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Matching with Externalities

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We incorporate externalities into the stable matching theory of two-sided markets. Extending the classical substitutes condition to markets with externalities, we establish that stable matchings exist when agent choices satisfy substitutability. We show that substitutability is a necessary condition for the existence of a stable matching in a maximal-domain sense and provide a characterization of substitutable choice functions. In addition, we extend the standard insights of matching theory, like the existence of side-optimal stable matchings and the deferred acceptance algorithm, to settings with externalities even though the standard fixed-point techniques do not apply.

Key words: Matching with externalities, Deferred acceptance algorithm, Substitutability, Labour markets with couples.

JEL Codes: C78, D47, D62

1. INTRODUCTION

Externalities are present in many two-sided markets. For instance, couples in a labour market pool their resources as do partners in legal or consulting partnerships. As a result, the preferences of an agent depend on contracts signed by partners. Likewise, a firm's hiring decisions are affected by how candidates compare to competitors' employees. Finally, because of technological requirements of interoperability, an agent's purchase decisions depend on other agents' decisions.[1]

In this article, we incorporate externalities into the stable matching theory of Gale and Shapley (1962).[2] We refer to the two sides of the market as buyers and sellers. Each buyer–seller pair can sign bilateral contracts. Furthermore, each agent is endowed with a choice function that selects a subset of contracts from any given set conditional on a reference set for the other agents. We build a theory of matching with externalities that both establish new insights and extend to the settings with externalities some of the key insights of the classical theory without externalities,

1. These markets are discussed in more detail in Section 3 and Supplementary Appendix D.

2. Even though we derive our results in a general many-to-many matching setting with contracts (cf. Hatfield and Milgrom, 2005; Klaus and Walzl, 2009; Hatfield and Kominers, 2017), the results are new in all special instances of our setting, including many-to-one and one-to-one matching problems.

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such as the existence of stable matchings and the role of the deferred acceptance (or cumulative offer) algorithm.[3]

Our theory is built on a substitutes condition that extends the classical substitutes condition to the setting with externalities. Our condition requires that each agent rejects more contracts from any set than its subsets conditional on the same reference set (as in the classical substitutes condition) and also that each agent rejects more contracts from a set X conditional on a reference set μ than set X conditional on a reference set μ' such that μ reflects better market conditions than μ' for her side of the market. The idea of better market condition extends the revealed preference idea of Blair (1988) to the setting with externalities. When there are no externalities, this substitutes condition reduces to the classical gross substitutes condition of Kelso and Crawford (1982). Our condition is satisfied by standard choice functions of households consisting of a primary and a secondary earner who pool resources; the pooling of resources implies that the choice function of a secondary earner depends on the income of the primary earner and hence exhibits externalities (see Section 3).

We first construct a version of the deferred acceptance algorithm that performs well despite the presence of externalities. This algorithm—which may be interpreted as a new ascending auction—may be useful in potential market-design applications. Because an agent's choice depends on others' contracts, our algorithm keeps track not only of which contracts are available but also of the reference sets that agents on each side use to condition their choice. The construction requires care because after the reference set changes an agent may want to go back to a contract that they already rejected. To ensure that this does not happen, we construct the initial reference sets in a preliminary phase of the algorithm. Relatedly, we cannot stop the algorithm as soon as the sets of available contracts converge: we need to continue until the reference sets converge as well. Our construction of initial reference sets ensures that subsequent reference sets change in a monotonic way with respect to the better market conditions pre-order, thus ensuring that from some point on the reference sets belong to the same equivalence class. While these equivalence classes might consist of many matchings, we further show that the algorithm converges to one of them and never cycles among the members of the same equivalence class.

Our main results show the existence of a stable matching when choice functions satisfy substitutability because the algorithm converges to one (Theorem 1), and that substitutability is necessary for the existence of a stable matching in a maximal-domain sense extending the insights of Hatfield and Milgrom (2005), Hatfield and Kojima (2010), and Hatfield and Kominers (2017) for the standard substitutability condition in settings without externalities (Theorem 2).

In addition to the main results, we show that every stable matching is Pareto efficient (Theorem 3) and an optimal stable matching exists for side θ under the additional assumption that there exists a matching that reflects better market conditions than any other matching that can be chosen for side θ (Theorem 4). This additional assumption is satisfied trivially in settings without externalities, where the existence of side-optimal stable matchings was established by Gale and Shapley (1962) for the marriage problem. Furthermore, we provide a characterization of substitutable choice functions (Theorem 5): a choice function satisfies the substitutes condition if, and only if, the choice from a set consists of the highest ranked contracts according to some ranking, where the set of allowed rankings is fixed for the choice function. This characterization is inspired by the decomposition result of Aizerman and Malishevski (1981) for

3. We focus on the classical short-sighted stability concept in which each agent assumes that other agents do not react to their choice. Our results, however, are applicable to many other stability concepts including far-sighted ones because we formulate the results in terms of agents' choice behaviour and not in terms of their preferences. See Remark 1 of the previous version of our article, which is available at http://dx.doi.org/10.2139/ssrn.2475468.

the setting without externalities.[4] In Supplementary Appendix C, we further establish comparative statics that show how the presence of externalities changes the set of stable matchings and we show that a general version of the rural hospitals theorem (McVitie and Wilson, 1970; Roth, 1984; Hatfield and Milgrom, 2005) holds true in matching with externalities.[5] In Supplementary Appendix D.3, we apply our results to the analysis of dynamic matching.

Many of our contributions—the substitutes condition, its characterization, results on efficiency, side-optimal stable matchings, and comparative statics—have no forerunners in the literature analysing externalities in matching. The prior matching literature studying externalities focused on the question of the existence of stable matchings and algorithms that find them.[6]

The literature analysing the existence and non-existence results largely builds on the seminal paper by Sasaki and Toda (1996), who showed that stable one-to-one matchings need not exist in the presence of externalities and proposed a weak stability concept that allows a pair of agents to block a matching only if they benefit from the block under all possible rematches of the remaining agents and showed that such weak stable matchings exist in one-to-one environments.[7] In contrast, our article uses the standard stability concept of Gale and Shapley (1962) and the literature on matching without externalities.[8]

Our contribution on the existence question is closest to the few papers that look at the standard stability in selected matching problems with externalities. Bando (2012, 2014) studies many-to-one matching allowing externalities in the choice behaviour of firms (agents who match with potentially many agents on the other side) but not of workers. Imposing several assumptions on firms' choice behaviour, he proves the existence of stable matchings and analyses the deferred acceptance algorithm; his assumptions ensure that the standard deferred acceptance algorithm that does not keep track of the reference sets finds a stable matching.[9] Our substitutes condition does not imply Bando's assumptions nor is implied by them; see Examples 1 and 2. An advantage of our approach is that it is equally valid in one-to-one, many-to-one, and many-to-many matching settings, while Bando's conditions do not guarantee the existence of stable many-to-many matchings even when there are externalities only on one side of the market.

4. For applications of such a decomposition result in settings without externalities, see Chambers and Yenmez (2017).

5. The rural hospital theorem states that each agent gets the same number of contracts in every stable matching in a many-to-one matching problem without externalities. Our generalization allows different contracts to have different weights that may depend on the quantity, price, or quality of the contracts. An agent's choice function satisfies *the law of aggregate demand* if the weight of contracts chosen from a set conditional on a reference set μ is greater than the weight of contracts chosen from a subset conditional on a reference set that has worse market conditions than μ. When there are no externalities, this law of aggregate demand reduces to the monotonicity condition of Fleiner (2003). We show that when choice functions satisfy the law of aggregate demand in addition to the aforementioned properties, all stable matchings have the same weight for every agent (see Theorem 7 in Supplementary Appendix).

6. See Bando, Kawasaki and Muto (2016) for a recent survey.

7. The subsequent literature—e.g., Chowdhury (2004), Hafalir (2008), Eriksson, Jansson and Vetander (2011), Chen (2013), Gudmundsson and Habis (2017), Salgado-Torres (2011a,b), and Bodine-Baron, Lee, Chong, Hassibi and Wierman (2011)—maintained the focus on the existence question and proposed a variety of weak stability concepts that modify Sasaki and Toda's by varying the degree to which the rematches of other agents penalize the blocking pair. Ray and Vohra (2015) placed the rematches in the context of von Neumann–Morgenstern stable set.

8. Even in the absence of externalities, an agent might be unwilling to block if they are concerned that doing so will trigger a chain of events that will lead them to lose a partner they block with. In the standard stability concept, agents ignore such chain reactions: they block if it benefits them in the absence of further reaction from the remaining agents. We contribute to the literature on stability defined through blocking chains by pointing out that agents' choice behaviour—which we take to be a primitive of our modelling—synthesizes both agents' preferences and assumptions on other agents' reactions to a block; see Footnote 3.

9. See also Uetake and Watanabe (2012) who use the deferred acceptance algorithm to estimate firm mergers in a matching model with externalities.

Two types of externalities attracted particular attention in the literature. Dutta and Massó (1997), Klaus and Klijn (2005), Kojima, Pathak and Roth (2013), and Ashlagi, Braverman and Hassidim (2014) study externalities within couples who have joint preferences over pairs of jobs. Complementing these studies, we illustrate our general theory by applying it to externalities within couples in local labour markets when a higher-paying job for one member of the couple might enable the other member to be more selective; see Section 3. Dutta and Massó (1997), Echenique and Yenmez (2007), Pycia (2012), and Hatfield and Kominers (2015) study peer effects among students matched to the same college and production complementarities among workers matched to the same firm.[10] We complement these studies by applying our general theory to benchmarking in admissions and hiring (see Supplementary Appendix D.2); benchmarking is an externality across colleges or firms.

Another important difference with the aforementioned papers is that they focused on sufficient conditions for existence, except for Pycia (2012) who also—like us—provided a corresponding necessity result. Within the confines of the college admission setting he studies, he showed that his preference alignment condition is not only sufficient but also necessary in a maximal-domain sense. Pycia's alignment condition is neither implied by nor implies standard substitutability as discussed in his article; for the same reasons, his condition is neither implied by nor implies our condition.[11]

The second focus area of the previous literature that allowed externalities is algorithms that lead to stable matchings (Echenique and Yenmez, 2007; Pycia, 2012; İnal, 2015). These studies of the algorithmic question restricted attention to settings in which the complementarities and peer effects are only among market participants matched to the same agent on the other side of the market. The deferred acceptance algorithm we proposed is not restricted in this way.[12]

2. MODEL

There is a finite set of agents $\mathcal{I}$ partitioned into buyers, $\mathcal{B}$, and sellers, $\mathcal{S}$, $\mathcal{B}\cup\mathcal{S}=\mathcal{I}$. The set of agents on the same side with agent i is denoted as $\theta(i)$. Therefore, $\theta(i)=\mathcal{B}$ if i is a buyer and $\theta(i)=\mathcal{S}$ if i is a seller. With a slight abuse of notation, θ also denotes one side of the market, so $\theta\in\{\mathcal{B},\mathcal{S}\}$. If θ is a side, then $-\theta$ is the other side, that is, $-\mathcal{B}\equiv\mathcal{S}$ and $-\mathcal{S}\equiv\mathcal{B}$. Agents interact with each other bilaterally through contracts. Each contract x specifies a buyer $b(x)$, a seller $s(x)$, and terms, which may specify price, quantity, and quality. There exists a finite set of contracts $\mathcal{X}$. For any $X\subseteq\mathcal{X}$, X_i denotes the set of contracts in X involving agent i, that is, $X_i\equiv\{x\in X : i\in\{b(x),s(x)\}\}$. Similarly, X_{-i} denotes the set of contracts not involving agent i, that is, $X_{-i}\equiv X\setminus X_i$.

Each agent i has a choice function c_i, where $c_i\left(X_i|\mu_{-i}\right)$ is the set of contracts that i chooses from a set X_i conditional on a reference set μ_{-i}, which is the set of contracts signed by the

10. Relatedly, Ostrovsky (2008) studies complementarities in a supply chain network and Sun and Yang (2006) study complementarities in an exchange economy.

11. In particular, his alignment condition generally fails in models with transfers because the receiver of the transfer prefers a higher payment while the sender prefers a lower payment (cf. Pycia, 2008). Mumcu and Saglam (2010) extend the alignment approach to analyse when all matchings in the non-empty collection of top matchings are stable and Teytelboym (2012) extends this approach to externalities among agents in a component of a network and shows that Pycia's alignment condition is then sufficient for the existence of a stable matching.

12. On the other hand, our algorithm cannot substitute for the earlier proposals in their applicability settings. For instance, in the environment they study, Echenique and Yenmez (2007) constructed an algorithm that finds all stable matching whenever stable matchings exist.

other agents on the same side.[13] The presence of externalities means that agents' choices are conditional on the state of the market, and to allow the conditioning, the state of the market should be observable by the agents. A natural observable is the matching that prevails on the market; and hence we condition the choices on the matching.

We expand the domain of the choice function so that, for any $X,\mu\subseteq\mathcal{X}$, $c_i(X|\mu)=c_i\left(X_i|\mu_{-i}\right)$. Choice function c_i has externalities if there exist $X,\mu,\mu'\subseteq\mathcal{X}$ such that $c_i(X|\mu)\neq c_i(X|\mu')$; otherwise, the choice function exhibits no externalities. Let $r_i(X|\mu)\equiv X_i\setminus c_i(X|\mu)$ be the set of contracts rejected by agent i from X conditional on a reference set μ. Similarly, define $C^{\theta}(X|\mu)\equiv\cup_{i\in\theta}c_i(X|\mu)$ to be the set of chosen contracts and $R^{\theta}(X|\mu)\equiv\cup_{i\in\theta}r_i(X|\mu)$ to be the set of rejected contracts from set X by side θ conditional on a reference set μ. Note that for any $X,\mu\subseteq\mathcal{X}$ and side θ, $C^{\theta}(X|\mu)$ and $R^{\theta}(X|\mu)$ form a partition of X since every contract involves exactly one agent from each side of the market and is either accepted or rejected by the agent. A *matching problem* is a tuple $(\mathcal{B},\mathcal{S},\mathcal{X},C^{\mathcal{B}},C^{\mathcal{S}})$.

We use the term *matching* to refer to any set of contracts. We embed any quota constraints, if they exist, in agents' choice behaviour. For instance, we model one-to-one matching markets by assuming that each agent chooses at most one contract from any set of contracts. Thus, examples of our setting include standard one-to-one and many-to-one matching problems with and without transfers.[14]

A matching μ is *individually rational* for agent i if $c_i(\mu_i|\mu_{-i})=\mu_i$. Less formally, conditional on the contracts of other agents on the same side, agent i wants to keep all of her contracts. A buyer i and seller j form a *blocking pair* for matching μ if there exists a contract $x\in\mathcal{X}_i\cap\mathcal{X}_j$ such that $x\notin\mu$ and $x\in c_i(\mu\cup\{x\}|\mu)\cap c_j(\mu\cup\{x\}|\mu)$. In words, a pair can block a matching μ if they both would like to sign a new contract conditional on μ. Matching μ is *stable* if it is individually rational for all agents and there are no blocking pairs. This stability concept is identical to pairwise stability studied in settings without externalities (Gale and Shapley, 1962). As in the standard settings without externalities, stability defined in terms of individual and pairwise blocking is equivalent to group stability when choice rules are substitutable; see Supplementary Appendix C.3.

2.1. *Properties of choice functions*

To guarantee the existence of stable matchings, we impose more structure on choice functions. First, we generalize two standard assumptions studied in the matching literature without externalities to our setting. Then, we introduce a new assumption, which is trivially satisfied when there are no externalities.

The first assumption is a basic rationality axiom we assume throughout the article.

Definition 1. *Choice function c_i satisfies the* irrelevance of rejected contracts *if for all $X_i,X_i'\subseteq\mathcal{X}_i$ and $\mu_{-i}\subseteq\mathcal{X}_{-i}$, we have*

$$c_i(X_i'|\mu_{-i})\subseteq X_i\subseteq X_i'\Longrightarrow c_i\left(X_i|\mu_{-i}\right)=c_i(X_i'|\mu_{-i}).$$

13. We could allow choice functions c_i to depend not only on X_i and μ_{-i} but also on μ_i (i.e. the set of contracts signed by i) with no change in our analysis except for claims entailing the restrictions of our conditions to subsets of agents, as in Footnote 19.

14. Without affecting any of the results, we could alternatively model one-to-one matching and other matching environments with quota constraints by assuming that only some sets of contracts are feasible matchings. This alternative route is straightforward if agents condition their choice behaviour on any sets of contracts rather than on feasible matchings. As is usual in models of matching with contracts, in applications with transfers, we assume that there is a lowest monetary unit.

If choice function c_i satisfies the irrelevance of rejected contracts, then excluding contracts that are not chosen does not change the chosen set.[15] This is a basic property of choice functions. It is equivalent to the *weak axiom of revealed preference* in settings without externalities (Alva, 2018). The irrelevance of rejected contracts has been recognized as an important property in the choice-function approach to matching by, e.g., Blair (1988) and Aygün and Sönmez (2013), who restricted attention to the case without externalities. The irrelevance of rejected contracts is satisfied in all applications and examples that we discuss.

The second assumption rules out complementarities between contracts of an agent.

Definition 2. *Choice function c_i satisfies* standard substitutability *if for all $X_i, X_i' \subseteq \mathcal{X}_i$ and $\mu_{-i} \subseteq \mathcal{X}_{-i}$,*

$$X_i' \supseteq X_i \Longrightarrow r_i\left(X_i'|\mu_{-i}\right) \supseteq r_i\left(X_i|\mu_{-i}\right).$$

A choice function satisfies standard substitutability if the corresponding rejection function is monotone for a fixed reference set, or equivalently, a contract that is chosen from a set is also chosen from any subset including that contract conditional on the same reference set. When there are no externalities, the choice behaviour does not depend on the reference set and this assumption reduces to the condition introduced by Kelso and Crawford (1982) for a matching market with transfers.[16]

Our third assumption captures the idea that not only a single agent's contracts are substitutable but also a similar substitutability of contracts obtains across agents on the same side of the market. Roughly speaking, the intuition is that when all agents on one side of the market choose from larger sets, then each agent on this side rejects more contracts.

To formalize the third assumption, we need the following definitions. A *binary relation* $\succsim_i$ on a domain $\mathcal{A}_i \subseteq 2^{\mathcal{X}_i}$ is a set of ordered pairs of matchings in $\mathcal{A}_i$; it is *reflexive* if for any $\mu_i \in \mathcal{A}_i$, $\mu_i \succsim_i \mu_i$; it is *transitive* if $\mu_i^1 \succsim_i \mu_i^2$ and $\mu_i^2 \succsim_i \mu_i^3$ imply $\mu_i^1 \succsim_i \mu_i^3$. A *pre-order* is a reflexive and transitive binary relation. We restrict our attention to pre-orders $\succsim_i$ that have the empty set in their domain, so $\varnothing \in \mathcal{A}_i$.[17] Given a pre-order $\succsim_i$ on a domain $\mathcal{A}_i \subseteq 2^{\mathcal{X}_i}$ for each agent i on side θ, we define the corresponding pre-order $\succsim^\theta$ for side θ on domain $\mathcal{A} = \{\mu \subseteq \mathcal{X} : \mu_i \in \mathcal{A}_i\} \subseteq 2^{\mathcal{X}}$ as follows: for every $\mu, \mu' \in \mathcal{A}$,

$$\mu' \succsim^\theta \mu \Longleftrightarrow \mu_i' \succsim_i \mu_i \forall i \in \theta.$$

Using pre-orders of individual agents, we define a similar pre-order $\succsim^{\theta'}$ for any set of agents $\theta' \subseteq \theta$.

An example of a pre-order is the *revealed-preference order*, defined for the case when there are no externalities: $\mu_i' \succsim_i \mu_i$ if, and only if, $c_i(\mu_i' \cup \mu_i) = \mu_i'$. In the matching context, this revealed-preference order was introduced by Blair (1988), and hence it is sometimes called Blair order (Echenique and Oviedo, 2006). In general, not all matchings can be compared using the revealed-preference order and the comparison is reflexive only on the set of the fixed points of the choice function, $\{\mu_i \subseteq \mathcal{X}_i : c_i(\mu_i) = \mu_i\}$. Likewise, in our general case, if a matching μ_i is not in the domain $\mathcal{A}_i \subseteq 2^{\mathcal{X}_i}$ of pre-order $\succsim_i$, we cannot compare it to any other matching. While in Blair's setting, the revealed-preference order is a partial order, that is an *antisymmetric* pre-order, where antisymmetry means that no two distinct matchings can be related in both directions, our analysis

15. All our assumptions on individual choice functions can equivalently be stated in terms of the side choice functions.

16. See also Roth (1984), Fleiner (2003), and Hatfield and Milgrom (2005). The substitutes condition is behind the monotonicity properties of the deferred acceptance algorithm when there are no externalities, and in this way underpins the standard matching analysis.

17. Instead of pre-orders, we can also work with a transitive binary relation satisfying $\varnothing \succsim_i \varnothing$.

requires us to use the more general concept of a pre-order because antisymmetry might fail in the presence of externalities (see Example 1). In particular, an agent's choice from a given set of contracts may depend on the reference set when there are externalities and as a result the comparison of two matchings with respect to a pre-order may go both ways.

As in the revealed-preference order, we only need to compare matchings that can be chosen. When the choice is conditional on the same reference set, we need to be able to compare the matching chosen from a set with any matching chosen from its subsets. When the choice is conditional on different reference sets, we need to be able to make comparisons implied by the following consistency assumption. A pre-order $\succsim^{\theta}$ for side θ is *consistent* with the side choice function C^{θ} if, for any $i \in \theta$ and $X, X', \mu, \mu' \subseteq \mathcal{X}$,

$$X'_i \supseteq X_i \text{ and } \mu'_{-i} \succsim^{\theta \setminus \{i\}} \mu_{-i} \Longrightarrow c_i\left(X'_i | \mu'_{-i}\right) \succsim_i c_i\left(X_i | \mu_{-i}\right).$$

Thus, consistency requires that if an agent has more contracts to choose from and if the reference set improves (is ranked higher) in the pre-order for the other agents on the same side, then the set of contracts chosen by the agent also improves in the pre-order. When $\mu' \succsim^{\theta} \mu$ for a consistent pre-order $\succsim^{\theta}$, we say that μ' reflects *better market conditions* than μ for side θ. As in the revealed-preference order, when there are more alternatives to choose from then the choice made reflects a better market condition than the choice made from fewer alternatives when the choice is conditional on the same or better reference set.

For every side choice function, there exists a pre-order that is consistent. For example, the pre-order that ranks every pair of matchings both ways is consistent. We focus on the consistent pre-order which is minimal in the following sense: a pre-order $\succsim^{\theta}$ is *minimal* if for every consistent pre-order $\tilde{\succsim}^{\theta}$, for any $\mu, \mu' \subseteq \mathcal{X}$, $\mu \succsim^{\theta} \mu' \Longrightarrow \mu \,\tilde{\succsim}^{\theta}\, \mu'$. We establish the existence and uniqueness of the minimal pre-order in Lemma 4 in Supplementary Appendix C.3.[18] For example, when there are no externalities and standard substitutability is satisfied, the revealed-preference pre-order is the minimal consistent pre-order. In the rest of the article, we denote the minimal consistent pre-order by $\succsim^{\theta}$ unless otherwise stated.

We are now ready to state our third, and main, assumption.

Definition 3. *Choice function C^{θ} satisfies* monotone externalities *if for all $i \in \theta$, $X_i \subseteq \mathcal{X}_i$, and $\mu_{-i}, \mu'_{-i} \subseteq \mathcal{X}_{-i}$,*

$$\mu'_{-i} \succsim^{\theta \setminus \{i\}} \mu_{-i} \succsim^{\theta \setminus \{i\}} \varnothing \Longrightarrow r_i(X_i | \mu'_{-i}) \supseteq r_i(X_i | \mu_{-i}).$$

The choice function of a side thus satisfies monotone externalities if every agent on this side rejects more contracts when the reference set reflects better market conditions.[19] This is a strong requirement. It is satisfied in some markets but not in others. We show that is satisfied in natural settings when agents pool their resources; for example, when couples share income and participate in a local labour market, one partner may be more selective in accepting an offer as their partner gets a higher-paying job (see Section 3 and Supplementary Appendix D.1; for more general resource sharing, see Supplementary Appendix D.4). The monotone externalities assumption is

18. Because in every pre-order $\varnothing \succsim^{\theta} \varnothing$, the minimal pre-order is non-empty. Furthermore, consistency implies that even the minimal pre-order relates some pairs of distinct matchings provided at least one agent $i \in \theta$ has at least one contract $x \in \mathcal{X}_i$ such that $c_i(\{x\} | \varnothing) = \{x\}$.

19. We extend the definitions of consistency and monotone externalities to any $C^{\theta'}$ where $\theta' \subseteq \theta$ by restricting the set of contracts to those associated only with agents in θ'. For any $\theta' \subseteq \theta$, if C^{θ} satisfies monotone externalities so does $C^{\theta'}$. In addition, if θ' has only one agent, say i, then $C^{\theta'}$ satisfies monotone externalities even if C^{θ} does not. The reason is our assumption that an agent i's choice conditional on a reference set μ is the same as the choice conditional on μ_{-i}.

also satisfied in settings where externalities are caused by benchmarking among competitors; e.g., a consulting firm may be more likely to reject a marginal job candidate when competing firms have stronger consultants (see Supplementary Appendix D.2). On the other hand, monotone externalities may fail when members of a couple participate in geographically dispersed labour markets and the externalities between them reflect the costs associated with substantially different geographic locations of their jobs; see Section 3. It may also fail due to the economies of scale when the better market conditions for a firm's competitors require the firm to scale up its production to remain competitive.

While monotone externalities is a novel property, it is importantly always satisfied when there are no externalities for a side because, in that case, the rejection function does not depend on the reference set. Thus, the setting with externalities that we study contains the standard substitutable setting when there are no externalities as a special case.

The conjunction of standard substitutability and monotone externalities is equivalent to the following property.

Definition 4. *Choice function C^θ satisfies* substitutability *if for all $i \in \theta$, $X_i, X_i' \subseteq \mathcal{X}_i$, and $\mu_{-i}, \mu'_{-i} \subseteq \mathcal{X}_{-i}$,*

$$X_i' \supseteq X_i \text{ and } \mu'_{-i} \succsim^{\theta\setminus\{i\}} \mu_{-i} \succsim^{\theta\setminus\{i\}} \varnothing \Longrightarrow r_i\left(X_i'|\mu'_{-i}\right) \supseteq r_i\left(X_i|\mu_{-i}\right).$$

We refer to this joint condition simply as substitutability because of the parallelism of the monotonicity ideas captured by its two components: standard substitutability captures the monotonicity of the rejection function with respect to an agent's own choice set, while monotone externalities proxies for such monotonicity with respect to other agents' choice sets.[20] While weaker than the conjunction of standard substitutability and no externalities, our substitutability assumption excludes complementarities. In Section 6, we address the question of which choice functions are allowed by providing a characterization of substitutable choice functions in terms of maximizing a set of complete preference orderings.

3. AN APPLICATION: COUPLES IN A LOCAL LABOUR MARKET

In this section, we discuss couples' (or households') labour provision in a local market.[21] Workers play the role of, say, sellers of their labour, and sign contracts with employers, who play the role of buyers. Workers are either single or members of exogenously married couples. As we focus on externalities within couples, we assume that there are no externalities for single workers.

Each worker prefers a higher-paying job to a lower-paying job. Furthermore, each worker has a *reservation wage*, which is the lowest wage at which a worker is indifferent between accepting a job and staying unemployed. For single workers, reservation wages are fixed and do not depend on market conditions. However, for married workers reservation wages depend on the income of their partner as follows. Within each couple, we distinguish between a primary earner and a secondary earner: the labour market participation of the secondary earner depends on the

20. Whenever substitutability (or monotone externalities) is satisfied for a consistent pre-order, then it is also satisfied for the minimal consistent pre-order $\succsim^\theta$. The reason is that the minimal pre-order $\succsim^\theta$ compares fewer pairs of reference sets, so substitutability (or monotone externalities) is weaker for the minimal pre-order compared to any other consistent pre-order.

21. We are grateful to Michael Ostrovsky for suggesting this application. Additional motivating applications—including relative rankings, dynamic matching, profit sharing, and add-ons—are provided in Supplementary Appendix D. More abstract illustrative examples are provided in Section 4.1.

wage of the primary earner.[22] When the primary earner receives a higher wage, the secondary earner becomes weakly more selective. More precisely, the reservation wage of the secondary earner goes up when the primary earner has a higher income. There are no externalities for primary earners, so their reservation wages are fixed and do not depend on the income of their partners.

This kind of externality arises in labour markets where members of a couple pool their incomes. For instance, suppose that any secondary earner's job imposes labour-provision disutility c and that the secondary earner is willing to accept a job if any only if it pays wage w such that $U(w+w_p)-c \geq U(w_p)$, where w_p is the wage of the primary earner and U is the concave utility function of income for the couple.[23] In these examples, only the wage earned by the primary earner impacts the choice behaviour of the secondary earner and the relative locations of the two jobs can be ignored; this is in line with our restriction to local labour markets.

To check substitutability, we define the pre-order $\succsim_i$ for primary earner i of a couple so that $\mu'_i \succsim_i \mu_i$ when the wage specified in contract μ'_i is weakly higher than the wage specified in contract μ_i. For any other worker i, let $\succsim_i$ be the trivial pre-order for which every pair of contracts is comparable.[24] The better market pre-order for workers is consistent with the choice behaviour because primary earners choose the contract with the highest wage from any set of contracts; the choice functions satisfy standard substitutability because workers have unit demand; their choice functions satisfy monotone externalities (and hence substitutability) because a secondary earner becomes weakly more selective whenever their partner gets a higher-paying job.

Supposing that employers' choice functions also satisfy substitutability—e.g., because their choice behaviour does not exhibit externalities and satisfies standard substitutability—the general theory we develop in subsequent sections implies that a stable job matching exists and is Pareto efficient. The theory also implies that all employers weakly prefer the stable job matching before some set of workers marry to a stable matching following the marriages, while all primary earners weakly prefer a job matching post marriages to the one before; an analogous comparative statics is also valid for divorces.[25]

The presence of income-driven externalities within couples has been studied since Becker (1973) and is well documented. The rich literature on the so-called *added worker effect* (e.g. Lundberg, 1985; Chiappori, 1992; Cullen and Gruber, 2000) finds that married women are more likely to take or search for paid employment when their husbands are unemployed. Studies based on more recent data—e.g., Kleven, Kreiner and Saez (2009)—relax the distinction between men

22. In this section, we maintain the assumption that the roles of primary earners and secondary earners are fixed and do not depend on market conditions. This assumption is empirically motivated; see the empirical labour market discussion below. We relax this assumption in Supplementary Appendix D.1.

23. The utility of income may represent the outcome of intra-household bargaining, as in, e.g., Manser and Brown (1980). The main driver of labour provision costs is hours worked, and the assumption that c is fixed means that different jobs considered by the secondary earner are equivalent in terms of hours worked. Thus the above example is a good approximation of labour markets in which the vast majority of jobs are full-time, as is true, e.g., in Eastern Europe and Russia. For instance, in Bulgaria, the country-wide proportion of full-time jobs was 98.4% in 2019, the most recent year with available OECD data. At the other extreme is, e.g., Switzerland, with only 73.1% of full-time jobs. Other than Russia, large economies are in between these two extremes, e.g., the proportion of full-time jobs in the US was 87.6%. The data are available at https://data.oecd.org/emp/part-time-employment-rate.htm.

24. It is easy to see that these binary relations are pre-orders.

25. For existence, see Theorem 1 in Section 4; for efficiency, see Theorem 3 in Section 5; for comparative statics, see Theorem 6 in Supplementary Appendix C.1. Note that we can analyse two sides of a market separately because we impose no assumptions relating the choice behaviour of agents across sides.

and women and, instead, like us, analyse couples composed of a primary earner who always participates in the labour market and a secondary earner who chooses whether to work or not.[26]

Finally, note that our restriction to local labour markets plays an important role in the above analysis by decoupling couple's or household's labour provision choices from their decision where to live. This assumption is generally satisfied in labour markets in which members of the working class (also called the middle class) and the poor participate: their costs of moving or accepting distant jobs are high relative to potential benefits as have been well documented in the empirical studies, see, e.g., Manning and Petrongolo (2017) for a discussion of the UK labour markets and Williams (2017) for an analysis of the US working class. As recognized in this literature, an exception to the ubiquitous locality of labour markets are markets for professional and some managerial jobs—a small fraction of jobs in the economy—which are not necessarily local. The externalities faced by the participants of non-local labour markets, are more complex than those studied in our model and the empirical literature on secondary earners' labour provision discussed above. For instance, the primary earner's choice between jobs in the UK and US, or between jobs on the East Coast and West Coast of the US, would affect the secondary earner's preferences between jobs in these countries or regions.[27]

4. STABLE MATCHINGS

As in classical matching theory, a key step in proving the existence of a stable matching is an algorithm akin to the deferred acceptance algorithm.

Our generalization of the deferred acceptance algorithm has two phases. First, we construct an auxiliary matching μ^* such that $C^{\mathcal{S}}(\mathcal{X}|\mu^*) \precsim^{\mathcal{S}} \mu^*$. Then, we use μ^* to construct a stable matching in a way resembling the classic deferred acceptance algorithm of Gale and Shapley (1962) and, particularly, its extension by Hatfield and Milgrom (2005): we run the algorithm in rounds, $t=1,2,\ldots$. In any round $t \geq 1$, we denote by $A^s(t)$ the set of contracts available to the sellers and $A^b(t)$ the set of contracts available to the buyers. Therefore, the set of contracts held at the beginning of each round is $A^s(t) \cap A^b(t)$. We also track the reference sets for each side: $\mu^s(t)$ is the seller reference set and $\mu^b(t)$ is the buyer reference set.[28]

Phase 1: Construction of an auxiliary matching μ^* such that $\mu^* \succsim^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$. Set $\mu_0 \equiv \varnothing$ and define recursively $\mu_k \equiv C^{\mathcal{S}}(\mathcal{X}|\mu_{k-1})$ for every $k \geq 1$. Since the number of contracts is finite, so is the number of sets of contracts. Therefore, there exist m and $n \leq m$ such that $\mu_{m+1} = \mu_n$. Let $m^* = \min\{m | \exists n \leq m \text{ s.t. } \mu_{m+1} = \mu_n\}$. Let $\mu^* \equiv \mu_{m^*}$. In the proof of Theorem 1, we establish that $\mu^* \succsim^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$.

Phase 2: Construction of a stable matching. Set $A^s(1) \equiv \mathcal{X}$ (all contracts are available to the sellers), $A^b(1) \equiv \varnothing$ (no contracts are available to the buyers), and $\mu^s(1) \equiv \mu^*$, and $\mu^b(1) \equiv \varnothing$.

26. Other related findings include Johnson and Skinner (1986) who find that women increase their labour supply prior to divorce; an evidence that their labour supply was lowered by high earnings of the spouse, an externality of the type we study.

27. For an analysis of location choices, see e.g. Costa and Kahn (2000) and Compton and Pollak (2007).

28. The tracking of reference sets has no counterpart in earlier formulations of the deferred acceptance algorithms of, among many others, Gale and Shapley (1962), Roth (1984), Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Echenique and Oviedo (2006), Echenique and Yenmez (2007), Ostrovsky (2008), Hatfield and Kojima (2010), and Bando (2014). In these papers, there is no need to track reference sets and the deferred acceptance algorithm terminates when there are no more rejections and no new offers. However, in our setting, the lack of rejections and new offers is not sufficient to stop the algorithm and we need to run it until the reference sets converge. We run the algorithm so that in each round agents on both sides respond to the offers and rejections from the previous round. This is formally different from the standard approach where agents on the proposing side respond to rejections from the earlier round but the agents on the accepting side respond to offers in the current round. This difference is not substantive: we could run the deferred acceptance algorithm in the latter manner with straightforward adjustments.

In each round $t=1,2,\ldots$, we update these sets and matchings as follows:

$$A^s(t+1) \equiv \mathcal{X}\setminus R^{\mathcal{B}}\big(A^b(t)|\mu^b(t)\big),$$
$$A^b(t+1) \equiv \mathcal{X}\setminus R^{\mathcal{S}}\big(A^s(t)|\mu^s(t)\big),$$
$$\mu^s(t+1) \equiv C^{\mathcal{S}}\big(A^s(t)|\mu^s(t)\big), \text{ and}$$
$$\mu^b(t+1) \equiv C^{\mathcal{B}}\big(A^b(t)|\mu^b(t)\big).$$

Thus, the buyers reject some of the contracts available in $A^b(t)$ conditional on the reference set $\mu^b(t)$ and the set of contracts not rejected by the buyers is available to the sellers in the next round, i.e., $A^s(t+1)=\mathcal{X}\setminus R^{\mathcal{B}}\big(A^b(t)|\mu^b(t)\big)$. Likewise, the sellers reject some contracts available in $A^s(t)$ conditional on the reference set $\mu^s(t)$ and the set of contracts that are not rejected by the sellers is available to the buyers in the next round, i.e., $A^b(t+1)=\mathcal{X}\setminus R^{\mathcal{S}}\big(A^s(t)|\mu^s(t)\big)$. We also update the reference sets: at the next round, the sellers' reference set is the set of contracts that sellers choose from $A^s(t)$ conditional on $\mu^s(t)$ and likewise for the buyers. We continue updating these sets until round T such that $A^s(T+1)=A^s(T)$, $A^b(T+1)=A^b(T)$, $\mu^s(T+1)=\mu^s(T)$, and $\mu^b(T+1)=\mu^b(T)$. The outcome of the algorithm is then $A^s(T)\cap A^b(T)$.

This is the seller-proposing version of the deferred acceptance algorithm. Like in the setting without externalities, we interpret $A^s(t)$ as the set of contracts not yet rejected by the buyers; this set contains all contracts in the beginning of the algorithm and in each round this set becomes weakly smaller (in the set inclusion sense). Similarly, we interpret $A^b(t)$ as the set of contracts already offered to the buyers; this set contains no contracts in the beginning of the algorithm and in each round this set becomes weakly larger (in the set inclusion sense).[29] The buyer-proposing version can be defined analogously.

The main result of this section establishes that the algorithm terminates at some round despite the presence of externalities and, furthermore, it produces a stable matching.

Theorem 1. *(Sufficiency) Suppose that the choice functions satisfy substitutability. Then, the algorithm terminates at some finite round T, its outcome $A^s(T)\cap A^b(T)$ is stable, and*

$$\mu^s(T)=\mu^b(T)=A^s(T)\cap A^b(T).$$

When there are no externalities, the proof of the existence of stable matchings might be conceptualized as constructing a function that maps a set of contracts already offered by one side and a set of contracts not yet rejected by the other side before a step of the deferred acceptance algorithm into such sets updated by offers and rejections made in the step of the algorithm.[30] Under the standard substitutes condition this function is monotonic: it increases (in the sense of set inclusion) the set of offered contracts and decreases the set of not yet rejected contracts. The resulting monotonic sequence of pairs of contract sets converges to a fixed point by the fixed-point theorem of Tarski (1955) and the fixed point corresponds to a stable matching. We adapt this idea to our setting with externalities (see Appendix A).

The second phase of our algorithm is similar to the standard deferred acceptance algorithm except that our agents condition their choices on reference sets of contracts. Our algorithm thus needs to keep track not only of the sets of offered contracts and not yet rejected contracts but also of

29. See Appendix A for proofs of these claims.

30. See Adachi (2000), Fleiner (2002), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), and the subsequent literature.

the reference sets. We use our better market condition pre-order to compare the reference sets, and we extend monotonicity to require that with each round the reference set for sellers reflects worse market conditions and the reference set for buyers reflects better market conditions. Extending the deferred acceptance idea in this way to the setting with externalities requires us to overcome two subtleties.

The first subtlety arises because, without the preparatory first phase, the reference sets would not necessarily be monotonically comparable. The monotonicity property could fail already in the initial step of the second phase of the algorithm if the reference sets were chosen to be the empty set (the set of already offered contracts) and the set of all contracts (the set of not yet rejected contracts); note that such an initial choice of these sets has become standard in earlier constructions of deferred acceptance. For the side whose initial reference set is empty (buyers in our formulation), the problem does not arise as $\mu^b(2)=C^{\mathcal{B}}(\mathcal{X}|\mu^b(1))\succsim^{\mathcal{B}}\mu^b(1)$ if $\mu^b(1)=\varnothing$. For the side whose initial reference set is the set of all contracts (sellers in our formulation), monotonicity would require that $\mu^s(1)\succsim^{\mathcal{S}}\mu^s(2)=C^{\mathcal{S}}(\mathcal{X}|\mu^s(1))$, and this comparison might fail if $\mu^s(1)=\mathcal{X}$.[31] The first phase of the algorithm constructs $\mu^s(1)$ satisfying this initial comparison. As in the standard deferred acceptance, our substitutes condition then guarantees that if this monotonicity property is satisfied in a step of the algorithm, then their analogues are satisfied in each subsequent step.

The second subtlety arises because we work with pre-orders rather than partial orders and the domain of the function that we analyse is not a lattice. The failure of these two properties, on which the standard analysis hinges, implies that Tarski's fixed-point theorem does not guarantee that the second phase of our algorithm has a fixed point. We resolve this issue by using the finiteness of the set of contracts to show that the iterative application in the second phase must have two rounds at which the reference sets are equivalent in the pre-order, $\mu^s\sim^{\mathcal{S}}\tilde{\mu}^s$ and $\mu^b\sim^{\mathcal{B}}\tilde{\mu}^b$, while the set of contracts available to the buyers and sellers are the same, $A^s=\tilde{A}^s$ and $A^b=\tilde{A}^b$. The substitutes condition then implies that $C^{\mathcal{S}}(A^s|\mu^s)=C^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s)$ and $C^{\mathcal{B}}(A^b|\mu^b)=C^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b)$, thereby both the set of available contracts and the reference sets have to be identical in the subsequent rounds implying that the second phase converges. Once the deferred acceptance algorithm converges, it produces a stable matching (see Appendix A).

We complement our Theorem 1 by showing that monotone externalities is necessary for the existence of a stable matching in a "maximal domain" sense when standard substitutability is satisfied. The necessity of standard substitutability for the existence of stable matchings was established by Hatfield and Kominers (2017) for many-to-many matching markets without externalities.

Theorem 2. *(Necessity) Let i be an agent on side θ whose choice function exhibits externalities and satisfies standard substitutability. Then there exist choice functions for other agents such that (i) no stable matching exists, (ii) the choice functions for agents in $\theta\setminus\{i\}$ are such that $C^{\theta\setminus\{i\}}$ satisfies substitutability, but C^{θ} fails substitutability, and (iii) the choice functions for agents on side $-\theta$ exhibit no externalities and satisfy standard substitutability.*

In this theorem, the choice function of agent i is fixed while choice functions of other agents are constructed. In the construction, $C^{\theta\setminus\{i\}}$ and $C^{-\theta}$ satisfy substitutability, but C^{θ} does not. To develop the intuition for the proof, consider a simple example with two workers i and j on side θ and one firm k on side $-\theta$. For each worker–firm pair, there is only one contract; in particular, each worker's choice satisfies standard substitutability. The firm wants to hire as many workers

31. For instance, in Example 1 below, this comparison holds for some consistent pre-orders but not for the minimal one.

as possible; the firm's choice thus exhibits no externalities and satisfies substitutability. Worker i's choice function exhibits externalities and thus whether worker i wants to work or not depends on whether worker j is hired by the firm or not. These externalities might take one of two forms.

One possibility is that worker i wants to work for the firm only when worker j also works for it. Let then worker j be willing to work only when worker i is not working; this choice of worker j is substitutable and, with the set of workers other than i having only one member, it satisfies monotone externalities (see Footnote 19). There is, however, no stable matching because worker j blocks the matching in which both workers are employed, worker i (or worker i and the firm) blocks the matching in which exactly one worker is employed, and worker j and the firm block the matching in which no workers are employed. The other possibility is that worker i wants to work for the firm only when worker j does not work for the firm. In this case, let worker j be willing to work only when worker i is working. The analysis of this case is analogous to the previous one.[32]

4.1. *Illustrative examples*

In this section, we provide two examples to illustrate the deferred acceptance algorithm. In Example 1, substitutability is satisfied, so the algorithm produces a stable matching. In Example 2, substitutability is not satisfied and a stable matching does not exist.

Like the standard deferred acceptance algorithm, in each round of Phase 2, substitutability implies that $A^s(t+1) \subseteq A^s(t)$ and $A^b(t+1) \supseteq A^b(t)$, i.e., the sellers make more offers to the buyers while the buyers reject more contracts with each passing round (Lemma 1). As a consequence, the sellers' reference set gets worse and the buyers' reference set gets better. Hence, both of these two sets converge at some round t; however, the algorithm does not necessarily terminate when $A^s(t+1) = A^s(t)$ and $A^b(t+1) = A^b(t)$. Indeed, because of externalities, the set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is not necessarily stable. Instead, the algorithm converges only when $A^s(t+1) = A^s(t)$, $A^b(t+1) = A^b(t)$, $\mu^s(t+1) = \mu^s(t)$, and $\mu^b(t+1) = \mu^b(t)$. The set of contracts held at such a round, $A^s(t) \cap A^b(t)$, is stable.

The next example illustrates this point and shows the steps of the algorithm. It also demonstrates that our algorithm can be viewed as an ascending auction in the presence of externalities.

Example 1. *Suppose that there are two sellers s_1 and s_2 and two buyers b_1 and b_2. Seller s_1 and buyer b_1 can sign contract x_1 and seller s_1 and buyer b_2 can sign contract x_2. Seller s_2 can sign contract x_3 with buyer b_2 only.*[33] *The contractual structure is demonstrated in Figure 1.*[34]

Seller choice functions do not have externalities. Seller s_1 always chooses one contract, if there exists one, and prefers contract x_2 over x_1 and seller s_2 chooses contract x_3 when it is available. Therefore, seller choice functions satisfy standard substitutability. They also satisfy monotone externalities because there are no externalities for sellers.

Buyer b_1 chooses contract x_1 regardless of the contracts signed by buyer b_2. Conditional on the empty set, buyer b_2 chooses one contract only and prefers contract x_3 to x_2. Conditional on the reference set $\{x_1\}$, buyer b_2 chooses contract x_3, if it is available, and rejects x_2, if it is

32. As in the first case, our assumptions are satisfied on the sub-market without worker i but there does not exist a stable matching.

33. This example is a special case of Application 1 with the following interpretation. Sellers are firms and buyers are workers. Buyers b_1 and b_2 are married. Buyer b_1 is a woman; her choice function does not have externalities. Buyer b_2 is a man and the outside option of not working is ranked higher whenever his wife works. In particular, contract x_2 is ranked below the outside option if the wife has a job.

34. In this example, our substitutes condition is satisfied, Bando's assumptions are not, and a stable matching exists.

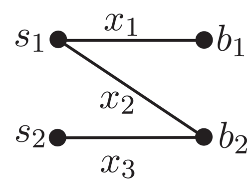

FIGURE 1
Contractual structure in Example 1.

TABLE 1
Choice function of buyer b_2 in Example 1

	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\|\{x_1\})$	$\{x_3\}$	$\{x_3\}$	$\varnothing$	$\varnothing$
$c_{b_2}(\cdot\|\varnothing)$	$\{x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$

Notes: Columns are indexed by the set of available contracts and rows are indexed by the reference set of contracts signed by buyer b_1.

available. Therefore, the only choice function that has externalities is that of buyer b_2, which is summarized in Table 1.

First let us construct the better market pre-order for buyers. Since buyer b_1 chooses contract x_1 whenever it is available, we have $\{x_1\}\succsim_{b_1}\varnothing$. For buyer b_2, using consistency on sets of contracts $\{x_2,x_3\}\supseteq\{x_2\}\supseteq\varnothing$ with the empty set as a reference set, we get $\{x_3\}\succsim_{b_2}\{x_2\}\succsim_{b_2}\varnothing$. In addition, since $\{x_1\}\succsim_{b_1}\varnothing$, $c_{b_2}(\{x_2\}|\{x_1\})=\varnothing$, and $c_{b_2}(\{x_2\}|\varnothing)=\{x_2\}$, we get $\varnothing\succsim_{b_2}\{x_2\}$. Therefore, for buyer b_2, $\{x_3\}\succsim_{b_2}\{x_2\}\sim_{b_2}\varnothing$. The better market pre-order for buyers $\succsim^{\mathcal{B}}$ is then defined as $\mu'\succsim^{\mathcal{B}}\mu\Leftrightarrow\mu'_{b_i}\succsim_{b_i}\mu_{b_i}$ for every $i\in\{1,2\}$. For example, $\{x_1,x_2\}\succsim^{\mathcal{B}}\{x_1\}$ because $\{x_1\}\succsim_{b_1}\{x_1\}$ and $\{x_2\}\succsim_{b_2}\varnothing$. Similarly, $\{x_1\}\succsim^{\mathcal{B}}\{x_2\}$ because $\{x_1\}\succsim_{b_1}\varnothing$ and $\varnothing\succsim_{b_2}\{x_2\}$.

It is easy to check that standard substitutability is satisfied for the buyers. To check monotone externalities, note that choice function of buyer b_1 does not have externalities, so it does not depend on the reference set and the choice function of buyer b_2 rejects more contracts when it is conditional on the reference set $\{x_1\}$ rather than the reference set $\varnothing$, where $\{x_1\}\succsim_{b_1}\varnothing$.

Since the choice functions satisfy substitutability, the deferred acceptance algorithm produces a stable matching (Theorem 1). We now show how it works in this example. In the first phase, we start with $\mu_0=\varnothing$. Then, $\mu_1=C^{\mathcal{S}}(\mathcal{X}|\mu_0)=\{x_2,x_3\}$ and $\mu_2=C^{\mathcal{S}}(\mathcal{X}|\mu_1)=\{x_2,x_3\}$. Since $\mu_1=\mu_2$, we set $\mu^*=\{x_2,x_3\}$.

In the first round of the second phase, all contracts are available to the sellers, so they choose $\{x_2,x_3\}$. However, no contract is available to the buyers, so they choose the empty set. Therefore, in the second round, the seller reference set is $\{x_2,x_3\}$ and the buyer reference set is the empty set. In addition, the set of contracts available to the buyers is the set of contracts not rejected by the sellers at the first round, which is $\{x_2,x_3\}$.

The algorithm continues to proceed in this way. Table 2 shows all the rounds. Notice that between the fourth and fifth rounds the sets of contracts available to the buyers and sellers are the same, i.e., $A^b(4)=A^b(5)$ and $A^s(4)=A^s(5)$. In the standard deferred acceptance algorithm, we could stop the algorithm here. In our setting, the deferred acceptance does not converge yet because the reference sets for the buyers are different at these two rounds. The algorithm eventually converges at the sixth round and produces the matching $A^s(6)\cap A^b(6)=\{x_1,x_3\}$, which is stable: it is individually rational for all agents. There is only one potential blocking pair (s_1,b_2) via contract x_2 but they do not block this matching because $x_2\notin c_{b_2}(\{x_2,x_3\}|\{x_1\})$.

TABLE 2
Rounds of the deferred acceptance algorithm in Example 1

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^{\mathcal{S}}(A^s(t)\|\mu^s(t))$	$C^{\mathcal{B}}(A^b(t)\|\mu^b(t))$
$t=1$	$\mathcal{X}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$
$t=2$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\{x_3\}$
$t=3$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_3\}$
$t=4$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=5$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=6$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$		

TABLE 3
Choice function of buyer b_2 in Example 2

	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\|\{x_1\})$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_2\}$	$\varnothing$
$c_{b_2}(\cdot\|\varnothing)$	$\{x_3\}$	$\{x_3\}$	$\varnothing$	$\varnothing$

Notes: Columns are indexed by the set of available contracts and rows are indexed by the reference set of contracts signed by buyer b_1.

Note that the set of contracts available to the sellers, $A^s(t)$, is shrinking and the set of contracts available to the buyers, $A^b(t)$, is expanding as the algorithm proceeds. Likewise, the seller reference set $\mu^s(t)$ is getting worse for the sellers and the buyer reference set $\mu^b(t)$ is getting better for the buyers. □

When choice functions satisfy standard substitutability, DA produces a stable matching *if* it converges even if monotone externalities is not satisfied (see Lemma 3 in Appendix A). However, when monotone externalities fails, it does not have to converge and a stable matching need not exist. We show these two claims with the following example.

Example 2. *We modify Example 1 by changing the choice function of buyer b_2. Buyer b_2 chooses all available contracts conditional on the reference set $\{x_1\}$. Furthermore, conditional on the empty set, she chooses contract x_3, if it is available, and rejects x_2, if it is available. Choice function of buyer b_2 is in Table 3.*[35]

As in the previous example, it is easy to check that standard substitutability is satisfied for buyers. However, monotone externalities fails. To see this, note that for the minimum consistent pre-order we need $\{x_1\}\succsim_{b_1}\varnothing$. But conditional on $\{x_1\}$, buyer b_2 accepts more contracts than conditional on the empty set when the available set of contracts is $\{x_2,x_3\}$, violating the monotone externalities condition.

While our general result implies that there exists a stable matching in Example 1, it is easy to see that there is no stable matching in Example 2: matchings $\varnothing$ and $\{x_3\}$ are blocked by seller s_1 and buyer b_1 via contract x_1. Matchings $\{x_1\}$ and $\{x_1,x_2\}$ are blocked by seller s_2 and buyer b_2 via contract x_3. Matchings $\{x_2\}$ and $\{x_2,x_3\}$ are not individually rational for buyer b_2. Matching $\{x_1,x_3\}$ is blocked by seller s_1 and buyer b_2 via contract x_2. The last remaining matching, $\mathcal{X}$, is not individually rational for seller s_1.

Now let us consider the deferred acceptance algorithm. The first phase works as in the previous example because sellers' choice functions remain the same. The algorithm starts diverging after

35. In this example, our substitutes condition is not satisfied, however, the assumptions in Bando (2012) are satisfied, and a stable matching does not exist.

TABLE 4
Rounds of the deferred acceptance algorithm in Example 2

	$A^s(t)$	$A^b(t)$	$\mu^s(t)$	$\mu^b(t)$	$C^{\mathcal{S}}(A^s(t)\mid\mu^s(t))$	$C^{\mathcal{B}}(A^b(t)\mid\mu^b(t))$
$t=1$	$\mathcal{X}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\varnothing$
$t=2$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\varnothing$	$\{x_2,x_3\}$	$\{x_3\}$
$t=3$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_3\}$
$t=4$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$
$t=5$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\{x_1,x_3\}$	$\mathcal{X}$
$t=6$	$\mathcal{X}$	$\mathcal{X}$	$\{x_1,x_3\}$	$\mathcal{X}$	$\{x_2,x_3\}$	$\mathcal{X}$
$t=7$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$
$t=8$	$\mathcal{X}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$
$t=9$	$\{x_1,x_3\}$	$\{x_2,x_3\}$	$\{x_2,x_3\}$	$\{x_3\}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		

round five of the second phase because conditional on the reference set $\mu^b(5)=\{x_1,x_3\}$, the buyers choose all contracts. Table 4 shows the first nine rounds of DA.

At Round 9, we get the same sets of contracts available to the buyers and sellers and the same reference sets as in Round 3. Therefore, the algorithm does not converge. This outcome is not surprising because we showed that there is no stable matching in this example. □

5. PROPERTIES OF STABLE MATCHINGS UNDER EXTERNALITIES

Two key normative insights in the standard theory of stable matchings are Pareto efficiency of stable matchings and the existence of side-optimal stable matchings (Gale and Shapley, 1962). In this section, we extend them to settings with externalities.

Theorem 3. *(Pareto efficiency) Suppose that the choice functions satisfy standard substitutability. If matching μ is stable then it is Pareto efficient in the following sense: there is no other matching $\nu\neq\mu$ such that $\nu_i=c_i(\nu\cup\mu|\mu)$ for every agent i.*

The proof is similar as in the case without externalities: if there is such a matching $\nu\neq\mu$ then $\nu(i)\neq\mu(i)$ for some agent i. Then, agent i prefers ν to μ in the choice sense, $\nu_i=c_i(\nu\cup\mu|\mu)$. Therefore, agent i and any one of the agents with whom i contracts in ν_i would form a blocking pair.[36]

The existence of the side-optimal stable matchings is more subtle in the setting with externalities.

Definition 5. *A stable matching μ is θ-*optimal *if $\mu\succsim^{\theta}\mu'$ for every stable matching μ', it is θ-pessimal if $\mu\precsim^{\theta}\mu'$ for every stable matching μ'.*

This concept subsumes its counterpart from matching theory without externalities, where side optimality is measured with respect to the revealed-preference order.

Theorem 4. *(Side optimality) Suppose that the choice functions satisfy substitutability and, in addition, for side θ there exists a set of contracts $\bar{\mu}^{\theta}$ such that for any $\mu,X\subseteq\mathcal{X}$, we have*

36. We strengthen the efficiency result to group stability in Proposition 1 in Supplementary Appendix C.3. Beyond stability, efficiency has been thoroughly studied in markets with externalities (cf. Pigou, 1920; Ray and Vohra, 2001; Ashlagi and Shi, 2014; Watson, 2014; Chade and Eeckhout, 2019; Vosooghi, Arvaniti and van der Ploeg, 2021).

$\bar{\mu}^\theta \succsim^\theta C^\theta(X|\mu)$. Then, there exists a θ-optimal stable matching, which is also a $-\theta$-pessimal stable matching.

In this result, in addition to substitutability, we assume that there exists a set of contracts $\bar{\mu}^\theta$ that reflects better market conditions for side θ than any set of contracts that can be chosen by this side. Therefore, despite possible externalities, agents on side θ agree what set of contracts would be best for all of them; this set does not need to be acceptable to the other side nor stable. In the absence of externalities, this assumption is automatically satisfied. Indeed, for this special case, we can take $\bar{\mu}^\theta$ to be $C^\theta(\mathcal{X})$. Then for any $X \subseteq \mathcal{X}$, $\mathcal{X} \supseteq \bar{\mu}^\theta \cup C^\theta(X) \supseteq \bar{\mu}^\theta = C^\theta(\mathcal{X})$, and the irrelevance of rejected contracts yields $C^\theta(\bar{\mu}^\theta \cup C^\theta(X)) = C^\theta(\mathcal{X}) = \bar{\mu}^\theta$. As the minimal consistent pre-order $\succsim^\theta$ is the revealed-preference order, the equality we derived means that $\bar{\mu}^\theta \succsim^\theta C^\theta(X)$ for any X. Consequently, Theorem 4 subsumes the standard insight that, in the absence of externalities and under the standard substitutes condition, the outcome of θ-proposing deferred acceptance algorithm is the θ-optimal and $-\theta$-pessimal stable matching with respect to the revealed-preference order.

The existence assumption we impose is also satisfied in all applications we discuss in Section 3 and Supplementary Appendix D. In the presence of externalities, this assumption is however not innocuous, and it might fail even when substitutability is satisfied as illustrated by the following example.

Example 3. *Suppose that there are two buyers b_1, b_2 and one seller, s_1. There is only one contract associated with every seller–buyer pair. Let the contract between b_1 and s_1 be x_1 and the contract between b_2 and s_1 be x_2. Since there is only one seller, there are no externalities for the seller side. The choice functions are as follows: seller s_1 chooses all contracts available. Buyer b_1 chooses x_1 conditional on the reference set $\{x_2\}$ and rejects x_1 conditional on the empty set. Buyer b_2 chooses x_2 conditional on the reference set $\{x_1\}$ and rejects x_2 conditional on the empty set. That is each buyer chooses their contract only if the other buyer has the other contract.*

The choice function of the seller satisfies substitutability. For buyers, the minimal pre-order $\succsim^{\mathcal{B}}$ with the domain $\{\varnothing\}$ is such that $\varnothing \succsim^{\mathcal{B}} \varnothing$ and no other pair of sets is comparable.[37] This pre-order is consistent because conditional on the empty set both buyers choose no contracts. In addition, the buyer-side choice function satisfies substitutability because the buyer-side rejection function is monotone conditional on the empty set.

The existence assumption in Theorem 4 fails in this setting because there exists no buyer-optimal set of contracts $\bar{\mu}^{\mathcal{B}}$ such that $\bar{\mu}^{\mathcal{B}} \succsim^{\mathcal{B}} C^{\mathcal{B}}(X|\mu)$ for all X and μ.

The above example also shows that the existence assumption is necessary in Theorem 4 because there is no buyer-optimal stable matching in the example: both the empty set and $\{x_1, x_2\}$ are stable, but they cannot be compared by the pre-order $\succsim^{\mathcal{B}}$.[38] The existence assumption thus plays a crucial role in the proof of Theorem 4. To understand how it guarantees the existence of side-optimal stable matchings, let us note that the θ-optimal (and $-\theta$-pessimal) stable matching is obtained by running the second phase of the θ-proposing deferred acceptance algorithm when the initial reference set for side θ is $\bar{\mu}^\theta$. Take any stable matching μ. By assumption, $\bar{\mu}^\theta$ reflects

37. We allow the domain of the pre-order to be smaller than the set of all matchings, which is the case in this example.

38. In addition, this example shows that in our setting the set of stable matchings does not need to have a lattice structure. This is in contrast to matching without externalities, where standard substitutability implies that the set of stable matchings is a lattice, cf. Hatfield and Milgrom (2005). A lattice structure may also exist in our setting under additional assumptions. We leave this question for future research.

better market conditions than μ and substitutability implies that this comparison with μ remains true in each step of the second phase of the deferred acceptance algorithm (cf. the analysis of deferred acceptance in the Proof of Theorem 1). Thus, the resulting stable matching reflects better market conditions than μ.

Remark 1. *Suppose that agents are members of coalitions and coordinate their choices. Examples include couples, sports teams, corporate divisions, single firms, or even multiple firms controlled by the same owner. If an outside observer is unaware that the coalition—rather than the agents—is the decision maker, the outside observer might infer that the choices of the coalition members exhibit externalities. The standard matching theory without externalities guarantees the existence of stable matchings and their properties among such coalitions provided coalitional choice functions satisfy the standard substitutes condition (Hatfield and Milgrom, 2005; Hatfield and Kominers, 2017). In particular, the standard theory guarantees the existence of stable matchings that are side-optimal for the coalitions. As the above example shows, in our framework, the existence of side-optimal stable matchings is not guaranteed, and indeed, the above example cannot be reinterpreted as coalitional choice where buyers form a coalition with a choice function $C^{\mathcal{B}}$ that has no externalities: to have $\{x_1,x_2\}$ as a stable matching in the example, we need the coalitional choice to satisfy $C^{\mathcal{B}}(\{x_1,x_2\})=\{x_1,x_2\}$. Then substitutability implies that $C^{\mathcal{B}}(X)=\{X\}$ for every $X\subseteq\{x_1,x_2\}$. Therefore, every matching is stable with this coalitional choice unlike the example above which has only two stable matchings.*

6. A CHARACTERIZATION OF SUBSTITUTABLE CHOICE FUNCTIONS

Which choice functions are substitutable? We establish a simple structure of substitutable choice functions. We describe the structure using the standard matching concept of truncation (see Roth and Rothblum (1999)). Linear order $\succ'$ over $\mathcal{X}_i\cup\{\varnothing\}$ is a *truncation* of linear order $\succ$ over $\mathcal{X}_i\cup\{\varnothing\}$ if, for all $x,y\in\mathcal{X}_i$ the following two implications hold true:

- $x\succ'\varnothing$ implies $x\succ\varnothing$, and
- $x\succ' y\succ'\varnothing$ implies $x\succ y\succ\varnothing$.

In words, any contract ranked above the empty set by the linear order $\succ'$ is also ranked above the empty set by the linear order $\succ$ and the relative ranking of any two contracts preferred to the empty set in the linear order $\succ'$ is the same as in the linear order $\succ$. Therefore, a truncation of a linear order moves the outside option $\varnothing$ higher in the ranking.

The next result characterizes choice functions satisfying our substitutability condition.

Theorem 5. *(Characterization of substitutability) Choice function C^θ satisfies substitutability if, and only if, for every agent $i\in\theta$ there is a non-empty set $\mathcal{J}$ and linear orders $\succ_j^{\mu_{-i}}$ over $\mathcal{X}_i\cup\{\varnothing\}$ indexed by $j\in\mathcal{J}$ and matching μ_{-i} that does not include i's contracts such that if $\mu'_{-i}\succsim^\theta\mu_{-i}\succsim^\theta\varnothing$ then for any $j\in\mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$. Furthermore, for any $X,\mu\subseteq\mathcal{X}$,*

$$c_i(X_i|\mu_{-i})=\bigcup_{j\in\mathcal{J}}\{x_j^{\mu_{-i}}\},$$

where $x_j^{\mu_{-i}}$ is the maximum element of $X_i\cup\{\varnothing\}$ in order $\succ_j^{\mu_{-i}}$.

This result is inspired by the Aizerman and Malishevski (1981) decomposition result for substitutable functions when there are no externalities. It states that the choice function can be constructed from a set of linear orders over individual contracts such that the choice from a set conditional on a reference set is the union of the most-preferred contracts with respect to these linear orders. In this representation, the linear orders depend on the reference set and as the reference set gets better with respect to the better market pre-order the linear orders are truncated.[39]

Theorem 5 takes a particularly simple form in the context of the local labour market model of Section 3. In the simplest version of this model, each couple in the labour market consists of a primary and a secondary earner. The choices of a primary earner exhibit no externalities and hence satisfy our substitutes condition. Choices of a secondary earner can exhibit externalities and the choice function of a secondary satisfies the substitutes condition if and only if it is represented by a family of rankings indexed by the contract of the primary earner. These rankings only differ in how being unemployed is ranked: the higher the wage of the primary earner is, the higher is the reservation wage of the secondary earner. For instance, in Example 1, which is a special case of the local labour market application, the choice function of buyer b_2 can be represented as choosing the maximal element with respect to the linear order $x_3 \succ_{b_2}^{\{x_1\}} x_2 \succ_{b_2}^{\{x_1\}} \varnothing$ when $\mu_{b_1} = \{x_1\}$, and with respect to the linear order $x_3 \succ_{b_2}^{\varnothing} \varnothing$ when $\mu_{b_1} = \varnothing$.

7. CONCLUSION

In this article, we have studied a two-sided matching problem with externalities where each agent's choice depends on other agents' contracts. For such settings, we have developed the theory of stable matchings by introducing a new substitutability condition when externalities are present. More explicitly, we have studied the existence of stable matchings, Pareto efficiency of stable matchings, side-optimal stable matchings, the deferred acceptance algorithm, comparative statics, and the rural hospitals theorem (the latter two in Supplementary Appendix C).

The standard substitutability condition can be weakened without affecting our results in two different ways. In the first approach, the reference set can be restricted to be a set that can be chosen by side θ. More formally, consider the minimal set of matchings $\mathcal{A}^\theta$ that contains the empty set and satisfies $C^\theta(X|\mu) \in \mathcal{A}^\theta$ whenever $X \subseteq \mathcal{X}$ and $\mu \in \mathcal{A}^\theta$. The minimal such domain is $\mathcal{A}^\theta \equiv \bigcup_{t=0,1,\ldots} \mathcal{A}_t^\theta$, where $\mathcal{A}_0^\theta \equiv \{\varnothing\}$ and $\mathcal{A}_t^\theta$ for $t \geq 1$ are defined recursively

$$\mathcal{A}_t^\theta \equiv \{C^\theta(X|\mu) : X \subseteq \mathcal{X}, \mu \in \mathcal{A}_{t-1}^\theta\} \cup \mathcal{A}_{t-1}^\theta.$$

Since there exists a finite number of contracts, $\mathcal{A}^\theta$ is well defined; it is the set of all matchings that can be reached from the empty set by applying the choice function C^θ. Standard substitutability can be weakened by imposing it only for reference sets in $\mathcal{A}^\theta$.

The second approach to weaken standard substitutability works only when agents on one side of the market have unit demand using the techniques developed in Hatfield and Kojima (2010), Hatfield and Kominers (2016), and Hatfield, Kominers and Westkamp (2020) when there are no externalities. These conditions usually proceed by restricting X' and X under which the standard

39. Can we interpret rankings $\succ_j^{\mu_{-i}}$ in this theorem as preferences of sub-agents for agent i? Such an interpretation runs into the problem that two or more of the sub-agents might rank the same contract x as their best contract from a choice set and, in general, it is not possible to designate one of these subagents to be the signatory of x. In fact, Remark 1 above shows that—despite Theorem 5—our conditions cannot be in general reinterpreted as coalitional choices. We would like to thank an anonymous referee for raising the question.

substitutability condition holds. Such conditions can also be studied in our setting when one side of the market can sign at most one contract. Furthermore, a combination of the two approaches can be used when agents on one side of the market have unit demand.

Our notion of substitutability may be useful to study other important questions in matching markets with externalities. For example, the relations between pairwise stability, group stability, core, and other stability concepts have been an important question in classical matching theory at least since Blair (1988). We analyse the relation between pairwise and group stability in Supplementary Appendix C.3, but many related questions remain open. The strategy proofness of the deferred acceptance algorithm (for the proposing side) has been another important question extensively studied since Dubins and Freedman (1981). We think that a deferred acceptance procedure remains strategy-proof in our setting provided we impose the law of aggregate demand à la Hatfield and Milgrom (2005); we leave an exploration of this question for future work. Furthermore, even though we have studied two-sided markets, we think that our techniques are applicable to more general markets such as the supply chain networks of Ostrovsky (2008) where externalities may naturally appear.[40]

Our techniques might also be applied to study one-sided allocation in the presence of externalities across agents. The earlier theoretical literature provided analysis of substitutes and complementarities among assigned goods but usually assumed the absence of externalities across agents; cf. Budish (2011), Budish and Cantillon (2012), and Miralles and Pycia (2020). The main exception is Baccara, Imrohoroglu, Wilson and Yariv (2012), who analyse stable one-sided allocations and, in addition to an in-depth empirical analysis of office allocation at a university, they prove that stable one-sided allocations exist in the presence of externalities provided these externalities have no impact on agents' choice behaviour; in contrast, we allow externalities that may affect behaviour.[41],[42]

Data Availability Statement

No new data were generated in support of this research.

A. APPENDIX: PROOF OF THEOREM 1 AND THE FIXED-POINT APPROACH TO STABILITY

The Proof of Theorem 1 builds on the fixed-point methods used in Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Bando (2014), and others. As the first two steps in the proof, we construct a function that mimics the iterative step of the deferred acceptance algorithm and study the properties of its fixed points.

Each iteration in the second phase of our deferred acceptance algorithm can be described as the following transformation function

$$f\left(A^s, A^b, \mu^s, \mu^b\right) \equiv \left(\mathcal{X}\backslash R^{\mathcal{B}}(A^b|\mu^b), \mathcal{X}\backslash R^{\mathcal{S}}(A^s|\mu^s), C^{\mathcal{S}}\left(A^s|\mu^s\right), C^{\mathcal{B}}(A^b|\mu^b)\right),$$

40. Subsequent to our work, some of these questions and related ones have been investigated in Fisher and Hafalir (2016), Ali (2016), Dur and Wiseman (2019), Rostek and Yoder (2019, 2020), Leshno (2021), Kumano and Marutani (2021), and Cox, Fonseca and Pakzad-Hurson (2021).

41. Hong and Park (2018) study externalities that have no impact on agents' behaviour in the context of house allocation; they assume that agents' preferences over objects do not exhibit externalities but allow agents to have lexicographically second-order preferences over economy-wide assignments. Since mechanisms based on the top trading cycles algorithm are non-bossy, these second-order preferences have no impact on agents' behaviour. Frys and Heller (2016) assume that agents are partitioned into groups of friends—any two friends have identical preferences and care about each other assignments, there are no externalities across friends—and study mechanisms based on the random serial dictatorship.

42. The empirical literature offered careful tests of whether peer effects affect schooling outcomes, that is, whether there are payoff externalities among students in schools, see, e.g., Sacerdote (2001), Duflo, Dupas and Kremer (2011); for a survey, see Angrist (2014).

where f is a function from $2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}} \times 2^{\mathcal{X}}$ into itself. Function f has two important properties, monotonicity and stability of its fixed points.

Lemma 1. *Suppose that the choice functions satisfy substitutability. Then function f is monotone increasing with respect to the pre-order $\sqsubsetsim$ defined as follows:*

$$(A^s, A^b, \mu^s, \mu^b) \sqsubsetsim (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b) \Longleftrightarrow A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \precsim^{\mathcal{S}} \tilde{\mu}^s, \mu^b \succsim^{\mathcal{B}} \tilde{\mu}^b.$$

Proof. Function f is monotonic in $\sqsubsetsim$ because for any $A^s \subseteq \tilde{A}^s, A^b \supseteq \tilde{A}^b, \mu^s \precsim^{\mathcal{S}} \tilde{\mu}^s, \mu^b \succsim^{\mathcal{B}} \tilde{\mu}^b$, substitutability implies that

$$\mathcal{X} \backslash R^{\mathcal{B}}(A^b|\mu^b) \subseteq \mathcal{X} \backslash R^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b),$$
$$\mathcal{X} \backslash R^{\mathcal{S}}(A^s|\mu^s) \supseteq \mathcal{X} \backslash R^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s),$$

and consistency implies that

$$C^{\mathcal{S}}(A^s|\mu^s) \precsim^{\mathcal{S}} C^{\mathcal{S}}(\tilde{A}^s|\tilde{\mu}^s),$$
$$C^{\mathcal{B}}(A^b|\mu^b) \succsim^{\mathcal{B}} C^{\mathcal{B}}(\tilde{A}^b|\tilde{\mu}^b).$$

Therefore, $(A^s, A^b, \mu^s, \mu^b) \sqsubsetsim (\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$ implies that $f(A^s, A^b, \mu^s, \mu^b) \sqsubsetsim f(\tilde{A}^s, \tilde{A}^b, \tilde{\mu}^s, \tilde{\mu}^b)$. □

The fixed points of f satisfy the following properties even when the choice functions do not satisfy substitutability or monotone externalities.

Lemma 2. *Let (A^s, A^b, μ^s, μ^b) be a fixed point of function f. Then $A^s \cup A^b = \mathcal{X}$ and*

$$\mu^s = \mu^b = A^s \cap A^b = C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{S}}(A^s|\mu^s).$$

Proof. $A^s \cup A^b = A^s \cup \left[\mathcal{X} \backslash R^{\mathcal{S}}(A^s|\mu^s))\right] \supseteq A^s \cup [\mathcal{X} \backslash A^s] = \mathcal{X}$, so

$$A^s \cup A^b = \mathcal{X}.$$

Similarly, $A^s \cap A^b = A^s \cap \left[\mathcal{X} \backslash R^{\mathcal{S}}(A^s|\mu^s))\right] = A^s \backslash R^{\mathcal{S}}(A^s|\mu^s) = C^{\mathcal{S}}(A^s|\mu^s)$, which implies $C^{\mathcal{S}}(A^s|\mu^s) = A^s \cap A^b$. Analogously for buyers, $C^{\mathcal{B}}(A^b|\mu^b) = A^s \cap A^b$. Finally, $\mu^s = C^{\mathcal{S}}(A^s|\mu^s)$ and $\mu^b = C^{\mathcal{B}}(A^b|\mu^b)$ imply

$$\mu^s = \mu^b = A^s \cap A^b = C^{\mathcal{B}}(A^b|\mu^b) = C^{\mathcal{S}}(A^s|\mu^s).$$

□

When choice functions satisfy standard substitutability, a matching is stable if, and only if, it can be supported as a fixed point of f.

Lemma 3. *(Characterization of stability) Suppose that the choice functions satisfy standard substitutability. Then a matching μ is stable if, and only if, there exist sets of contracts $A^s, A^b \subseteq \mathcal{X}$ such that (A^s, A^b, μ, μ) is a fixed point of function f.*

Proof. First, suppose that (A^s, A^b, μ, μ) is a fixed point of f. Claim 1 below shows that μ is a stable matching.

Claim 1. Suppose that the choice functions satisfy standard substitutability. Then matching μ is stable.

Proof. Suppose, for contradiction, that μ is not stable. Then, there are three possibilities, all of which we proceed to rule out.

1. Matching μ is not individually rational for some seller j, that is, $c_j(\mu|\mu) \subsetneq \mu_j$. Since (A^s, A^b, μ, μ) is a fixed point of f, $C^{\mathcal{S}}(A^s|\mu) = \mu$ and $A^s \supseteq \mu$. But standard substitutability and $c_j(\mu|\mu) \subsetneq \mu_j$ imply that there is a contract $x \in \mu_j$ rejected out of A^s by agent j, that is $x \notin C^{\mathcal{S}}(A^s|\mu)$, a contradiction.
2. Matching μ is not individually rational for some buyer i, that is, $c_i(\mu|\mu) \subsetneq \mu_i$. This is analogous to the previous case since f treats buyers and sellers symmetrically.
3. There exists a blocking pair $i \in \mathcal{B}$ and $j \in \mathcal{S}$ with contract $x \in \mathcal{X}_i \cap \mathcal{X}_j$ such that $x \notin \mu$ and $x \in c_i(\mu \cup \{x\}|\mu) \cap c_j(\mu \cup \{x\}|\mu)$. Since (A^s, A^b, μ, μ) is a fixed point of f, by Lemma 2, $A^s \cup A^b = \mathcal{X}$. Therefore, without loss of generality, assume that $x \in A^b$. Again, since (A^s, A^b, μ, μ) is a fixed point of f, by Lemma 2, $C^{\mathcal{B}}(A^b|\mu) = \mu$, which implies that $c_i(A^b|\mu) = \mu_i$. By the irrelevance of rejected contracts, for any set Y such that $A^b \supseteq Y \supseteq \mu$, $c_i(Y|\mu) = \mu_i$. In particular, for $Y = \mu \cup \{x\}$, $c_i(\mu \cup \{x\}|\mu) = \mu_i$, which is a contradiction because $x \in c_i(\mu \cup \{x\}|\mu) \backslash \mu$.

To finish the proof of the theorem, we need to show that if matching μ is stable then there exist sets of contracts A^s, A^b such that (A^s, A^b, μ, μ) is a fixed point of f. The following is useful in our construction of A^s and A^b.

Claim 2. Suppose that the choice functions satisfy standard substitutability. Then, the function $M^\theta(\mu) \equiv \max\{X \subseteq \mathcal{X} | C^\theta(X|\mu) = \mu\}$, where the maximum is with respect to set inclusion, is well defined. Moreover, for any contract $x \notin M^\theta(\mu)$, $x \in C^\theta(M^\theta(\mu) \cup x|\mu)$.

Proof. If there are two sets M' and M'' such that $C^\theta(M'|\mu) = C^\theta(M''|\mu) = \mu$, then (by standard substitutability)

$$\begin{aligned} C^\theta(M' \cup M''|\mu) &= (M' \cup M'') \setminus R^\theta(M' \cup M''|\mu) = [M' \setminus R^\theta(M' \cup M''|\mu)] \cup [M'' \setminus R^\theta(M' \cup M''|\mu)] \\ &\subseteq [M' \setminus R^\theta(M'|\mu)] \cup [M'' \setminus R^\theta(M''|\mu)] = \mu. \end{aligned}$$

If $C^\theta(M' \cup M''|\mu)$ was a proper subset of μ, then the irrelevance of rejected contracts would imply that $C^\theta(M'|\mu) = C^\theta(M''|\mu) = C^\theta(M' \cup M''|\mu)$, which is a contradiction. Therefore, $M^\theta(\mu)$ is well defined. Let $x \notin M^\theta(\mu)$. If $x \notin C^\theta(M^\theta(\mu) \cup x|\mu)$, then $C^\theta(M^\theta(\mu) \cup x|\mu) = C^\theta(M^\theta(\mu)|\mu)$ by the irrelevance of rejected contracts. But, this implies $C^\theta(M^\theta(\mu) \cup x|\mu) = \mu$, which contradicts maximality of $M^\theta(\mu)$. Hence, $x \in C^\theta(M^\theta(\mu) \cup x|\mu)$.

Claim 3. Suppose that matching μ is stable and the choice functions satisfy standard substitutability. Then, there exist sets of contracts A^s and A^b such that (A^s, A^b, μ, μ) is a fixed point of f.

Proof. By Claim 2, there exists the largest set $M^\theta(\mu) = \max\{X \subseteq \mathcal{X} | C^\theta(X|\mu) = \mu\}$. Let $A^s \equiv M^{\mathcal{S}}(\mu)$ and $A^b \equiv \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)$. By construction of $M^{\mathcal{S}}(\mu)$, $\mu = C^{\mathcal{S}}(A^s|\mu)$. Thus, we get $A^s \cap A^b = A^s \cap (\mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)) = C^{\mathcal{S}}(A^s|\mu) = \mu$. To finish the proof, we need to show $\mu = C^{\mathcal{B}}(A^b|\mu)$ and $A^s = \mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu)$.

Note that $A^b = \mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup C^{\mathcal{S}}(A^s|\mu) = (\mathcal{X} \setminus A^s) \cup \mu$. Therefore, $A^b \supseteq \mu$. If $Y \equiv C^{\mathcal{B}}(A^b|\mu) \neq \mu$, there are two cases, both of which contradict stability of μ. First, if $Y \subsetneq \mu$, then the irrelevance of rejected contracts implies $C^{\mathcal{B}}(\mu|\mu) = Y$, implying that μ is not individually rational for some buyers, contradicting stability. Second, if $Y \nsubseteq \mu$, then there exists $y \in Y \setminus \mu$, and $y \in C^{\mathcal{B}}(\mu \cup \{y\}|\mu)$ by standard substitutability since $y \in C^{\mathcal{B}}(A^b|\mu)$ and $A^b \supseteq \mu \cup \{y\}$. But we also have that $y \in C^{\mathcal{S}}(A^s \cup \{y\}|\mu)$ by Claim 2. Then, the agents associated with $\{y\}$ block μ, contradicting stability. Thus, the only case consistent with stability is $C^{\mathcal{B}}(A^b|\mu) = \mu$.

Finally, we show that $A^s = \mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu) = \mathcal{X} \setminus R^{\mathcal{B}}(\mathcal{X} \setminus R^{\mathcal{S}}(A^s|\mu)|\mu)$. Since $C^{\mathcal{B}}(A^b|\mu) = \mu$, then $\mathcal{X} \setminus R^{\mathcal{B}}(A^b|\mu) = \mathcal{X} \setminus (A^b \setminus \mu) = \mathcal{X} \setminus (((\mathcal{X} \setminus A^s) \cup \mu) \setminus \mu) = \mathcal{X} \setminus (\mathcal{X} \setminus A^s) = A^s$ and we have the result. □

Proof of Theorem 1. First, let us consider the first phase of the algorithm and check that $\mu^* \succsim^{\mathcal{S}} C^{\mathcal{S}}(\mathcal{X}|\mu^*)$. Since $C^{\mathcal{S}}(\mathcal{X}|\mu_{k-1}) = \mu_k$, by the irrelevance of rejected contracts, we get $C^{\mathcal{S}}(\mu_k|\mu_{k-1}) = \mu_k$ for every $k \geq 1$. We show that $\mu_k \succsim^{\mathcal{S}} \mu_{k-1}$ for every $k \geq 1$. The proof is by mathematical induction on k. For the base case when $k = 1$, note that $\mathcal{X} \supseteq \varnothing$ and consistency imply that

$$\mu_1 = C^{\mathcal{S}}(\mathcal{X}|\varnothing) \succsim^{\mathcal{S}} C^{\mathcal{S}}(\varnothing|\varnothing) = \varnothing = \mu_0.$$

For the general case, $\mu_k \succsim^{\mathcal{S}} \mu_{k-1}$ and $\mathcal{X} \supseteq \mu_k$ imply that (by consistency)

$$\mu_{k+1} = C^{\mathcal{S}}(\mathcal{X}|\mu_k) \succsim^{\mathcal{S}} C^{\mathcal{S}}(\mu_k|\mu_{k-1}) = \mu_k.$$

Therefore, $\{\mu_k\}_{k \geq 1}$ is a monotone sequence with respect to the pre-order $\succsim^{\mathcal{S}}$. Since the number of contracts is finite, there exists n and $m \geq n$ such that $\mu_{m+1} = \mu_n$; we take the minimum m satisfying this property and set $\mu^* = \mu_m$. Then,

$$C^{\mathcal{S}}(\mathcal{X}|\mu_m) = \mu_{m+1} = \mu_n \precsim^{\mathcal{S}} \mu_m,$$

where the monotonicity comparison follows because $\precsim^{\mathcal{S}}$ is transitive.

It remains to show that the second phase converges and that the resulting matching is stable. It is easy to see that $f(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsubsetsim (\mathcal{X}, \varnothing, \mu^*, \varnothing)$ because $C^{\mathcal{S}}(\mathcal{X}|\mu^*) \precsim^{\mathcal{S}} \mu^*$ by construction and $C^{\mathcal{B}}(\varnothing|\varnothing) = \varnothing \succsim^{\mathcal{B}} \varnothing$ by reflexivity of $\succsim^{\mathcal{B}}$. By Lemma 1, f is monotone increasing, so we can repeatedly apply it to $f(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsubsetsim (\mathcal{X}, \varnothing, \mu^*, \varnothing)$ to get $f^k(\mathcal{X}, \varnothing, \mu^*, \varnothing) \sqsubsetsim f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing)$ for every $k \geq 1$. We consider two separate possibilities. Suppose, first that this sequence converges. Therefore, there exists k such that $f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing) = f^k(\mathcal{X}, \varnothing, \mu^*, \varnothing)$. As a result, $f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing)$ is a fixed point of f. Let $(\hat{A}^s, \hat{A}^b, \hat{\mu}^s, \hat{\mu}^b) \equiv f^{k-1}(\mathcal{X}, \varnothing, \mu^*, \varnothing)$. By Lemma 2, $\hat{\mu}^s = \hat{\mu}^b = \hat{A}^s \cap \hat{A}^b$ and, by Theorem 3, $\hat{A}^s \cap \hat{A}^b$ is a stable matching.

Otherwise, if the sequence does not converge, there exists a subsequence

$$f^n(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq f^{n+1}(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq \ldots \sqsupseteq f^m(\mathcal{X},\varnothing,\mu^*,\varnothing) \sqsupseteq f^{m+1}(\mathcal{X},\varnothing,\mu^*,\varnothing) = f^n(\mathcal{X},\varnothing,\mu^*,\varnothing)$$

because the number of contracts is finite. By transitivity of the pre-order $\sqsupseteq$ and the previous inequality, we get $f^n(\mathcal{X},\varnothing,\mu^*,\varnothing)=f^{m+1}(\mathcal{X},\varnothing,\mu^*,\varnothing)\sqsupseteq f^m(\mathcal{X},\varnothing,\mu^*,\varnothing)\sqsupseteq f^n(\mathcal{X},\varnothing,\mu^*,\varnothing)$. Let $f^n(\mathcal{X},\varnothing,\mu^*,\varnothing)=(A_1^s,A_1^b,\mu_1^s,\mu_1^b)$ and $f^m(\mathcal{X},\varnothing,\mu^*,\varnothing)=(A_2^s,A_2^b,\mu_2^s,\mu_2^b)$. By definition of $\sqsupseteq$, we get that $A_1^s=A_2^s$, $A_1^b=A_2^b$, $\mu_1^s\sim^s\mu_2^s$, and $\mu_1^b\sim^b\mu_2^b$. Now, by construction $C^{\mathcal{S}}(A_2^s|\mu_2^s)=\mu_1^s$ and by monotone externalities $C^{\mathcal{S}}(A_2^s|\mu_2^s)=C^{\mathcal{S}}(A_1^s|\mu_1^s)$, which imply that $C^{\mathcal{S}}(A_1^s|\mu_1^s)=\mu_1^s$. Similarly, we get that $C^{\mathcal{S}}(A_1^s|\mu_1^b)=\mu_1^b$. Furthermore, by monotone externalities, $\mathcal{X}\backslash R^{\mathcal{B}}(A_2^b|\mu_2^b)=\mathcal{X}\backslash R^{\mathcal{B}}(A_1^b|\mu_1^b)$ and, by construction, $\mathcal{X}\backslash R^{\mathcal{B}}(A_2^b|\mu_2^b)=A_1^b$, which imply $\mathcal{X}\backslash R^{\mathcal{B}}(A_1^b|\mu_1^b)=A_1^b$. Similarly, we get $\mathcal{X}\backslash R^{\mathcal{S}}(A_1^s|\mu_1^s)=A_1^s$. Therefore, $(A_1^s,A_1^b,\mu_1^s,\mu_1^b)$ is a fixed point of f. This shows that the sequence converges as in the previous paragraph, which is a contradiction. Therefore, there exists a stable matching. □

B. APPENDIX: REMAINING PROOFS

B.1. *Proof of Theorem 2*

Since agent i's choice function c_i has externalities, there exist $X,\mu,\mu'\subseteq\mathcal{X}$ such that $c_i(X|\mu')\neq c_i(X|\mu)$. This implies, without loss of generality, that there exists a contract $x\in X_i$ such that $x\in c_i(X_i|\mu'_{-i})$ and $x\notin c_i(X_i|\mu_{-i})$. We construct choice functions of agents other than i satisfying the stated properties such that no stable matching exists.

The choice functions of agents on side $-\theta$ exhibit no externalities. Furthermore, each agent chooses all the contracts in $\mu_{-i}\cup\mu'_{-i}\cup X_i$ that are associated with them whenever they are available. No other contracts are chosen. The choice functions of agents on side θ other than i depend on whether the reference set has contract x or not. When contract x is in the reference set, each agent chooses contracts in μ_{-i} associated with them. When contract x is not in the reference set, then each agent chooses contracts in μ'_{-i} associated with them. Otherwise, no contracts are chosen.

We first check that the properties in the statement of this result are satisfied. The agents on side $-\theta$ have choice functions that have no externalities. Furthermore, $C^{-\theta}$ satisfies substitutability and the irrelevance of rejected contracts. Now, consider the minimum consistent pre-order $\succsim^{\theta\setminus\{i\}}$ for $C^{\theta\setminus\{i\}}$. Any reference set μ in the domain of $\succsim^{\theta\setminus\{i\}}$ does not include contract x because, for every agent $j\in\theta\setminus\{i\}$, $\succsim_j$ is a pre-order with a domain that is a subset of $2^{\mathcal{X}_j}$, so no matching in this domain includes contract x. Therefore, for any $X\subseteq\mathcal{X}$, $C^{\theta\setminus\{i\}}(X|\mu)$ is the same for all μ in the domain of $\succsim^{\theta\setminus\{i\}}$ because μ does not have contract x, implying that monotone externalities is satisfied. Furthermore, by construction, standard substitutability and the irrelevance of rejected contracts are also satisfied. Hence, $C^{\theta\setminus\{i\}}$ satisfies substitutability.

Suppose, for contradiction, that there exists a stable matching Y. We consider two possibilities:

Case 1: Consider the case when $x\in Y$. If a contract in μ_{-i} is not in Y, then the agents associated with the contract form a blocking pair. Thus, every contract in μ_{-i} must be signed, so $\mu_{-i}\subseteq Y_{-i}$. Furthermore, $Y_{-i}\setminus\mu_{-i}$ cannot have a contract as Y would not be individually rational for agents on side θ. Therefore, $\mu_{-i}=Y_{-i}$. Likewise, there cannot be any contract in $Y_i\setminus X_i$ because of individual rationality for agents on side $-\theta$. This implies that $Y_i\subseteq X_i$. If there exists a contract $x'\in c_i(X_i|\mu_{-i})\setminus Y_i$, then agents associated with contract x' block Y because $x'\in c_i(Y_i\cup\{x'\}|\mu_{-i})$ by standard substitutability. Therefore, $Y_i\supseteq c_i(X_i|\mu_{-i})$. By the irrelevance of rejected contracts, $c_i(Y_i|\mu_{-i})=c_i(X_i|\mu_{-i})$, which is a contradiction since $x\in Y_i=c_i(Y_i|\mu_{-i})$ by individual rationality of Y and $x\notin c_i(X_i|\mu_{-i})$ by construction.

Case 2: Consider the case when $x\notin Y$. As in the previous case, it is easy to see that $Y_{-i}=\mu'_{-i}$. Likewise, $Y_i\subseteq X_i$. Since $x\in c_i(X_i|\mu'_{-i})$ by construction, $x\in c_i(Y_i\cup\{x\}|\mu'_{-i})$ by standard substitutability. But, this is a contradiction because $x\notin Y$ implies that the agents associated with contract x form a blocking pair.

Therefore, there exists no stable matching. □

Proof of Theorem 4

Without loss of generality assume that $\theta=\mathcal{S}$. For any $A^s,A^b\subseteq\mathcal{X}$, $\mu^s\subseteq\mathcal{X}$ that can be chosen by sellers, and $\mu^b\subseteq\mathcal{X}$ that can be chosen by buyers, we have $(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)\sqsupseteq(A^s,A^b,\mu^s,\mu^b)$. Therefore, $(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)\sqsupseteq f(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)$. By Lemma 1, function f is monotone increasing, so we can repeatedly apply it to the last inequality to get $f^{k-1}(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)\sqsupseteq f^k(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)$ for every $k\geq 1$. Since $2^{\mathcal{X}}\times 2^{\mathcal{X}}\times 2^{\mathcal{X}}\times 2^{\mathcal{X}}$ is a finite set, this sequence converges at some point as in the Proof of Theorem 1, so there exists k such that $f^{k-1}(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)=f^k(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)$. Therefore, $f^{k-1}(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)$ is a fixed point of f. By Lemma 2, there is $\left(\hat{A}^s,\hat{A}^b,\hat{\mu},\hat{\mu}\right)$ that is equal to $f^{k-1}(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)$. Theorem 3 tells us that $\hat{\mu}$ is a stable matching, which is the outcome of the seller-proposing deferred acceptance algorithm.

We next show that $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. Let μ be any stable matching. By Theorem 3, there exist A^s and A^b such that (A^s,A^b,μ,μ) is a fixed point of f. Since $(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)\sqsupseteq(A^s,A^b,\mu,\mu)$ and f is monotonic increasing, f can be applied repeatedly while preserving the order. Therefore, $f^k(\mathcal{X},\varnothing,\bar{\mu}^s,\varnothing)\sqsupseteq f^k(A^s,A^b,\mu,\mu)$

for every k, which implies $\left(\hat{A}^s,\hat{A}^b,\hat{\mu},\hat{\mu}\right)\sqsupseteq(A^s,A^b,\mu,\mu)$. Therefore, $\hat{\mu}\succsim^{\mathcal{S}}\mu$ and $\hat{\mu}\precsim^{\mathcal{B}}\mu$, so $\hat{\mu}$ is a seller-optimal and buyer-pessimal stable matching. □

Proof of Theorem 5

We first show the necessity that when C^θ satisfies substitutability, then, for each agent $i\in\theta$, there exists a list of preferences with the stated properties.

For any μ_{-i}, we can construct a list of preferences as follows. Let $x_1\in c_i(\mathcal{X}|\mu_{-i})$, $x_2\in c_i(\mathcal{X}\setminus\{x_1\}|\mu_{-i})$, $x_3\in c_i(\mathcal{X}\setminus\{x_1,x_2\}|\mu_{-i})$, ... , $x_k\in c_i(\mathcal{X}\setminus\{x_1,\ldots,x_{k-1}\}|\mu_{-i})$, and $c_i(\mathcal{X}\setminus\{x_1,\ldots,x_k\}|\mu_{-i})=\varnothing$. This sequence creates an incomplete preference ranking over $\mathcal{X}_i\cup\{\varnothing\}$: $x_1\succ^{\mu_{-i}}\ldots\succ^{\mu_{-i}}x_k\succ^{\mu_{-i}}\varnothing$. Consider all such preference rankings $(\succ_j^{\mu_{-i}})_{j\in\mathcal{J}}$. We need the following:

Claim: For any $X,\mu\subseteq\mathcal{X}$, $c_i(X|\mu_{-i})=\bigcup_{j\in\mathcal{J}}\{x_j^{\mu_{-i}}\}$, where $x_j^{\mu_{-i}}=\max_{\succ_j^{\mu_{-i}}}(X\cup\{\varnothing\})$.[43]

Let $x\in c_i(X|\mu_{-i})$. We show that $x=x_j^{\mu_{-i}}$ for some $j\in\mathcal{J}$ when X is the set of contracts. If $x\in c_i(\mathcal{X}|\mu_{-i})$, then $x=x_j^{\mu_{-i}}$ for some j. Suppose that $x\notin c_i(\mathcal{X}|\mu_{-i})$. If $c_i(X|\mu_{-i})\supseteq c_i(\mathcal{X}|\mu_{-i})$, then the irrelevance of rejected contracts would imply $c_i(X|\mu_{-i})=c_i(\mathcal{X}|\mu_{-i})$, which is a contradiction because $x\in c_i(X|\mu_{-i})\setminus c_i(\mathcal{X}|\mu_{-i})$. Therefore, there exists $x_1\in c_i(\mathcal{X}|\mu_{-i})\setminus c_i(X|\mu_{-i})$. Standard substitutability implies that $x_1\notin X$. Consider preference rankings in $\mathcal{J}$ that have x_1 as their maximal contract. If $x\in c_i(\mathcal{X}\setminus\{x_1\}|\mu_{-i})$, then we are done since x_1 would be the maximal element of X with respect to a preference ranking since $x_1\notin X$ and there would be a preference ranking in $\mathcal{J}$ such that $x_1\succ x\succ\ldots$. Suppose that $x\notin c_i(\mathcal{X}\setminus\{x_1\}|\mu_{-i})$. By the irrelevance of rejected contracts, we cannot have $c_i(X|\mu_{-i})\supseteq c_i(\mathcal{X}\setminus\{x_1\}|\mu_{-i})$. Therefore, there exists $x_2\in c_i(\mathcal{X}\setminus\{x_1\}|\mu_{-i})\setminus c_i(X|\mu_{-i})$. Standard substitutability implies that $x_2\notin X$. Repeat this argument. Suppose, for contradiction, that $x\notin c_i(\mathcal{X}\setminus\{x_1,\ldots,x_j\}|\mu_{-i})$ for all j. But, there must exist some j^* for which $\mathcal{X}\setminus\{x_1,\ldots,x_{j^*}\}\subseteq X$. Then $x\in c_i(X|\mu_{-i})$ and standard substitutability imply that $x\in c_i(\mathcal{X}\setminus\{x_1,\ldots,x_{j^*}\}|\mu_{-i})$. This is a contradiction. Therefore, $x\in c_i(\mathcal{X}\setminus\{x_1,\ldots,x_{j^*}\}|\mu_{-i})$ for some j^*, which implies that $x=x_j^{\mu_{-i}}$ for some $j\in\mathcal{J}$ because $\{x_1,\ldots,x_{j^*}\}\cap X=\varnothing$. Since $x\in c_i(X|\mu_{-i})$ implies $x=x_j^{\mu_{-i}}$ for some $j\in\mathcal{J}$, we get $c_i(X|\mu_{-i})\subseteq\bigcup_{j\in\mathcal{J}}\{x_j^{\mu_{-i}}\}$.

Now let $x=x_j^{\mu_{-i}}$ for some j. This implies that for every $y\succ_j^{\mu_{-i}}x$, we have $y\notin X$. By construction, $x\in c_i(\mathcal{X}\setminus\bigcup_{y:y\succ_j^{\mu_{-i}}x}\{y\}|\mu_{-i})$. Standard substitutability and the fact that $\mathcal{X}\setminus\bigcup_{y:y\succ_j^{\mu_{-i}}x}\{y\}\supseteq X$ imply that $x\in c_i(X|\mu_{-i})$. This argument proves that $\bigcup_{j\in\mathcal{J}}\{x_j^{\mu_{-i}}\}\subseteq c_i(X|\mu_{-i})$. Therefore, $\bigcup_{j\in\mathcal{J}}\{x_j^{\mu_{-i}}\}=c_i(X|\mu_{-i})$, which concludes the proof of the claim.

Next we prove that, for any $\mu'_{-i}\succsim^\theta\mu_{-i}\succsim^\theta\varnothing$ and $j\in\mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$.

Take $\mu=\varnothing$ and construct the list of preferences $(\succ_j^{\varnothing})_{j\in\mathcal{J}}$ as above. For any $\mu_{-i}\succsim^\theta\varnothing$ and $X\subseteq\mathcal{X}$, $c_i(X|\mu_{-i})\subseteq c_i(X|\varnothing)$ by monotone externalities. Thus, for each j, we can truncate the preference ranking $\succ_j^{\varnothing}$ to get a sequence as constructed above, call it $\succ_j^{\mu_{-i}}$.

For each $\mu-i\succsim^\theta\varnothing$, $c_i(X|\mu_{-i})=\bigcup_{j\in\mathcal{J}}\{x_j^{\mu_{-i}}\}$ where $x_j^{\mu_{-i}}=\max_{\succ_j^{\mu_{-i}}}(X\cup\{\varnothing\})$ by construction. Furthermore, for any $\mu'_{-i}\succsim^\theta\mu_{-i}\succsim^\theta\varnothing$ and $X\subseteq\mathcal{X}$, $c_i(X|\mu'_{-i})\subseteq c_i(X|\mu_{-i})$ by monotone externalities. Therefore, for any j, $\succ_j^{\mu'_{-i}}$ and $\succ_j^{\mu_{-i}}$ are both truncations of $\succ_j^{\varnothing}$ such that $\succ_j^{\mu'_{-i}}$ is truncated at a weakly more-preferred contract than $\succ_j^{\mu_{-i}}$. Therefore, we get the conclusion that for any $j\in\mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$.

Finally, we show the sufficiency that when there exists a list of preferences with the desired properties, then C^θ satisfies substitutability. Standard substitutability follows from the decomposition result of Aizerman and Malishevski (1981). To show monotone externalities, suppose that $\mu'\succsim^\theta\mu\succsim^\theta\varnothing$, we need $R^\theta(X|\mu')\supseteq R^\theta(X|\mu)$ for every $X\subseteq\mathcal{X}$. Equivalently, we need that $r_i(X_i|\mu'_{-i})\supseteq r_i(X_i|\mu_{-i})$ for every $i\in\theta$ and $X\subseteq\mathcal{X}$. By the definition of $\succsim^\theta$, $\mu'\succsim^\theta\mu\succsim^\theta\varnothing$ implies $\mu'_{-i}\succsim^\theta\mu_{-i}\succsim^\theta\varnothing$ for every $i\in\theta$. By construction, there exists a list of preference rankings $(\succ_j^{\mu_{-i}})_{j\in\mathcal{J}}$ and $(\succ_j^{\mu'_{-i}})_{j\in\mathcal{J}}$ such that for every $j\in\mathcal{J}$, $\succ_j^{\mu'_{-i}}$ is a truncation of $\succ_j^{\mu_{-i}}$. Therefore, $r_i(X_i|\mu'_{-i})\supseteq r_i(X_i|\mu_{-i})$ is satisfied. □

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43. For an analogue of this claim in the setting without externalities, see Chambers and Yenmez (2017).

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Supplementary Data

Supplementary data are available at *Review of Economic Studies* online.

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