DE GRUYTER Adv. Geom. 2023; 23(1):151–152

Madeline Brandt and Paul Alexander Helminck*

Erratum for "Tropical superelliptic curves"

DOI 10.1515/advgeom-2022-0029. Received 24 September, 2022

Abstract: We correct two errors in our paper *Tropical superelliptic curves* published in Advances in Geometry 20 (2020), 527–551. These corrections do not change the main results of the paper.

1 First Erratum

In [1] on page 532, in the proof of Proposition 3.12(2), the product of two algebras is assumed to give the base change of the two algebras. The authors cannot directly say whether this is true here. We note however that Proposition 3.12(2) is a consequence of the main theorems on strict Henselizations (or étale local rings) as described in [2, Section 0BSK]. Indeed, the strict Henselization A^{sh} is a filtered colimit of étale algebras and the given algebra B^H is étale at $\mathfrak{q} \cap B^H$, so we obtain an injection of the fraction field of B^H into the fraction field of the strict Henselization K^{sh}. The statement then follows from infinite Galois theory and the fact that the Galois group of $K^{\mathrm{sh}} \subset K^{\mathrm{sep}}$ is the absolute inertia group. More details can be found in Section 3.

2 Second Erratum

On page 538, in the statement of Lemma 4.2, the formula for the order of the decomposition group in this generality is wrong. It is correct however if the inertia group is trivial, which is exactly the case we need in the paper. The proof in that case is moreover as published.

3 Details for the First Erratum

We give some extra details here for the proof of Proposition 3.12(2). To prove the inclusion $H \supseteq I_{\mathfrak{q}}$, we use the material in [2, Section 0BSK] on strict Henselizations. We first note that we can localize A and assume that it is local with maximal ideal $\mathfrak{p} = \mathfrak{q} \cap A$. The strict Henselization of the triple $(A, \mathfrak{p}, k(\mathfrak{p}))$ is then given by $A^{\mathrm{sh}} := \lim_{(S,\mathfrak{p}_S,\alpha)} S$, where the limit is taken over the filtered category of all triples $(S, \mathfrak{p}_S, \alpha)$, where $A \to S$ is an étale ring map, $\mathfrak{p}_S$ is a prime ideal in S lying above $\mathfrak{p}$ and $\alpha : k(\mathfrak{p}_S) \to k(\mathfrak{p})^{\mathrm{sep}}$ is an embedding of residue fields, see [2, Lemma 04GN]. Now consider the integral closure A^{sep} of A in the separable closure K^{sep}. We choose a prime ideal $\mathfrak{q}^{\mathrm{sep}}$ in A^{sep} lying over $\mathfrak{q} \in B$. The localization $A^{\mathrm{sep}}_{\mathfrak{q}^{\mathrm{sep}}}$ is then a strictly Henselian ring, and this induces an inclusion $A^{\mathrm{sh}} \to A^{\mathrm{sep}}_{\mathfrak{q}^{\mathrm{sep}}}$, see the beginning of [2, Section 0BSD]. As in the proof of [2, Lemma 0BSW], we then have

$$A^{\mathrm{sh}} = (A^{\mathrm{sep}}_{\mathfrak{q}^{\mathrm{sep}}})^{I_{\mathfrak{q}^{\mathrm{sep}}}}, \tag{1}$$

where $I_{\mathfrak{q}^{\mathrm{sep}}}$ is the absolute inertia group.

We now relate this material to the subgroups $H \subset G$. In terms of infinite Galois theory, H corresponds to a closed (and open) subgroup $G_L \subset \overline{H} \subset G_K$ such that $\overline{H}/G_L = H$. Similarly, we have a natural surjective map

$$I_{\mathfrak{q}^{\mathrm{sep}}} \to I_{\mathfrak{q}} \tag{2}$$

obtained by restricting automorphisms to L, see [2, Lemma 0BSX]. The kernel of this map is $G_L \cap I_{\mathfrak{q}^{\mathrm{sep}}}$, inducing the isomorphism $I_{\mathfrak{q}} = I_{\mathfrak{q}^{\mathrm{sep}}}/(G_L \cap I_{\mathfrak{q}^{\mathrm{sep}}})$. By our assumption on H, we have that $B^H \supset A$ is étale at $\mathfrak{q}_H := \mathfrak{q} \cap B^H$.

Madeline Brandt, University of California, Berkeley, USA, email: brandtm@berkeley.edu
***Corresponding author: Paul Alexander Helminck,** Durham University, UK, email: paul.a.helminck@durham.ac.uk

This means that there exists a $g \in B^H \backslash \mathfrak{q}_H$ such that the localization $A \to (B^H)_g$ is étale. After choosing an embedding $k(\mathfrak{q}_H) \to k(\mathfrak{p}^{\mathrm{sep}})$, we then obtain maps

$$A \to (B^H)_g \to A^{\mathrm{sh}} = (A^{\mathrm{sep}}_{\mathfrak{q}^{\mathrm{sep}}})^{I_{\mathfrak{q}^{\mathrm{sep}}}}. \tag{3}$$

These are easily seen to be injective. Since the fraction field of $(B^H)_g$ is the invariant field of $\overline{H}$, we have $\overline{H} \supset I_{\mathfrak{q}^{\mathrm{sep}}}$ by infinite Galois theory. We then obtain an inclusion

$$I_{\mathfrak{q}} = I_{\mathfrak{q}^{\mathrm{sep}}}/(G_L \cap I_{\mathfrak{q}^{\mathrm{sep}}}) \subseteq \overline{H}/G_L = H, \tag{4}$$

as desired.

References

[1] M. Brandt, P. A. Helminck, Tropical superelliptic curves. *Adv. Geom.* **20** (2020), 527–551. MR4160287 Zbl 1460.14144
[2] The Stack Project Authors, Stacks Project. 2022. http://stacks.math.columbia.edu