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Inferential Survival Analysis for Type II Censored Truncated Exponential Topp Leone Exponential Distribution with Application to Engineering Data

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Abstract: This study focuses on estimating the unknown parameters of the truncated exponential Topp-Leon distribution using a type II scheme. We estimate the unknown parameters, survival, and hazard functions using maximum likelihood estimation methods. Additionally, we derive the approximate variance covariance matrix and asymptotic confidence intervals. Furthermore, we compute Bayesian estimates of the unknown parameters under squared error and linear loss functions. To generate samples from the posterior density functions, we use the Metropolies-Hastings algorithm. We demonstrate the effectiveness of the proposed distribution by applying it to two data sets: Monte Carlo simulation and real data set. Our results show that the proposed distribution provides accurate estimates of the unknown parameters and performs well in fitting the data. Our findings also indicate that Bayesian estimation can provide more precise estimates with narrower confidence intervals compared to maximum likelihood estimation method. In summary, the study provides a comprehensive analysis of the estimation of the unknown parameters for the truncated exponential Topp-Leone distribution using a type II scheme. Also, the results demonstrate the potential of this distribution in modeling real data and the usefulness of both maximum likelihood and Bayesian estimation methods in obtaining accurate parameter estimates.

Keywords: Bayesian estimation, Censoring schemes, Metropolis–Hastings algorithm, Truncated exponential Distribution.

1 Introduction

Censoring can occur in industrial and clinical experiments when items are lost or removed from the experiment before the event of interest occurs. In such cases where complete information about the event of interest is not available for all experimental items, the resulting data is referred to as censored data. Pre-planned censoring can help save on the total test time and reduce the overall cost of the experiment. One of the most common censored schemes is type II censored scheme. It can be described as follows: Let we have a collection of experimental items and we denote their lifetimes as $X_1, X_2, ..., X_n$. Let's assume that the X's, $i = 1, ..., n$, are independently and identically distributed using the PDF $f(x_i, \phi)$ and CDF $F(x_i, \phi)$. Let m represents the number of failures, and it is determined before initialing the experiment. once m failures are observed, the life testing experiment is terminated, resulting in a type II censored sample consisting of $x_1, x_2, ..., x_m$. According to [1], The likelihood function (LF) for a type II censored scheme was defined in the following manner:

$$L(\nu) = \frac{n!}{(n-m)!} \prod_{i=1}^{m} f(x_i, \nu)(1 - F(x_{(m)}, \nu))^{n-m} \tag{1}$$

where ν is the unknown parameter. Complete data result from (1) when $m = n$. Recent studies have extended the use of censoring schemes to truncated distributions. One such study, referenced as [6] focused on statistical inference for a truncated normal distribution using progressive type I censoring scheme. They employed various estimation techniques such as maximum likelihood (ML) and Bayesian methods ,and constructed confidence intervals . Another referenced as [24] focused on the truncated weibull distribution and its application in reliability analysis. They examined the properties of the distribution explored various methods for parameter estimation, and conducted comparisons with alternative

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models. In yet another study referenced as [7] statistical inference for a truncated normal distribution with a progressive type II censoring scheme was investigated. The researchers utilized ML and Bayesian methods to estimate the unknown parameters and presented both point and interval estimates. Specifically concentrating on the estimation of points and intervals for a left-truncated normal distribution under progressive first failure censoring scheme. The study [28] derived ML estimators and constructed confidence intervals using asymptotic normality and bootstrap method. For Bayesian estimation, Lindley approximation method was employed, taking into account both symmetric and asymmetric loss functions. The study [26] directed their attention to the statistical inference of a truncated normal distribution utilizing generalized progressive hybrid censored scheme. They derived ML estimators and constructed asymptotic confidence intervals. They also derived the evaluation of Bayes estimates (BEs) under various loss functions and the constructions of Bayesian credible intervals. Finally, study [25] conducted parameter estimation for the truncated Weibull Rayleigh distribution while employing progressive type II censoring scheme. ML and Bayesian methods were utilized, and confidence intervals were constructed as part of their analysis. This paper employs ML and Bayesian estimation techniques to obtain point and interval estimates of unknown parameters of the LTETL-ex distribution under a type II censored scheme. When both parameters are unknown, Bayes estimates are obtained based on the squared error (SE) and linear exponential (LINEX) loss functions. Since closed-form solutions for the ML and Bayes estimates are not feasible, they are evaluated numerically using Markov chain Monte Carlo (MCMC) techniques such as the Metroplies-Hasting (M-H) algorithm. The performance of various estimates is compared through a Monte Carlo (MC) simulation study, based on their root mean squared error (RMEs), mean absolute biased (MABs), average confidence lengths (ACLs), and coverage probabilities (CPs). To demonstrate the effectiveness of the proposed methods, numerical examples are presented using real data sets and MCMC data sets. Lastly, the paper aims to fill the scholarly gap of estimating the unknown parameters of the LTETL-ex distribution under a censoring scheme by utilizing a type II censored sample. The paper is structured into seven sections, with In Section 2, outlines the method for deriving the MLEs of the unknown parameters under a type II censored scheme. In Section 3, Bayesian estimates of the unknown parameters using MCMC and the SE and LINEX loss functions are presented. Section 4 exhibits the simulation results study that compares the results of different estimation methods in terms of their RMSEs, MABs, ACLs, and CPs. Section 5 and 6presents computational results that demonstrate the application of the proposed methods using both a real dataset and MCMC dataset. Finally, Section 7 concludes the paper by providing some concluding remarks discussed.

2 Model description

Truncated distribution (TD) for a continuous random variable is a key research topic in both theoretical and applied contexts. It arises in various problems of probabilistic modeling in areas such as engineering, insurance forecasting, lifetime data analysis, and reliability analysis, among others, as noted by [17]. The left truncated exponential Topp Leone exponential (LTETL-ex) distribution is introduced by [19], which is a sub-model of the truncated exponential Topp Leone family. The probability density function (PDF) of the LTETL-ex distribution, with truncation limits$(0,\infty)$, is expressed as follows:

$$f(x;\alpha,\theta,\lambda)=\frac{2\alpha\lambda\theta e^{-2\theta x}}{(1-e^{-\lambda})}e^{-\lambda(1-e^{-2\theta x})^{\alpha}}(1-e^{-2\theta x})^{\alpha-1},\qquad \alpha,\theta,\lambda>0,x>0, \tag{2}$$

where α denotes shape parameter. θ and λ denote two scale parameters, respectively. The corresponding cumulative distribution function (CDF) is

$$F(x;\alpha,\theta,\lambda)=\frac{(1-e^{-\lambda(1-e^{-2\theta x})^{\alpha}})}{(1-e^{-\lambda})},\qquad \alpha,\theta,\lambda>0,x>0, \tag{3}$$

The survival and the hazard functions are

$$S(t)=\frac{(e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{-\lambda})}{(1-e^{-\lambda})},\qquad \alpha,\lambda,\theta>0,t>0, \tag{4}$$

and

$$h(t)=\frac{2\lambda\alpha\theta e^{-2\theta t}e^{-\lambda(1-e^{-2\theta t})^{\alpha}}(1-e^{-2\theta t})^{\alpha-1}}{e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{-\lambda}},\qquad \alpha,\theta,\lambda>0,t>0. \tag{5}$$

respectively.

3 Maximum likelihood estimation

In this section, MLEs of unknown parameters of LTETL-ex distribution based on type II censoring scheme are derived. Suppose that $\mathbf{x} = (x_{(1)}, \ldots, x_{(m)})$ denote a type II censored sample drawn from the LTETL-ex distribution. The PDF and CDF of this distribution are denoted by (2) and (3), respectively. By utilizing these equations, along with (5), we can express the LF as follows:

$$L(\alpha,\theta,\lambda) \propto \lambda^m \alpha^m \theta^m e^{-2\theta\sum_{i=1}^m x_{(i)}} \prod_{i=1}^m (1-e^{-2\theta x_{(i)}})^{(\alpha-1)} e^{-\lambda(n-m)(1-e^{-2\theta x_{(m)}})^\alpha} (1-e^{-\lambda})^{-n} (1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^\alpha)})^{(n-m)} \tag{6}$$

The logarithm of LF, denoted as $\ln L(\alpha,\theta,\lambda)$, can be obtained from (6) as follows

$$\begin{aligned} \ln L(\alpha,\theta,\lambda) \propto\ & m\ln\alpha + m\ln\lambda + m\ln\theta - n\ln(1-e^{-\lambda}) - 2\theta\sum_{i=1}^m x_{(i)} - \lambda\sum_{i=1}^m (1-e^{-2x_{(i)}\theta})^\alpha + (\alpha-1)\sum_{i=1}^m \ln(1-e^{-2x_{(i)}\theta}) \\ & -(n-m)\lambda(1-e^{-2\theta x_{(m)}})^\alpha + (n-m)\ln(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^\alpha)}) \end{aligned} \tag{7}$$

Calculating the first partial derivatives of (7) with respect to α, θ, and λ, and equating each of them to zero; the result is:

$$\begin{aligned} & \frac{m}{\hat\alpha} - \hat\lambda\sum_{i=1}^m (1-e^{-2\hat\theta x_{(i)}})^{\hat\alpha}\ln(1-e^{-2\hat\theta x_{(i)}}) + \sum_{i=1}^m \ln(1-e^{-2\hat\theta x_{(i)}}) \\ & -(n-m)(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha}\ln(1-e^{-2\hat\theta x_{(m)}}) - \hat\lambda(n-m)(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha} \\ & \frac{\ln(1-e^{-2\hat\theta x_{(m)}})e^{-\hat\lambda(1-(1-e^{-2\hat\theta x_{(m)}}))}}{(1-e^{-\hat\lambda(1-(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha})})} = 0, \end{aligned} \tag{8}$$

$$\frac{m}{\hat\lambda} - \sum_{i=1}^m (1-e^{-2\hat\theta x_{(i)}})^{\hat\alpha} - \frac{ne^{-\hat\lambda}}{(1-e^{-\hat\lambda})} + (n-m)\frac{(1-(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha})e^{-\hat\lambda(1-(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha})}}{(1-e^{-\hat\lambda(1-(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha})})} = 0, \tag{9}$$

and

$$\begin{aligned} & \frac{m}{\hat\theta} - 2\sum_{i=1}^m x_{(i)} + 2\hat\alpha\hat\lambda(n-m)x_{(m)}e^{-2\hat\theta x_{(m)}}(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha-1} \\ & -2\hat\alpha\hat\lambda\sum_{i=1}^m e^{-2\hat\theta x_{(i)}}(1-e^{-2x_{(i)}\hat\theta})^{\hat\alpha-1} + 2(\hat\alpha-1)\sum_{i=1}^m \frac{x_{(i)}e^{-2\hat\theta x_{(i)}}}{(1-e^{-2\hat\theta x_{(i)}})} \\ & -\hat\alpha\hat\lambda(n-m)\frac{e^{-2\hat\theta x_{(m)}}e^{-\hat\lambda(1-(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha})}(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha-1}}{(1-e^{-\hat\lambda(1-(1-e^{-2\hat\theta x_{(m)}})^{\hat\alpha})})} = 0. \end{aligned} \tag{10}$$

Since (8), (9), and (10) can not have closed form solutions, they could be solved numerically using any iteration procedure to get the MLEs of of α, θ, and λ. Once the MLEs $\hat\alpha$, $\hat\theta$, and $\hat\lambda$ are obtained, the MLEs $\hat S(t)$ and $\hat h(t)$ of $S(t)$ and $h(t)$ can be found using the invariance property of MLEs, where α, θ, and λ are replaced by their MLEs $\hat\alpha$, $\hat\theta$, and $\hat\lambda$, respectively, as

$$\hat S(t) = \frac{(e^{-\hat\lambda(1-e^{-2\hat\theta t})^{\hat\alpha}} - e^{-\hat\lambda})}{(1-e^{-\hat\lambda})}, \qquad t>0$$

$$\hat h(t) = \frac{2\hat\lambda\hat\alpha\hat\theta e^{-2\hat\theta t}e^{-\hat\lambda(1-e^{-2\hat\theta t})^{\hat\alpha}}(1-e^{-2\hat\theta t})^{\hat\alpha}}{(e^{-\hat\lambda(1-e^{-2\hat\theta t})^{\hat\alpha}} - e^{-\hat\lambda})}, \qquad t>0$$

The variances and covariances $[V]$ of the MLEs are obtained by the elements of the inverse of Fisher information matrix (FIM). FIM denoted as $I(\alpha,\theta,\lambda)$, can be expressed as follows:

$$I(\alpha,\theta,\lambda) = -E\begin{bmatrix} \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\alpha^2} & \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\alpha\partial\lambda} & \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\alpha\partial\theta} \\ \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\theta\partial\alpha} & \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\theta^2} & \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\theta\partial\lambda} \\ \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\lambda\partial\alpha} & \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\lambda\partial\theta} & \frac{\partial^2 \ln L(\alpha,\theta,\lambda)}{\partial\lambda^2} \end{bmatrix}$$

The elements of the FIM can be derived from the logarithm of the LF (7) are as follows

$$\frac{\partial^2 \ln L(\alpha,\lambda,\theta)}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \lambda\sum_{i=1}^{m}(1-e^{-2\theta x_{(i)}})^{\alpha}(\ln(1-e^{-2\theta x_{(i)}}))^2 - \lambda(n-m)(1-e^{-2\theta x_{(m)}})^{\alpha}((\ln 1-e^{-2\theta x_{(m)}}))^2,$$

$$\frac{\partial^2 \ln L(\alpha,\lambda\theta)}{\partial\lambda^2} = \frac{m}{\lambda^2} + ne^{-\lambda}((1-e^{-\lambda})^{-1}) + \frac{e^{-\lambda}}{(1-e^{-\lambda})^2} - (n-m)e^{-\lambda(1-(1-e^{2\theta x_{(m)}})^{\alpha})}(1-(1-e^{-2\theta x_{(m)}})^{\alpha})$$
$$((1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})^{-1} - e^{-\lambda(1-(1-e^{2\theta x_{(m)}}))}),$$

$$\frac{\partial^2 \ln L(\alpha,\lambda,\theta)}{\partial\alpha\partial\lambda} = -(n-m)(1-e^{-2\theta x_{(m)}})^{\alpha}\frac{\ln(1-e^{-2\theta x_{(m)}})e^{-2\lambda(1-(1-e^{-2\theta x_{(m)}}))^2}}{(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha}})^2} - \sum_{i=1}^{m}(1-e^{-2\theta x_{(i)}})^{\alpha}\ln(1-e^{-2\theta x_{(i)}}),$$

$$\frac{\partial^2 \ln L(\alpha,\lambda,\theta)}{\partial\theta^2} = \frac{m}{\theta^2} - 4\alpha\lambda\sum_{i=1}^{m}x_{(i)}(1-e^{-2\theta x_{(i)}})^{\alpha-1}e^{-2\theta x_{(i)}}((\alpha-1)e^{-2\theta x_{(i)}}(1-1-e^{-2\theta x_{(i)}})^{-1})$$
$$+4(\alpha-1)\sum_{i=1}^{m}x_{(i)}e^{-2\theta x_{(i)}}\frac{(1-e^{-2\theta x_{(i)}})-e^{-2\theta x_{(i)}}}{(1-e^{-2\theta x_{(i)}})^2} - 4\alpha\lambda(n-m)x_{(m)}e^{-2\theta x_{(m)}}$$
$$\frac{(1-e^{-2\theta x_{(m)}})^{\alpha-1}}{(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})^2}(e^{-2\theta x_{(m)}}e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})}(1-e^{-2\theta x_{(m)}})^{\alpha-1})$$
$$+((\alpha-1)(1-e^{-2\theta x_{(m)}})^{\alpha-1}-1)$$

$$\frac{\partial^2 \ln L(\alpha,\lambda,\theta)}{\partial\alpha\partial\theta} = 2\lambda\sum_{i=1}^{m}e^{-2\theta x_{(i)}}(1-e^{-2\theta x_{(i)}})^{\alpha-1}(1+\alpha\ln(1-e^{-2\theta x_{(i)}}))$$
$$+2\sum_{i=1}^{m}\frac{x_{(i)}e^{-2\theta x_{(i)}}}{(1-e^{-2\theta x_{(i)}})} + 2\lambda(n-m)x_{(m)}e^{-2\theta x_{(m)}}(1-e^{-2\theta x_{(m)}})^{\alpha-1}$$
$$(1+\alpha\ln(1-e^{-2\theta x_{(m)}})) + \frac{\lambda(n-m)e^{-2\theta x_{(m)}}}{(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})^2}e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha}})$$
$$(1-e^{-2\theta x_{(m)}})^{\alpha-1}\ln(1-e^{-2\theta x_{(m)}})(1+\lambda(1-e^{-2\theta x_{(m)}})^{\alpha})(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha}})$$
$$-\lambda e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha}}(1-e^{-2\theta x_{(m)}})^{\alpha}$$

and

$$\frac{\partial^2 \ln L(\alpha,\lambda,\theta)}{\partial\theta\partial\lambda} = 2\alpha(n-m)x_{(m)}e^{-2\theta x_{(m)}}(1-e^{2\theta x_{(m)}})^{\alpha-1} - 2\alpha\sum_{i=1}^{m}e^{2\theta x_{(i)}}$$
$$(1-e^{2\theta x_{(i)}})^{\alpha-1} - \frac{\alpha(n-m)e^{-2\theta x_{(m)}}(1-e^{2\theta x_{(m)}})^{\alpha-1}}{(1-e^{-\lambda(1-(1-e^{2\theta x_{(m)}})^{\alpha})})^2}$$
$$((1-e^{-\lambda(1-(1-e^{2\theta x_{(m)}})^{\alpha})})e^{-\lambda(1-(1-e^{2\theta x_{(m)}})^{\alpha})}$$
$$(1-(1-e^{2\theta x_{(m)}})^{\alpha}) - \lambda e^{-2\lambda(1-(1-e^{2\theta x_{(m)}})^{\alpha})}(1-(1-e^{2\theta x_{(m)}})^{\alpha}))$$

Since the mathematical expectation of FIM can not be easily obtained, The observed information matrix (IM) $I(\hat{\alpha},\hat{\lambda},\hat{\theta})$ is used. It is obtained by replacing the expected value of the second derivatives of logarithm of LF by their MLEs. Hence, the asymptotic variance-covariance $[\hat{V}]$ matrix for the MLEs is obtained by inverting the observed $I(\hat{\alpha},\hat{\lambda},\hat{\theta})$ as it becomes

$$[\hat{V}] \cong I^{-1}(\hat{\alpha},\hat{\theta},\hat{\lambda}) \cong \begin{bmatrix} V(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta}) & Cov(\hat{\alpha},\hat{\lambda}) \\ Cov(\hat{\theta},\hat{\alpha}) & V(\hat{\theta}) & Cov(\hat{\theta},\hat{\lambda}) \\ Cov(\hat{\lambda},\hat{\alpha}) & Cov(\hat{\lambda},\hat{\theta}) & V(\hat{\lambda}) \end{bmatrix}$$

According to [11], the asymptotic distribution of the MLEs $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\lambda}$ is approximately distributed as multivariate normal distribution i.e, $\hat{\alpha} \sim N(\alpha, V(\hat{\alpha}))$, $\hat{\theta} \sim N(\theta, V(\hat{\theta}))$, and $\hat{\lambda} \sim N(\lambda, V(\hat{\lambda}))$. Thus, the $(1-p)100\%$ two sides approximate confidence intervals for α, θ, and λ are obtained respectively as:

$$(\hat{\alpha} \mp Z_{\frac{p}{2}} \sqrt{V(\hat{\alpha})}), \quad (\hat{\theta} \mp Z_{\frac{p}{2}} \sqrt{V(\hat{\theta})}) \quad \text{and} \quad (\hat{\lambda} \mp Z_{\frac{p}{2}} \sqrt{V(\hat{\lambda})})$$

respectively, where $V(\hat{\alpha})$, $V(\hat{\theta})$,and $V(\hat{\lambda})$ are the elements on the main diagonal of variance covarince matrix. $Z_{\frac{p}{2}}$ is the percentile of standard normal distribution with write tail probability $\frac{p}{2}$. To find out the variances of the survival and hazard functions, needed to construct their approximate confidence intervals, delta method (developed by [?]) is used. Thereby, the variance of $S(t)$ and $h(t)$ can be obtained as follows:

$$\hat{V}(\hat{S}(t)) = [\nabla \hat{S}(t)]^T [\hat{V}] [\nabla \hat{S}(t)]$$

and

$$\hat{V}(\hat{h}(t)) = [\nabla \hat{h}(t)]^T [\hat{V}] [\nabla \hat{h}(t)]$$

respectively, where T represents the transpose operation, $\Delta \hat{S}(t)$ and $\Delta \hat{h}(t)$ represent the gradient vector of first partial derivatives of $S(t)$ and $h(t)$ with respect to α, θ, and λ. They are evaluated at $\alpha = \hat{\alpha}, \theta = \hat{\theta}$ and $\lambda = \hat{\lambda}$. This is defined as follows

$$[\Delta \hat{S}(t)]^T = \left[\frac{\partial \Delta \hat{S}(t)}{\partial \alpha} \ \frac{\partial \Delta \hat{S}(t)}{\partial \theta} \ \frac{\partial \Delta \hat{S}(t)}{\partial \lambda} \right]_{(\alpha=\hat{\alpha}, \theta=\hat{\theta}, \lambda=\hat{\lambda})},$$

and

$$[\Delta \hat{h}(t)]^T = \left[\frac{\partial \Delta \hat{h}(t)}{\partial \alpha} \ \frac{\partial \Delta \hat{h}(t)}{\partial \theta} \ \frac{\partial \Delta \hat{h}(t)}{\partial \lambda} \right]_{(\alpha=\hat{\alpha}, \theta=\hat{\theta}, \lambda=\hat{\lambda})},$$

where,

$$\frac{\partial S(t)}{\partial \alpha} = -\frac{\lambda (1-e^{-2\theta t})^{\alpha} e^{-\lambda (1-e^{-2\theta t})^{\alpha}} \ln (1-e^{-2\theta t})}{(1-e^{-\lambda})},$$

$$\frac{\partial S(t)}{\partial \lambda} = e^{-\lambda} \left(\frac{(1-(1-e^{-2\theta t}))exp{-}\lambda (1-e^{-2\theta t})^{\alpha}}{(1-e^{-\lambda})} - \frac{e^{-\lambda}(1-exp{-}\lambda(1-e^{-2\theta t})^{\alpha})-1}{(1-e^{-\lambda})^2} \right),$$

$$\frac{\partial S(t)}{\partial \theta} = \left(-\frac{2\alpha\lambda t e^{-2\theta t}(1-e^{-2\theta t})^{\alpha-1} e^{-\lambda}(1-exp{-}\lambda(1-e^{-2\theta t})^{\alpha}}{(1-e^{-\lambda})^2} \right),$$

$$\begin{aligned}\frac{\partial h(t)}{\partial \alpha} =& \frac{2\lambda e^{-2\theta t}}{(1-e^{-2\theta t})} \left(\frac{e^{-\lambda(1-e^{-2\theta t})^{\alpha}}(1-e^{2\theta t})^{\alpha}(1+\alpha \ln(1-e^{-2\theta t}))(1-\lambda(1-e^{-2\theta t})^{\alpha})}{(e^{p}-\lambda(1-e^{-2\theta t})^{\alpha}-e^{\lambda})} \right. \\ & \left. \frac{\lambda \alpha e^{-2\theta t}(1-e^{-2\theta t})^{2\alpha} \ln (1-e^{-2\theta t})}{(e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{\lambda})^2} \right)\end{aligned}$$

$$\begin{aligned}\frac{\partial h(t)}{\partial \lambda} =& 2\lambda e^{-2\theta t}(1-e^{-2\theta t})^{(\alpha-1)} \left(\frac{e^{-\lambda(1-e^{-2\theta t})^{\alpha}}(1-\lambda(1-e^{-2\theta t})^{\alpha})}{(e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{\lambda})} \right. \\ & \left. - \frac{\lambda e^{-\lambda(1-e^{-2\theta t})^{\alpha}+1}(1-e^{-\lambda(1-e^{-2\theta t})^{\alpha}-1}(1-e^{-2\theta t})^{\alpha})}{(e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{\lambda})} \right)\end{aligned}$$

$$\begin{aligned}\frac{\partial h(t)}{\partial \theta} =& 2\lambda\alpha \left(\frac{(1-\alpha\lambda(1-e^{-2\theta t})^{\alpha}-(\alpha-1)e^{-2\theta t}(1-e^{-2\theta t})^{-1})}{(e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{\lambda})} \left(2\lambda\alpha t e^{-2\theta t}(1-e^{-2\theta t})^{\alpha-1}\right) \right. \\ & \left. -2\theta t e^{-2\theta t} e^{-\lambda(1-e^{-2\theta t})^{\alpha}}(1-e^{-2\theta t})^{\alpha-1} + \frac{2\alpha\lambda t e^{-2\lambda(1-e^{-2\theta t})^{\alpha}}(1-e^{-2\theta t})^{2(\alpha-1)}}{(e^{-\lambda(1-e^{-2\theta t})^{\alpha}}-e^{\lambda})^2} \right),\end{aligned}$$

respectively, the $(1-p)100\%$ approximate confidence intervals for $S(t)$ and $h(t)$ are given as follows:

$$\hat{S}(t) \mp Z_{\frac{p}{2}} \sqrt{\hat{V}(\hat{S}(t))} \qquad \text{and} \qquad \hat{h}(t) \mp Z_{\frac{p}{2}} \sqrt{\hat{V}(\hat{h}(t))}$$

respectively.

4 Bayesian estimation

In this section, BEs of unknown parameters α, θ, and λ, as well as for the survival function $S(t)$ and hazard function $h(t)$ are derived. Let the prior distributions for the unknown parameters α, θ, and λ, are independent Gamma distributions, as follows

$$\pi(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \qquad \alpha > 0, a_1, b_1 > 0,$$

$$\pi(\theta) \propto \theta^{a_2-1}e^{-b_2\theta}, \qquad \theta > 0, a_2, b_2 > 0, \tag{11}$$

and

$$\pi(\lambda) \propto \lambda^{a_3-1}e^{-b_3\lambda}, \qquad \lambda > 0, a_3, b_3 > 0.$$

where, a_1, a_2, a_3, b_1, b_2 ,and b_3 denote hyper parameter that are known and non-negative. From (11), the joint prior distribution for α, θ, and λ can be written as follows :

$$\pi(\alpha,\theta,\lambda) \propto \alpha^{a_1-1}\lambda^{a_3-1}\theta^{a_2-1}e^{-(b_1\alpha+b_2\theta+b_3\lambda)}, \qquad ,\alpha,\lambda,\theta > 0, a_1, a_2, a_3, b_1, b_2, b_3 > 0, \tag{12}$$

The joint posterior distribution (PD) $\pi(\alpha,\theta,\lambda|\mathbf{x})$ of the unknown parameters α, θ, and λ can be obtained by utilizing the LF (6) and the joint prior distribution (12) as follows:

$$\begin{aligned}\pi(\alpha,\theta,\lambda|\mathbf{x}) = K^{-1}\alpha^{a_1+m-1}\theta^{a_2+m-1}\lambda^{a_3+m-1}e^{-\lambda(b_3+b_2\theta+((n-m)(1-e^{-2\theta x_{(m)}})^{\alpha})}\\ (1-e^{-\lambda})^{-n}(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})\prod_{i=1}^{m}(1-e^{-2\theta x_{(i)}})^{\alpha-1}\end{aligned} \tag{13}$$

where K is the normalizing constant equal to

$$\begin{aligned}K^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{a_1+m-1}\theta^{a_2+m-1}\lambda^{a_3+m-1}e^{-\lambda(b_3+b_2\theta+((n-m)(1-e^{-2\theta x_{(m)}})^{\alpha})}(1-e^{-\lambda})^{-n}\\ (1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})\prod_{i=1}^{m}(1-e^{-2\theta x_{(i)}})^{\alpha-1}d\alpha d\theta d\lambda\end{aligned}$$

The loss function determines the effects of financial loss resulting from an inaccurate estimate of the unknown parameter. A commonly used loss function is SE loss function; a symmetrical loss function that assigns equal losses to overestimation and underestimation. Let $\tilde{\varphi}(\nu)$ represent an estimate of $\varphi(\nu)$, a function of the unknown parameter ν. The SE loss function is defined as:

$$\mathscr{L}_{SE}(\tilde{\varphi}(\nu),\varphi(\nu)) = (\tilde{\varphi}(\nu)-\varphi(\nu))^2$$

where $\mathscr{L}_{SE}(.)$ is SE loss function. Bayes estimate $\tilde{\varphi}(\nu)$ of $\varphi(\nu)$ under this loss function is considered the mean of the PD. Hence, BE of any function of α ,θ and λ, say $\varphi(\alpha,\theta,\lambda)$ under SE loss function is

$$\tilde{\varphi}_{SE}(\alpha,\theta,\lambda) = E(\varphi(\alpha,\theta,\lambda)|\mathbf{x})$$

where

$$E(\varphi(\alpha,\theta,\lambda)|\mathbf{x}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \varphi(\alpha,\theta,\lambda)\pi(\alpha,\theta,\lambda|\underline{x})d\alpha d\theta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\lambda,\theta|\underline{x})d\alpha d\theta d\lambda} \tag{14}$$

The LINEX loss function, originally proposed by [14], is a specific example of an asymmetric loss function. The LINEX loss function is given by

$$\mathscr{L}_{LINEX}(\tilde{\varphi}(\nu),\varphi(\nu)) = e^{c(\tilde{\varphi}(\nu)-\varphi(\nu))} - c(\tilde{\varphi}(\nu)-\varphi(\nu)) - 1, \quad c \neq 0$$

where c denotes a constant term. Using this loss function, Bayes estimate of any function of α, θ, and λ, denoted as $\varphi(\alpha,\theta,\lambda)$, can be derived as

$$\tilde{\varphi}_{LINEX}(\alpha,\theta,\lambda) = -\frac{1}{c}\ln(E(e^{-c\varphi(\alpha,\theta,\lambda)}|\mathbf{x}))$$

where

$$E(e^{-c\varphi(\alpha,\theta,\lambda)}|\mathbf{x}) = -\frac{1}{c}\ln(\frac{(\int_0^\infty\int_0^\infty\int_0^\infty e^{-c\varphi(\alpha,\theta,\lambda)}\pi(\alpha,\theta,\lambda|\mathbf{x})d\alpha d\theta d\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha,\lambda,\theta|\underline{x})d\alpha d\theta d\lambda}) \tag{15}$$

Obviously, the calculation of integrals given by (14) and (15). cannot be obtained in a closed form. To obtain Bayes estimates of α, θ, λ, $S(t)$, and $h(t)$, the MCMC algorithm is used to generate samples from joint PD, detail discussions about MCMC technique can be traced in [23], [21], [13] and [18] and for example. From (13), it is possible to derive the conditional PD of α given θ, λ, and $\mathbf{x}$ as

$$\pi(\alpha|\lambda,\theta,\mathbf{x}) \propto \alpha^{a_1+m-1}e^{-b\alpha}e^{-\lambda((n-m)(1-e^{-2\theta x_{(m)}})^{\alpha})}(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})\prod_{i=1}^{m}(1-e^{-2\theta x_{(i)}})^{\alpha-1}, \tag{16}$$

the conditional PD of θ given α, λ and $\mathbf{x}$, can be obtained as follows

$$\pi(\theta|\alpha,\lambda,\mathbf{x}) \propto \theta^{a_3+m-1}e^{-\lambda(((n-m)(1-e^{-2\theta x_{(m)}})^{\alpha})+b_3\theta}(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})})\prod_{i=1}^{m}(1-e^{-2\theta x_{(i)}})^{\alpha-1} \tag{17}$$

and the conditional PD of λ given α, θ and $\mathbf{x}$, can be expressed as

$$\pi(\lambda|\alpha,\theta,\mathbf{x}) \propto \lambda^{a_2+m-1}e^{-\lambda(b_2+((n-m)(1-e^{-2\theta x_{(m)}})^{\alpha})}(1-e^{-\lambda})^{-n}(1-e^{-\lambda(1-(1-e^{-2\theta x_{(m)}})^{\alpha})}), \tag{18}$$

From equations (16), (17), and (18), it is apparent that the conditional PDs of α, θ, and λ do not follow the standard form of statistical distributions. In order to tackle this issue, M-H algorithm with a normal proposal distribution is employed. The steps of M-H algorithm are as follow

Step 1. Begin by choosing initial estimates of α, θ and λ as $\alpha^{(0)}$, $\theta^{(0)}$ and $\lambda^{(0)}$respectively
Step 2. Put $j = 1$
Step 3. Generate $\alpha^{(j)}$, $\theta^{(j)}$ and $\lambda^{(j)}$ from (14) , (15) and (16), respectively, with normal proposal distributions $N(\alpha^{(j-1)},V(\alpha))$, $N(\theta^{(j-1)},V(\theta))$and $N(\lambda^{(j-1)},V(\lambda))$, respectively, where $V(\alpha)$,$V(\theta)$,and $V(\lambda)$ can be obtained from the main diagonal in an asymptotic variance-covariance matrix.
Step 4. Generate a proposal α^* from $N(\alpha^{(j-1)},V(\alpha))$, θ^* from $N(\theta^{(j-1)},V(\theta))$ and λ^* from $N(\lambda^{(j-1)},V(\lambda))$, respectively.
Step 5. Determine the acceptance probabilities

$$\eta_\alpha^* = \min\left(1, \frac{\pi(\alpha^*|\theta^{(j-1)},\lambda^{(j-1)},\mathbf{x})}{\pi(\alpha^{(j-1)}|,\theta^{(j-1)},\lambda^{(j-1)},\underline{x})}\right), \quad \eta_\theta^* = \min\left(1, \frac{\pi(\theta^*|\alpha^{(j)},\lambda^{(j-1)},\mathbf{x})}{\pi(\theta^{(j-1)}|\alpha^{(j)},\lambda^{(j-1)},\underline{x})}\right)$$

and

$$\eta_\lambda^* = \min\left(1, \frac{\pi(\lambda^*|\alpha^{(j)},\theta^{(j)},\mathbf{x})}{\pi(\lambda^{(j-1)}|\alpha^{(j)},\theta^{(j)},\mathbf{x})}\right).$$

Step 6. Generate u_1, u_2 ,and u_3 from uniform distribution.

i.If $u_1 < \eta_\alpha^*$ the proposal is accepted and set $\alpha^* = \alpha^{(j)}$, otherwise put $\alpha^{(j)} = \alpha^{(j-1)}$.
ii.If $u_2 < \eta_\theta^*$ accept the proposal and set $\theta^* = \theta^{(j)}$, otherwise set $\theta^{(j)} = \theta^{(j-1)}$.
iii.If $u_3 < \eta_\lambda^*$ accept the proposal and put $\lambda^* = \lambda^{(j)}$, otherwise put $\lambda^{(j)} = \lambda^{(j-1)}$.

Step 7. For a given time t, obtain the survival function $S^{(j)}(t)$ and hazard function $h^{(j)}(t)$ as

$$S^{(j)}(t) = \frac{1}{(1-e^{-\lambda^{(j)}})}(e^{-\lambda^{(j)}(1-e^{-2\theta^{(j)}t})^{\alpha^{(j)}}} - e^{-\lambda^{(j)}}), \qquad \alpha,\theta,\lambda > 0, t > 0,$$

and

$$h^{(j)}(t) = \frac{2\lambda^{(j)}\alpha^{(j)}\theta^{(j)}e^{-2\theta^{(j)}t}e^{-\lambda^{(j)}(1-e^{-2\theta^{(j)}t})^{\alpha^{(j)}}}(1-e^{-2\theta^{(j)}t})^{\alpha^{(j)}-1}}{e^{-\lambda^{(j)}(1-e^{-2\theta^{(j)}t})^{\alpha^{(j)}}} - e^{-\lambda^{(j)}}}, \qquad \alpha,\theta,\lambda > 0, t > 0.$$

Step 8. Set $j = j+1$.
Step 9. Repeat steps 3-7 M times and get $\alpha^{(j)}$, $\theta^{(j)}$, $\lambda^{(j)}$, $j = 1,2,...,M$. In order to guarantee the convergence and to avoid the bias of the selection of initial guess, the first simulated M^* are discarded. Only the selected samples $\varphi^{(j)} = (\alpha^{(j)},\theta^{(j)},\lambda^{(j)},S^{(j)}(t),h^{(j)}(t))$ for $j = M^*+1,...,M$ are used. When M is sufficiently large, the selected samples

$\alpha^{(j)}, \theta^{(j)}, \lambda^{(j)}, S^{(j)}(t)$ and $h^{(j)}(t)$ for $j = M^* + 1, ..., M$. Bayes estimates of α, θ, λ,, $S(t)$, and $h(t)$ are obtained based on the SE loss function as

$$\tilde{\alpha}_{SE} = \frac{1}{M - M^*} \sum_{j=M^*+1}^{M} \alpha^{(j)}, \quad \tilde{\theta}_{SE} = \frac{1}{M - M^*} \sum_{j=M^*+1}^{M} \theta^{(j)}, \quad \tilde{\lambda}_{SE} = \frac{1}{M - M^*} \sum_{j=M^*+1}^{M} \lambda^{(j)},$$

$$\tilde{S}(t)_{SE} = \frac{1}{M - M^*} \sum_{j=M^*+1}^{M} S^{(j)}(t), \quad \text{and} \quad \tilde{h}(t)_{SE} = \frac{1}{M - M^*} \sum_{j=M^*+1}^{M} h^{(j)}(t).$$

Using the LINEX loss function, Bayes estimates of α, θ,λ, $S(t)$, and $h(t)$ are computed as:

$$\tilde{\alpha}_{LINEX} = -\frac{1}{c} \ln \left(\frac{1}{M - M^*} \sum_{j=M^*+1}^{M} e^{-c\alpha(j)} \right), \quad \tilde{\theta}_{LINEX} = -\frac{1}{c} \ln \left(\frac{1}{M - M^*} \sum_{j=M^*+1}^{M} e^{-c\theta(j)} \right),$$

$$\tilde{\lambda}_{LINEX} = -\frac{1}{c} \ln \left(\frac{1}{M - M^*} \sum_{j=M^*+1}^{M} e^{-c\lambda(j)} \right), \quad \tilde{S}(t)_{LINEX} = \frac{1}{c} \ln \left(\frac{1}{M - M^*} \sum_{j=M^*+1}^{M} e^{-cS^{(j)}(t)} \right),$$

and

$$\tilde{h}(t)_{LINEX} = -\frac{1}{c} \ln \left(\frac{1}{M - M^*} \sum_{j=M^*+1}^{M} e^{-ch^{(j)}(t)} \right).$$

respectively.

5 Simulation study

In this section, extensive MC simulations are implemented with the purpose of comparing the performance of the previously proposed estimators of unknown parameters α, λ and θ ,and examine the reliability characteristics of the survival function $S(t)$ and hazard rate function $h(t)$ of the LTETL-ex distribution under a large sample of 2,000 Type-II censored observations. Two different sets of true values for the LTETL-ex parameters$(\alpha, \lambda, \theta)$ are considered: Set-1: $(0.8, 0.1, 0.4)$, and Set-2: $(1.5, 0.3, 0.8)$. The corresponding actual values of $S(t)$ and $h(t)$ at a specified mission time of t are 0.8207 and 0.3223 for Set-1, and 0.8181 and 0.5424 for Set-2, respectively. Four different choices of the total number of test items, namely 50, 100, 150, and 200, are considered. The effective sample size is determined based on different failure percentages (FPs), represented as $(m/n) \times 100\%$, where m is the number of failed items and n is the total number of items. Three different FPs are considered for each n: 30%, 70%, and 100%. When the FP reaches 100%, it means that the estimates are obtained under complete sampling.To assign values to the hyper-parameters of the conjugate gamma priors, two different informative sets of a_i and b_i, $i = 1, 2, 3$, are used for each set of α, θ , and λ, namely:

- Taking $b_i = 5$ for $i = 1, 2, 3$ Prior-I $(a_1, a_2, a_3) = (4, 0.5, 2)$ for Set-1 and Prior-I:$(a_1, a_2, a_3) = (7.5, 1.5, 4)$ for Set-2.
- Taking $b_i = 10$ for $i = 1, 2, 3$, Prior-II: $(a_1, a_2, a_3) = (8, 1, 4)$ for Set-1 and Prior-II: $(a_1, a_2, a_3) = (15, 3, 8)$ for Set-2.

Following [11], the values of hyper-parameters of α, θ, and λ are assigned in such a way that the prior mean becomes the expected value of the target parameter. It is worth noting that when a vague gamma prior is used, the resulting PD is similar to the proportional LF. Therefore, in cases where no prior information is available on the unknown parameters of interest, it is preferable to use ML estimates rather than BEs since the latter can be computationally more complex. For Bayesian computations, we utilize the M-H algorithm to generate 12,000 MCMC samples, discarding the first 2,000 values. From the remaining 10,000 MCMC samples, we obtain the average BEs and their corresponding 95% credible intervals using SE and LINEX functions for $c = (-3, 0.03, +3)$. For each setting, the MCMC sampler is run using the respective ML estimates of α, θ, and λ as initial values, and we observe that the MCs reach the stationary condition very quickly. For computational efficiency, we represent the average estimate (either ML or Bayes) of α, θ, λ, $S(t)$ or $h(t)$ as $\overline{\hat{\varphi}}$ as

$$\overline{\hat{\varphi}}_\tau = \frac{1}{G} \sum_{j=1}^{G} \hat{\varphi}_\tau^{*(j)}, \quad \tau = 1, 2, 3, 4, 5,$$

Where, G denotes the number of replications, τ denotes the sample size, $\hat{\varphi}^*$ is the desired estimate of φ^*, and $\hat{\varphi}_\tau^{(j)}$ denotes the calculated estimate of φ^* at the jth sample. The components of φ_j^* are $\varphi_1^* = \alpha$, $\varphi_2^* = \theta$, $\varphi_3^* = \lambda$, $\varphi_4^* = S(t)$, and

$\varphi_5^* = h(t)$. To compare different point estimate values, Using two criteria; RMSE and MAB, different point estimates values are compared according to the following formulae

$$\text{RMSE}(\hat{\varphi}_\tau^*) = \sqrt{\frac{1}{G}\sum_{j=1}^{G}(\hat{\varphi}_\tau^{*(j)} - \varphi_\tau^*)^2} \quad \tau = 1,2,3,4,5,$$

and

$$\text{MAB}(\hat{\varphi}_\tau^*) = \frac{1}{G}\sum_{j=1}^{G}\left|\hat{\varphi}_\tau^{*(j)} - \varphi_\tau^*)^2\right| \quad \tau = 1,2,3,4,5,$$

respectively. Moreover, the performance of 95% asymptotic/credible intervals estimates is compared using their ACLs and CPs respectively as.

$$\text{ACL}_{(1-p)}(\hat{\varphi}_\tau^*) = \tfrac{1}{G}\textstyle\sum_{j=1}^{G}\left(U^*(\hat{\varphi}_\tau^{*(j)}) - L^*(\hat{\varphi}_\tau^{*(j)})\right), \quad \tau = 1,2,3,4,5, \text{ and}$$
$$\text{CP}_{(1-p)}(\varphi_\tau^*) = \tfrac{1}{G}\textstyle\sum_{j=1}^{G} I^*\left(L^*(\hat{\varphi}_\tau^{*(j)}), U^*(\hat{\varphi}_\tau^{*(j)})\right), \quad \tau = 1,2,3,4,5,$$

respectively, $I(.)$ denotes the indicator function, and $L(.)$ and $U^*(.)$ are the lower and upper limits of the $(1-p)100\%$ asymptotic or credible interval. The computational algorithms were implemented using the R statistical programming language version 4.1.2, utilizing two packages: the Coda package by [12] and the MaxLik package by [5], which were recommended by [3]. Tables 1-10 present the simulation results for α, θ, λ, $S(t)$ and $h(t)$, including the average point estimates, along with their associated RMSEs and MABs. The first, second, and third lines in each table correspond to the average estimates, RMSEs, and MABs, respectively. Tables 15-21 provide the ACLs and CPs for 95% asymptotic and credible interval estimates of α, θ, λ, $S(t)$, and $h(t)$. Based on the results presented in Tables 1-14, the following observations can be made:

–The smallest RMSE, MAB, and ACL and the highest CPs are advantages of both ML and BEs of all unknown parameters α, θ, and λ, $S(t)$ and $h(t)$ of LTETL-ex distribution.
–The higher the true values of α, θ, and λ are, the higher the associated RMSE, MAB, and ACL of all proposed estimates and the lower the associated CPs are.
–The lower the RMSEs, MABs, and ACLs of all estimates and the higher the CPs are, as expected, when n is higher. For more accurate estimation results, the effective sample size should be large.
–Bayes MCMC estimates perform better than those obtained from the LF since BEs of any unknown parametric function include gamma prior information than the classical estimates. A similar pattern is observed in the case of Bayesian CI compared to asymptotic CI for all unknown parameters.
–BEs and the associated credible intervals based on Prior II become even better than others since the variance of Prior II is lower than the variance of Prior I.
–Estimates of all unknown parameters based on LINEX function perform better than SE loss function in terms of lowest RMSEs and MABs. This result is due to the fact that the use of SE loss function gives equal weight to underestimation and overestimation according to its symmetrical nature.

Table 1: AEs, RMSEs and MABs of α for Set-1.

n	FB	MLE	SE		LINEX					
Prior			I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
50	30%	0.9023	0.7911	0.7960	0.7938	0.7911	0.7856	0.7987	0.7961	0.7907
		0.2705	0.0526	0.0524	0.0095	0.0148	0.0254	0.0080	0.0126	0.0227
		0.1896	0.0142	0.0098	0.0074	0.0099	0.0154	0.0062	0.0089	0.0144
	70%	0.8679	0.7935	0.8000	0.7960	0.7935	0.7886	0.8028	0.8001	0.7944
		0.1865	0.0524	0.0513	0.0093	0.0146	0.0254	0.0075	0.0123	0.0226
		0.1356	0.0130	0.0092	0.0059	0.0086	0.0141	0.0051	0.0079	0.0130
	100%	0.8605	0.7913	0.7959	0.7941	0.7914	0.7859	0.7986	0.7959	0.7906
		0.1760	0.0510	0.0508	0.0077	0.0127	0.0224	0.0071	0.0123	0.0225
		0.1330	0.0096	0.0078	0.0050	0.0077	0.0127	0.0040	0.0065	0.0115
100	30%	0.8413	0.7923	0.8013	0.7951	0.7923	0.7867	0.8040	0.8014	0.7962
		0.1559	0.0525	0.0520	0.0092	0.0143	0.0244	0.0078	0.0122	0.0222
		0.1177	0.0135	0.0096	0.0072	0.0091	0.0146	0.0061	0.0086	0.0140
	70%	0.8282	0.7923	0.8013	0.7951	0.7923	0.7867	0.8040	0.8014	0.7962
		0.1227	0.0523	0.0505	0.0081	0.0135	0.0234	0.0072	0.0120	0.0218
		0.0958	0.0104	0.0090	0.0049	0.0077	0.0138	0.0048	0.0077	0.0125
	100%	0.8328	0.7900	0.7897	0.7926	0.7900	0.7850	0.7924	0.7897	0.7844
		0.1215	0.0505	0.0499	0.0076	0.0125	0.0222	0.0071	0.0118	0.0215
		0.0939	0.0091	0.0075	0.0050	0.0075	0.0122	0.0038	0.0063	0.0113
150	30%	0.8357	0.7892	0.7883	0.7917	0.7892	0.7843	0.7909	0.7883	0.7832
		0.1438	0.0521	0.0516	0.0093	0.0141	0.0242	0.0075	0.0118	0.0220
		0.1077	0.0117	0.0092	0.0073	0.0090	0.0143	0.0061	0.0088	0.0133
	70%	0.8289	0.7960	0.7897	0.7989	0.7960	0.7902	0.7922	0.7897	0.7847
		0.1041	0.0513	0.0506	0.0080	0.0134	0.0224	0.0072	0.0117	0.0213
		0.0821	0.0093	0.0090	0.0046	0.0074	0.0127	0.0048	0.0075	0.0127
	100%	0.8257	0.7922	0.7973	0.7950	0.7923	0.7867	0.8000	0.7973	0.7919
		0.1016	0.0501	0.0494	0.0074	0.0112	0.0220	0.0071	0.0112	0.0208
		0.0789	0.0088	0.0072	0.0048	0.0072	0.0121	0.0038	0.0063	0.0105
200	30%	0.8284	0.7918	0.7892	0.7945	0.7919	0.7865	0.7918	0.7892	0.7840
		0.1242	0.0519	0.0514	0.0091	0.0138	0.0240	0.0068	0.0116	0.0216
		0.0937	0.0094	0.0093	0.0070	0.0084	0.0141	0.0057	0.0081	0.0131
	70%	0.8217	0.7785	0.7894	0.7809	0.7785	0.7738	0.7920	0.7894	0.7842
		0.0904	0.0508	0.0502	0.0076	0.0136	0.0221	0.0065	0.0115	0.0206
		0.0715	0.0091	0.0089	0.0042	0.0072	0.0113	0.0043	0.0072	0.0118
	100%	0.8191	0.7874	0.7933	0.7899	0.7875	0.7828	0.7960	0.7933	0.7878
		0.0899	0.0502	0.0491	0.0072	0.0105	0.0213	0.0061	0.0111	0.0200
		0.0715	0.0086	0.0070	0.0047	0.0071	0.0107	0.0036	0.0061	0.0102

6 Illustrative Examples

This section examines both actual and simulated data sets to demonstrate how the suggested methodology can be utilized to address real-world phenomena.

6.1 Example 1: (Simulated data)

In this particular example, three Type-II censored samples were generated using the MC simulation method when $n = 40$ and $m = 10, 20, 30$ from LTETL-ex distribution when $\alpha = 0.7$, $\theta = 0.2$, and $\lambda = 0.5$ with different choices of n and m.All simulated samples are reported in Table 16. For each generated sample, the proposed point and interval estimates of α, λ,θ,$S(t)$ and $h(t)$ at distinct time $t = 1$ are computed. BEs are obtained against SE and LINEX for $c = (-5,+5)$ using three different sets of the hyper-parameter values $a_i, b_i, i = 1,2,3$ such as: $a_i, b_i = 0, i = 1,2,3$ (Prior-I); $(a_1,a_2,a_3) = (1.4,0.4,1)$ and $b_i = 2$ for $i = 1,2,3$ (Prior-II); $(a_1,a_2,a_3) = (3.5,1,2.5)$ and $b_i = 5$ for $i = 1,2,3$ (Prior-III). To obtain BEs and their associated Bayesian credible intervals (BCIs), a total of 30,000 MCMC samples were generated while disregarding the first 5,000 iterations. For calculation purpose, the corresponding ML estimates of α, θ and λ are used as initial guess values to run the MCMC algorithm. Using Table 20, the ML and BEs of α, λ,θ,$S(t)$ and $h(t)$ with their standard errors are calculated and reported in Table 21. Furthermore, the 95% ACIs and BCIs with their lengths of the same unknown parameters are also computed and presented in Table 18. According to Table 17, BEs of all unknown parameters performed satisfactorily in terms of the lowest standard error values than the ML estimates do. As reported in Table 23, the HPD intervals show good behavior compared to the other intervals in terms of the shortest confidence length obtained.

6.2 Example 2: (Real data)

To indicate how the different proposed estimators can be applied in practical situations. The failure times of 18 electronic devices, reported by [10] study. are used. The ordered survival times of these devices are: 5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321, 330, 350, 420. To evaluate the effectiveness and versatility of the LTETL-ex lifetime model, its performance is compared to several other competitive models with varying hazard rates. These models include the Topp-Leone Lomax (TLL) by [22], Topp-Leone Weibull (TLW) by [8], Topp-Leone generalized exponential (TLGE) by [27], Topp-Leone Nadarajah-Haghighi (TLNH) by [16], and Topp-Leone Bur XII (TLBXII) by [15] distributions, using Wang's dataset. To check the validity of the LTETL-ex distribution as well as with other competing models, different methods of testing the goodness-of-fit are applied, namely: Akaike's information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (K-S) with its p-value. To evaluate both the unknown parameters of the considered distribution and the goodness-of-fit measures, the ML estimation method is considered. The computations of these criteria are conducted using R statistical software by implementing 'Adequacy Model' package proposed by [20]. The distribution with the lowest values of AIC, CAIC, BIC, HQIC, and K-S and the highest p-value is considered superior. The calculated estimates of the model parameters with their SEs are provided in Table 24. The corresponding goodness-of-fit measures are computed and listed in Table 22 which shows that the LTETL-ex distribution fits the life Wang's data set quite satisfactorily. Moreover, it is the best choice among other competing models because it has the smallest K-S value with the highest p-value. Quantile-quantile (Q-Q) plots of all competitive distributions using Wang's data set are graphed, see Figure 3. Furthermore, the relative histogram of the Wang's data set and the fitted densities, as well as the plot of fitted and empirical reliability functions are graphed in Figures 3. Figures 3-2 support the numerical findings.

Table 2: Three simulated Type-II censored samples from TETL-ex distribution

m	Type-II censored data
10	0.00099, 0.00441, 0.03066, 0.04715 0.05282, 0.06014, 0.07332, 0.15447, 0.17034, 0.21821
20	0.00099, 0.00441, 0.03066, 0.04715, 0.05282, 0.06014, 0.07332, 0.15447, 0.17034, 0.21821, 0.22433, 0.25342, 0.25455, 0.31528, 0.37252, 0.43877, 0.46927, 0.59769, 0.60526, 0.62401
30	0.00099, 0.00441, 0.03066, 0.04715, 0.05282, 0.06014, 0.07332, 0.15447, 0.17034, 0.21821, 0.22433, 0.25342, 0.25455, 0.31528, 0.37252, 0.43877, 0.46927, 0.59769, 0.60526, 0.62401, 0.71540, 0.73065, 0.75778, 0.79123, 0.82152, 0.85569, 0.99451, 1.02664, 1.19687, 1.37432

Table 3: Summary fit for the competing life models under Wang's data.

Model	α		θ		λ	
	Estimate	standard errors	Estimate	standard errors	Estimate	standard errors
TETLE	0.5462	0.9789	0.0038	0.0009	2.9597	5.7961
TLL	2.1496	1.1838	51.694	61.1107	0.6038	0.2752
TLW	1.2358	0.4246	0.0049	0.0038	0.9305	0.1266
TLGE	0.9862	2.0247	0.0033	0.0013	1.1291	1.9321
TLNH	1.1318	0.3277	0.0025	0.0007	1.2174	0.3277
TLBXII	43.529	46.124	0.3895	0.1904	1.0786	0.6540

Table 4: Selection fit for the competing life models under Wang's data.

Model	AIC	BIC	HQIC	K-S	p-value
TETLE	226.778	229.4495	227.147	0.1085	0.968
TLL	234.283	236.9544	234.652	0.1791	0.551
TLW	227.528	230.1987	227.896	0.1281	0.893
TLGE	227.180	229.8516	227.549	0.1269	0.899
TLNH	226.913	229.5846	227.282	0.1205	0.929
TLBXII	237.166	239.8369	237.534	0.2008	0.408

–Extending the results of [19] in the case of LTETL-ex distribution under complete sampling to incomplete (type-II censoring) sampling may be obtained as a special case by setting $m = n$.
–To sum up, Bayes (point/interval) estimates using M-H algorithm of the unknown parameters and the reliability characteristics of the LTETL-ex lifetime model are recommended.

Table 5: AEs, RMSEs and MABs of α for Set-2.

n			SE		LINEX					
Prior	FB	MLE	I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
50	30%	1.8106	1.4861	1.4910	1.4928	1.4862	1.4649	1.4977	1.4912	1.4701
		0.7773	0.0910	0.0901	0.0123	0.0248	0.0658	0.0120	0.0245	0.0640
		0.5090	0.0183	0.0158	0.0092	0.0156	0.0363	0.0081	0.0144	0.0351
	70%	1.6158	1.4883	1.4949	1.4948	1.4884	1.4682	1.5017	1.4950	1.4732
		0.4054	0.0899	0.0892	0.0115	0.0243	0.0629	0.0094	0.0220	0.0624
		0.2843	0.0148	0.0147	0.0082	0.0146	0.0360	0.0072	0.0138	0.0341
	100%	1.5718	1.4855	1.4843	1.4919	1.4856	1.4659	1.4908	1.4844	1.4640
		0.3228	0.0896	0.0888	0.0099	0.0226	0.0627	0.0092	0.0213	0.0617
		0.2403	0.0142	0.0139	0.0061	0.0126	0.0337	0.0052	0.0116	0.0318
100	30%	1.6150	1.4873	1.4964	1.4941	1.4875	1.4657	1.5029	1.4965	1.4759
		0.4063	0.0903	0.0898	0.0122	0.0247	0.0652	0.0118	0.0242	0.0621
		0.2983	0.0154	0.0149	0.0090	0.0152	0.0352	0.0077	0.0142	0.0348
	70%	1.5271	1.4847	1.4847	1.4912	1.4848	1.4646	1.4913	1.4848	1.4638
		0.2420	0.0893	0.0890	0.0104	0.0232	0.0628	0.0093	0.0210	0.0615
		0.1861	0.0144	0.0139	0.0080	0.0142	0.0354	0.0069	0.0135	0.0336
	100%	1.5172	1.4985	1.4943	1.5054	1.4986	1.4760	1.5009	1.4944	1.4732
		0.2091	0.0892	0.0884	0.0098	0.0207	0.0624	0.0090	0.0206	0.0595
		0.1633	0.0136	0.0134	0.0059	0.0125	0.0333	0.0051	0.0114	0.0315
150	30%	1.5658	1.4847	1.4834	1.4911	1.4848	1.4648	1.4899	1.4835	1.4629
		0.3049	0.0902	0.0898	0.0121	0.0246	0.0634	0.0113	0.0240	0.0620
		0.2252	0.0146	0.0145	0.0089	0.0151	0.0349	0.0071	0.0135	0.0341
	70%	1.5208	1.4916	1.4846	1.4985	1.4917	1.4693	1.4911	1.4847	1.4646
		0.1970	0.0893	0.0889	0.0101	0.0231	0.0617	0.0091	0.0208	0.0614
		0.1548	0.0143	0.0140	0.0074	0.0135	0.0343	0.0069	0.0133	0.0334
	100%	1.5053	1.4872	1.4921	1.4939	1.4873	1.4657	1.4987	1.4922	1.4709
		0.1731	0.0891	0.0882	0.0094	0.0205	0.0611	0.0084	0.0201	0.0590
		0.1359	0.0133	0.0130	0.0055	0.0122	0.0332	0.0051	0.0111	0.0310
200	30%	1.5468	1.4866	1.4843	1.4933	1.4867	1.4657	1.4908	1.4844	1.4635
		0.2592	0.0901	0.0891	0.0118	0.0237	0.0624	0.0105	0.0231	0.0614
		0.1939	0.0141	0.0140	0.0077	0.0133	0.0343	0.0068	0.0126	0.0335
	70%	1.5052	1.4733	1.4844	1.4796	1.4734	1.4540	1.4909	1.4845	1.4636
		0.1670	0.0884	0.0881	0.0098	0.0223	0.0618	0.0089	0.0201	0.0603
		0.1326	0.0137	0.0134	0.0071	0.0126	0.0336	0.0062	0.0125	0.0326
	100%	1.5029	1.4824	1.4883	1.4887	1.4825	1.4631	1.4950	1.4884	1.4670
		0.1559	0.0886	0.0880	0.0091	0.0201	0.0607	0.0082	0.0200	0.0588
		0.1248	0.0130	0.0125	0.0054	0.0117	0.0319	0.0050	0.0107	0.0303

Using the complete Wang's data set based on different choices of m the proposed estimates are carried out. Three different type-II censored samples are generated and presented in Table 22. Because of lack of prior information about the LTETL-ex parameters, BEs under SE and LINEX for$c = (-5,5)$ are approximated by MCMC sampler under gamma improper, i.e., a_i and b_i are set to 0 for $i = 1,2,3$. Using the M-H algorithm, 30,000 MCMC samples are generated with ignoring the first 5,000 iterations of the simulated variates as burn-in. Using Table 24, both ML and BEs with their standard errors of the unknown parameters of α, λ, and θ,as well as the survival characteristics $S(t)$ and $h(t)$ (at a given mission time $t = 0.5$) are calculated and reported in Table 23. Two-sided 95% ACI and BCI estimates with their lengths are also listed in Table 24. It shows that the point estimates of the unknown parameters α, λ, θ, $S(t)$ and $h(t)$ obtained by ML and Bayesian estimation methods are quite close to each other. Furthermore, the associated interval estimates of α, λ, θ, $S(t)$ and $h(t)$ obtained by 95% asymptotic/credible intervals are also similar. To judge how quickly the MCMC converges at each iteration with the iteration (x-axis) and the sampled values (y-axis), trace plots of the simulated 25,000 samples of α, λ, θ, $S(t)$ and $h(t)$ based on Sample 1 data as an example) are displayed in Figure 3. Each trace plot displays the sample mean as a horizontal solid line (-), while the lower and upper bounds of the 95% credible intervals are represented by dashed $(---)$ horizontal lines.This indicates that the MCMC procedure converges well also it shows that discarding the first 5,000 samples as burn-in is appropriate size to erase the effect of the initial values. Using the Gaussian kernel under 25,000 chain values, the marginal posterior density estimates of α, λ, θ, $S(t)$ and $h(t)$ with their histograms are represented in Figure 4. Similarly, in each histogram plot, the sample mean of any unknown parameter is represented by a

Table 6: AEs, RMSEs and MABs of θ for Set-1.

n Prior c	FB	MLE	SE		LINEX					
			I	II	I			II		
					-3	-0.03	3	-3	-0.03	3
50	30%	0.1129	0.1108	0.1136	0.1318	0.1109	0.1042	0.1348	0.1137	0.1069
		0.0775	0.0908	0.0905	0.0679	0.0313	0.0197	0.0666	0.0247	0.0110
		0.0574	0.0241	0.0153	0.0437	0.0242	0.0177	0.0350	0.0137	0.0069
	70%	0.0976	0.1114	0.1086	0.1325	0.1115	0.1047	0.1305	0.1087	0.1018
		0.0390	0.0904	0.0903	0.0656	0.0244	0.0108	0.0637	0.0233	0.0098
		0.0318	0.0152	0.0150	0.0350	0.0132	0.0064	0.0331	0.0129	0.0065
	100%	0.0981	0.1086	0.1128	0.1299	0.1088	0.1020	0.1340	0.1129	0.1061
		0.0291	0.0900	0.0899	0.0645	0.0226	0.0094	0.0632	0.0226	0.0093
		0.0228	0.0149	0.0138	0.0340	0.0129	0.0063	0.0327	0.0128	0.0061
100	30%	0.0947	0.1076	0.1132	0.1276	0.1078	0.1012	0.1350	0.1133	0.1064
		0.0523	0.0904	0.0901	0.0651	0.0245	0.0109	0.0638	0.0233	0.0098
		0.0421	0.0156	0.0138	0.0350	0.0133	0.0067	0.0342	0.0132	0.0064
	70%	0.0917	0.1079	0.1132	0.1279	0.1080	0.1015	0.1350	0.1133	0.1064
		0.0364	0.0898	0.0882	0.0649	0.0241	0.0107	0.0599	0.0224	0.0091
		0.0292	0.0148	0.0136	0.0348	0.0131	0.0064	0.0328	0.0122	0.0054
	100%	0.0962	0.1241	0.1083	0.1437	0.1242	0.1177	0.1292	0.1084	0.1017
		0.0232	0.0881	0.0879	0.0623	0.0220	0.0089	0.0598	0.0216	0.0088
		0.0177	0.0135	0.0132	0.0339	0.0128	0.0060	0.0326	0.0117	0.0050
150	30%	0.0947	0.1150	0.1153	0.1360	0.1151	0.1084	0.1358	0.1154	0.1087
		0.0576	0.0904	0.0900	0.0645	0.0242	0.0101	0.0634	0.0230	0.0095
		0.0469	0.0154	0.0133	0.0348	0.0130	0.0064	0.0336	0.0131	0.0061
	70%	0.0927	0.1079	0.1116	0.1308	0.1080	0.1009	0.1326	0.1117	0.1050
		0.0344	0.0890	0.0880	0.0636	0.0237	0.0099	0.0592	0.0221	0.0088
		0.0277	0.0141	0.0132	0.0339	0.0128	0.0060	0.0324	0.0121	0.0051
	100%	0.0979	0.1140	0.1090	0.1350	0.1142	0.1074	0.1306	0.1091	0.1023
		0.0197	0.0874	0.0796	0.0614	0.0218	0.0086	0.0590	0.0214	0.0086
		0.0149	0.0132	0.0130	0.0320	0.0125	0.0048	0.0320	0.0115	0.0048
200	30%	0.0909	0.1174	0.1113	0.1379	0.1175	0.1108	0.1320	0.1115	0.1048
		0.0545	0.0903	0.0896	0.0642	0.0241	0.0099	0.0629	0.0222	0.0092
		0.0452	0.0142	0.0123	0.0345	0.0127	0.0062	0.0322	0.0117	0.0052
	70%	0.0904	0.1090	0.1116	0.1286	0.1091	0.1026	0.1322	0.1117	0.1050
		0.0332	0.0876	0.0874	0.0630	0.0233	0.0095	0.0586	0.0213	0.0086
		0.0266	0.0136	0.0122	0.0331	0.0125	0.0053	0.0320	0.0114	0.0050
	100%	0.0987	0.1136	0.1136	0.1347	0.1137	0.1070	0.1347	0.1138	0.1070
		0.0167	0.0865	0.0792	0.0611	0.0215	0.0084	0.0584	0.0210	0.0081
		0.0127	0.0128	0.0118	0.0309	0.0122	0.0045	0.0307	0.0111	0.0044

Table 7: Three Type-II censored samples generated from Wang's data set.

Sample	m	Type II censored data
1	10	5, 11, 21, 31, 46, 75, 98, 122, 145, 165
2	12	5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224
3	15	5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321

vertical dash-dotted line (:). It is evident from the estimates that all the generated posteriors are fairly symmetric. Lastly, the results of the proposed estimates under complete Wang's data set give a good explanation to the proposed model.

Table 8: AEs, RMSEs and MABs of θ for Set-1.

n			SE		LINEX					
Prior	FB	MLE	I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
50	30%	0.1129	0.1108	0.1136	0.1318	0.1109	0.1042	0.1348	0.1137	0.1069
		0.0775	0.0908	0.0905	0.0679	0.0313	0.0197	0.0666	0.0247	0.0110
		0.0574	0.0241	0.0153	0.0437	0.0242	0.0177	0.0350	0.0137	0.0069
	70%	0.0976	0.1114	0.1086	0.1325	0.1115	0.1047	0.1305	0.1087	0.1018
		0.0390	0.0904	0.0903	0.0656	0.0244	0.0108	0.0637	0.0233	0.0098
		0.0318	0.0152	0.0150	0.0350	0.0132	0.0064	0.0331	0.0129	0.0065
	100%	0.0981	0.1086	0.1128	0.1299	0.1088	0.1020	0.1340	0.1129	0.1061
		0.0291	0.0900	0.0899	0.0645	0.0226	0.0094	0.0632	0.0226	0.0093
		0.0228	0.0149	0.0138	0.0340	0.0129	0.0063	0.0327	0.0128	0.0061
100	30%	0.0947	0.1076	0.1132	0.1276	0.1078	0.1012	0.1350	0.1133	0.1064
		0.0523	0.0904	0.0901	0.0651	0.0245	0.0109	0.0638	0.0233	0.0098
		0.0421	0.0156	0.0138	0.0350	0.0133	0.0067	0.0342	0.0132	0.0064
	70%	0.0917	0.1079	0.1132	0.1279	0.1080	0.1015	0.1350	0.1133	0.1064
		0.0364	0.0898	0.0882	0.0649	0.0241	0.0107	0.0599	0.0224	0.0091
		0.0292	0.0148	0.0136	0.0348	0.0131	0.0064	0.0328	0.0122	0.0054
	100%	0.0962	0.1241	0.1083	0.1437	0.1242	0.1177	0.1292	0.1084	0.1017
		0.0232	0.0881	0.0879	0.0623	0.0220	0.0089	0.0598	0.0216	0.0088
		0.0177	0.0135	0.0132	0.0339	0.0128	0.0060	0.0326	0.0117	0.0050
150	30%	0.0947	0.1150	0.1153	0.1360	0.1151	0.1084	0.1358	0.1154	0.1087
		0.0576	0.0904	0.0900	0.0645	0.0242	0.0101	0.0634	0.0230	0.0095
		0.0469	0.0154	0.0133	0.0348	0.0130	0.0064	0.0336	0.0131	0.0061
	70%	0.0927	0.1079	0.1116	0.1308	0.1080	0.1009	0.1326	0.1117	0.1050
		0.0344	0.0890	0.0880	0.0636	0.0237	0.0099	0.0592	0.0221	0.0088
		0.0277	0.0141	0.0132	0.0339	0.0128	0.0060	0.0324	0.0121	0.0051
	100%	0.0979	0.1140	0.1090	0.1350	0.1142	0.1074	0.1306	0.1091	0.1023
		0.0197	0.0874	0.0796	0.0614	0.0218	0.0086	0.0590	0.0214	0.0086
		0.0149	0.0132	0.0130	0.0320	0.0125	0.0048	0.0320	0.0115	0.0048
200	30%	0.0909	0.1174	0.1113	0.1379	0.1175	0.1108	0.1320	0.1115	0.1048
		0.0545	0.0903	0.0896	0.0642	0.0241	0.0099	0.0629	0.0222	0.0092
		0.0452	0.0142	0.0123	0.0345	0.0127	0.0062	0.0322	0.0117	0.0052
	70%	0.0904	0.1090	0.1116	0.1286	0.1091	0.1026	0.1322	0.1117	0.1050
		0.0332	0.0876	0.0874	0.0630	0.0233	0.0095	0.0586	0.0213	0.0086
		0.0266	0.0136	0.0122	0.0331	0.0125	0.0053	0.0320	0.0114	0.0050
	100%	0.0987	0.1136	0.1136	0.1347	0.1137	0.1070	0.1347	0.1138	0.1070
		0.0167	0.0865	0.0792	0.0611	0.0215	0.0084	0.0584	0.0210	0.0081
		0.0127	0.0128	0.0118	0.0309	0.0122	0.0045	0.0307	0.0111	0.0044

7 Conclusion

In conclusion, this paper has examined the estimation of parameters for the LTETL-ex distribution under a type II censoring scheme. The estimation of both the unknown parameters and the survival and hazard functions was performed utilizing both ML and Bayesian estimation methods. The approximate asymptotic variance-covariance matrix and CIs based on the asymptotic normality of the classical estimators were obtained. Several Bayes estimates of the unknown parameters under SE and LINEX loss functions were computed, using a gamma prior distribution. The BEs were calculated using the M–H algorithm of MCMC techniques. Credible intervals were also constructed using MCMC method samples. MC simulations were carried out to assess the performance of the different estimation methods by analyzing their RMSEs, MABs, ACLs, and CPs.Additionally, a numerical example that employed both a real data set and MCMC data sets was presented to demonstrate the effectiveness of the proposed methods. The paper has contributed to the scholarly gap of estimating the unknown parameters of the LTETL-ex distribution under a censoring scheme by utilizing a type II censored sample. The proposed estimation methods have demonstrated better performance than existing estimation methods, as evidenced by the simulation results. The Bayesian estimation method was also shown to be a viable alternative to the ML method, especially when prior information is available. Overall, the proposed estimation methods provide useful tools for researchers and practitioners in fields such as public health, engineering, medicine, survival analysis, and biological science, where the LTETL-ex distribution is a flexible distribution that can be applied. Further research can be conducted to extend the proposed methods to other distributions and censoring schemes.

Table 9: AEs, RMSEs and MABs of θ for Set-2.

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
50	30%	0.4071	0.3191	0.3219	0.4039	0.3195	0.3047	0.4070	0.3223	0.3075
		0.2503	0.1567	0.1564	0.2039	0.0482	0.0216	0.2002	0.0416	0.0119
		0.1787	0.0335	0.0266	0.1146	0.0338	0.0194	0.1081	0.0232	0.0088
	70%	0.3327	0.3207	0.3172	0.4054	0.3211	0.3063	0.4039	0.3175	0.3026
		0.1122	0.1559	0.1553	0.2028	0.0432	0.0138	0.1997	0.0397	0.0104
		0.0900	0.0291	0.0238	0.1108	0.0247	0.0098	0.1077	0.0220	0.0075
	100%	0.3069	0.3109	0.3131	0.3975	0.3113	0.2963	0.3976	0.3135	0.2988
		0.0935	0.1558	0.1549	0.2006	0.0417	0.0122	0.1983	0.0396	0.0103
		0.0733	0.0246	0.0221	0.1099	0.0223	0.0076	0.1060	0.0220	0.0074
100	30%	0.3540	0.3148	0.3213	0.3973	0.3151	0.3006	0.4081	0.3217	0.3068
		0.1607	0.1553	0.1548	0.1981	0.0408	0.0114	0.1980	0.0404	0.0110
		0.1242	0.0245	0.0231	0.1080	0.0239	0.0096	0.1054	0.0217	0.0068
	70%	0.3152	0.3335	0.3164	0.4146	0.3338	0.3194	0.4008	0.3167	0.3020
		0.0893	0.1545	0.1543	0.1971	0.0394	0.0105	0.1944	0.0381	0.0101
		0.0728	0.0236	0.0229	0.1072	0.0216	0.0068	0.1052	0.0211	0.0063
	100%	0.2989	0.3243	0.3173	0.4108	0.3247	0.3098	0.4038	0.3176	0.3027
		0.0738	0.1540	0.1527	0.1960	0.0381	0.0102	0.1913	0.0373	0.0099
		0.0566	0.0220	0.0218	0.1070	0.0215	0.0067	0.1041	0.0207	0.0060
150	30%	0.3352	0.3229	0.3235	0.4077	0.3232	0.3085	0.4072	0.3239	0.3092
		0.1332	0.1552	0.1545	0.1978	0.0404	0.0105	0.1977	0.0398	0.0100
		0.1030	0.0233	0.0230	0.1071	0.0236	0.0092	0.1050	0.0209	0.0062
	70%	0.3069	0.3147	0.3194	0.4041	0.3150	0.2999	0.4041	0.3197	0.3050
		0.0867	0.1535	0.1534	0.1965	0.0392	0.0098	0.1939	0.0376	0.0095
		0.0720	0.0231	0.0227	0.1068	0.0212	0.0065	0.1042	0.0202	0.0061
	100%	0.2939	0.3256	0.3177	0.4097	0.3259	0.3112	0.4037	0.3181	0.3032
		0.0717	0.1534	0.1525	0.1956	0.0378	0.0099	0.1907	0.0365	0.0093
		0.0547	0.0217	0.0215	0.1067	0.0205	0.0061	0.1034	0.0203	0.0057
200	30%	0.3276	0.3253	0.3197	0.4088	0.3257	0.3110	0.4036	0.3200	0.3054
		0.1222	0.1550	0.1541	0.1971	0.0400	0.0102	0.1963	0.0393	0.0099
		0.0949	0.0230	0.0222	0.1068	0.0231	0.0090	0.1038	0.0201	0.0056
	70%	0.3036	0.3176	0.3200	0.3989	0.3180	0.3035	0.4039	0.3203	0.3057
		0.0790	0.1526	0.1522	0.1957	0.0386	0.0095	0.1931	0.0370	0.0091
		0.0669	0.0228	0.0220	0.1060	0.0207	0.0057	0.1035	0.0200	0.0054
	100%	0.2980	0.3211	0.3218	0.4062	0.3215	0.3067	0.4066	0.3221	0.3074
		0.0595	0.1519	0.1510	0.1925	0.0370	0.0091	0.1901	0.0361	0.0088
		0.0455	0.0210	0.0207	0.1009	0.0201	0.0055	0.1006	0.0200	0.0052

Table 10: AEs, RMSEs and MABs of λ for Set-1.

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
50	30%	1.1074	0.3938	0.3976	0.3954	0.3938	0.3910	0.3992	0.3976	0.3948
		1.5995	0.0405	0.0399	0.0102	0.0131	0.0183	0.0093	0.0108	0.0167
		1.1302	0.0148	0.0102	0.0088	0.0101	0.0129	0.0085	0.0101	0.0127
	70%	0.9411	0.3977	0.3935	0.3993	0.3977	0.3950	0.3952	0.3935	0.3908
		1.2921	0.0395	0.0386	0.0101	0.0130	0.0181	0.0091	0.0105	0.0145
		0.9357	0.0101	0.0075	0.0084	0.0100	0.0127	0.0084	0.0100	0.0124
	100%	0.8386	0.3930	0.3977	0.3947	0.3930	0.3902	0.3993	0.3977	0.3949
		1.2556	0.0384	0.0382	0.0066	0.0094	0.0145	0.0059	0.0090	0.0143
		0.7910	0.0077	0.0070	0.0046	0.0062	0.0090	0.0044	0.0059	0.0087

Table 11: *

Continued Table.10

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
		1.1746	0.0385	0.0377	0.0064	0.0093	0.0142	0.0060	0.0090	0.0140
		0.7008	0.0071	0.0066	0.0038	0.0053	0.0079	0.0044	0.0060	0.0083
150	30%	1.2076	0.3986	0.3966	0.4002	0.3987	0.3960	0.3983	0.3967	0.3939
		1.6025	0.0389	0.0384	0.0073	0.0090	0.0140	0.0063	0.0094	0.0147
		1.2113	0.0089	0.0074	0.0052	0.0067	0.0093	0.0041	0.0057	0.0084
	70%	0.9222	0.3831	0.3984	0.3846	0.3831	0.3805	0.4001	0.3984	0.3957
		1.2573	0.0387	0.0386	0.0071	0.0087	0.0136	0.0062	0.0092	0.0142
		0.8938	0.0086	0.0072	0.0046	0.0065	0.0085	0.0045	0.0052	0.0083
	100%	0.6462	0.3987	0.3968	0.4004	0.3987	0.3958	0.3985	0.3968	0.3939
		0.9890	0.0382	0.0369	0.0059	0.0085	0.0134	0.0060	0.0084	0.0139
		0.5933	0.0069	0.0061	0.0042	0.0052	0.0076	0.0036	0.0050	0.0081
200	30%	1.2529	0.3967	0.3934	0.3983	0.3968	0.3942	0.3950	0.3934	0.3907
		1.6314	0.0376	0.0374	0.0068	0.0084	0.0135	0.0061	0.0083	0.0129
		1.2337	0.0084	0.0071	0.0051	0.0062	0.0084	0.0040	0.0055	0.0082
	70%	0.9499	0.3982	0.3944	0.3998	0.3982	0.3956	0.3960	0.3944	0.3918
		1.2799	0.0378	0.0375	0.0063	0.0082	0.0131	0.0059	0.0080	0.0126
		0.9083	0.0082	0.0071	0.0045	0.0063	0.0075	0.0040	0.0051	0.0072
	100%	0.5722	0.3992	0.3932	0.4009	0.3992	0.3962	0.3948	0.3932	0.3905
		0.8238	0.0375	0.0370	0.0054	0.0080	0.0130	0.0056	0.0078	0.0124
		0.4972	0.0066	0.0060	0.0038	0.0048	0.0072	0.0032	0.0045	0.0066

Table 12: AEs, RMSEs and MABs of λ for Set-2.

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
50	30%	2.2185	0.7904	0.7942	0.7944	0.7905	0.7813	0.7981	0.7943	0.7852
		3.9930	0.0669	0.0660	0.0124	0.0197	0.0373	0.0098	0.0177	0.0363
		2.0497	0.0142	0.0108	0.0099	0.0138	0.0229	0.0059	0.0098	0.0189
	70%	0.8045	0.7940	0.7902	0.7978	0.7941	0.7852	0.7941	0.7902	0.7811
		1.4374	0.0657	0.0654	0.0123	0.0192	0.0358	0.0096	0.0172	0.0351
		1.0264	0.0137	0.0107	0.0097	0.0137	0.0223	0.0054	0.0093	0.0189
	100%	1.1161	0.7969	0.7907	0.8010	0.7970	0.7873	0.7947	0.7907	0.7812
		0.9723	0.0650	0.0637	0.0089	0.0160	0.0343	0.0080	0.0155	0.0332
		0.8364	0.0127	0.0105	0.0058	0.0099	0.0187	0.0054	0.0092	0.0183
100	30%	1.9284	0.7862	0.8014	0.7901	0.7862	0.7771	0.8055	0.8015	0.7918
		1.5259	0.0659	0.0655	0.0097	0.0168	0.0346	0.0091	0.0155	0.0342
		1.1504	0.0139	0.0102	0.0076	0.0114	0.0210	0.0058	0.0094	0.0185
	70%	0.8312	0.7926	0.7902	0.7964	0.7926	0.7840	0.7942	0.7902	0.7807
		1.3243	0.0647	0.0639	0.0095	0.0170	0.0333	0.0088	0.0161	0.0328
		0.9011	0.0123	0.0104	0.0056	0.0095	0.0183	0.0056	0.0091	0.0181
	100%	1.0307	0.7863	0.7959	0.7901	0.7863	0.7777	0.7999	0.7960	0.7866
		1.0124	0.0642	0.0631	0.0085	0.0159	0.0330	0.0080	0.0153	0.0327
		0.8450	0.0105	0.0099	0.0054	0.0090	0.0173	0.0050	0.0087	0.0170
150	30%	1.8555	0.7954	0.7932	0.7992	0.7954	0.7865	0.7971	0.7933	0.7842
		1.3403	0.0655	0.0645	0.0085	0.0154	0.0326	0.0084	0.0150	0.0316
		0.9733	0.0136	0.0100	0.0067	0.0100	0.0189	0.0052	0.0090	0.0180
	70%	0.9473	0.7794	0.7951	0.7833	0.7795	0.7706	0.7990	0.7951	0.7859
		1.0068	0.0645	0.0640	0.0082	0.0155	0.0328	0.0082	0.0149	0.0308
		0.8894	0.0122	0.0099	0.0053	0.0092	0.0180	0.0051	0.0089	0.0169
	100%	1.0885	0.7953	0.7934	0.7992	0.7953	0.7862	0.7974	0.7934	0.7839
		0.9119	0.0642	0.0629	0.0080	0.0147	0.0323	0.0078	0.0140	0.0229
		0.8217	0.0103	0.0096	0.0049	0.0086	0.0168	0.0047	0.0084	0.0161

Table 13: *

Continued Table.11

n			SE		LINEX					
Prior	FB	MLE	I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
200	30%	1.6406	0.7931	0.7900	0.7969	0.7931	0.7844	0.7939	0.7901	0.7811
		1.1115	0.0644	0.0633	0.0081	0.0151	0.0322	0.0081	0.0147	0.0307
		0.8673	0.0127	0.0099	0.0063	0.0097	0.0175	0.0049	0.0088	0.0169
	70%	0.9105	0.7949	0.7911	0.7987	0.7949	0.7862	0.7950	0.7912	0.7823
		0.9796	0.0640	0.0633	0.0080	0.0148	0.0324	0.0079	0.0145	0.0304
		0.8380	0.0123	0.0095	0.0050	0.0090	0.0170	0.0051	0.0088	0.0165
	100%	0.9541	0.7961	0.7900	0.8001	0.7961	0.7865	0.7939	0.7901	0.7811
		0.8954	0.0641	0.0617	0.0077	0.0140	0.0321	0.0075	0.0137	0.0224
		0.7235	0.0101	0.0093	0.0048	0.0084	0.0165	0.0045	0.0081	0.0158

Table 14: AEs, RMSEs and MABs of $S(t)$ for Set-1.

n			SE		LINEX					
Prior	FB	MLE	I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
50	30%	0.8215	0.8107	0.8089	0.8162	0.8108	0.7953	0.8143	0.8089	0.7937
		0.0472	0.0804	0.0801	0.0104	0.0210	0.0506	0.0099	0.0206	0.0506
		0.0372	0.0174	0.0155	0.0081	0.0134	0.0292	0.0066	0.0120	0.0276
	70%	0.8252	0.8107	0.8174	0.8162	0.8108	0.7957	0.8230	0.8175	0.8015
		0.0443	0.0800	0.0796	0.0098	0.0203	0.0503	0.0092	0.0198	0.0498
		0.0355	0.0145	0.0126	0.0064	0.0118	0.0273	0.0061	0.0119	0.0270
	100%	0.8244	0.8138	0.8099	0.8193	0.8139	0.7982	0.8153	0.8100	0.7947
		0.0438	0.0795	0.0791	0.0091	0.0196	0.0491	0.0091	0.0186	0.0488
		0.0350	0.0136	0.0120	0.0054	0.0107	0.0260	0.0048	0.0101	0.0252
100	30%	0.8192	0.8154	0.8112	0.8209	0.8155	0.7999	0.8167	0.8113	0.7962
		0.0327	0.0801	0.0795	0.0098	0.0183	0.0482	0.0096	0.0179	0.0463
		0.0259	0.0155	0.0139	0.0068	0.0126	0.0272	0.0064	0.0110	0.0257
	70%	0.8203	0.8150	0.8112	0.8206	0.8151	0.7995	0.8167	0.8113	0.7962
		0.0318	0.0796	0.0792	0.0086	0.0180	0.0482	0.0081	0.0176	0.0445
		0.0255	0.0152	0.0122	0.0064	0.0117	0.0261	0.0053	0.0106	0.0251
	100%	0.8220	0.7927	0.8134	0.7979	0.7928	0.7788	0.8189	0.8135	0.7981
		0.0313	0.0794	0.0792	0.0080	0.0176	0.0481	0.0078	0.0172	0.0439
		0.0251	0.0138	0.0118	0.0051	0.0102	0.0249	0.0046	0.0097	0.0241
150	30%	0.8186	0.8041	0.8034	0.8094	0.8042	0.7894	0.8087	0.8035	0.7886
		0.0276	0.0794	0.0790	0.0096	0.0177	0.0478	0.0094	0.0168	0.0456
		0.0219	0.0146	0.0140	0.0066	0.0117	0.0270	0.0064	0.0106	0.0245
	70%	0.8216	0.8177	0.8087	0.8234	0.8178	0.8013	0.8141	0.8088	0.7938
		0.0268	0.0789	0.0783	0.0082	0.0172	0.0476	0.0081	0.0165	0.0440
		0.0212	0.0143	0.0119	0.0064	0.0116	0.0259	0.0062	0.0101	0.0241
	100%	0.8229	0.8066	0.8155	0.8121	0.8067	0.7913	0.8211	0.8156	0.8000
		0.0253	0.0773	0.0769	0.0079	0.0171	0.0472	0.0077	0.0163	0.0431
		0.0203	0.0139	0.0106	0.0050	0.0100	0.0246	0.0048	0.0096	0.0238
200	30%	0.8190	0.8022	0.8092	0.8077	0.8023	0.7872	0.8146	0.8093	0.7941
		0.0237	0.0787	0.0779	0.0094	0.0173	0.0471	0.0092	0.0165	0.0449
		0.0176	0.0142	0.0126	0.0065	0.0113	0.0265	0.0063	0.0102	0.0241
	70%	0.8209	0.8073	0.8089	0.8126	0.8074	0.7926	0.8143	0.8089	0.7937
		0.0224	0.0785	0.0777	0.0081	0.0170	0.0470	0.0080	0.0160	0.0430
		0.0189	0.0141	0.0118	0.0064	0.0111	0.0252	0.0061	0.0099	0.0238
	100%	0.8220	0.8052	0.8079	0.8105	0.8053	0.7907	0.8134	0.8080	0.7927
		0.0223	0.0770	0.0762	0.0078	0.0166	0.0466	0.0076	0.0159	0.0428
		0.0179	0.0135	0.0105	0.0051	0.0098	0.0240	0.0049	0.0095	0.0236

Table 15: AEs, RMSEs and MABs of $S(t)$ for Set-2.

n			SE		LINEX					
Prior	FB	MLE	I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
50	30%	0.8149	0.8073	0.8062	0.8134	0.8074	0.7894	0.8122	0.8063	0.7883
		0.0532	0.0851	0.0849	0.0103	0.0221	0.0569	0.0100	0.0218	0.0567
		0.0405	0.0166	0.0149	0.0073	0.0130	0.0310	0.0067	0.0125	0.0308
	70%	0.8230	0.8064	0.8109	0.8124	0.8065	0.7887	0.8170	0.8110	0.7927
		0.0441	0.0848	0.0846	0.0100	0.0215	0.0563	0.0089	0.0204	0.0557
		0.0356	0.0148	0.0137	0.0066	0.0125	0.0305	0.0057	0.0116	0.0294
	100%	0.8211	0.8128	0.8113	0.8189	0.8129	0.7948	0.8173	0.8114	0.7934
		0.0439	0.0846	0.0844	0.0088	0.0202	0.0548	0.0086	0.0196	0.0543
		0.0354	0.0129	0.0123	0.0059	0.0118	0.0297	0.0052	0.0111	0.0291
100	30%	0.8152	0.8109	0.8075	0.8170	0.8110	0.7929	0.8135	0.8076	0.7896
		0.0352	0.0846	0.0845	0.0098	0.0201	0.0550	0.0098	0.0199	0.0545
		0.0275	0.0159	0.0139	0.0071	0.0122	0.0302	0.0059	0.0110	0.0293
	70%	0.8194	0.7961	0.8090	0.8020	0.7962	0.7791	0.8150	0.8091	0.7911
		0.0314	0.0843	0.0846	0.0097	0.0198	0.0548	0.0086	0.0195	0.0536
		0.0253	0.0147	0.0131	0.0064	0.0118	0.0294	0.0052	0.0107	0.0277
	100%	0.8203	0.8068	0.8103	0.8129	0.8069	0.7886	0.8164	0.8104	0.7922
		0.0312	0.0842	0.0840	0.0089	0.0192	0.0547	0.0082	0.0181	0.0529
		0.0249	0.0128	0.0121	0.0052	0.0112	0.0289	0.0050	0.0101	0.0271
150	30%	0.8166	0.8039	0.8032	0.8098	0.8040	0.7863	0.8092	0.8033	0.7856
		0.0279	0.0841	0.0838	0.0097	0.0200	0.0545	0.0095	0.0197	0.0540
		0.0222	0.0152	0.0137	0.0071	0.0120	0.0298	0.0057	0.0103	0.0290
	70%	0.8197	0.8128	0.8065	0.8190	0.8129	0.7942	0.8124	0.8066	0.7887
		0.0266	0.0839	0.0837	0.0096	0.0196	0.0540	0.0085	0.0190	0.0531
		0.0210	0.0140	0.0128	0.0065	0.0117	0.0283	0.0053	0.0101	0.0273
	100%	0.8204	0.8025	0.8096	0.8085	0.8026	0.7848	0.8157	0.8097	0.7916
		0.0253	0.0837	0.0835	0.0088	0.0190	0.0538	0.0081	0.0178	0.0528
		0.0203	0.0122	0.0118	0.0051	0.0111	0.0287	0.0050	0.0098	0.0269
200	30%	0.8170	0.8027	0.8065	0.8087	0.8028	0.7850	0.8125	0.8066	0.7887
		0.0234	0.0840	0.0836	0.0095	0.0197	0.0541	0.0093	0.0194	0.0535
		0.0187	0.0151	0.0127	0.0070	0.0116	0.0293	0.0056	0.0100	0.0288
	70%	0.8190	0.8049	0.8062	0.8108	0.8050	0.7874	0.8122	0.8063	0.7884
		0.0225	0.0837	0.0835	0.0095	0.0195	0.0537	0.0083	0.0189	0.0528
		0.0178	0.0138	0.0123	0.0065	0.0114	0.0280	0.0052	0.0098	0.0270
	100%	0.8195	0.8046	0.8059	0.8106	0.8047	0.7872	0.8120	0.8060	0.7881
		0.0224	0.0835	0.0831	0.0087	0.0190	0.0535	0.0080	0.0175	0.0526
		0.0177	0.0120	0.0113	0.0051	0.0106	0.0277	0.0050	0.0099	0.0264

Table 16: AEs, RMSEs and MABs of $h(t)$ for Set-1.

n			SE		LINEX					
Prior	FB	MLE	I	II	I			II		
c					-3	-0.03	3	-3	-0.03	3
50	30%	0.3422	0.3482	0.3538	0.5289	0.3489	0.3266	0.5343	0.3544	0.3322
		0.0943	0.2136	0.2117	0.3881	0.0577	0.0137	0.3817	0.0557	0.0114
		0.0670	0.0395	0.0356	0.2194	0.0332	0.0116	0.2130	0.0316	0.0097
	70%	0.3260	0.3490	0.3381	0.5281	0.3497	0.3274	0.5253	0.3388	0.3161
		0.0601	0.2118	0.2116	0.3836	0.0567	0.0124	0.3816	0.0548	0.0112
		0.0473	0.0377	0.0325	0.2120	0.0321	0.0098	0.2088	0.0289	0.0092
	100%	0.3257	0.3422	0.3516	0.5245	0.3429	0.3204	0.5325	0.3523	0.3300
		0.0599	0.2106	0.2115	0.3831	0.0537	0.0122	0.3778	0.0523	0.0106
		0.0465	0.0332	0.0309	0.2101	0.0299	0.0077	0.2061	0.0278	0.0068
100	30%	0.3336	0.3390	0.3509	0.5143	0.3396	0.3177	0.5343	0.3516	0.3291
		0.0594	0.2122	0.2103	0.3876	0.0567	0.0109	0.3777	0.0506	0.0103
		0.0450	0.0378	0.0326	0.2155	0.0311	0.0101	0.2119	0.0292	0.0096
	70%	0.3275	0.3397	0.3509	0.5151	0.3403	0.3184	0.5343	0.3516	0.3291
		0.0444	0.2077	0.2101	0.3666	0.0508	0.0105	0.3658	0.0507	0.0102
		0.0349	0.0351	0.0324	0.2152	0.0307	0.0097	0.2082	0.0292	0.0067
	100%	0.3262	0.3842	0.3423	0.5519	0.3848	0.3634	0.5212	0.3429	0.3208
		0.0426	0.2076	0.2100	0.3662	0.0505	0.0101	0.3557	0.0503	0.0100
		0.0338	0.0326	0.0305	0.2095	0.0294	0.0084	0.2075	0.0291	0.0056

Table 17: *

Continued Table. 14

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
150	30%	0.3348	0.3609	0.3619	0.5382	0.3615	0.3395	0.5374	0.3625	0.3406
		0.0576	0.2113	0.2101	0.3813	0.0516	0.0102	0.3703	0.0499	0.0100
		0.0451	0.0347	0.0319	0.2118	0.0302	0.0101	0.2051	0.0285	0.0095
	70%	0.3266	0.3366	0.3516	0.5312	0.3373	0.3141	0.5298	0.3523	0.3302
		0.0378	0.2071	0.2051	0.3405	0.0486	0.0103	0.3377	0.0483	0.0100
		0.0298	0.0332	0.0316	0.2079	0.0284	0.0096	0.2044	0.0280	0.0096
	100%	0.3240	0.3569	0.3409	0.5366	0.3575	0.3353	0.5249	0.3415	0.3189
		0.0351	0.2022	0.2025	0.3243	0.0471	0.0098	0.3143	0.0465	0.0095
		0.0277	0.0318	0.0300	0.2042	0.0272	0.0082	0.2029	0.0270	0.0078
200	30%	0.3320	0.3657	0.3506	0.5420	0.3663	0.3440	0.5280	0.3512	0.3292
		0.0502	0.2111	0.2098	0.3606	0.0464	0.0100	0.3563	0.0458	0.0098
		0.0375	0.0332	0.0290	0.2107	0.0294	0.0101	0.2007	0.0281	0.0088
	70%	0.3263	0.3501	0.3512	0.5195	0.3507	0.3290	0.5289	0.3519	0.3299
		0.0327	0.2035	0.2020	0.3393	0.0432	0.0099	0.3269	0.0421	0.0096
		0.0255	0.0312	0.0288	0.1973	0.0280	0.0094	0.1966	0.0275	0.0095
	100%	0.3241	0.3581	0.3546	0.5349	0.3587	0.3367	0.5352	0.3553	0.3331
		0.0310	0.2009	0.2003	0.3198	0.0429	0.0096	0.3140	0.0418	0.0094
		0.0246	0.0309	0.0276	0.1925	0.0254	0.0080	0.1918	0.0249	0.0080

Table 18: AEs, RMSEs and MABs of $h(t)$ for Set-2.

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
50	30%	0.6160	0.5849	0.5898	1.0509	0.5865	0.5469	1.0567	0.5915	0.5518
		0.2430	0.3393	0.3386	0.9098	0.0924	0.0164	0.9013	0.0910	0.0136
		0.1605	0.0618	0.0570	0.5215	0.0522	0.0154	0.5199	0.0509	0.0129
	70%	0.5428	0.5883	0.5763	1.0532	0.5900	0.5503	1.0499	0.5780	0.5380
		0.1051	0.3406	0.3382	0.9047	0.0906	0.0135	0.8997	0.0873	0.0122
		0.0825	0.0541	0.0521	0.5143	0.0498	0.0106	0.5108	0.0476	0.0093
	100%	0.5428	0.5678	0.5721	1.0377	0.5694	0.5294	1.0356	0.5737	0.5342
		0.1027	0.3383	0.3361	0.8981	0.0850	0.0115	0.8898	0.0834	0.0114
		0.0816	0.0491	0.0483	0.5125	0.0491	0.0094	0.5105	0.0462	0.0090
100	30%	0.5777	0.5737	0.5876	1.0330	0.5753	0.5360	1.0585	0.5892	0.5493
		0.1388	0.3325	0.3307	0.8828	0.0845	0.0139	0.8611	0.0820	0.0123
		0.1023	0.0606	0.0511	0.5167	0.0492	0.0123	0.5101	0.0486	0.0116
	70%	0.5414	0.6190	0.5792	1.0706	0.6206	0.5817	1.0429	0.5809	0.5414
		0.0735	0.3384	0.3270	0.8742	0.0829	0.0125	0.8574	0.0816	0.0111
		0.0586	0.0531	0.0484	0.5082	0.0482	0.0105	0.5014	0.0472	0.0079
	100%	0.5375	0.5902	0.5780	1.0643	0.5919	0.5518	1.0498	0.5797	0.5398
		0.0715	0.3344	0.3229	0.8618	0.0786	0.0111	0.8487	0.0775	0.0104
		0.0567	0.0480	0.0472	0.5019	0.0475	0.0094	0.4886	0.0460	0.0067
150	30%	0.5632	0.5952	0.5968	1.0596	0.5968	0.5573	1.0578	0.5984	0.5591
		0.1030	0.3290	0.3278	0.8634	0.0821	0.0128	0.8578	0.0754	0.0119
		0.0772	0.0529	0.0476	0.5120	0.0474	0.0119	0.5085	0.0456	0.0110
	70%	0.5412	0.5699	0.5871	1.0537	0.5716	0.5309	1.0514	0.5888	0.5493
		0.0627	0.3242	0.3228	0.8456	0.0802	0.0118	0.8384	0.0750	0.0105
		0.0496	0.0513	0.0473	0.5050	0.0473	0.0103	0.5039	0.0448	0.0079
	100%	0.5366	0.6001	0.5796	1.0638	0.6018	0.5622	1.0494	0.5812	0.5413
		0.0583	0.3217	0.3195	0.8349	0.0779	0.0100	0.8255	0.0742	0.0081
		0.0466	0.0472	0.0451	0.4814	0.0469	0.0093	0.4771	0.0441	0.0069

Table 19: *

Continued Table. 15

n Prior c	FB	MLE	SE I	SE II	LINEX I -3	LINEX I -0.03	LINEX I 3	LINEX II -3	LINEX II -0.03	LINEX II 3
200	30%	0.5580	0.5994	0.5869	1.0612	0.6010	0.5614	1.0495	0.5885	0.5491
		0.0848	0.3186	0.3171	0.8596	0.0790	0.0117	0.8456	0.0735	0.0114
		0.0638	0.0507	0.0469	0.5101	0.0470	0.0116	0.5051	0.0446	0.0107
	70%	0.5404	0.5883	0.5878	1.0407	0.5899	0.5510	1.0505	0.5895	0.5501
		0.0535	0.3129	0.3113	0.8365	0.0771	0.0115	0.8313	0.0721	0.0104
		0.0419	0.0505	0.0452	0.4983	0.0466	0.0102	0.4975	0.0431	0.0077
	100%	0.5385	0.5922	0.5898	1.0560	0.5939	0.5544	1.0562	0.5915	0.5519
		0.0513	0.3085	0.3091	0.8244	0.0762	0.0097	0.8222	0.0712	0.0081
		0.0412	0.0468	0.0448	0.4759	0.0456	0.0091	0.4736	0.0426	0.0071

Table 20: ACLs and CPs for 95% asymptotic and credible intervals of α

Set Prior	n	FB	ACI ACL	ACI CP	BCI I ACL	BCI I CP	BCI II ACL	BCI II CP
1	50	30%	0.9357	0.950	0.1154	0.954	0.1116	0.955
		70%	0.7328	0.952	0.1132	0.955	0.1062	0.957
		100%	0.6968	0.960	0.1115	0.957	0.1041	0.959
	100	30%	0.6171	0.952	0.1152	0.955	0.1059	0.957
		70%	0.5299	0.954	0.1109	0.958	0.1058	0.960
		100%	0.5018	0.962	0.1048	0.959	0.1039	0.964
	150	30%	0.5586	0.952	0.1148	0.958	0.1058	0.959
		70%	0.4591	0.954	0.1100	0.958	0.1047	0.962
		100%	0.4254	0.962	0.1047	0.962	0.1038	0.965
	200	30%	0.4923	0.957	0.1147	0.964	0.1040	
		70%	0.3944	0.963	0.1061	0.965	0.1040	0.967
		100%	0.3814	0.974	0.1026	0.969	0.1016	0.970
2	50	30%	2.4508	0.934	0.1860	0.942	0.1790	0.945
		70%	1.5069	0.939	0.1779	0.944	0.1864	0.949
		100%	1.3014	0.941	0.1787	0.947	0.1812	0.953
	100	30%	1.4741	0.935	0.1899	0.943	0.1808	0.947
		70%	1.0175	0.940	0.1808	0.948	0.1809	0.950
		100%	0.9054	0.947	0.1862	0.951	0.1807	0.954
	150	30%	1.1565	0.938	0.1801	0.946	0.1789	0.947
		70%	0.8440	0.953	0.1890	0.949	0.1766	0.951
		100%	0.7559	0.955	0.1905	0.952	0.1809	0.953
	200	30%	1.0010	0.940	0.1903	0.947	0.1797	0.951
		70%	0.7279	0.956	0.1853	0.950	0.1789	0.951
		100%	0.6578	0.960	0.1800	0.955	0.1791	0.956

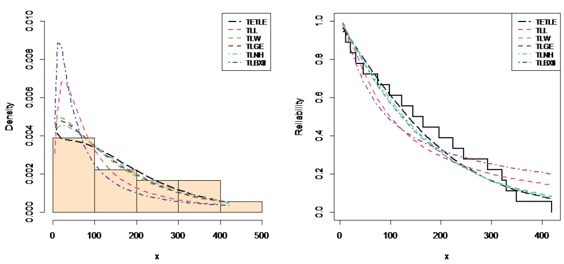

Fig. 1: Fitted PDFs (left-panel) and fitted RFs (right-panel) under Wang's data.

Table 21: ACLs and CPs for 95% asymptotic and credible intervals of θ

Set Prior	n	FB	ACI		BCI			
					I		II	
			ACL	CP	ACL	CP	ACL	CP
1	50	30%	0.6508	0.910	0.1894	0.933	0.1871	0.935
		70%	0.2465	0.935	0.1861	0.945	0.1833	0.948
		100%	0.1831	0.949	0.1809	0.957	0.1313	0.960
	100	30%	0.5783	0.913	0.1877	0.936	0.1833	0.937
		70%	0.2446	0.939	0.1852	0.949	0.1831	0.951
		100%	0.1830	0.952	0.1804	0.962	0.0950	0.963
	150	30%	0.5400	0.935	0.1837	0.940	0.1826	0.954
		70%	0.2201	0.949	0.1829	0.953	0.1804	0.955
		100%	0.1827	0.956	0.1793	0.964	0.0779	0.975
	200	30%	0.5085	0.940	0.1814	0.945	0.1803	0.955
		70%	0.1915	0.954	0.1798	0.959	0.1764	0.957
		100%	0.1797	0.962	0.1771	0.968	0.0679	0.977
2	50	30%	1.8712	0.829	0.3162	0.875	0.3114	0.882
		70%	0.7700	0.866	0.3116	0.883	0.3081	0.902
		100%	0.3918	0.887	0.3108	0.902	0.3075	0.915
	100	30%	1.4994	0.839	0.3153	0.881	0.3084	0.888
		70%	0.6102	0.874	0.3116	0.887	0.3079	0.907
		100%	0.3081	0.902	0.3063	0.908	0.2971	0.922
	150	30%	1.3157	0.852	0.3127	0.899	0.3059	0.909
		70%	0.5573	0.894	0.3082	0.906	0.3054	0.915
		100%	0.3079	0.912	0.3052	0.919	0.2554	0.928
	200	30%	1.3130	0.860	0.3119	0.904	0.3043	0.922
		70%	0.5069	0.910	0.3064	0.917	0.3021	0.927
		100%	0.3056	0.929	0.3015	0.930	0.2283	0.934

Table 22: ACLs and CPs for 95% asymptotic and credible intervals of $h(t)$

Set Prior	n	FB	ACI		BCI			
					I		II	
			ACL	CP	ACL	CP	ACL	CP
1	50	30%	0.4455	0.933	0.4408	0.935	0.3229	0.944
		70%	0.4411	0.945	0.4324	0.946	0.2485	0.953
		100%	0.4391	0.953	0.4284	0.952	0.2431	0.961
	100	30%	0.4452	0.941	0.4284	0.947	0.2211	0.952
		70%	0.4375	0.950	0.4277	0.951	0.1825	0.959
		100%	0.4358	0.954	0.4226	0.955	0.1740	0.966
	150	30%	0.4355	0.950	0.4237	0.950	0.2199	0.958
		70%	0.4261	0.951	0.4247	0.954	0.1531	0.961
		100%	0.4260	0.955	0.4154	0.958	0.1428	0.970
	200	30%	0.4340	0.951	0.4214	0.953	0.1893	0.961
		70%	0.4260	0.954	0.4204	0.958	0.1327	0.964
		100%	0.4242	0.960	0.4128	0.962	0.1261	0.976
2	50	30%	0.7216	0.931	0.6815	0.933	0.6906	0.936
		70%	0.6868	0.935	0.6762	0.936	0.4183	0.945
		100%	0.6833	0.948	0.6697	0.950	0.4050	0.947
	100	30%	0.6907	0.934	0.6738	0.943	0.4686	0.944
		70%	0.6818	0.938	0.6719	0.944	0.2975	0.952
		100%	0.6809	0.952	0.6689	0.954	0.2852	0.958
	150	30%	0.6856	0.938	0.6695	0.947	0.3713	0.948
		70%	0.6800	0.943	0.6679	0.948	0.2473	0.956
		100%	0.6708	0.955	0.6664	0.956	0.2360	0.960
	200	30%	0.6779	0.942	0.6654	0.950	0.3178	0.954
		70%	0.6707	0.949	0.6610	0.953	0.2149	0.961
		100%	0.6697	0.956	0.6588	0.957	0.2061	0.974

Table 23: ACLs and CPs for 95% asymptotic and credible intervals of λ

Set Prior	n	FB	ACI		BCI			
					I		II	
			ACL	CP	ACL	CP	ACL	CP
1	50	30%	4.7638	0.851	0.0912	0.887	0.0892	0.901
		70%	3.6343	0.868	0.0888	0.890	0.0820	0.909
		100%	3.0748	0.873	0.0897	0.895	0.0818	0.910
	100	30%	3.4952	0.898	0.0881	0.902	0.0892	0.902
		70%	2.7205	0.905	0.0875	0.916	0.0817	0.917
		100%	2.0495	0.910	0.0850	0.923	0.0809	0.925
	150	30%	2.6383	0.906	0.0864	0.909	0.0820	0.911
		70%	1.8004	0.911	0.0839	0.919	0.0793	0.922
		100%	1.2801	0.913	0.0829	0.926	0.0785	0.927
	200	30%	1.1672	0.909	0.0831	0.921	0.0785	0.922
		70%	0.9702	0.912	0.0829	0.923	0.0781	0.926
		100%	0.8730	0.915	0.0822	0.928	0.0779	0.931
2	50	30%	6.7632	0.824	0.1482	0.876	0.1395	0.880
		70%	5.2059	0.845	0.1395	0.882	0.1326	0.891
		100%	4.8664	0.857	0.1363	0.889	0.1319	0.893
	100	30%	5.2264	0.844	0.1415	0.891	0.1353	0.896
		70%	4.7011	0.867	0.1388	0.895	0.1321	0.901
		100%	3.3843	0.878	0.1346	0.900	0.1316	0.904
	150	30%	3.5877	0.862	0.1359	0.903	0.1319	0.903
		70%	2.8983	0.886	0.1339	0.907	0.1309	0.910
		100%	2.3131	0.891	0.1320	0.909	0.1281	0.911
	200	30%	2.7717	0.882	0.1342	0.905	0.1293	0.908
		70%	1.5873	0.907	0.1322	0.910	0.1285	0.912
		100%	1.1941	0.913	0.1320	0.910	0.1281	0.914

Table 24: ACLs and CPs for 95% asymptotic and credible intervals of $S(t)$

Set Prior	n	FB	ACI		BCI			
					I		II	
			ACL	CP	ACL	CP	ACL	CP
1	50	30%	0.1762	0.929	0.1745	0.935	0.1699	0.947
		70%	0.1734	0.935	0.1715	0.942	0.1623	0.976
		100%	0.1706	0.939	0.1699	0.949	0.1629	0.978
	100	30%	0.1706	0.935	0.1661	0.939	0.1261	0.975
		70%	0.1692	0.943	0.1623	0.945	0.1243	0.980
		100%	0.1671	0.945	0.1607	0.955	0.1223	0.983
	150	30%	0.1695	0.939	0.1639	0.942	0.1058	0.981
		70%	0.1671	0.947	0.1612	0.949	0.1015	0.984
		100%	0.1618	0.950	0.1597	0.958	0.0998	0.990
	200	30%	0.1681	0.941	0.1600	0.946	0.0916	0.984
		70%	0.1666	0.949	0.1584	0.951	0.0881	0.986
		100%	0.1612	0.965	0.1572	0.967	0.0867	0.992
2	50	30%	0.1853	0.929	0.1755	0.933	0.1724	0.943
		70%	0.1756	0.931	0.1724	0.937	0.1698	0.951
		100%	0.1730	0.935	0.1717	0.946	0.1690	0.951
	100	30%	0.1756	0.932	0.1708	0.935	0.1305	0.948
		70%	0.1736	0.935	0.1692	0.943	0.1237	0.962
		100%	0.1721	0.937	0.1684	0.951	0.1227	0.966
	150	30%	0.1738	0.937	0.1684	0.940	0.1063	0.954
		70%	0.1726	0.944	0.1682	0.948	0.1014	0.962
		100%	0.1718	0.946	0.1676	0.953	0.1005	0.972
	200	30%	0.1717	0.944	0.1673	0.944	0.0918	0.961
		70%	0.1692	0.947	0.1667	0.950	0.0881	0.965
		100%	0.1683	0.961	0.1660	0.966	0.0873	0.971

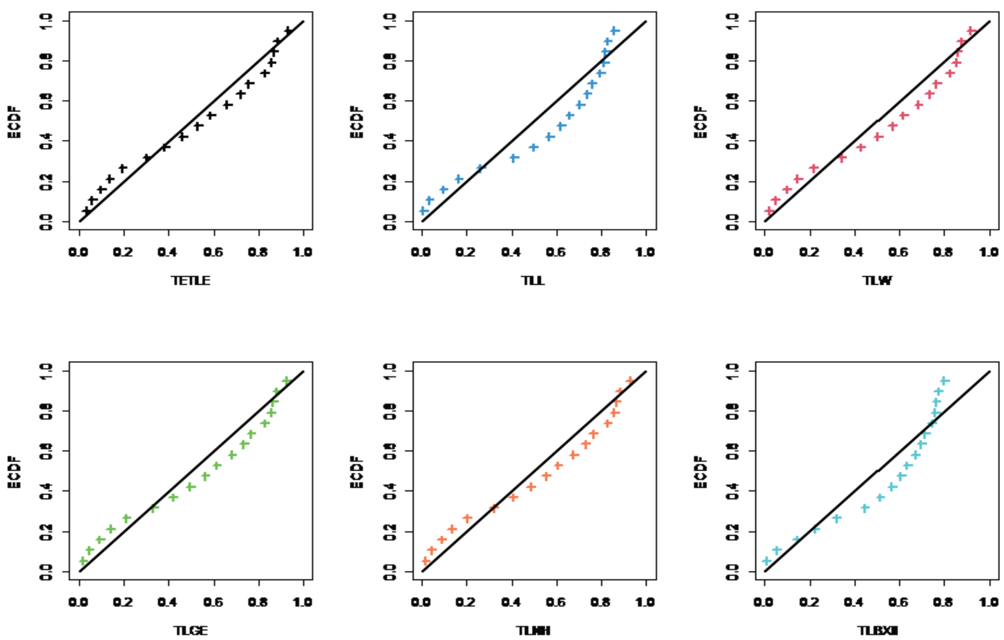

Fig. 2: The Q–Q plots of TETLE distribution and its competing models under Wang's data.

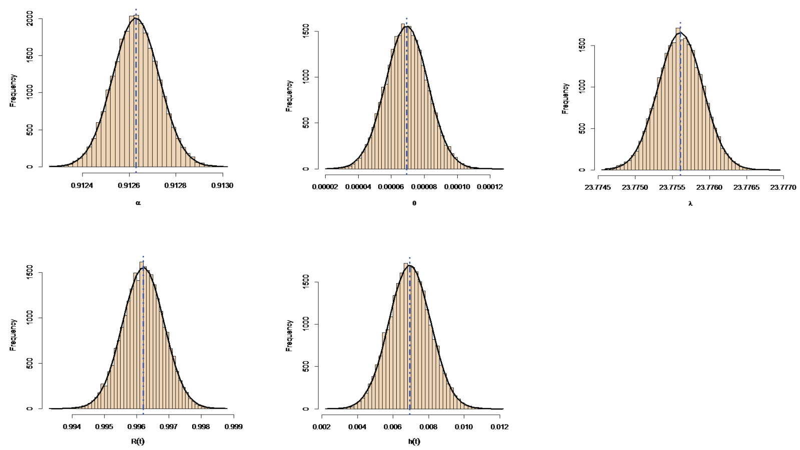

Fig. 3: Histogram and kernel density estimates of α, θ, λ , $S(t)$ and $h(t)$ from Sample 1

Table 25: The point estimates with their standard errors and interval estimates with their [lengths].

Sample c	Parameter	MLE	SEL	LINEX -5	LINEX +5	ACI	BCI
1	α	0.91264	0.91262	0.91263	0.91262	(0.3152,1.5101)	(0.9124,0.9128)
		(3.05×10^{-1})	(6.28×10^{-7})	(7.24×10^{-8})	(7.27×10^{-8})	[1.1949]	[0.0004]
	θ	0.00007	0.00007	0.00007	0.00007	(0.0000,0.0003)	(0.00004,0.00009)
		(1.10×10^{-4})	(8.10×10^{-5})	(4.25×10^{-5})	(4.25×10^{-5})	[0.0003]	[0.00005]
	λ	23.7757	23.7756	23.7756	23.7756	(11.215,36.336)	(23.775,23.776)
		(0.64×10^{-1})	(1.90×10^{-6})	(5.80×10^{-7})	(5.83×10^{-7})	[25.122]	[0.0012]
	$R(0.5)$	0.99598	0.99621	0.99621	0.99621	(0.9827,0.9992)	(0.9949,0.9974)
		(6.79×10^{-3})	(4.05×10^{-6})	(2.50×10^{-7})	(2.37×10^{-7})	[0.0165]	[0.0025]
	$h(0.5)$	0.00736	0.00694	0.00694	0.00693	(0.0000,0.0270)	(0.0047,0.0093)
		(1.01×10^{-2})	(7.41×10^{-6})	(4.24×10^{-7})	(4.67×10^{-6})	[0.0270]	[0.0046]
2	α	0.92927	0.92925	0.92925	0.92925	(0.2991,1.5595)	(0.9291,0.9295)
		(3.22×10^{-1})	(6.33×10^{-7})	(6.33×10^{-7})	(6.75×10^{-8})	[1.2604]	[0.0004]
	θ	0.00005	0.00005	0.00005	0.00005	(0.0000,0.0002)	(0.00003,0.00006)
		(7.46×10^{-5})	(5.26×10^{-5})	(5.26×10^{-5})	(1.14×10^{-5})	[0.0002]	[0.00003]
	λ	38.1945	38.1944	38.1944	38.1944	(11.494,64.895)	(38.194,38.195)
		(1.36×10^{-1})	(1.90×10^{-6})	(1.90×10^{-6})	(5.59×10^{-7})	[53.401]	[0.0012]
	$R(0.5)$	0.99632	0.99629	0.99629	0.99629	(0.9832,0.9999)	(0.9951,0.9974)
		(6.72×10^{-2})	(3.76×10^{-6})	(3.76×10^{-6})	(8.43×10^{-7})	[0.0167]	[0.0023]
	$h(0.5)$	0.00685	0.00690	0.00690	0.00691	(0.0000,0.0268)	(0.0048,0.0091)
		(1.02×10^{-2})	(7.01×10^{-6})	(7.01×10^{-6})	(1.51×10^{-6})	[0.0268]	[0.0043]
3	α	0.99348	0.99347	0.99347	0.99347	(0.3353,1.6516)	(0.9933,0.9937)
		(3.36×10^{-1})	(6.30×10^{-7})	(6.23×10^{-7})	(6.26×10^{-8})	[1.3163]	[0.0004]
	θ	0.00004	0.00004	0.00005	0.00004	(0.0000,0.0001)	(0.00002,0.00005)
		(6.25×10^{-5})	(4.52×10^{-5})	(1.21×10^{-5})	(1.21×10^{-5})	[0.0001]	[0.00003]
	λ	66.4978	66.4977	66.4977	66.4977	(47.369,85.627)	(66.497,66.498)
		(0.97×10^{-1})	(1.89×10^{-6})	(5.74×10^{-7})	(5.77×10^{-7})	[38.258]	[0.0011]
	$R(0.5)$	0.99735	0.99730	0.99730	0.99729	(0.9871,0.9999)	(0.9963,0.9982)
		(5.24×10^{-3})	(3.19×10^{-6})	$(8.63\times 10-7)$	(8.54×10^{-7})	[0.0128]	[0.0019]
	$h(0.5)$	0.00528	0.00538	0.00538	0.00537	(0.0000,0.0222)	(0.0035,0.0074)
		(8.67×10^{-3})	(6.35×10^{-6})	(1.69×10^{-6})	(1.72×10^{-6})	[0.0222]	[0.0039]

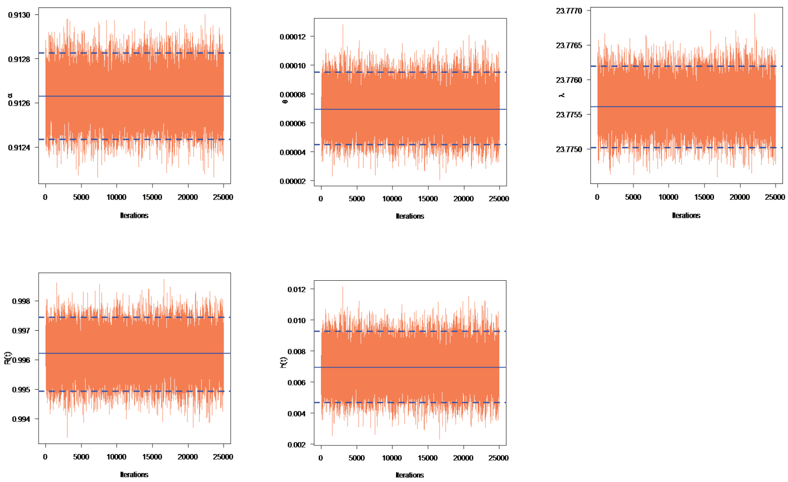

Fig. 4: MCMC trace plots of α, θ, λ , $S(t)$ and $h(t)$ from Sample 1

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