ $\hat{\beta}_2 = 7.75088$ $\hat{\theta} = 0.62665$	78.79	−147.59	−145.17	−137.59	−148.49
BNTK	$\hat{\alpha}_1 = 1.326095$ $\hat{\alpha}_2 = 0.307437$ $\hat{\beta}_1 = 1.55858$ $\hat{\beta}_2 = 2.909156$ $\hat{\theta} = 0.9115225$	87.008	−164.02	−161.592	−154.016	−164.914

3.6 Conclusion and Discussion

In this paper, the Bivariate Kumaraswamy Proportional Hazard distribution (BKPH) is proposed and studied via several characteristics: Properties, Estimation, Simulation, and Applications. A related copula belonging to the Archimedean family was obtained and its dependence properties were examined. Point estimation of parameters is done and examined via 3 types of estimation, MLE, MME and IFM methods. IFM method describes the best method for the marginal's parameter estimation followed by MLE method. For the dependence parameter, the MME method is the best. For a real-life practical application concerned with toxic absorption in fish bodies, we conclude that BKPH gives a flexible and good model for dependency between mercury absorption in *Trachurus trachurus* &*Trachurus mediterraneus* fish bodies and their length compared with some competitor-bivariate models.

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