# Numerical analysis of nonlinear interaction between a gas bubble and free surface in a viscous compressible liquid 

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Liquid viscosity has a potential e ect on bubble dynamics．This paper is concerned with bubble dynamics in a compressible viscous liquid near the free surface．The liquid－gas ow is modeled using the Eulerian nite element method（EFEM）coupled with the volume of uid method（VOF）． The numerical results have been shown to be in excellent agreement with those from the spherical bubble theory and experiment．Parametric studies are carried out regarding the Reynolds number Re and the stand－o parameter ${ }_{d}$ ．It clearly demonstrated that the liquid viscosity inhibits bubble pulsation，jet ow，free surface jet，and bubble splitting．Quantitatively，as Reynolds number Re decreases，the maximum bubble volume，jet tip velocity，free surface spike and crown height decrease， and the toroidal bubble splitting weakens．As the stand－o parameter ${ }_{d}$ increases，the maximum bubble volume，jet velocity，and bubble average pressure peak increase while the height of the free surface spike decreases．Close observation reveals that the free surface crown tends to disappear at small Re or large ${ }_{d}$ ，further indicating the complex mechanism behind the crown spike evolution．

## I．INTRODUCTION

Pulsating bubble dynamics have been widely applied in marine engineering，biomedicine and other elds，such as underwater explosion ${ }^{1-8}$ ，ultrasonic bubble cleaning ${ }^{9-13}$ ，microbubble ultrasound contrast agents ${ }^{14-16}$ ，and digital printing technique ${ }^{17-19}$ ．When the initial bubble pressure is much greater than the pressure of the ow eld，it will expand and start to oscillate under the action of the pressure gradient on both sides of the interface．At the same time，bubbles oscillating near the boundary（such as the free surface） can exhibit non－spherical dynamics such as jet generation due to the in uence of asymmetric forces ${ }^{20-27}$ ． For the pulsating bubble near the free surface，the oscillation and migration of the bubble have strong nonlinear characteristics due to the in uence of the deformation of the free surface ${ }^{28-32}$ ．If liquid viscosity is considered，the nonlinear characteristics of the bubble，such as jets and tears，will be changed，which will a ect the application of the bubble in engineering．Thus it is necessary for us to study the interaction of pulsating bubbles and the free surface in viscous uids．

Due to the common phenomenon of the pulsating bubble jet near the free surface in nature，many scholars have studied this phenomenon and found that depth，buoyancy，and pressure are the key factors a ecting bubble dynamics and the free surface motion ${ }^{33-47}$ ．Theoretical models such as the Rayleigh－Plesset equation ${ }^{48}$ ， and the Keller－Miksis equation ${ }^{49}$ can accurately predict the pulsating bubbles in free eld，but they cannot include the in uence of free surface and bubble migration．Recently，Zhang et al．${ }^{50}$ established a new oscillating bubble dynamics uni ed equation with an elegant mathematical form，considering various factors such as boundary，gravity，viscosity，compressibility，and bubble migration．This equation expands the applicability of bubble theory and provides a new idea for subsequent research．However，most of the theories are based on the assumption of spherical bubbles and cannot account for bubble jets and splitting，numerical and experimental methods are still widely used to study bubble dynamics．Blake and Gibson ${ }^{33,34}$ observed the movement of bubbles adjacent to the free surface through the experiment of spark－generated bubbles． They found that the bubble is repelled during the collapse and generates downward liquid jets．Supponen ${ }^{36}$ studied the bubble jet＇s generation conditions through many laser－induced bubble experiments and proposed an anisotropy parameter to describe the three states of the bubble jet，which can comprehensively consider the in uence of the buoyancy of the bubble and the distance to the free surface．Zhang ${ }^{37}$ used the discharge spark generator to study the interaction of the bubble and the free surface at di erent initial depths．Furthermore，

[^0]they summarized the laws of bubble shapes and pulsation periods, heights of the water skirts and the free surface spikes, and jet tip velocities and identi ed six distinctive patterns of the free surface. Cerbus ${ }^{42}$ observed the motion characteristics of the free surface jet and analyzed the formation mechanism of the second jet and the e ect of the two control parameters. Unfortunately, the e ect of liquid viscosity on the interaction between the pulsating bubble and the free surface has rarely been mentioned in previous experiments. For numerical techniques, the boundary element method (BEM) has been widely employed in studying non-spherical bubble dynamics because of low computational cost ${ }^{51-55}$. However, the traditional BEM was developed based on the potential ow theory, thus it is not easy to calculate the e ect of the viscosity of the liquid on bubble dynamics. Based on the traditional BEM, Miksis ${ }^{56}$ proposed Boundary Layer theory to deal with viscous bubble dynamics. Then Lunderen ${ }^{57}$ and Boulton-Stone ${ }^{58}$ extended the theory and obtained the continuous expression of normal and tangential components. Based on these studies, $L^{54}$ used the Boundary Layer theory to study the e ects of the Reynolds number on the movement of two bubbles close to the free surface and found that viscosity would depress the bubble movement. Lind ${ }^{53}$ paid attention to the in uence of viscoelasticity on bubble dynamics in the vicinity of the free surface. Moreover, Lind also found that viscoelasticity a ected the bubbles' jet and changed the shape of collapsing bubbles. However, the boundary layer theory only calculates in a thin layer at the bubble boundary, which cannot accurately calculate the in uence of viscous dissipation on the liquid outside the boundary layer. It is also challenging to consider the compressibility of the liquid at the same time.

Many numerical multiphase ow models based on domain methods are also widely used to analyze bubble dynamics, which can obtain complete ow data. $\mathrm{Li}^{38}$ used the open-source software OpenFoam to numerically study the bubble bursting properties close to the free surface and found that gas ow plays an important role in the re-closure of burst bubbles. Singh ${ }^{45}$ studied the internal gas dynamics of burst bubbles using the VOF method. Liu ${ }^{39}$ combined FVM and FTM to investigate the dynamical characteristics of oscillating bubbles near the free surface at di erent depths and buoyancy. Saade ${ }^{40}$ used the FVM solver to simulate the crown of bubble-induced generation, analyzed the in uence of non-dimensional parameters such as Reynolds number and stand-o parameter on crown formation, and explained the mechanism of crown formation. Saade aimed to obtain stable free surface jets for application in the LIFT process, and therefore a lower initial pressure of the bubble was chosen. However, this is not suitable for simulating higher initial pressure bubbles such as underwater explosions and high-voltage spark bubbles. Bempedelis ${ }^{41}$ numerically studied the dynamics of a bubble-free surface coupling system, focused on the dynamic characteristics and deformation of the free surface, and characterized the process of bubble-induced atomization. From the above analysis, previous researchers hardly consider the e ects of liquid compressibility and viscosity simultaneously when analyzing the coupling e ect between high-pressure bubbles and the free surface. However, compressibility and liquid viscosity have essential e ects on the dynamic characteristics of high-pressure pulsating bubbles. For more accurate results, the compressibility e ect should be considered ${ }^{59}$. The EFEM numerically solves the Navier-Stokes equation and uid equation of state based on the domain approach. It can consider both the strong compressibility, viscosity, and pressure distribution of the uid. The EFEM uses the operator splitting algorithm to separate the governing equations, which allows it to be exibly combined with other numerical techniques to improve accuracy and applicability. At the same time, the combination of VOF for interface capture and reconstruction can accurately and exibly simulate the splitting and merging process of two-phase interface deformation and topological relationship changes. This method can accurately simulate the complete stage of underwater explosion and the interaction between the bubble and the free surface.

The present paper establishes an axisymmetric dynamics model of the interaction between the free surface and bubbles using EFEM. The VOF is used to capture the uid interface and study the e ect of liquid viscosity on bubbles and free surface motion. The main contents are as follows: the second section describes the physical model, governing equations, and initial conditions of the problem, establish the bubble dynamics model, and veri es the accuracy of the numerical results. In section 3, the e ects of liquid viscosity and distance on the bubble-free surface coupling dynamics are studied, and the mechanical mechanism is analyzed in detail. The fourth section draws some critical conclusions.

## II. THEORETICAL AND NUMERICAL MODELS

## A. Problem assumptions and governing equations

This study mainly focuses on the interaction between bubbles and the free surface. A sketch is shown in Fig.1(a) to describe the problem. An initial spherical bubble is placed in a static viscous liquid at a depth $d$ below the free surface. The bubble can be induced by spark discharge, underwater explosion, or
air-gun, and the initial pressure is higher than the surrounding uid. The bubble expands rapidly under pressure and interacts with the free surface. The non-re ection boundary condition is used for the edge of the computational domain ${ }^{60}$. An axisymmetric coordinate system is established as shown in Fig.1(a), where $O, r, z$, and represent the coordinate origin, radial coordinate, axial coordinate, and angular coordinate, respectively, and the computational domain size is $w \times h$. The gravity points to the negative direction of the $z$ axis.


FIG. 1. (a)Con guration for the interaction between the single bubble and the free surface; (b)comparison of bubble radius in di erent calculation domain sizes $2 R_{m} \times 4 R_{m}, 4 R_{m} \times 8 R_{m}, 5 R_{m} \times 10 R_{m}, 6 R_{m} \times 12 R_{m}$, and $8 R_{m} \times 16 R_{m}$.

In this paper, the e ects of liquid viscosity on the interaction of a pulsating bubble and the nearby free surface were studied, while surface tension and heat conduction could be neglected ${ }^{34,61-64}$. In the current cylindrical coordinate system, the following governing equations can be established:

$$
\left\{\begin{array}{l}
\frac{{ }^{i}}{t}+\nabla \cdot\left({ }_{i} \boldsymbol{v}\right)={ }_{i} \frac{\bar{K}}{K_{i}} \nabla \cdot \boldsymbol{v}  \tag{1}\\
\frac{{ }_{i} i_{i}}{t}+\nabla \cdot\left({ }_{i}{ }_{i} \boldsymbol{v}\right)+\frac{i_{i} v_{r}}{r}=0 \\
\frac{-\boldsymbol{v}}{t}+\nabla \cdot\left({ }^{-} \boldsymbol{v} \otimes \boldsymbol{v}\right)+\frac{-v_{r} \boldsymbol{v}}{r}+\nabla p=-\mathbf{g}+\nabla \cdot \boldsymbol{\tau}+\frac{\boldsymbol{\tau}_{r}}{r} \\
\frac{{ }_{i}{ }_{i} e_{i}}{t}+\nabla \cdot\left({ }_{i}{ }_{i} e_{i} \boldsymbol{v}\right)+p{ }_{i} \frac{\bar{K}}{K_{i}}\left(\nabla \cdot \boldsymbol{v}+\frac{v_{r}}{r}\right)+\frac{{ }_{i} i_{i} e_{i}}{r}={ }_{i}
\end{array}\right.
$$

which represent the volume fraction, mass conservation, momentum conservation in $r$ and $z$ directions, and energy conservation equations, respectively. Here, the subscript ' $i$ ' represents the type of uid, represents volume fraction which is the volume ratio of the uid phase to the element and $\sum_{i}=1$, is the density of the uid and ${ }^{-}=1_{1}+_{2} 2_{2}$ is the average density, $K=c^{2}$ is uid bulk modulus, $c$ is the sound speed of the uid, $p$ is the pressure, and $e$ is the speci c internal energy of the uid, $\boldsymbol{v}=\left(v_{r} v_{z}\right)^{T}$ is the velocity vector of the uid, $\mathbf{g}$ is gravity vector. The subscripts $r z$ indicate the components of the vector in the $r$ and $z$ directions, respectively. $\boldsymbol{\tau}$ is the viscous stress tensor, and $\boldsymbol{\tau}_{r}=\left({ }_{r r}-\quad r_{r z}\right)^{T}$ is a component of tensor $\boldsymbol{\tau}$. The symbol ' $\otimes$ ' is the tensor product operator, and $\nabla=\left(\bar{\tau}_{r} \bar{z}_{z}\right)$ is the gradient operator. Besides, in the mixing element, the average bulk modulus $\bar{K}$ can be determined by

$$
\begin{equation*}
\bar{K}=\frac{K_{1} K_{2}}{{ }_{1} K_{2}+{ }_{2} K_{1}} \tag{2}
\end{equation*}
$$

In a Newtonian uid, the viscous stress tensor $\boldsymbol{\tau}$ can be expressed as ${ }^{38}$

$$
\left\{\begin{align*}
\boldsymbol{\tau}= & { }_{i} \boldsymbol{\tau}_{i}=\quad{ }_{i} i\left[\nabla \boldsymbol{v}+\nabla \boldsymbol{v}^{T}-\frac{2}{3}\left(\nabla \cdot \boldsymbol{v}+\frac{v_{r}}{r}\right) \mathbf{I}\right]  \tag{3}\\
& =\quad i_{i}\left[2 \frac{v_{r}}{r}-\frac{2}{3}\left(\nabla \cdot \boldsymbol{v}+\frac{v_{r}}{r}\right)\right]
\end{align*}\right.
$$

where is the dynamic viscosity coe cient, and $\mathbf{I}$ is the unit matrix. For the energy conservation equation,

$$
\begin{equation*}
{ }_{i}=\boldsymbol{\tau}_{i}: \nabla \boldsymbol{v}={ }_{r r_{-} i} \frac{v_{r}}{r}+{ }_{r z_{-} i} \frac{v_{r}}{z}+{ }_{z r_{-} i} \frac{v_{z}}{r}+{ }_{z z_{-} i} \frac{v_{z}}{z}+\quad{ }_{-i} \frac{v_{r}}{r} \tag{4}
\end{equation*}
$$

which represents the viscous dissipation energy. The equation of state(Eos) of the uid is used to close the governing equations of the compressible uid. In present work, the Tammann equation ${ }^{65}$ is used to describe the uid state:

$$
\begin{equation*}
p=e(-1)-P_{w} \tag{5}
\end{equation*}
$$

In this paper, the initial density of liquid and gas is $1000.0 \mathrm{~kg} / \mathrm{m}^{3}$ and $1.29 \mathrm{~kg} / \mathrm{m}^{3}$, respectively. is the speci c heat ratio of uid, which is 7.15 and 1.25 for liquid and gas, respectively. $P_{w}=3309 \mathrm{e} 8 \mathrm{~Pa}$ is the pressure constant of water. For gas, $P_{w}=0 \mathrm{~Pa}$ is chosen to represent the ideal gas.

## B. Eulerian Finite Element Method(EFEM)

The EFEM has been maturely applied and has obtained reliable results in solving underwater explosion bubble dynamics ${ }^{66-69}$. The operator split technique is the key to EFEM, and the previous literature ${ }^{60,62,66,70,71}$ has described the operator split method in detail. The operator split method separates the convection term from the governing equation to divide the equation into two parts for calculation, called the Lagrangian phase and the Eulerian phase. The explicit nite element method calculates the equation without the convection term in the Lagrangian phase. Thus, the uid material follows the mesh movement in a single time increment. The convection term in the equation is calculated in the Eulerian phase, where the uid remains stationary, the mesh returns to its original position, and material transport occurs between adjacent elements. A two-step calculation is performed in a time increment to move the uid material while the mesh remains stationary. The equation solved in the Lagrangian phase does not contain a convection term and can be solved by the FEM. Combining the mass conservation equation and Gauss formula, the following momentum integral form can be obtained:

$$
\begin{equation*}
\iint_{\Omega}-\frac{\boldsymbol{v}}{t} \mathrm{~d} s=\iint_{\Omega}\left(p \nabla+{ }^{-} \mathbf{g}-\boldsymbol{\tau} \cdot \nabla+\frac{\boldsymbol{\tau}_{r}}{r}\right) \mathrm{d} s-\int_{\Omega} p \boldsymbol{n} \mathrm{~d} l+\int_{\Omega} \boldsymbol{\tau} \cdot \boldsymbol{n} \mathrm{d} l \tag{6}
\end{equation*}
$$

where is the outer boundary of the element volume, is a weight function determined by the element shape, and $\boldsymbol{n}=\left(n_{r} n_{z}\right)$ is the interface unit normal vector. The variables on the right-hand side of Eq.(6) are known, and the element nodal acceleration can be calculated according to Newton's second law of motion:

$$
\begin{equation*}
\mathbf{M A}_{\text {node }}=\mathbf{F}_{\text {node }} \tag{7}
\end{equation*}
$$

where $\mathbf{M}=\iint_{\Omega}^{-} \mathrm{d} s$ is the nodal mass matrix, $\mathbf{A}_{\text {node }}$ is the nodal acceleration vector, and $\mathbf{F}_{\text {node }}$ is the nodal force, which can be calculated from the right-hand side of Eq.(6). From this, the nodal velocity and displacement can be calculated by the explicit nite element method, and Tang ${ }^{67}$ gave the calculation formula for any node:

$$
\begin{gather*}
\boldsymbol{v}^{k+12}=\boldsymbol{v}^{k-12}+\boldsymbol{a}^{k} t  \tag{8}\\
\boldsymbol{x}^{k+1}=\boldsymbol{x}^{k}+\boldsymbol{v}^{k+12} \quad t \tag{9}
\end{gather*}
$$

where $t$ is the time increment, $k$ is the time count, $\boldsymbol{a}$ is the nodal acceleration vector obtained from Eq.(7). Therefore a new node position is obtained, where the mesh and the uid material move together, and the material inside the element at the current stage has not changed. However, the expansion and contraction
of the element cause change in the material's physical properties. The changes in volume, mass, and energy are calculated according to the continuity equation and energy equation after operator splitting:

$$
\left\{\begin{array}{l}
\frac{i}{t}=-{ }_{i} \nabla \cdot \boldsymbol{v}+{ }_{i} \frac{\bar{K}}{K_{i}} \nabla \cdot \boldsymbol{v}  \tag{10}\\
\frac{i_{i} i}{t}=-{ }_{i}{ }_{i} \nabla \cdot \boldsymbol{v}-\frac{i_{i} v_{r}}{r} \\
\frac{i_{i} e_{i}}{t}={ }_{i}-\left({ }_{i}{ }_{i} e_{i}+p \quad{ }_{i} \frac{\bar{K}}{K_{i}}\right)\left(\nabla \cdot \boldsymbol{v}+\frac{v_{r}}{r}\right)
\end{array}\right.
$$

At this point, the calculation of the Lagrangian phase is completed, and the uid deforms with the mesh.
In the Eulerian phase, the material is xed, and the mesh returns to its original position. During this process, volume transport occurs between adjacent elements. The convection volume can be obtained by integrating over the edges of the mesh. For single-phase elements, the uid transport volume equals the mesh convection volume. However, for multiphase elements, the uid transport volume on each mesh edge needs to be determined based on the location of the uid interface within the element. In this paper, the second-order algorithm Piecewise Line Interface Calculation(PLIC) is applied to construct a linear uid interface ${ }^{72}$, and the uid's transport of various variables is calculated by combining it with Monotone Upwind Schemes for Conservation Laws (MUSCL) ${ }^{73}$. The Eulerian phase of EFEM is described in detail by Tian ${ }^{66,72}$, and these numerical techniques will not be repeated again. After the Eulerian phase, the element's mass, momentum, and energy are updated. After the material has been transported, the uid pressure $p$ needs to be updated using Eq.5. At this point, we have completed the calculation of a time increment.

This model must satisfy the CFL condition to ensure numerical stability. Therefore, the time increment $t$ is limited by

$$
\begin{equation*}
t=C_{o}\left(\frac{l}{\left|\boldsymbol{v}_{\max }\right|+c}\right)^{\min } \tag{11}
\end{equation*}
$$

where $l$ is the characteristic size of the mesh, and the superscript min indicates the minimum value of all elements. The Courant number $C_{o}=03$ is chosen in this paper. The sound speed $c$ can be expressed as ${ }^{66}$

$$
\begin{equation*}
c^{2}=\underline{\left(p+P_{w}\right)} \tag{12}
\end{equation*}
$$

when the Tammann equation is given.

## C. Initial conditions and dimensionless parameters

Fig.1(a) illustrates the calculation model in this paper. At the initial moment, the bubble is located in a static viscous liquid, the depth from the free surface is $d$, the initial bubble radius is $R_{0}$, and the bubble pressure is $P_{0}$, respectively. The atmospheric pressure of the air above the free surface is $\mathrm{P}_{\mathrm{atm}}=101 \mathrm{kPa}$.

In order to avoid introducing errors due to di erent units or the numerical accuracy of the computer in the study of the same problem, dimensionless variables are used to simulate bubble dynamics. The expected maximum radius $R_{m}$ of the spherical pulsation of the bubble in the inviscid and incompressible free eld is used as the length scale. Moreover, the initial density of the viscous liquid is used as the density scale ${ }_{r e f}={ }_{l}$, and the ambient pressure $P_{\text {ref }}=\mathrm{P}_{\mathrm{atm}}+|\mathbf{g}| d_{\text {ref }}$ is used as the pressure scale. Therefore, the scales of velocity, time, and acceleration can be represented separately, as shown in Table I. According to these dimensionless scales, the case parameters of bubbles can be dimensionless in Table II. In this paper, the buoyancy parameter represents the gravity e ect, the strength parameter represents the initial pressure, the stand-o parameter ${ }_{d}$ represents the dimensionless inception depth, the Mach number Ma represents the compressibility of the liquid ( $c_{\infty}$ is the sound speed in the liquid at in nity), and the Reynolds number Re represents the viscosity e ect. The dimensionless radius and time are denoted as $R^{*}=R R_{m}$ and $t^{*}=t\left[R_{m}\left(\text { ref } \mathrm{P}_{r e f}\right)^{1}{ }^{2}\right]$ respectively, and other variables are expressed in the same way, in which the superscript ' $*$ ' represents dimensionless. Unless otherwise speci ed, all subsequent analyses are performed using dimensionless variables.

TABLE I. Basic variables for dimensionless scales.

| $\operatorname{Velocity}\left(V_{r e f}\right)$ | $\operatorname{Time}\left(T_{\text {ref }}\right)$ | Acceleration $\left(A_{r e f}\right)$ |
| :---: | :---: | :---: |
| $\sqrt{\frac{P_{r e f}}{r e f}}$ | $R_{m} \sqrt{\frac{r e f}{P_{r e f}}}$ | $\frac{P_{\text {ref }}}{R_{m r e f}}$ |

TABLE II. Critical dimensionless case parameters.

| 2 |  |  |  | Ma |
| :---: | :---: | :---: | :---: | :---: |

## D. Validation and convergence test

Before starting the veri cation, we rst analyze the convergence of the computational domain. The computational domain sizes of $2 R_{m} \times 4 R_{m}, 4 R_{m} \times 8 R_{m}, 5 R_{m} \times 10 R_{m}, 6 R_{m} \times 12 R_{m}$, and $8 R_{m} \times 16 R_{m}$ are selected for numerical simulation. The radius time history is shown in Fig.1(b), from which it can be found that the results are approximately the same except for $2 R_{m} \times 4 R_{m}$, indicating that the current calculation results are plausible. Although the di erence in the radius curves in Fig.1(b) is not particularly obvious, in our numerical results, the bubble shape appears nonphysically distorted in the $2 R_{m} \times 4 R_{m}$, which indicates that the boundary a ects the bubble in a smaller computational domain. To avoid the error caused by the free surface jet impacting the boundary of the computational domain, we select $6 R_{m} \times 12 R_{m}$ for calculation. Then we compare the model with theoretical and experimental results to verify the accuracy of the bubble dynamics model. First, we a rm the validity of the viscosity in our numerical model. The uni ed equation for bubble dynamics proposed by Zhang ${ }^{50}$ can be used to describe bubble pulsation and migration in the free eld of a viscous uid. Taking a single spherical pulsating bubble in an in nite domain as the research object and ignoring the in uence of initial velocity and surface tension, the bubble pulsation equation and migration equation of the simpli ed uni ed equation can be expressed as:

$$
\begin{align*}
\left(\frac{c-R}{R}+\frac{\mathrm{d}}{\mathrm{~d} t}\right)\left[\frac{R^{2}}{c}\left(\frac{1}{2} R^{2}+\frac{1}{4} \boldsymbol{v}_{m}^{2}+\frac{P_{b}-P_{a}}{}\right)\right] & =2 R R^{2}+R^{2} R  \tag{13}\\
C_{a}\left(R \boldsymbol{v}_{m}+3 R \boldsymbol{v}_{m}\right)-\mathbf{g} R+\frac{3}{8} C_{d} \quad\left(\boldsymbol{v}_{m}\right) & =0 \tag{14}
\end{align*}
$$

where $R$ is the bubble radius, $c$ is the sound speed of the external uid, $P_{a}$ is the ambient pressure at the bubble center, $P_{b}=P_{g a s}-4 R R$ is the uid pressure on the outer bubble surface, $P_{g a s}$ is the internal bubble pressure, $C_{a}$ is the added mass coe cient, $C_{d}$ is the drag coe cient, $\boldsymbol{v}_{m}$ is the migration velocity,
$(\cdot)=(\cdot)|\cdot|$ is a signed square operator, and the top dot represents the rst or second derivatives of the variable for time. When the migration of bubbles is not considered, Eq. 13 can be further degenerated to the Keller-Miksis ${ }^{49}$ equation. The KM(Keller-Miksis) equation improves the RP equation and includes the e ect of uid compressibility. Therefore, the KM equation is also used to compare the results of this paper.

Select the strength parameter $=100$ (which is considered a reasonable value ${ }^{74}$ ), the buoyancy parameter
$\approx 01$, the gravity acceleration $|\mathbf{g}|=98 \mathrm{~m} \mathrm{~s}^{2}$, the speed of sound $c=1536 \mathrm{~m} / \mathrm{s}$ (according to Eq.12), the added mass coe cient $C_{a}=10$, the drag coe cient $C_{d}=05$, and ignore the surface tension, assume that the maximum dimensionless bubble radius is 1.0 in incompressible and inviscid cases. According to the relationship between the bubble radius and initial bubble pressure given by Klaseboer ${ }^{2}$, the dimensionless initial radius $R_{0}^{*}=0.149$ is obtained. The numerical results are compared with Zhang et al. ${ }^{50}$ and the KM equations at $\mathrm{Re}=100$ as shown in Fig.2, where $\mathrm{Re}=\infty$ represents the inviscid case. The bubble migration distance of KM equation is zero because gravity is not considered. It can be found that the maximum bubble radius decreases signi cantly when $\operatorname{Re}=100$. Due to the liquid compressibility e ect, the bubble does not reach the expected maximum radius $R_{m}$. However, the bubble radius evolution and migration curves of numerical simulation are in good agreement with the theoretical model, indicating that the EFEM bubble dynamics model can accurately predict the e ect of viscosity on bubble motion.


FIG. 2. Comparison of bubble radius evolution(a) and bubble migration(b) between EFEM and theoretical model at $\operatorname{Re}=100$; (a)convergence analysis of equivalent radius for di erent mesh sizes $l=0005 R_{m}, l=001 R_{m}$, and $l=002 R_{m}$.

In order to eliminate the in uence of the mesh size on the model calculation results, it is necessary to carry out the convergence analysis. Set the calculation mesh size to $0.005 R_{m}, 0.01 R_{m}$, and $0.02 R_{m}$, respectively. The evolution of bubble radius with time is shown in Fig.2(a). As the mesh size decreases, the bubble radius curve converges to a theoretical solution. In order to take into account the calculation e ciency, the grid size is selected as $l=001 R_{m}$ in this paper. In addition, the numerical model is compared with the spark discharge experiment in glycerol. The experiment was carried out in a cube tank with a side length of 300 mm and a glycerol depth of 200 mm . The density of glycerol $=1261 \mathrm{~kg} \mathrm{~m}^{3}$ and the dynamic viscosity $=1499$ $\mathrm{mPa} \cdot \mathrm{s}\left(20^{\circ} \mathrm{C}\right)$, respectively. Bubbles were generated by spark discharge using a high voltage device of 1800 V. The motion of bubbles and the free surface was captured by a high speed camera(Phantom V711). In the numerical model, a bubble with an initial radius of 4.1 mm is placed 26.1 mm below the free surface, and the initial pressure is set to $100 P_{\infty}\left(P_{\infty}\right.$ represents the liquid pressure at in nity at the same depth as the bubble). According to Zhang et al. ${ }^{50}$, the expected maximum radius of the bubble was $R_{m}=27.5 \mathrm{~mm}$, thus the dimensionless inception depth was ${ }_{d}=095$, and the Reynolds number $\operatorname{Re}=183$. The interaction of the bubble with the free surface is simulated using the EFEM model.

The main results are shown in Fig.3(a). The numerical results show the changes in the uid pressure (contour map) and the bubble boundary (black line). At $t=26 \mathrm{~ms}$, the bubble reaches a radius of 26.67 mm , lower than the expected maximum bubble radius $(27.5 \mathrm{~mm})$, mainly caused by the liquid viscosity and the free surface e ect. The kinetic viscosity of glycerol is strongly dependent on temperature. Spark discharge causes a change in the temperature of the surrounding uid temperature, which leads to a change in viscosity. Unevenly distributed uid viscosity can lead to errors between numerical and experimental results. There is a slight di erence in the period caused by complicated reasons. First, the initial bubble formed by the spark discharge has complex dynamical properties rather than a spherical static bubble. Second, the gas inside the bubble in the experiment is water vapor rather than ideal gas. In addition, Unevenly distributed uid viscosity a ects the motion of the bubble. The bubble and the free surface shapes calculated by the numerical model agree with the experiments, indicating that this paper's model can simulate the motion of bubbles and the free surface in viscous uids. The bottom of Fig.3(b) also compares the interfaces with viscous $(\mathrm{Re}=183)$ and non-viscous $(\mathrm{Re}=\infty)$ uids in the same case. It can be found that the liquid viscosity hinders the rise of the free surface, while the e ect on the bubble position is not signi cant. This is because the free surface jet velocity is much smaller than the bubble expansion velocity. The free surface has a smaller local Reynolds number, resulting in a more signi cant viscous e ect. The e ect of viscosity on the bubble is mainly re ected in the bubble jet and shape. For example, in the fourth sub gure of 3(b), bubbles split in a viscosity-free liquid but not in a viscous liquid. It indicates that viscosity has an essential e ect on the bubble-free surface coupled system.

The above results show that the EFEM bubble dynamics model can accurately simulate the interaction of bubbles and the free surface in compressible viscous uids. This paper will use this model to study the e ects of di erent parameters.


FIG. 3. (a)Comparison of EFEM bubble dynamics model(right) with experimental results(left) in glycerol at $\mathrm{t}=0.70 \mathrm{~ms}, 2.60 \mathrm{~ms}, 4.45 \mathrm{~ms}, 5.60 \mathrm{~ms}$; (b)comparison of results for viscous(right) and non-viscous(left) uids. The black line represents the bubble boundary and the free surface boundary; the colored contour represents the pressure eld.

## III. RESULTS AND DISCUSSION

Scholars have extensively studied the dynamics of bubbles in non-viscous uids. However, when small bubbles are moving in highly viscous liquids, the e ect of viscosity on bubble dynamics must be considered, which has potential value in the elds of the food-chemical industry. This section will discuss the bubble and the free surface coupling system's dynamic characteristics at di erent viscosity and depths. In the real problem, the in uence of surface tension is feeble. In order to study only the e ect of viscosity, the bubble surface tension e ect is not considered. Therefore, our numerical results are not applicable to the bubble dynamics problem with dominant surface tension. In addition, due to the small Reynolds number, it is not necessary to solve the turbulence problem in this paper. The maximum bubble radius $R_{m}=002 \mathrm{~m}$ and the dimensionless initial radius $R_{0}^{*}=0149$ were selected in the simulation. In order to avoid boundary e ects, the computational domain is set to $6 R_{m} \times 12 R_{m}$, and other dimensionless parameters are $=100$ and $=0044$, respectively. The initial dimensionless depth and Reynolds number are determined according to the cases, and the grid size $l=001 R_{m}$ was used for simulation.

## A. Bubble dynamics with di erent viscosity

In order to analyze the viscous e ects, the computations are carried out for $\operatorname{Re}=50100200400 \infty$ and ${ }_{d}=05$, where $\operatorname{Re}=\infty$ for the inviscid case. Fig. 4 shows the evolution results of bubbles and the free surface for $\operatorname{Re}=\infty, 100$, and 50 , respectively.

In Fig.4, (a)-(d) display the bubbles at the maximum volume, the jet penetration, the minimum volume, and the second maximum volume. Under initial high pressure, the bubble expands rapidly and radiates pressure waves outwards. Due to the in uence of the free surface and gravity, the bubble expands faster upwards. Meanwhile, the free surface bulges under the push of the bubble. Fig.4(a) shows that when the


FIG. 4. $\operatorname{Re}=\infty 10050, \quad{ }_{d}=05$ bubbles at four important moments of pressure and velocity contours: (a) maximum volume; (b) jet penetration; (c) minimum volume; (d) maximum volume in the second cycle of pulsation. The black line represents the bubble boundary and the free surface boundary; the left colored contour represents the pressure eld; the arrow and the right colored contour represent the velocity eld.
bubble reaches maximum radius, the liquid on both sides ows towards the top of the bubble and generates a high-pressure region. Due to the high-pressure region, the top interface collapses faster, forming a jet that points to the inside and pushes the free surface to move upward rapidly to form a sharp jet. Under the continuous action of pressure, the top jet of the bottom forms a toroidal bubble, and this impact will create a high-pressure region, as shown in Fig.4(b). Due to the small stand-o parameter $d_{d}$, the bubble has not yet collapsed the minimum volume when the jet penetrates. The bubble continues to shrink and generate an annular sideways jet on the side wall pointing to the inside of the bubble, which will cause the bubble to split into two smaller toroidal bubbles, as shown in Fig.4(c), called toroidal bubble splitting. After the bubble collapses to the minimum volume, the internal pressure is greater than the external pressure, and the bubble begins to expand again into the second pulsation period. With the pulsation of the toroidal bubble, a low water skirt is formed around the free surface, also called a crown spike because of its shape resembling a crown. The formation mechanism of the crown is complex. Youssef ${ }^{40}$ pointed out that crown is caused by the combined action of ow focusing induced by pressure distortion over the curved interface and ow reversal in the secondary expansion process of bubbles.

Comparing the bubble and the free surface motion under di erent Reynolds numbers in Fig.4, the dimensionless times for the bubble to expand to the maximum radius for the three Reynolds numbers in Fig.4(a)


FIG. 5. Time histories of (a)equivalent radius of bubbles ; (b)average pressure in bubbles with di erent Reynolds numbers $\operatorname{Re}=50100200400 \infty$, and stand-o parameter ${ }_{d}=05$, respectively.
are $t^{*}=0581,0.576$ and 0.566 , respectively. Due to the hindrance of the liquid viscosity, the bubble expands to the maximum radius earlier when $\operatorname{Re}=50$. Moreover, the high-pressure region also decreases with the Reynolds number decrease. The main reason is that the liquid velocity owing to this area on both sides of the bubble is slowed down by viscous retardation, and the impact e ect is weakened. When the viscous e ect is considered, the bubble jet velocity also changes. From the contours in Fig.4(b), the jet velocity decreases with the Reynolds number, and the shape of the top of the bubble jet is also smoother at small Re. The viscous friction is related to the velocity gradient, so the kinetic energy dissipates rapidly near the jet region, which leads to a smaller jet velocity when $\operatorname{Re}=50$. Because of the reduced velocity, the impact pressure formed by the small Re bubble jet penetrating the bottom is smaller than that in the inviscid case. The di erent bubble shapes di ered signi cantly when the bubbles shrunk to the minimum volume. The bubble with $\mathrm{Re}=50$ has almost no toroidal bubble splitting (generated by an annular sideways jet ), as shown in Fig.4(c). It is noteworthy that the ambient pressure increases with the decrease of the Reynolds number, mainly because viscous dissipation transforms liquid kinetic energy into internal energy, thus causing pressure increase. Due to the obstruction of viscosity, the crown height caused by the second cycle pulsation of low Reynolds number bubbles is much smaller than that under the inviscid condition in Fig.4(d).

Signi cant di erences exist in the maximum bubble radius and period at di erent Reynolds numbers. Fig.5(a) shows the evolution of the equivalent radius of the bubble pulsation with time. During the rst quarter of the pulsation period, the initial internal pressure pushes the bubble to expand rapidly, resulting in a large instantaneous Reynolds number. At this time, due to the small velocity gradient, the e ect of liquid viscosity is relatively weak, and there is no noticeable di erence in the radius of bubbles with di erent Reynolds numbers. It is necessary to know that the viscous force always exists and continuously accumulates in the bubble. As the Reynolds number decreased, the maximum equivalent radius of the bubble decreased, and the period also decreased. After the internal bubble pressure is less than the liquid eld, it continues to expand under inertia, and the viscosity dissipates the kinetic energy so that the bubble cannot reach a larger radius. Unlike BEM or BIM, the pressure inside the bubble calculated by the EFEM model is not evenly distributed. Since the unevenly distributed pressure is di cult to be compared, the average pressure variation of the gas inside the bubble is calculated in this paper, as shown in Fig.5(b). It can be found that during most of the rst period of bubble pulsation, the average pressure in the bubble has little di erence under di erent Reynolds number conditions. Comparing the second pressure peak (corresponding to the minimum volume of the bubble), as shown in Fig.5(b), the average pressure peak shows a non-monotonic variation as the Reynolds number decreases, with the peak rst increasing and then decreasing. However, it is worth noting that this law does not apply to the stand-o parameter $d=10,1.2$, and 1.5 . In these cases, the average pressure peak in the bubble decreases with the decrease of the Reynolds number. At the same time, the minimum equivalent radius increases with decreasing Reynolds number, which is caused by the strength of the bubble-free surface interaction.

Fig.6(a) shows the position of the top and bottom of the bubble at the axis. As the air above the free surface always maintains atmospheric pressure, the liquid pressure around the top of the bubble is less than the bottom when the bubble expands, leading to a faster expansion speed of the top, showing the phenomenon that the free surface attracts the bubble. When the bubble begins to collapse, the top is more


FIG. 6. Time histories of (a)the bubble top and bottom locations at $\operatorname{Re}=\infty$ and $50, \gamma_{d}=0.5$; (b)the free surface spike height at different Reynolds numbers $\operatorname{Re}=50,100,200,400, \infty$, and stand-off parameter $\gamma_{d}=0.5$, respectively.
violently affected by the high-pressure region than the bottom. The jet penetrates the bottom at about $t^{*}=1.03$, while the $\operatorname{Re}=50$ bubble jet penetrates at $t^{*}=1.13$. The slope of the displacement curve of the bubble in Fig.6(a) represents the jet velocity. It can be found that compared with the bubble with $\mathrm{Re}=50$, the jet velocity is faster without considering the viscosity of the liquid. Meanwhile, compared with Fig.4(a), it can also be found that the inviscid bubble has formed an obvious top jet before the bubble reaches the maximum volume.

The evolution of the free surface spike height at different Reynolds numbers is shown in Fig.6(b). In the initial expansion stage of the bubble, due to the relatively long distance from the free surface, the change of the free surface with different Re is the same. It can be seen from Fig.6(b) that as the Re decreases, the viscous effect increases and the height of the jet on the free surface decreases significantly. The main reason is the significant rate of change of tangential velocity caused by the jet, which leads to sizeable viscous friction and faster kinetic energy loss.

The pulsation of the bubble in the second period induces a low axisymmetric water skirt on both sides of the free-surface jet, called the crown spike. Fig.7(a) shows the shape of the crown spike under different Reynolds numbers Re when the bubble expands to the maximum volume in the second pulsation cycle. With the increase in Reynolds number, the phenomenon of a crown spike is more pronounced, and the height of the water skirt is also increasing. And the shape is sharp, and when $\operatorname{Re}=50$, the free surface almost does not form a crown spike, indicating that the viscous effect weakens the evolution of the free surface. Fig.7(b) shows the shapes of bubbles when jet impact with different Reynolds numbers. The $\operatorname{Re}=\infty$ bubble jet has a bulge at the front, called a mushroom-shaped jet. Koukouvinis ${ }^{26}$ believes that the jet is caused by interface instability. However, compared with an inviscid bubble, the jet tip at the top of low Reynolds number bubbles is smoother, and no mushroom-shaped jet appears. Meanwhile, the volume of jet penetration is smaller, and the velocity and impact strength of the bubble jet is weak, which is reflected in the contours in Fig. 4.

The bubble expands under the initial pressure. With the increase of liquid viscosity, the Reynolds number under the corresponding conditions decreases. The retardation effect of viscosity on bubble expansion becomes stronger and stronger, and the maximum radius of the bubble decreases. At the same time, the bubble jet is also affected by the viscosity, the jet velocity decreases with the decrease of the Reynolds number, and the jet penetration time is also delayed. The viscosity of the liquid affects the height and speed of the free surface jet: the viscosity increases, and the speed and height of the free surface jet decrease. When the viscosity increases to a specific value, the free surface crown spike generated by the pulsation of the bubbles will disappear, and the viscosity of the liquid weakens the interaction between bubbles and the free surface.

## B. Bubble dynamics with different stand-off parameters

In the same fluid, the viscosity coefficient of the liquid has been determined. When a pulsating bubble is generated using the same method, the effect of viscosity on bubbles is unchanged. The parameter that affects the dynamics of bubbles and the free surface is the distance $d$ (as shown in Fig.1(a)), and the dimensionless parameter $\gamma_{d}$ represents the inception depth of the bubble. Set $\operatorname{Re}=100$, other parameters remain the same


FIG. 7. (a)Shape of the free surface; (b)shape of the bubble when the jet impact at different Reynolds numbers $\operatorname{Re}=50,100,200,400, \infty$, and stand-off parameter $\gamma_{d}=0.5$, respectively.
as the previous section, and study the bubble dynamics for $\gamma_{d}=0.5-1.5$.
Fig. 8 shows the bubble and the free surface shapes, pressure, and velocity contours at three important moments when Reynolds number $\mathrm{Re}=100$ and stand-off parameter $\gamma_{d}=0.5,1.0$ and 1.5, respectively, corresponding to maximum bubble volume, jet impact, and maximum volume in the second pulsation period. The dimensionless time for the bubble to expand to the maximum radius increases with $\gamma_{d}, t^{*}=0.576,0.775$, and 0.848 , respectively, as shown in Fig.8. The bubble jet is already formed in the expansion phase when $\gamma_{d}=0.5$, while the bubble jet in the other two cases starts to form in the contraction phase. Compared with the other two conditions, the bubble and the free surface jet with smaller $\gamma_{d}$ are also more slender, and the jet impacts the bottom earlier. As shown in Fig.8(b), bubble jets of $\gamma_{d}=1.0$ and 1.5 impacting the bottom occurred near the moment when bubbles collapsed to the minimum volume. In contrast, bubble jets of $\gamma_{d}=0.5$ had penetrated through the bottom in the contraction stage, and a local high-pressure region was formed when the jets penetrated through the bottom. It can be seen from Fig.8(c) that the bubble with $\gamma_{d}=0.5$ produces toroidal splitting because the jet penetrates the bottom earlier, and the annular sideways jet is generated when the bubble continues to collapse. This phenomenon is noticeable at large Reynolds numbers, and bubbles with larger stand-off parameters do not produce apparent toroidal splitting. At the same time, with the increase of $\gamma_{d}$, the free surface jet becomes lower, and the phenomenon of a crown spike is less obvious. The free surface does not produce a crown spike during the secondary pulsation of the bubble when $\gamma_{d}=1.5$.

Fig.9(a) shows the shape of the bubble when the jet penetrates with different $\gamma_{d}$. It can be seen from the figure that with the increase of the stand-off parameter, the volume of bubble jet impact continues to decrease. When the inception depth is small, the impact time of the jet is earlier than the minimum volume time due to the strong interaction between bubbles and the free surface. The bubble jet width also varies with the change of the stand-off parameter. When the $\gamma_{d}$ increases from 0.5 to 1.0 , the jet width gradually increases; when the $\gamma_{d}$ increases from 1.0 to 1.5 , the jet width gradually decreases, which is the same as the law obtained by $\mathrm{Li}^{38}$ without considering viscosity. With the increase of $\gamma_{d}$, the water jet penetration time is delayed, and the jet width changes due to the different degrees of bubble shrinkage. It can also be seen from the velocity contours in Fig.8(b) that the velocity of the jet impinges increases with the increase of $\gamma_{d}$. Fig.9(b) shows the shape of the free surface when the jet penetrates. With the increase of $\gamma_{d}$, the jet height of the free surface decreases while the jet width gradually increases. From the pressure contours in Fig.8(a), there is no high-pressure region below the free surface when the $\gamma_{d}$ is large, which is not conducive to the generation of the bubble and the free surface jet. At the same time, the effect of bubble expansion on the free surface is weakened, which is also the reason for the low height of the free surface jet. Fig. 10 shows the position changes of the top and bottom of the bubble on the axis, which can vividly see the changing trend of the axial length of the bubble. Meanwhile, the velocity of the bubble surface at the axis can also be obtained by calculating the slope of the curve. In the early stage of bubble expansion, the displacement of the upper surface of bubbles with different $\gamma_{d}$ is different, and the displacement decreases with the increase


FIG. 8. Pressure and velocity contours at $d=051015, \mathrm{Re}=100$, respectively:(a) maximum volume; (b) jet impact; (c) bubble second pulsation period. The black line represents the bubble boundary and the free surface boundary; the left colored contour represents the pressure eld; the arrow and the right colored contour represent the velocity eld.
of ${ }_{d}$. However, the displacement of the lower surface changes almost the same, mainly because the existence of the free surface a ects the movement of the upper surface of bubbles. It is worth noting that at ${ }_{d}=15$, the axial bubble jet does not directly penetrate the bottom, but generates a temporary air cushion, as shown in Fig.8(b) and Fig.9(a). This is mainly because the bubble jet at impact is very wide and penetrates the side walls of the bubble rst rather than the bottom. According to the slope analysis of the bubble top displacement curve in Fig.11(a), it can be found that the jet with ${ }_{d}=15$ is always in the acceleration stage from bubble contraction to jet slamming, which is the reason why the bubble jet velocity increases with the increase of $d$.

The time histories curve of the equivalent radius of bubble pulsation with di erent stand-o parameters $d$ is shown in Fig.11(a). With the increase of the bubble-free surface distance, the maximum equivalent radius of the bubble increases, the minimum equivalent radius decreases, and the in uence of the free surface on the bubble gradually decrease. The diamonds in Fig.11(a) represent the moment when the liquid jet penetrates the bubbles' bottom. With the decrease of the stand-o parameter, the time of jet penetration is advanced.


FIG. 9. The shape of (a)the bubble; and (b)the free surface before the jet penetrates the bottom at ${ }_{d}=0.5,0.7,1.0$, 1.2 , and $1.5, \operatorname{Re}=100$, respectively.


FIG. 10. Time histories of the bubble top and bottom locations at ${ }_{d}=0.5,1.0,1.5, \operatorname{Re}=100$, respectively.

For ${ }_{d}=05$ and 0.7 , the jet penetration occurs in the contraction phase of the rst pulsation period. In contrast, the bubble jet penetration for $\quad d=10,1.2$, and 1.5 occurs in the second pulsation period. Because the buoyancy parameter is relatively small, the primary in uence on the jet is the free surface. In the case with a greater stand-o distance, the repulsion e ect of the free surface on the bubble is weaker and a delay in the jet penetration is observed.

Fig.11(b) shows the variation of the average pressure in the bubble with time. The average internal pressure of bubbles with di erent ${ }_{d}$ has little di erence in the rst pulsation cycle and is at a small value most of the time. The second pressure peak (corresponding to the bubble's collapse to the minimum volume) increases with increasing $d$, and the bubble radius at the time corresponding to the peak pressure decreases with increasing $d$. Because bubbles with small distance parameters do more work on the free surface and lose more energy, thus resulting in smaller internal pressure of bubbles.

## IV. CONCLUSIONS

A dynamic model is established for the interaction between a bubble and a free surface in a viscous compressible liquid based on the EFEM coupled with the VOF. The numerical model is veri ed by the spherical bubble theories and experiments with excellent agreement. Bubble dynamics and free surface evolution are analyzed in terms of the Reynolds number Re and the dimensionless inception depth ${ }_{d}$ of the bubble. The following features are observed:


FIG. 11. Time histories of (a)the equivalent radius of bubbles; (b)average pressure in bubbles with different stand-off parameters $\gamma_{d}=0.5,0.7,1.0,1.2$, and $1.5, \operatorname{Re}=100$, respectively.
(I) As Re decreases, the bubble oscillation amplitude and period decrease, the bubble jet shape is blunter, the jet velocity decrease, and the impact time is delayed. At a lower Re, the jet is weaker and associated with less kinetic energy, resulting in earlier collapse. The viscosity also reduces the toroidal bubble splitting during pulsation, and the bubble with $\mathrm{Re}=50$ hardly splits.
(II) As $\gamma_{d}$ increases, the maximum bubble volume increases, the minimum bubble volume decreases, and the second pressure peak of the bubble increases. When $\gamma_{d}$ decreases from 1.5 to 0.5 , the jet becomes sharper and impacts the opposite bubble surface earlier.
(III) As Re decreases, the height and velocity of the free surface jet decrease, and the crown spike height decreases or even disappears in the second pulsation cycle. As $\gamma_{d}$ increases, the free surface jet's height decreases, the jet's width increases, and the crown spike tends to be low or even disappear.

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[^1]${ }^{12}$ D. S. Fatyukhin, R. I. Nigmetzyanov, V. M. Prikhodko, A. V. Sukhov, and S. K. Sundukov, "A comparison of the effects of ultrasonic cavitation on the surfaces of 45 and 40kh steels," Metals 12, 138 (2022).
${ }^{13}$ J. Mustonen, O. Tommiska, A. Holmström, T. Rauhala, P. Moilanen, M. Gritsevich, A. Salmi, and E. Hæggström, "Fem-based time-reversal enhanced ultrasonic cleaning," Ultrasonics Sonochemistry 79, 105798 (2021).
${ }^{14}$ E. Stride and N. S. Ari, "Microbubble ultrasound contrast agents: a review," Proceedings of the Institution of Mechanical Engineers Part H Journal of Engineering in Medicine 217, 429-47 (2003).
${ }^{15}$ P. G. Durham and P. A. Dayton, "Applications of sub-micron low-boiling point phase change contrast agents for ultrasound imaging and therapy," Current Opinion in Colloid \& Interface Science 56, 101498 (2021).
${ }^{16}$ H. Yusefi and B. Helfield, "Ultrasound contrast imaging: Fundamentals and emerging technology," Frontiers in Physics, 100 (2022).
${ }^{17}$ C. B. Arnold, P. Serra, and A. Piqué, "Laser direct-write techniques for printing of complex materials," Mrs Bulletin 32, 23-31 (2007).
${ }^{18}$ M. Jalaal, M. K. Schaarsberg, C.-W. Visser, and D. Lohse, "Laser-induced forward transfer of viscoplastic fluids," Journal of fluid mechanics 880, 497-513 (2019).
${ }^{19}$ P. Serra and A. Piqué, "Laser-induced forward transfer: fundamentals and applications," Advanced Materials Technologies 4, 1800099 (2019).
${ }^{20}$ R. N. Cui, S. Li, S. P. Wang, and A. M. Zhang, "Pulsating bubbles dynamics near a concave surface," Ocean Engineering 250, 110989 (2022).
${ }^{21}$ Q. X. Wang, "Non-spherical bubble dynamics of underwater explosions in a compressible fluid," Physics of Fluids 25, 072104 (2013).
${ }^{22}$ A. M. Zhang, P. Cui, J. Cui, and Q. X. Wang, "Experimental study on bubble dynamics subject to buoyancy," Journal of Fluid Mechanics 776, 137-160 (2015).
${ }^{23}$ P. Koukouvinis, M. Gavaises, O. Supponen, and M. Farhat, "Numerical simulation of a collapsing bubble subject to gravity," Physics of Fluids 28, 032110 (2016).
${ }^{24}$ S. Li, B. C. Khoo, A. M. Zhang, and S. P. Wang, "Bubble-sphere interaction beneath a free surface," Ocean Engineering 169, 469-483 (2018).
${ }^{25}$ L. T. Liu, X. L. Yao, A. M. Zhang, and Y. Y. Chen, "Numerical analysis of the jet stage of bubble near a solid wall using a front tracking method," Physics of Fluids 29, 012105 (2017).
${ }^{26}$ P. Koukouvinis, M. Gavaises, O. Supponen, and M. Farhat, "Simulation of bubble expansion and collapse in the vicinity of a free surface," Physics of Fluids 28, 052103 (2016).
${ }^{27}$ A.-M. Zhang, S.-M. Li, P. Cui, S. Li, and Y.-L. Liu, "Interactions between a central bubble and a surrounding bubble cluster," Theoretical and Applied Mechanics Letters, 100438 (2023).
${ }^{28}$ L. Liu, J. Wang, and K. Tang, "Coupling characteristics of bubbles with a free surface initially disturbed by water waves," Physics of Fluids 34, 042117 (2022).
${ }^{29}$ A. Pearson, E. Cox, J. R. Blake, and S. R. Otto, "Bubble interactions near a free surface," Engineering Analysis with Boundary Elements 28, 295-313 (2004).
${ }^{30}$ Q. Wang, K. Yeo, B. Khoo, and K. Lam, "Nonlinear interaction between gas bubble and free surface," Computers \& fluids 25, 607-628 (1996).
${ }^{31}$ Y. L. Liu, Q. X. Wang, S. P. Wang, and A. M. Zhang, "The motion of a 3d toroidal bubble and its interaction with a free surface near an inclined boundary," Physics of Fluids 28, 122101 (2016).
${ }^{32}$ Y. Sun, Z. Yao, H. Wen, Q. Zhong, and F. Wang, "Cavitation bubble collapse in a vicinity of a rigid wall with a gas entrapping hole," Physics of Fluids 34, 073314 (2022).
${ }^{33}$ J. R. Blake and D. C. Gibson, "Growth and collapse of a vapour cavity near a free surface," J. Fluid Mech. 111, 123-123 (1981).
${ }^{34}$ J. R. Blake and D. C. Gibson, "Cavitation bubbles near boundaries," Annual Review of Fluid Mechanics 19, 99-123 (1987).
${ }^{35} \mathrm{~J} . \mathrm{Li}$ and J. L. Rong, "Bubble and free surface dynamics in shallow underwater explosion," Ocean Engineering 38, 1861-1868 (2011).
${ }^{36}$ O. Supponen, D. Obreschkow, M. Tinguely, P. Kobel, N. Dorsaz, and M. Farhat, "Scaling laws for jets of single cavitation bubbles," Journal of Fluid Mechanics 802, 263-293 (2016).
${ }^{37}$ S. Zhang, S. Wang, and A. Zhang, "Experimental study on the interaction between bubble and free surface using a high-voltage spark generator," Physics of Fluids 28, 032109 (2016).
${ }^{38}$ T. Li, A. M. Zhang, S. P. Wang, S. Li, and W. T. Liu, "Bubble interactions and bursting behaviors near a free surface," Physics of Fluids 31, 042104 (2019).
${ }^{39}$ L. T. Liu, X. B. Chen, W. Q. Zhang, and A. M. Zhang, "Study on the transient characteristics of pulsation bubble near a free surface based on finite volume method and front tracking method," Physics of Fluids 32, 052107 (2020).
${ }^{40}$ Y. Saade, M. Jalaal, A. Prosperetti, and D. Lohse, "Crown formation from a cavitating bubble close to a free surface," Journal of Fluid Mechanics 926 (2021).
${ }^{41}$ N. Bempedelis, J. Zhou, M. Andersson, and Y. Ventikos, "Numerical and experimental investigation into the dynamics of a bubble-free-surface system," Physical Review Fluids 6, 013606 (2021).
${ }^{42}$ R. T. Cerbus, H. Chraibi, M. Tondusson, S. Petit, D. Soto, R. Devillard, J. P. Delville, and H. Kellay, "Experimental and numerical study of laser-induced secondary jetting," Journal of Fluid Mechanics 934 (2022).
${ }^{43}$ T.-H. Phan, V.-T. Nguyen, and W.-G. Park, "Numerical study on strong nonlinear interactions between spark-generated underwater explosion bubbles and a free surface," International Journal of Heat and Mass Transfer 163, 120506 (2020).
${ }^{44}$ S. -M. Li, A.-M. Zhang, and N.-N. Liu, "Effect of a rigid structure on the dynamics of a bubble beneath the free surface," Theoretical and Applied Mechanics Letters 11, 100311 (2021).
${ }^{45}$ D. Singh and A. K. Das, "Dynamics of inner gas during the bursting of a bubble at the free surface," Physics of Fluids $\mathbf{3 3}$ (2021), 052105.
${ }^{46}$ Q. Wang, W. Liu, C. Corbett, and W. R. Smith, "Microbubble dynamics in a viscous compressible liquid subject to ultrasound," Physics of fluids, 34 (2022).
${ }^{47}$ Z. Wang, R. Duan, L. Liu, and H. Yang, "Jetting behavior as a bubble bursts in free space," Physics of Fluids 33, 023304 (2021).
${ }^{48}$ M. S. Plesset, "The dynamics of cavitation bubbles," J.appl.mech 16, 277-282 (1949).
${ }^{49}$ J. B. Keller and M. Miksis, "Bubble oscillations of large amplitude," The Journal of the Acoustical Society of America 68, 628-633 (1980).
${ }^{50}$ A.-M. Zhang, S.-M. Li, P. Cui, S. Li, and Y.-L. Liu, "A unified theory for bubble dynamics," Physics of Fluids 35, 033323 (2023).
${ }^{51}$ Q. Wang, K. Yeo, B. Khoo, and K. Lam, "Strong interaction between a buoyancy bubble and a free surface," Theoretical and Computational Fluid Dynamics 8, 73-88 (1996).
${ }^{52}$ A. M. Zhang and B. Y. Ni, "Three-dimensional boundary integral simulations of motion and deformation of bubbles with viscous effects," Computers \& Fluids 92, 22-33 (2014).
${ }^{53}$ S. J. Lind and T. N. Phillips, "The effect of viscoelasticity on the dynamics of gas bubbles near free surfaces," Physics of Fluids 25, 022104 (2013).
${ }^{54} \mathrm{~S}$. Li and B. Y. Ni, "Simulation on the interaction between multiple bubbles and free surface with viscous effects," Engineering Analysis with Boundary Elements 68, 63-74 (2016).
${ }^{55}$ B. Y. Ni, A. M. Zhang, and G. X. Wu, "Simulation of a fully submerged bubble bursting through a free surface," European Journal of Mechanics - B/Fluids 55, 1-14 (2016).
${ }^{56}$ M. J. Miksis, J. M. Vanden Broeck, and J. B. Keller, "Rising bubbles," Journal of Fluid Mechanics 123, 31-41 (1982).
${ }^{57}$ T. Lundgren and N. Mansour, "Oscillations of drops in zero gravity with weak viscous effects," Journal of Fluid Mechanics 194, 479-510 (1988).
${ }^{58}$ J. M. Boulton-Stone, "The effect of surfactant on bursting gas bubbles," Journal of Fluid Mechanics 302, 231-257 (1995).
${ }^{59}$ S. Popinet and S. Zaleski, "Bubble collapse near a solid boundary: a numerical study of the influence of viscosity," Journal of Fluid Mechanics 464, 137-163 (2002).
${ }^{60}$ Y. L. Liu, A. M. Zhang, Z. L. Tian, and S. P. Wang, "Investigation of free-field underwater explosion with eulerian finite element method," Ocean Engineering 166, 182-190 (2018).
${ }^{61}$ L. T. Liu, X. L. Yao, N. N. Liu, and F. L. Yu, "Toroidal bubble dynamics near a solid wall at different reynolds number," International Journal of Multiphase Flow 100, 104-118 (2018).
${ }^{62}$ Y. L. Liu, A. M. Zhang, Z. L. Tian, and S. P. Wang, "Dynamical behavior of an oscillating bubble initially between two liquids," Physics of Fluids 31, 092111 (2019).
${ }^{63}$ S. Li, Y. Saade, D. van der Meer, and D. Lohse, "Comparison of boundary integral and volume-of-fluid methods for compressible bubble dynamics," International Journal of Multiphase Flow 145, 103834 (2021).
${ }^{64}$ O. Supponen, D. Obreschkow, P. Kobel, and M. Farhat, "Detailed jet dynamics in a collapsing bubble," Journal of Physics: Conference Series 656, 012038 (2015).
${ }^{65}$ M. Ivings, D. Causon, and E. Toro, "On riemann solvers for compressible liquids," International Journal for Numerical Methods in Fluids 28, 395-418 (1998).
${ }^{66}$ Z. L. Tian, Y. L. Liu, A. M. Zhang, and S. P. Wang, "Analysis of breaking and re-closure of a bubble near a free surface based on the eulerian finite element method," Computers \& Fluids 170, 41-52 (2018).
${ }^{67}$ H. Tang, Y. L. Liu, P. Cui, and A. M. Zhang, "Numerical study on the bubble dynamics in a broken confined domain," Journal of Hydrodynamics 32, 1029-1042 (2020).
${ }^{68}$ Z. L. Tian, Y. L. Liu, A. M. Zhang, L. B. Tao, and L. Chen, "Jet development and impact load of underwater explosion bubble on solid wall," Applied Ocean Research 95, 102013 (2020).
${ }^{69}$ H. Tang, Z.-L. Tian, X.-Y. Ju, J.-T. Feng, Y.-L. Liu, and A.-M. Zhang, "Experimental and numerical investigations on the explosions nearby a free surface from both sides," Ocean Engineering 278, 114372 (2023).
${ }^{70}$ D. J. Benson and S. Okazawa, "Contact in a multi-material eulerian finite element formulation," Computer Methods in Applied Mechanics and Engineering 193, 4277-4298 (2004).
${ }^{71}$ W. T. Liu, A. M. Zhang, X. H. Miao, F. R. Ming, and Y. L. Liu, "Investigation of hydrodynamics of water impact and tail slamming of high-speed water entry with a novel immersed boundary method," Journal of Fluid Mechanics 958, A42 (2023)
${ }^{72}$ Z. L. Tian, Y. L. Liu, A. M. Zhang, and L. B. Tao, "Energy dissipation of pulsating bubbles in compressible fluids using the eulerian finite-element method," Ocean Engineering 196, 106714 (2020).
${ }^{73}$ D. J. Benson, "Computational methods in lagrangian and eulerian hydrocodes," Computer Methods in Applied Mechanics and Engineering 99, 235-394 (1992).
${ }^{74}$ R. Han, A. M. Zhang, S. Tan, and S. Li, "Interaction of cavitation bubbles with the interface of two immiscible fluids on multiple time scales," Journal of Fluid Mechanics 932 (2022).


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[^1]:    ${ }^{1}$ A. M. Zhang, W. B. Wu, Y. L. Liu, and Q. X. Wang, "Nonlinear interaction between underwater explosion bubble and structure based on fully coupled model," Physics of Fluids 29, 082111 (2017).
    ${ }^{2}$ E. Klaseboer, K. C. Hung, C. Wang, C. W. Wang, B. C. Khoo, P. Boyce, S. Debono, and H. Charlier, "Experimental and numerical investigation of the dynamics of an underwater explosion bubble near a resilient/rigid structure," Journal of Fluid Mechanics 537, 387-413 (2005).
    ${ }^{3}$ X. L. Yao, A. M. Zhang, and Y. C. Liu, "Interaction of two three-dimensional explosion bubbles," Journal of Marine Science and Application 6, 7 (2007).
    ${ }^{4}$ A. M. Zhang and Y. L. Liu, "Improved three-dimensional bubble dynamics model based on boundary element method," Journal of Computational Physics 294, 208-223 (2015).
    ${ }^{5}$ A. Tatlısuluoğlu and S. Beji, "Blast pressure measurements of an underwater detonation in the sea," Journal of Marine Science and Application, 1-8 (2021).
    ${ }^{6}$ N. Zhang, Z. Zong, and W. P. Zhang, "Dynamic response of a surface ship structure subjected to an underwater explosion bubble," Marine Structures 35, 26-44 (2014).
    ${ }^{7}$ G. Li, D. Shi, L. Wang, and K. Zhao, "Measurement technology of underwater explosion load: A review," Ocean Engineering 254, 111383 (2022).
    ${ }^{8}$ S. S. Emamzadeh, "Nonlinear dynamic response of a fixed offshore platform subjected to underwater explosion at different distances," Journal of Marine Science and Application 21, 168-176 (2022).
    ${ }^{9}$ Q. X. Wang and K. Manmi, "Three dimensional microbubble dynamics near a wall subject to high intensity ultrasound," Physics of Fluids 26, 032104 (2014).
    ${ }^{10}$ H. Lais, P. S. Lowe, T. H. Gan, and L. C. Wrobel, "Numerical modelling of acoustic pressure fields to optimize the ultrasonic cleaning technique for cylinders," Ultrasonics Sonochemistry 45, 7-16 (2018).
    ${ }^{11}$ W. D. Song, M. H. Hong, B. Lukyanchuk, and T. C. Chong, "Laser-induced cavitation bubbles for cleaning of solid surfaces," Journal of Applied Physics 95, 2952-2956 (2004).

