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A novel constraint-tightening approach for robust data-driven predictive control

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Abstract

In this paper, we present a data-driven model predictive control (MPC) scheme that is capable of stabilizing unknown linear time-invariant systems under the influence of process disturbances. To this end, Willems' lemma is used to predict the future behavior of the system. This allows the entire scheme to be set up using only a priori measured data and knowledge of an upper bound on the system order. First, we develop a state-feedback MPC scheme, based on input-state data, which guarantees closed-loop practical exponential stability and recursive feasibility as well as closed-loop constraint satisfaction. The scheme is extended by a suitable constraint tightening, which can also be constructed using only data. In order to control a priori unstable systems, the presented scheme contains a prestabilizing controller and an associated input constraint tightening. We first present the proposed data-driven MPC scheme for the case of full state measurements, and also provide extensions for obtaining similar closed-loop guarantees in case of output feedback. The presented scheme is applied to a numerical example.

KEYWORDS

data-driven MPC, robust MPC

1 | INTRODUCTION

In recent years, there has been significant interest in designing data-driven model predictive control (MPC) schemes, in which predictions are not based on a parametric model of the system, but rather directly on a priori collected input/output data from the system, thus circumventing the challenging intermediate step of finding an accurate model. This is done by employing the so-called Willems' fundamental lemma,[1] which states that for a controllable linear system, all possible system trajectories can be parameterized in terms of linear combinations of time-shifts of one single, persistently exciting, trajectory.

A direct data-based MPC scheme based on Willems' lemma was first considered by Yang et al.[2] and Coulson et al.[3] Guarantees for recursive feasibility, stability, and robustness (in the presence of measurement noise) of the closed loop were first proven by Berberich et al.[4] In recent years, various further properties and extensions of this data-driven MPC framework have been studied, compare, for example, the works by Coulson et al.,[5] Huang et al.,[6] Yin et al.,[7,8] Xue and Matni,[9] Furieri et al.,[10] Berberich et al.,[11] Fiedler and Lucia,[12] Breschi et al.,[13] Klädtke et al.,[14] and the overview paper by Markovsky and Dörfler.[15]

One of the major strengths of MPC is its ability to take constraints in the optimization problem into account, and therefore, guarantee their satisfaction in closed-loop operation. For data-driven MPC schemes, achieving constraint satisfaction is similarly important. However, in practice we typically only have access to noisy data. Thus, to achieve closed-loop constraint satisfaction, a suitable constraint tightening is required, similar to model-based MPC.[16-18] Such a constraint tightening takes into account possible (worst-case) disturbances as well as their influence on the system dynamics in order to ensure that no disturbance that may occur in the future can result in constraint violation.

For data-driven MPC schemes relying on an a priori identification of the system model, there already exist schemes that provide a proper constraint tightening even in the case of additive process noise.[19,20] However, in the direct data-driven setting—based on Willems' lemma—this problem has not been conclusively solved so far. In Berberich et al.,[21] closed-loop constraint satisfaction is shown in case of measurement noise; however, no process noise and no input constraint tightening is considered, resulting in the fact that the proposed scheme can only be applied (without being overly conservative) to open-loop stable systems. Process disturbances acting additively on the dynamics have been considered by Huang et al.[22] and Umenberger et al.[23] However, both schemes lack the aforementioned closed-loop guarantees, and moreover, rely on the knowledge of a priori measured disturbances. Recently, Liu et al.[24] proposed a scheme guaranteeing closed-loop stability and recursive feasibility in the presence of process disturbances, which, however, lacks of guarantees for closed-loop constraint satisfaction.

In this paper, we propose robust data-driven MPC schemes based on Willem's lemma. Throughout the course of this paper, by "robust" we mean that all presented closed-loop guarantees, that is, recursive feasibility, constraint satisfaction, and practical exponential stability, hold despite disturbances in the online and offline measurements. We start by proposing an MPC scheme for the case that full state measurement is available. Moreover, with some extensions and adaptations, a similar scheme can be set up for the case of output measurements, which is shown later in the paper. In order to deliver the aforementioned robust closed-loop guarantees, we bound the influence of the process disturbance on the online and offline measurements by means of a bound on the error propagation through the system dynamics. Using this bound, we set up a state/output constraint tightening inspired by the one of Berberich et al.[21] As for open-loop unstable systems the achievable disturbance bounds diverge, we make use of an input parameterization that prestabilizes the system dynamics. Furthermore, in order to still guarantee input constraint satisfaction, despite the use of this underlying pre-stabilizing controller, we introduce a novel input constraint tightening. We bound the error between the predicted state/output sequence and an undisturbed state/output trajectory resulting from the application of the open-loop optimal input sequence which enables us to deliver the aforementioned closed-loop guarantees. All system constants used for the MPC schemes can be (over-)approximated using only given input-output data which are perturbed by process disturbances, knowledge of an upper bound on the system order, and an upper bound on the disturbance.

The remainder of the paper is structured as follows. In Section 2 the problem setup and preliminaries, such as the concept of persistency of excitation and Willems' lemma, are introduced. Next, in Section 3, we set up the first proposed data-driven MPC scheme based on state measurements. To this end, we elaborate the data-driven parameterization of the constraint tightening, explain the MPC scheme and prove the aforementioned closed-loop guarantees. Thereafter, in Section 4 we present data-driven output-feedback MPC based on the case where only output measurements are available. We apply the proposed scheme to a numerical example in Section 5, and end with some concluding remarks in Section 6.

1.1 Notation

For a sequence $\{z_k\}_{k=0}^{N-1}$, we define the Hankel matrix of depth L as

$$H_L(z) = \begin{bmatrix} z_0 & z_1 & \cdots & z_{N-L} \\ z_1 & z_2 & \cdots & z_{N-L+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{L-1} & z_L & \cdots & z_{N-1} \end{bmatrix},$$

and the stacked window from time instant a to b as

$$z_{[a,b]} = \begin{bmatrix} z_a \\ \vdots \\ z_b \end{bmatrix}.$$

We write $\|x\|_P = \sqrt{x^\top Px}$ for a vector $x \in \mathbb{R}^n$ and a positive definite matrix $P \in \mathbb{R}^{n\times n}$. Furthermore, for a vector $x \in \mathbb{R}^n$ we denote the ℓ_1-, ℓ_2-, and ℓ_∞-norm by $\|x\|_1$, $\|x\|_2$, and $\|x\|_\infty$, respectively. Moreover, for a matrix $M \in \mathbb{R}^{n\times m}$ we denote the respective induced norms by $\|M\|_1$, $\|M\|_2$, and $\|M\|_\infty$. We write $F \succcurlyeq 0$ if $F \in \mathbb{R}^{n\times n}$ is a symmetric and positive semi-definite matrix.

2 | PROBLEM SETUP AND PRELIMINARIES

In this paper, we consider the discrete-time multi-input multi-output LTI system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k, \\ y_k &= Cx_k + Du_k, \end{aligned} \tag{1}$$

with the state $x_k \in \mathbb{R}^n$, the input $u_k \in \mathbb{R}^m$, the output $y_k \in \mathbb{R}^p$ and the process disturbance $w_k \in \mathbb{R}^n$. The setup can be extended to include measurement noise as well, see Remark 1 below. Throughout the paper, we assume that (1) is a minimal realization, that is, that the pair (A, B) is controllable and the pair (A, C) is observable. Moreover, we consider the matrices A, B, C, D as being unknown and the only knowledge about the system available being its order n.

Moreover, we assume that the process disturbances belongs to a hypercube (precise definitions will be given in Sections 3 and 4). The goal of this paper is to construct a data-driven MPC scheme that stabilizes the origin and ensures input and state constraint satisfaction (cf. Section 3) or output constraint satisfaction (cf. Section 4), where the respective constraint sets are given by hypercubes.

To this end, we apply a persistently exciting (p.e.) input sequence to the system, and measure the resulting state/output sequence, where a persistently exciting sequence is defined as follows.

Definition 1. A sequence $\{u_k\}_{k=0}^{N-1}$, with $u_k \in \mathbb{R}^m$, is persistently exciting of order L if $\operatorname{rank}(H_L(u)) = mL$.

We want to make use of Willems' fundamental lemma for the prediction in an MPC problem.

Lemma 1 (Willems' lemma[1]). *Suppose $\{u_k, \hat{y}_k\}_{k=0}^{N-1}$ is a trajectory of the controllable system*

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k, \\ \hat{y}_k &= C\hat{x}_k + Du_k, \end{aligned} \tag{2}$$

and u is persistently exciting of order $L + n$. Then, $\{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1}$ is a trajectory of System (2) if and only if there exists $\alpha \in \mathbb{R}^{N-L+1}$ such that

$$\begin{bmatrix} H_L(u) \\ H_L(\hat{y}) \end{bmatrix} \alpha = \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}. \tag{3}$$

This lemma states that in the absence of disturbances, that is, if $w_k = 0$ for all $k \geq 0$, all trajectories of system (1) can be parameterized by linear combinations of time shifts of a priori measured, sufficiently exciting input/output trajectories. In the following two sections, we set up MPC schemes that use these trajectories for the prediction of the systems behavior.

3 | DATA-DRIVEN STATE-FEEDBACK PREDICTIVE CONTROL

In this section, we present a robust data-driven state-feedback MPC scheme with closed-loop guarantees on stability and constraint satisfaction in the presence of process noise. In Section 3.1, we introduce the data-driven MPC scheme for the case of available state measurements. Thereafter, in Section 3.2 we prove the closed-loop guarantees of the introduced control scheme. Finally, in Section 3.3 we show how the system constants, which are used to set up the constraint tightening of the MPC scheme, can be approximated purely from data.

3.1 | Proposed MPC scheme

For the first data-driven predictive control scheme, we consider the availability of full state measurement, that is, $C = I$, $D = 0$ in (1). Moreover, we assume that the process disturbance belongs to the hypercube $w_t \in \mathbb{W} =$

$\{w \in \mathbb{R}^n | \; \|w\|_\infty \leq w_{\max}\}$ for all $t \geq 0$, where $w_{\max} \geq 0$ is known, and the input and state constraint sets are given by the hypercubes $u_t \in \mathbb{U} = \{u \in \mathbb{R}^m | \; \|u\|_\infty \leq u_{\max}\}$ and $x_t \in \mathbb{X} = \{x \in \mathbb{R}^n | \; \|x\|_\infty \leq x_{\max}\}$ for some $u_{\max} > 0$, $x_{\max} > 0$. As will become clear later in this section, it is crucial for the construction of a proper constraint tightening that the prediction model is stable. If this is not the case a priori (i.e., A is unstable), then this can be enforced via a prestabilizing input parameterization

$$u_k = Kx_k + v_k, \tag{4}$$

as it is common, for example, in tube-based MPC.[16,17] The state feedback matrix K is chosen such that all eigenvalues of $A_K = A + BK$ strictly lie inside the unit disc. Such a prestabilizing controller can be computed purely from data, e.g., following the approaches by Berberich et al.[25] or van Waarde et al.[26] Throughout this paper, we assume that such a controller is known a priori. In case of a stable system, the following scheme can be applied with $K = 0$.

To make use of Lemma 1 for the prediction of open-loop state sequences, we consider the input v_k of the pre-stabilized system

$$x_{k+1} = A_K x_k + B v_k + w_k. \tag{5}$$

Note that, starting at an initial state x_0 and applying the input sequence $v_{[0,k-1]}$ to system (5), the state after k time steps is given by

$$x_k = A_K^k x_0 + \sum_{i=0}^{k-1} A_K^{k-1-i} B v_i + \sum_{i=0}^{k-1} A_K^{k-1-i} w_i, \tag{6}$$

where $w_{[0,k-1]}$ is the disturbance sequence acting on the system. Thus we can denote the disturbance propagated k steps through the system dynamics by

$$d_k := \sum_{i=0}^{k-1} A_K^{k-1-i} w_i. \tag{7}$$

As discussed in the introduction, closed-loop constraint satisfaction of a data-driven MPC scheme could so far only be shown by Berberich et al.[21] for the case of output measurement noise, that is, additive noise v of the form $\tilde{x} = x + v$. However, taking a look at (7) clearly shows that due to the process noise acting on the state through the system dynamics, these dynamics now have to be taken into account to set up a proper robust MPC scheme. Thus, throughout the paper, we make use of the disturbance bound

$$\overline{d}_k \geq \sum_{i=0}^{k-1} \left\| A_K^{k-1-i} \right\|_\infty w_{\max} \geq \|d_k\|_\infty . \tag{8}$$

As will be discussed in Section 3.3, a suitable over-approximation (8) of the past disturbances acting on the state can be obtained purely from data. For unstable system matrices, the sum in (8) grows exponentially with increasing k. Therefore, it becomes clear why the use of a prestabilizing controller is crucial in order to obtain a suitable error-bound and, later on, a feasible constraint tightening, which is analogous to model-based robust MPC (cf. e.g., References 16-18).

Remark 1. Note that the following results can be easily extended to the case where, apart from the process disturbance w_k, also additive measurement noise on the state measurements is present. In this case, the disturbance sequence in (7) has to be extended by the actual measurement noise instant occurring at time k and, in case of $K \neq 0$, the past measurement noise instants which are fed back into and propagated through the system dynamics. More precisely, instead of (7) the disturbance sequence $d'_k := \sum_{i=0}^{k-1} A_K^{k-1-i} w_i + \sum_{i=0}^{k-1} A_K^{k-1-i} BK v_i + v_k$ needs considered in the arguments below, where $v_{[0,k]}$ is the measurement noise sequence. If an upper bound $\|v_k\| \leq v_{\max}$ for all $k \geq 0$ is known, a bound on $\left\| d'_k \right\|_\infty$ can be over-approximated purely from data, following similar steps as for the over-approximation of (8), discussed in this paper. Therefore, for the sake of simplicity, we consider only process disturbances throughout the paper.

We now apply a p.e. input sequence $\{v_k^d\}_{k=0}^{N-1}$ of length N to System (5), and measure the associated disturbed state sequence $\{x_k^d\}_{k=0}^{N}$, where the superscript "d" denotes a priori collected data.

Assumption 1. The input sequence $\{v_k^d\}_{k=0}^{N-1}$ is persistently exciting of order $L + n + 1$.

Starting at the initial state x_0^d, similar to (7), we to denote the cumulated disturbance influencing the collected data at time k as

$$d_k^d = \sum_{i=0}^{k-1} A_K^{k-1-i} w_i^d, \tag{9}$$

and the undisturbed state at as

$$\hat{x}_k^d = x_k^d - d_k^d. \tag{10}$$

Note that with the error bound $\bar{d}_k$ for $k = 0, \dots, N$, we can also upper bound the disturbances in the offline state measurements, that is, $\bar{d}_k \geq \left\| d_k^d \right\|_\infty$ for $k = 0, \dots, N$.

With these a priori generated data sequences, we are now able to set up the following optimal control problem (OCP), given the measured state x_t at time t and with the prediction horizon L

$$J_L^*(x_t) = \min_{\substack{\alpha(t),\sigma(t),\\ \bar{v}(t),\bar{x}(t)}} \sum_{k=0}^{L-1} \left(\|\bar{v}_k(t)\|_R^2 + \|\bar{x}_k(t)\|_Q^2 \right) + \lambda_\alpha w_{\max} \|\alpha(t)\|_2^2 + \frac{\lambda_\sigma}{w_{\max}} \|\sigma(t)\|_2^2 \tag{11a}$$

$$\text{s.t.} \begin{bmatrix} \bar{v}(t) \\ \bar{x}(t) + \sigma(t) \end{bmatrix} = \begin{bmatrix} H_L(v^d) \\ H_{L+1}(x^d) \end{bmatrix} \alpha(t), \tag{11b}$$

$$\bar{x}_0(t) = x_t, \tag{11c}$$

$$\bar{x}_L(t) = 0, \tag{11d}$$

$$\left\|\bar{x}_k(t)\right\|_\infty + a_{u,k} \left\|\bar{v}(t)\right\|_1 + a_{\alpha,k} \left\|\alpha(t)\right\|_1 + a_{\sigma,k} \left\|\sigma_k(t)\right\|_\infty + a_{c,k} \leq x_{\max}, \tag{11e}$$

$$\left\|\bar{v}_k(t)\right\|_\infty + b_{u,k} \left\|\bar{v}(t)\right\|_1 + b_{\alpha,k} \left\|\alpha(t)\right\|_1 + b_{\sigma,k} \left\|\sigma_k(t)\right\|_\infty + b_{c,k} + \left\|K\bar{x}_k(t)\right\|_\infty \leq u_{\max}, \tag{11f}$$

$$\forall k = 0, \dots, L-1. \tag{11g}$$

We denote the optimal solution of (11) at time t by $\bar{v}^*(t)$, $\bar{x}^*(t)$, $\alpha^*(t)$, $\sigma^*(t)$. Note that (11) is a strictly convex quadratic program and can thus be solved efficiently. In (11b), we make use of Lemma 1 for the prediction of future state sequences of the system. Note that, a Hankel matrix of depth $L+1$ is used in the second block row of (11b), since the predicted state sequence contains $L+1$ elements (from $k=0$ to $k=L$), whereas the predicted input sequence only contains L elements (from $k=0$ to $k=L-1$). Moreover, note that, as it is common in predictive control based on Willems' lemma, we make use of a slack variable $\sigma(t)$ (first introduced by Coulson et al.[3]) that renders (11b) feasible, even in the presence of disturbances. The slack variable σ as well as the variable α are regularized in (11a). This leads to smaller values of σ and α, improving the prediction accuracy and reducing the influence of disturbances in the Hankel matrices. For further discussion on these issues, see also section IV.A by Berberich et al.[4] and section IV by Dörfler et al.[27] Note that the regularization of α is scaled with $w_{\max}$ and the regularization of σ with $\frac{1}{w_{\max}}$ as introduced by Bongard et al.[28] This is needed in the proof of Theorem 1 in order to obtain an upper bound on $\|\alpha^*(t)\|_1$ and $\|\sigma^*(t)\|_\infty$. The scaling of the regularization of σ can also be dropped. However, this comes at the cost of an additional non-convex constraint as it was shown by Berberich et al.[4] Problem (11) contains the tightened state and input constraints (11e) and (11f). Note that both constraints depend on $\bar{u}(t)$, $\alpha(t)$ and $\sigma(t)$. Together with suitably defined coefficients $a_{u,k}$, $a_{\alpha,k}$, $a_{\sigma,k}$, $a_{c,k}$, and $b_{u,k}$, $b_{\alpha,k}$, $b_{\sigma,k}$, $b_{c,k}$, which will be defined later on, this constrained tightening ensures recursive feasibility and closed-loop constraint satisfaction (cf. Theorem 1), that is, $\|x_t\|_\infty \leq x_{\max}$ and $\|u_t\|_\infty = \|Kx_t + v_t\|_\infty \leq u_{\max}$ for all $t \geq 0$. Constraint (11f) can be dropped if no prestabilizing controller is used, that is, $K = 0$. Problem (11) is similar to the one by Berberich et al.,[21] which, however, does not consider process disturbances and the input constraint tightening (11f). The predictive control scheme is used in an n-step receding horizon manner, that is, at time t we solve (11) and choose $v_{t+k} = \bar{v}_k^*(t)$ in (4) for $k = 0, \dots, n-1$.

In the following, we introduce the coefficients used to set up the tightened state and input constraints (11e) and (11f). To this end, we first introduce some system constants. First, we define the constant $c_{pe} = \left\| H_{v\hat{x}}^\dagger \right\|_1$, with

$$H_{v\hat{x}} = \begin{bmatrix} H_L\left(v^d\right) \\ H_1\left(\hat{x}_{[0,N-L-1]}^d\right) \end{bmatrix}, \tag{12}$$

where $H_{v\hat{x}}^{\dagger}$ is the Moore–Penrose inverse of $H_{v\hat{x}}$. Using $\overline{d}_k$ as well as the over-approximation $\rho_{A,k} \geq \left\| A_K^k \right\|_\infty$, we define

$$c_{\alpha,k} = \rho_{A,k}\overline{d}_{N-L} + \overline{d}_{N-L+k}, \quad c_{\sigma,k} = \rho_{A,k} + 1, \tag{13}$$

for $k = 0, \dots, L$. Moreover, we introduce the controllability constant $\Gamma > 0$, which is chosen such that, starting at any x_0, we can find an input sequence $v_{[0,n-1]}$ steering the state of the prestabilized system (5) without disturbances to the origin in n steps and satisfying

$$\left\| v_{[0,n-1]} \right\|_1 \leq \Gamma \left\| x_0 \right\|_\infty . \tag{14}$$

Note that such a constant exists as the pair (A, B) is controllable.

We are now ready to define the coefficients of the state and input constraint tightening as

$$\begin{aligned} &a_{u,k} = 0,\ a_{\alpha,k} = c_{\alpha,k},\ a_{\sigma,k} = c_{\sigma,k},\ a_{c,k} = \overline{d}_k, \\ &b_{u,k} = 0,\ b_{\alpha,k} = \overline{K}c_{\alpha,k},\ b_{\sigma,k} = \overline{K}c_{\sigma,k},\ b_{c,k} = \overline{K}\overline{d}_k, \end{aligned} \tag{15}$$

for $k = 0, \dots, n-1$, and

$$\begin{aligned} a_{u,k+n} &= a_{u,k} + a_{\alpha,k}c_{pe} + a_{\sigma,k}c_{pe}\overline{d}_{N-1}, \\ a_{\alpha,k+n} &= a_{u,k+n}\Gamma c_{\alpha,L} + c_{\alpha,k+n}, \\ a_{\sigma,k+n} &= a_{u,k+n}\Gamma c_{\sigma,L} + c_{\sigma,k+n}, \\ a_{c,k+n} &= a_{c,k} + a_{\alpha,k}c_{pe}\left(nx_{\max} + n\overline{d}_n\right) + a_{\sigma,k}\left(\overline{d}_{N-1}c_{pe}\left(nx_{\max} + n\overline{d}_n\right) + \overline{d}_n\right) + \overline{d}_n, \\ b_{u,k+n} &= b_{u,k} + b_{\alpha,k}c_{pe} + b_{\sigma,k}c_{pe}\overline{d}_{N-1}, \\ b_{\alpha,k+n} &= b_{u,k+n}\Gamma c_{\alpha,L} + \overline{K}c_{\alpha,k+n}, \\ b_{\sigma,k+n} &= b_{u,k+n}\Gamma c_{\sigma,L} + \overline{K}c_{\sigma,k+n}, \\ b_{c,k+n} &= b_{c,k} + b_{\alpha,k}c_{pe}\left(nx_{\max} + n\overline{d}_n\right) + b_{\sigma,k}\left(\overline{d}_{N-1}c_{pe}\left(nx_{\max} + n\overline{d}_n\right) + \overline{d}_n\right) + \overline{K}\overline{d}_n, \end{aligned} \tag{16}$$

for $k = 0, \dots, L-n-1$, where $\overline{K} = \|K\|_\infty$. Note that $\overline{d}_k$ and $\rho_{A,k}$ grow exponentially if A_K has eigenvalues outside the unit disc. This is the main motivation for the usage of the prestabilizing controller (4), as diverging $\overline{d}_k$ and $\rho_{A,k}$ would also lead to diverging $c_{\alpha,k}$ and $c_{\sigma,k}$ and, therefore, to large coefficients (15) and (16). This would in general yield an infeasible OCP (11) even for small prediction horizons L. In order to set up the coefficients above, the system constants Γ, c_{pe}, $\rho_{A,k}$ for $k = 0, \dots, L$, and d_k for $k = 0, \dots, N-1$ have to be known. All of these constants can be approximated from data as will be shown in Section 3.3.

3.2 | Theoretical guarantees

Firstly, we denote the undisturbed state at time $t+k$ resulting from an open-loop application of $\overline{v}^*(t)$ as

$$\hat{x}_{t+k}^* := A_K^k x_t + \sum_{i=0}^{k-1} A_K^{k-1-i} B\overline{v}_i^*(t). \tag{17}$$

An upper bound for the prediction error between this undisturbed open-loop state trajectory $\hat{x}^*$ and the predicted optimal state sequence $\overline{x}^*(t)$ at time t can be derived by the following lemma.

Lemma 2. *If (11) is feasible at time t, then*

$$\left\| \hat{x}_{t+k}^* - \overline{x}_k^*(t) \right\|_\infty \leq c_{\alpha,k} \left\| \alpha^*(t) \right\|_1 + c_{\sigma,k} \left\| \sigma^*(t) \right\|_\infty , \tag{18}$$

holds for all $k = 0, \dots, L$.

Proof. Similar to the proof of Lemma 2 in the work of Berberich et al.,[4] we start by considering the error between the undisturbed open-loop state and the state prediction resulting from undisturbed data in the Hankel matrix

$$x^{-}_{[t,t+L]} := \hat{x}^{*}_{[t,t+L]} - H_{L+1}\left(\hat{x}^{d}\right)\alpha^{*}(t) = \hat{x}^{*}_{[t,t+L]} - \left(H_{L+1}\left(x^{d}\right) - H_{L+1}\left(d^{d}\right)\right)\alpha^{*}(t). \tag{19}$$

Note that $\hat{x}^{*}_{[t,t+L-1]}$ and $H_L\left(\hat{x}^{d}\right)\alpha^{*}(t)$ are trajectories of the undisturbed system (5) resulting from the application of $\overline{v}^{*}(t)$ with different initial states. Due to (11c) and (11b), the initial condition of $H_{L+1}\left(\hat{x}^{d}\right)\alpha^{*}(t)$ is

$$H_1\left(\hat{x}^{d}_{[0,N-L]}\right)\alpha^{*}(t) = x_t + \sigma_0(t) - H_1\left(\hat{d}^{d}_{[0,N-L]}\right)\alpha^{*}(t). \tag{20}$$

Thus, for the difference between both trajectories, it holds that

$$x^{-}_{t+k} = A_K^k\left(H_1\left(d^{d}_{[0,N-L]}\right)\alpha^{*}(t) - \sigma_0(t)\right), \tag{21}$$

for $k = 0, \dots, L$. To show (18), note that

$$\left\|\hat{x}^{*}_{t+k} - \overline{x}^{*}_{k}(t)\right\|_\infty \leq \left\|x^{-}_{t+k}\right\|_\infty + \left\|\sigma^{*}_{k}(t) - H_1\left(d^{d}_{[k,N-L+k]}\right)\alpha^{*}(t)\right\|_\infty \tag{22}$$

$$\leq \left(\left\|A_K^k\right\|_\infty \overline{d}_{N-L} + \overline{d}_{N-L-1}\right)\|\alpha^{*}(t)\|_1 + \left\|A_K^k\right\|_\infty \|\sigma^{*}_0(t)\|_\infty + \left\|\sigma^{*}_k(t)\right\|_\infty, \tag{23}$$

where the second inequality holds due to

$$\left\|H_1\left(d^{d}_{[k,N-L+k]}\right)\alpha^{*}(t)\right\|_\infty = \max_{i=1,\dots,n}\left|e_i^\top H_1\left(d^{d}_{[k,N-L+k]}\right)\alpha^{*}(t)\right|, \tag{24}$$

$$\leq \left\|d^{d}_{[k,N-L+k]}\right\|_\infty \|\alpha^{*}(t)\|_1, \tag{25}$$

$$\leq \overline{d}_{N-L+k}\|\alpha^{*}(t)\|_1, \tag{26}$$

for $k = 0, \dots, L$, where e_i is the ith unit vector in $\mathbb{R}^n$. Therefore, we obtain (18) with $c_{\alpha,k} = \rho_{A,k}\overline{d}_{N-L} + \overline{d}_{N-L+k}$, $c_{\sigma,k} = \rho_{A,k} + 1$, and $\rho_{A,k} \geq \left\|A_K^k\right\|_\infty$. ■

The main difference of this proof to the proof of lemma 2 by Berberich et al.[4] is that we consider process disturbances acting on the state through the system dynamics. Therefore, the error bounds have to account for all the past disturbances propagated through the system dynamics. To this end, we bound the disturbances d_k^d in the offline collected data (see also the discussion below (6)), thus, achieving similar error bounds to Berberich et al.[4]

Using the result of Lemma 2, we can now state our main result, which establishes recursive feasibility, practical exponential stability, and input and state constraint satisfaction of the closed-loop system, assuming that the initial state is feasible for Problem (11) and the disturbance bound is sufficiently small.

Theorem 1. *Suppose that Assumption 1 holds. Then, for any $V_{\mathrm{ROA}} > 0$, there exist $\underline{\lambda}_\alpha$, $\overline{\lambda}_\alpha$, $\underline{\lambda}_\sigma$, $\overline{\lambda}_\sigma$ such that for all λ_α, λ_σ satisfying*

$$\underline{\lambda}_\alpha \leq \lambda_\alpha \leq \overline{\lambda}_\alpha, \quad \underline{\lambda}_\sigma \leq \lambda_\sigma \leq \overline{\lambda}_\sigma, \tag{27}$$

there exist $\overline{w}_{\max}$, $\overline{c}_{pe} > 0$ as well as a continuous, strictly increasing function $\beta : [0, \overline{w}_{\max}] \to [0, V_{\mathrm{ROA}}]$ with $\beta(0) = 0$, such that for all $w_{\max}$ and c_{pe} satisfying

$$w_{\max} \leq \min\left\{\overline{w}_{\max}, \frac{\overline{c}_{pe}}{c_{pe}}\right\}, \tag{28}$$

the following holds for the closed loop resulting from an application of the n-step MPC scheme:

(i) If $J_L^*(x_0) \leq V_{\mathrm{ROA}}$, then OCP (11) is feasible at any time $t \geq 0$.
(ii) For any initial condition satisfying $J_T^*(x_0) \leq V_{\mathrm{ROA}}$ it holds that $x_t \in \mathbb{X}$ and $u_t \in \mathbb{U}$ for all $t \geq 0$, and $J_L^*(x_t)$ converges exponentially to $J_L^*(x_t) \leq \beta(\overline{w}_{\max})$.

The proof of this result makes use of a similar candidate solution as the one of theorem 10 by Berberich et al.[21] Together with the prediction error in Lemma 2, showing constraint satisfaction and recursive feasibility of the tightened state constraint (11e) as well as practical exponential stability works similar to the aforementioned reference. However, due to the pre-stabilizing input, we require the input constraint tightening (11f). Thus, showing that the aforementioned closed-loop properties also hold for this constraint is the main difficulty of this proof. Note that (ii) only shows exponential convergence of x_t to a neighborhood of $x = 0$; however, it is possible to establish a suitable lower as well as an upper bound on $J_L^*(x_t)$ analogous to lemma 1 by Berberich et al.,[4] thus, resulting in practical exponential stability. For a detailed discussion on the influence of the parameters λ_α, λ_σ, and c_{pe} on the stability properties, we refer to the work of Berberich et al.[4] In short, the region of attraction increases for smaller disturbance bounds or better persistency of excitation of the input signal, the latter being expressed by a decrease of c_{pe}.

Proof. To show (i), we construct a candidate solution for (11) at time $t + n$ and show that (11b)–(11f) hold for this candidate. Therefore, we define the candidate solution over the first $L - n$ steps via the previously optimal input shifted by n steps, that is, $\overline{v}'_k(t + n) = \overline{v}^*_{k+n}(t)$ for $k = 0, \dots, L - n - 1$. Moreover, the state candidate is chosen as

$$\overline{x}'_{[0,L-n]}(t + n) = \begin{bmatrix} x_{t+n} \\ \hat{x}^*_{[t+n+1,t+L]}, \end{bmatrix} \tag{29}$$

for the first $L - n + 1$ steps, with $\hat{x}^*$ defined as in (17). Note that, $c_{\alpha,k}$ scales linearly with $w_{\max}$ and

$$\|\alpha^*(t)\|_1 \leq \sqrt{N - L + 1}\, \|\alpha^*(t)\|_2 \leq \sqrt{(N - L + 1)\, \frac{V_{\text{ROA}}}{\lambda_\alpha w_{\max}}}, \tag{30}$$

which implies that the first term on the right-hand side of (18) becomes arbitrarily small for sufficiently small $w_{\max}$. Furthermore, the same holds true for the second term on the right-hand side of (18), since $c_{\sigma,k}$ is uniformly upper bounded for all k and

$$\|\sigma^*(t)\|_\infty \leq \|\sigma(t)\|_2 \leq \sqrt{\frac{V_{\text{ROA}} w_{\max}}{\lambda_\sigma}}. \tag{31}$$

Due to this, Lemma 2, and $\overline{x}^*_L(t) = 0$, the state $\overline{x}'_{L-n}(t + n) = \hat{x}^*_{t+L}$ becomes arbitrarily small for sufficiently small $w_{\max}$. Thus, for a sufficiently small $w_{\max}$, by controllability, there exists an input sequence $\overline{v}'_{[L-n,L-1]}(t + n)$ that steers $\overline{x}'_{[L-n,L]}(t + n)$ from $\hat{x}^*_{t+L}$ to 0 in n steps. Furthermore, we choose the candidate solution for $\alpha'(t + n)$ and $\sigma'(t + n)$ as

$$\alpha'(t + n) = H^\dagger_{v\hat{x}} \begin{bmatrix} \overline{v}'(t + n) \\ \hat{x}^*_{t+n} \end{bmatrix}, \tag{32}$$

and

$$\begin{aligned} \sigma'(t + n) &= H_{L+1}\left(x^d\right) \alpha'(t + n) - \overline{x}'(t + n), \\ &= \begin{bmatrix} \hat{d}_{t+n} \\ H_L\left(d^d_{[1,N]}\right) \alpha'(t + n) \end{bmatrix}, \end{aligned} \tag{33}$$

where $\hat{d}_{t+k}$ satisfies $x_{t+k} = \hat{x}^*_{t+k} + \hat{d}_{t+k}$. Thus, the candidate solution satisfies (11b)–(11d).

It remains to be shown that also (11e) and (11f) hold for the candidate solution. In order to show the satisfaction of (11f), we note that

$$\begin{aligned} \left\|\alpha'(t + n)\right\|_1 &\overset{(32)}{\leq} c_{pe}\left(\left\|\overline{v}'(t + n)\right\|_1 + \|\hat{x}^*_{t+n}\|_1\right) \\ &\leq c_{pe}\left(\left\|\overline{v}'(t + n)\right\|_1 + n x_{\max} + n\overline{d}_n\right), \end{aligned} \tag{34}$$

holds. Furthermore, it holds that

$$\begin{aligned}\left\|\sigma'(t+n)\right\|_\infty &\overset{(32)}{\leq} \bar{d}_{N-1}\left\|\alpha'(t+n)\right\|_1 + \bar{d}_n,\\ &\overset{(34)}{\leq} \bar{d}_{N-1}c_{pe}\left(\left\|\bar{v}'(t+n)\right\|_1 + nx_{\max} + n\bar{d}_n\right) + \bar{d}_n. \end{aligned} \tag{35}$$

Using the same arguments as in proposition 8 (Inequality (18)) by Berberich et al.,[21] we can bound the norm of the candidate input by

$$\left\|\bar{v}'(t+n)\right\|_1 \leq \left\|\bar{v}^*_{[n,L-1]}(t)\right\|_1 + \left\|\bar{v}'_{[L-n,L-1]}(t+n)\right\|_1, \tag{36}$$

$$\leq \left\|\bar{v}^*(t)\right\|_1 + \Gamma\left\|\hat{x}^*_{t+L}\right\|_\infty, \tag{37}$$

$$\leq \left\|\bar{v}^*(t)\right\|_1 + \Gamma c_{\alpha,L-1}\left\|\alpha^*(t)\right\|_1 + \Gamma c_{\sigma,L-1}\left\|\sigma^*(t)\right\|_\infty, \tag{38}$$

where the first inequality holds due to the definition of $\bar{v}'(t+n)$ and the triangle inequality, the second inequality holds due to (14), and the last inequality holds due to the terminal condition (11d) and the prediction error bound (18). Since (11f) holds for the optimal solution at time t and due to $\left\|\bar{v}'_k(t+n)\right\|_\infty = \left\|\bar{v}^*_{k+n}\right\|_\infty$ for $k = 0, \ldots, L-n-1$, we obtain

$$\left\|\bar{v}'_k(t+n)\right\|_\infty \leq u_{\max} - \left(b_{u,k+n}\left\|\bar{v}^*(t)\right\|_1 + b_{\alpha,k+n}\left\|\alpha^*(t)\right\|_1 + b_{\sigma,k+n}\left\|\sigma^*(t)\right\|_\infty + b_{c,k+n} + \left\|K\bar{x}^*_{k+n}(t)\right\|_\infty\right). \tag{39}$$

Plugging (38) and the coefficients in (16) into (39) yields for $k = 0, \ldots, L-n-1$

$$\begin{aligned}\left\|\bar{v}'_k(t+n)\right\|_\infty &\overset{(38)}{\leq} u_{\max} - b_{u,k}\left\|\bar{v}'(t+n)\right\|_1 - b_{\alpha,k}\left(c_{pe}\left\|\bar{v}'(t+n)\right\|_1 + c_{pe}(nx_{\max} + n\bar{d}_n)\right)\\ &\quad - b_{\sigma,k}\left(\bar{d}_{N-1}c_{pe}\left\|\bar{v}'(t+n)\right\|_1 + \bar{d}_{N-1}c_{pe}(nx_{\max} + n\bar{d}_n) + \bar{d}_n\right) - b_{c,k}\\ &\quad - \bar{K}c_{\alpha,k+n}\left\|\alpha^*(t)\right\|_1 - \bar{K}c_{\sigma,k+n}\left\|\sigma^*(t)\right\|_\infty - \left\|K\bar{x}^*_{k+n}(t)\right\|_\infty - \bar{K}\bar{d}_n, \end{aligned} \tag{40}$$

$$\begin{aligned}&\overset{(34),(35)}{\leq} u_{\max} - b_{u,k}\left\|\bar{v}'(t+n)\right\|_1 - b_{\alpha,k}\left\|\alpha'(t+n)\right\|_1 - b_{\sigma,k}\left\|\sigma'(t+n)\right\|_\infty - b_{c,k}\\ &\quad - \bar{K}c_{\alpha,k+n}\left\|\alpha^*(t)\right\|_1 - \bar{K}c_{\sigma,k+n}\left\|\sigma^*(t)\right\|_\infty - \left\|K\bar{x}^*_{k+n}(t)\right\|_\infty - \bar{K}\bar{d}_n, \end{aligned} \tag{41}$$

$$\begin{aligned}&\overset{(18)}{\leq} u_{\max} - b_{u,k}\left\|\bar{v}'(t+n)\right\|_1 - b_{\alpha,k}\left\|\alpha'(t+n)\right\|_1 - b_{\sigma,k}\left\|\sigma'(t+n)\right\|_\infty - b_{c,k}\\ &\quad - \bar{K}\left\|\hat{x}^*_{t+k+n} - \bar{x}^*_{k+n}(t)\right\|_\infty - \left\|K\bar{x}^*_{k+n}(t)\right\|_\infty - \bar{K}\bar{d}_n, \end{aligned} \tag{42}$$

$$\leq u_{\max} - b_{u,k}\left\|\bar{v}'(t+n)\right\|_1 - b_{\alpha,k}\left\|\alpha'(t+n)\right\|_1 - b_{\sigma,k}\left\|\sigma'(t+n)\right\|_\infty - b_{c,k} - \left\|K\bar{x}'_k(t+n)\right\|_\infty, \tag{43}$$

where the last inequality holds due to $\bar{x}'_k(t+n) = \hat{x}^*_{t+n+k}$ for $k = 1, \ldots, L-n-1$, and

$$\left\|\bar{x}'_0(t+n) - \hat{x}^*_{t+n}\right\|_\infty = \left\|x_{t+n} - \hat{x}^*_{t+n}\right\|_\infty \leq \bar{d}_n. \tag{44}$$

Therefore, the candidate solution satisfies (11f) for $k = 0, \ldots, L-n-1$. Showing that also (11e) holds for $k = 0, \ldots, L-n-1$ can be done following the analogous steps as above.

To show that (11e) and (11f) are also satisfied for $k = L-n, \ldots, L-1$, we recall from above that $\bar{x}'_{L-n}(t+n)$ becomes arbitrarily small for sufficiently small $w_{\max}$. Thus, due to controllability and (14) also $\bar{v}'_{[L-n,L-1]}(t+n)$ and, therefore, $\bar{x}'_{[L-n+1,L-1]}(t+n)$ become arbitrarily small. Moreover, due to (13) and (16) the coefficients $a_{\alpha,k}$ and $b_{\alpha,k}$ depend linearly on $w_{\max}$, and thus, they also become arbitrarily small for sufficiently small disturbance bounds $w_{\max}$. Moreover, the same holds for $a_{u,k}$, $a_{c,k}$, $b_{u,k}$, and $b_{c,k}$. The coefficients $a_{\sigma,k}$, $b_{\sigma,k}$ converge to constant values for $w_{\max} \to 0$.Finally, due to (35), $\sigma'(t+n)$ becomes arbitrarily small if $w_{\max}$ is sufficiently small. Hence, all terms on the left-hand side of (11e) and (11f) (except for $\bar{x}_k(t)$ and $\bar{v}_k(t)$, respectively) become arbitrary small. Therefore, since $x_{\max}$, $u_{\max} > 0$, there exists a sufficiently small bound $\bar{w}_{\max} > 0$ such that (11e) and (11f) are also satisfied for $k = L-n, \ldots, L-1$.

To show (ii), we can follow the same arguments as in theorem 3 by Berberich et al.[4] to conclude invariance of the sublevel set $J_L^*(x_t) \leq V_{\mathrm{ROA}}$ and exponential convergence of $J_L^*(x_t)$ to $J_L^*(x_t) \leq \beta(\overline{w}_{\max})$. This is possible since the candidate solution used in the first part of our proof is constructed analogous to the one used in the proof of the above reference. Thus, the closed-loop scheme is recursively feasible. Closed-loop state constraint satisfaction follows from the same arguments used in the proof of theorem 10 by Berberich et al.[21] In order to show closed-loop input constraint satisfaction, we note that the optimal solution at time t satisfies (11f) for the first n steps, that is,

$$u_{\max} \geq \left\|\overline{v}_k^*(t)\right\|_\infty + b_{u,k}\left\|\overline{v}^*(t)\right\|_1 + b_{\alpha,k}\left\|\alpha^*(t)\right\|_1 + b_{\sigma,k}\left\|\sigma^*(t)\right\|_\infty + b_{c,k} + \left\|K\overline{x}_k^*(t)\right\|_\infty, \tag{45}$$

$$= \left\|\overline{v}_k^*(t)\right\|_\infty + \overline{K}c_{\alpha,k}\left\|\alpha^*(t)\right\|_1 + \overline{K}c_{\sigma,k}\left\|\sigma^*(t)\right\|_\infty + \overline{K}\overline{d}_k + \left\|K\overline{x}_k^*(t)\right\|_\infty, \tag{46}$$

for $k = 0, \dots, n-1$, where the second inequality follows from plugging in the coefficients from (15). With (18), we finally obtain for $k = 0, \dots, n-1$

$$\begin{aligned} u_{\max} &\geq \left\|\overline{v}_k^*(t)\right\|_\infty + \overline{K}\left\|\hat{x}_{t+k}^* - \overline{x}_k^*(t)\right\|_\infty + \overline{K}\overline{d}_k + \left\|K\overline{x}_k^*(t)\right\|_\infty \\ &\geq \left\|\overline{v}_k^*(t)\right\|_\infty + \overline{K}\left\|\hat{x}_{t+k}^* - \overline{x}_k^*(t)\right\|_\infty + \overline{K}\left\|x_{t+k} - \hat{x}_{t+k}^*\right\|_\infty + \left\|K\overline{x}_k^*(t)\right\|_\infty \\ &\geq \left\|\overline{v}_k^*(t) + Kx_{t+k}\right\|_\infty, \end{aligned}$$

where the second inequality holds due to $\left\|x_{t+k} - \hat{x}_{t+k}^*\right\|_\infty \leq \overline{d}_k$ and the third inequality due to the triangle inequality. Thus, the input constraints are satisfied in closed loop. ∎

3.3 | Data-driven estimation of system constants

In order to set up the constraint tightening, we need to compute the coefficients (15) and (16). These depend on several system constants. While $\overline{K}$, $x_{\max}$, and n are known a priori, the system constants $\rho_{A,k}$, $\overline{d}_k$, c_{pe}, and Γ have to be estimated from data. In the following, we provide corresponding estimation procedures.

An approach for the computation of Γ was shown by Berberich et al.,[21] where, however, the availability of the undisturbed data is assumed. To extend this approach for the approximation of Γ in the presence of disturbances in the data, we adapt the respective optimization problem by including a slack variable as well as a regularization, similar to (11). Thus, we set up the optimization problem

$$\max_{x_0} \min_{\substack{\alpha,\sigma,\\ \overline{v},\overline{x}}} \quad \left\|\overline{v}_{[0,n-1]}\right\|_1 + \lambda'_\alpha w_{\max}\|\alpha\|_2^2 + \frac{\lambda'_\sigma}{w_{\max}}\|\sigma\|_2^2, \tag{47a}$$

$$\text{s.t.} \quad \left\|\overline{x}_0\right\|_\infty \leq x_{\max}, \tag{47b}$$

$$\overline{x}_n = 0, \tag{47c}$$

$$\begin{bmatrix} \overline{v} \\ \overline{x} + \sigma \end{bmatrix} = \begin{bmatrix} H_n(v^d) \\ H_{n+1}(x^d) \end{bmatrix} \alpha, \tag{47d}$$

which can be solved by solving the inner minimization problem for all vertices of $\mathbb{X}$. Then, we choose $\Gamma \approx \frac{\left\|\overline{v}_{[0,n-1]}^*\right\|_1}{x_{\max}}$, where $\overline{v}_{[0,n-1]}^*$ is the optimal solution of (47). Moreover, we approximate $c_{pe} \approx \left\|H_{vx}^\dagger\right\|_1$, with

$$H_{vx} = \begin{bmatrix} H_L\left(v^d\right) \\ H_1\left(x_{[0,N-L-1]}^d\right) \end{bmatrix}. \tag{48}$$

This approximation is possible, as for small disturbances also the error between H_{vx} and $H_{v\hat{x}}$ is small. So far, both procedures mentioned above only yield approximations of the real constants Γ and c_{pe}, without being guaranteed overapproximations of these constants. However, as was also confirmed in our numerical examples, the error between the real values and our estimates remains small for the considered disturbance level. Obtaining guaranteed overapproximations of the constants Γ and c_{pe} even in the presence of noise is an interesting subject for future research.

In order to estimate the overapproximations $\rho_{A,k} \geq \left\|A_K^k\right\|_\infty$ and $\overline{d}_k \geq \sum_{i=0}^{k-1} \left\|A_K^{k-1-i}\right\|_\infty w_{\max}$ for $i = 0, \dots, L-1$ we use algorithm 1 by Wildhagen et al.,[29] which makes use of the S-lemma.[26] Note that the setup in the aforementioned reference considers a bound on the 2-norm of the disturbance. However, by noting that $\|w\|_2 \leq \sqrt{n}\,\|w\|_\infty$, we can easily adapt the algorithm to our setting.

4 | DATA-DRIVEN OUTPUT-FEEDBACK PREDICTIVE CONTROL

In this section, we construct a robust data-driven predictive control scheme, in case only output measurements are available. First, in Section 4.1 we set up the MPC scheme and prove similar theoretical properties to the ones shown for the state-feedback case. Thereafter, in Section 4.2 we show how the coefficients used for the constraint tightening can be computed only from data.

4.1 | Proposed MPC scheme and theoretical guarantees

In contrast to the previous section, we consider the case where no full state measurements are available, but only output measurements. To this end, we consider the difference operator form

$$y_k = -A_n y_{k-1} - \dots - A_2 y_{k-n+1} - A_1 y_{k-n} + D u_k + B_n u_{k-1} + \dots + B_2 u_{k-n+1} + B_1 u_{k-n} + \tilde{w}_k, \tag{49}$$

with the process disturbance $\tilde{w}_k \in \tilde{\mathbb{W}} = \{\tilde{w} \in \mathbb{R}^p |\, \|\tilde{w}\|_\infty \leq \tilde{w}_{\max}\}$. The input and output constraints are given by $u_t \in \mathbb{U} = \{u \in \mathbb{R}^m |\, \|u\|_\infty \leq u_{\max}\}$ and $y_t \in \mathbb{Y} = \left\{y \in \mathbb{R}^p |\, \|y\|_\infty \leq y_{\max}\right\}$ for some $u_{\max} > 0$, $y_{\max} > 0$, similar to the setup in Section 3. Note that the following results also hold if only an upper bound on the system order is known, in which case n needs to be replaced by this upper bound. Furthermore, note that (49) is an equivalent characterization of the input–output behavior of (1), with $C \neq I$ or $D \neq 0$. Moreover, we can transform (49) into the nonminimal realization

$$\begin{aligned} \xi_{k+1} &= \tilde{A}\xi_k + \tilde{B}u_k + \tilde{E}\tilde{w}_k, \\ y_k &= \tilde{C}\xi_k + Du_k + \tilde{w}_k, \end{aligned} \tag{50}$$

with the extended state $\xi_k = \begin{bmatrix} u_{k-n}^\top & \dots & u_{k-1}^\top & y_{k-n}^\top & \dots & y_{k-1}^\top \end{bmatrix}^\top$, compare Reference 30. Similar to the state feedback case in Section 3, we want to make use of a prestabilizing controller in case of an unstable system. Therefore, we introduce the control law

$$u_k = \tilde{K}\xi_k + v_k, \tag{51}$$

where the stabilizing feedback matrix $\tilde{K}$ can be computed purely from data, for example, following the approach by Berberich et al.[31] However, for simplicity it is assumed that such a prestabilizing controller is known a priori. Thus, v_k is the input to the stabilized system

$$\begin{aligned} \xi_{k+1} &= \tilde{A}_K\xi_k + \tilde{B}v_k + \tilde{E}\tilde{w}_k, \\ y_k &= \tilde{C}_K\xi_k + Dv_k + \tilde{w}_k. \end{aligned} \tag{52}$$

We note that, starting at an initial state ξ_0 and applying the input sequence $v_{[0,k]}$ to system (52), the output after k time steps is given by

$$y_k = \tilde{C}_K\tilde{A}_K^k\xi_0 + \tilde{C}_K\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{B}v_i + Dv_k + \tilde{C}_K\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{E}\tilde{w}_i + \tilde{w}_k. \tag{53}$$

Thus, similar to (7) we denote the disturbance affecting the output after being propagated k steps through the system dynamics as

$$\delta_k^y := \tilde{C}_K\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{E}\tilde{w}_i + \tilde{w}_k, \tag{54}$$

for $k > 1$. Due to (51), also the pre-stabilizing input contains previously occurred disturbances. Therefore, using the same initial state and input sequence as in (53), the prestabilizing input at time k is given by

$$u_k = \tilde{K}\tilde{A}_K^k \xi_0 + \tilde{K}\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{B}v_i + v_k + \tilde{K}\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{E}\tilde{w}_i. \tag{55}$$

Similar to (54) we define the disturbance affecting the input after k steps as

$$\delta_k^u := \tilde{K}\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{E}\tilde{w}_i, \tag{56}$$

for $k > 1$. In order to bound the prediction error we will make use of the upper bounds $\overline{\delta}_k^y \geq \left\|\delta_k^y\right\|_\infty$ and $\overline{\delta}_k^u \geq \left\|\delta_k^u\right\|_\infty$, which, similar to the error bounds in Section 3, can be be overapproximated purely from data as will be discussed in Section 4.2.

Again, we apply a p.e. input sequence $\left\{v_k^d\right\}_{k=0}^{N-1}$ of length N to system (52), and measure the associated disturbed output sequence $\{y_k^d\}_{k=0}^{N-1}$ and the disturbed prestabilizing input sequence $\left\{u_k^d\right\}_{k=0}^{N-1}$, respectively. Note that, in order to apply the first input via the prestabilization (51), we need n additional initial data points $\xi_0^d = \begin{bmatrix} u_{[-n,-1]}^{d\top} & y_{[-n,-1]}^{d\top}\end{bmatrix}^\top$. In practice this initial sequence can be generated by applying an arbitrary input sequence to (50) and measuring the resulting output. As the OCP, introduced in the following, now contains n additional steps to fix the initial state, the following assumption is needed.

Assumption 2. The input sequence $\left\{v_k^d\right\}_{k=0}^{N-1}$ is persistently exciting of order $L + 2n$.

Using the a priori collected data sequences, we set up the OCP for the output-feedback predictive control problem as

$$J_L^*(\overline{v}_{[0,n-1]}(t-n), y_{[t-n,t-1]}) = \min_{\substack{\alpha(t),\sigma(t),\\ \overline{v}(t),\overline{y}(t)}} \sum_{k=0}^{L-1}\left(\left\|\overline{v}_k(t)\right\|_R^2 + \left\|\overline{y}_k(t)\right\|_Q^2\right) + \lambda_\alpha \tilde{w}_{\max}\|\alpha(t)\|_2^2 + \frac{\lambda_\sigma}{\tilde{w}_{\max}}\|\sigma(t)\|_2^2 \tag{57a}$$

$$\text{s.t.} \begin{bmatrix} \overline{v}(t) \\ \begin{bmatrix}\overline{y}(t)\\ \overline{u}(t)\end{bmatrix} + \sigma(t)\end{bmatrix} = \begin{bmatrix} H_{L+n}(v^d) \\ H_{L+n}(y^d) \\ H_{L+n}(u^d)\end{bmatrix}\alpha(t), \tag{57b}$$

$$\begin{bmatrix} \overline{v}_{[-n,-1]}(t) \\ \overline{y}_{[-n,-1]}(t) \\ \overline{u}_{[-n,-1]}(t)\end{bmatrix} = \begin{bmatrix} \overline{v}^*_{[0,n-1]}(t-n) \\ y_{[t-n,t-1]} \\ u_{[t-n,t-1]}\end{bmatrix}, \tag{57c}$$

$$\begin{bmatrix} \overline{v}_{[L-n,L-1]}(t) \\ \overline{y}_{[L-n,L-1]}(t) \\ \overline{u}_{[L-n,L-1]}(t)\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix}, \tag{57d}$$

$$\left\|\overline{y}_k(t)\right\|_\infty + f_{u,k}\left\|\overline{v}(t)\right\|_1 + f_{\alpha,k}\left\|\alpha(t)\right\|_1 + f_{\sigma,k}\left\|\sigma(t)\right\|_\infty + f_{c,k} \leq y_{\max}, \tag{57e}$$

$$\left\|\overline{u}_k(t)\right\|_\infty + g_{u,k}\left\|\overline{v}(t)\right\|_1 + g_{\alpha,k}\left\|\alpha(t)\right\|_1 + g_{\sigma,k}\left\|\sigma(t)\right\|_\infty + g_{c,k} \leq u_{\max}, \tag{57f}$$

$$\forall k = 0, \ldots, L-n-1. \tag{57g}$$

Problem (57) is similar to (11) in the state-feedback case. The key difference is that, due to the term $H_{L+n}(u^d)\alpha(t)$ in (57b), also a prediction for the prestabilizing input is incorporated in the OCP. This is necessary as an input constraint tightening analogous to (11f) is not possible in the output feedback case. The reason for this is that a prediction of the extended state is not available, due to the fact that the extended state of the prestabilized system incorporates the prestabilizing input u. However, making use of the prediction $\overline{u}(t)$ an input constraint tightening (57f) with the same structure as the output constraint tightening (57e) can be set up. Moreover, the initial constraint (57c) and the terminal constraint (57d) hold over n steps. This implies that the internal state of the underlying minimal realization corresponding to the prediction coincides with the initial state and the terminal state, respectively (cf, the work of Markovsky and Rapisarda[32]). Furthermore, note that due to the initial condition of the prestabilizing input the slack variable $\sigma = \begin{bmatrix} \sigma_y^\top & \sigma_u^\top \end{bmatrix}^\top \in \mathbb{R}^{L(p+m)}$ consists of an output and an input slack.

In order to set up the coefficients of the tightened input and output constraints, we have to define further system constants, similar to the ones of the state-feedback case. Again, all of these constants can be (over-)approximated from data as will be discussed in Section 4.2. Firstly, we consider the constant $c_{pe}^{\xi} = \left\| H_{u\hat{\xi}}^{\dagger} \right\|_1$ with

$$H_{v\hat{\xi}} = \begin{bmatrix} H_{L+n}\left(v^d\right) \\ H_1\left(\hat{\xi}^d_{[0,N-L-n]}\right) \end{bmatrix}, \tag{58}$$

where $\hat{\xi}_k^d = \begin{bmatrix} \hat{u}^{d\top}_{[k-n,k-1]} & \hat{y}^{d\top}_{[k-n,k-1]} \end{bmatrix}^\top$ consists of the undisturbed input/output data, that is, $\hat{u}_k^d = u_k^d - \delta_k^{u,d}$, and $\hat{y}_k^d = y_k^d - \delta_k^{y,d}$ with $\delta_k^{u,d} = \tilde{K}\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{E}\tilde{w}_i^d$ and $\delta_k^{y,d} = \tilde{C}_K\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{E}\tilde{w}_i^d + \tilde{w}_k^d$, respectively. Moreover, we introduce a controllability constant Γ_{uy} which has a similar interpretation as Γ in (14). However, in the output feedback case, the constant is defined by considering two different input/output trajectories $\left\{v_k', y_k'\right\}_{k=-n}^{2n-1}$ and $\left\{v_k'', y_k''\right\}_{k=-n}^{n-1}$ of the undisturbed system (52). Both trajectories start with the same initial condition, that is, $v'_{[-n,-1]} = v''_{[-n,-1]}$ and $y'_{[-n,-1]} = y''_{[-n,-1]}$. While $v''_{[0,n-1]}$ is zero, $v'_{[0,n-1]}$ steers the system to the origin in n steps, that is, $v'_{[n,2n-1]} = 0$ and $y'_{[n,2n-1]} = 0$. The controllability constant Γ_{uy} is now defined such that

$$\left\|v'_{[0,n-1]}\right\|_1 \leq \Gamma_{uy}\left\|y''_{[0,n-1]}\right\|_\infty. \tag{59}$$

is satisfied for all trajectories according to the above specifications. Furthermore, we will make use of the constant

$$\xi_{\max} := \max_{\xi\in\mathbb{U}^n\times\mathbb{Y}^n} \|\xi\|_1. \tag{60}$$

Finally, we introduce the constants $\rho_k^y \geq \left\|\tilde{C}_K\tilde{A}_K^k\right\|_\infty$ and $\rho_k^u \geq \left\|\tilde{K}\tilde{A}_K^k\right\|_\infty$ for $k = 0, \dots, L-1$.

Using these system constants together with the definitions

$$c_{\alpha,k}^y := \rho_k^y\bar{\delta}_{N-L-1} + \bar{\delta}^y_{N-L+k}, \quad c_{\sigma,k}^y := \rho_k^y + 1, \quad c_{\alpha,k}^u := \rho_k^u\bar{\delta}_{N-L-1} + \bar{\delta}^u_{N-L+k}, \quad c_{\sigma,k}^u := \rho_k^u + 1, \tag{61}$$

for $k = 0, \dots, L-1$, where $\bar{\delta}_{N-L-1} := \max\left(\bar{\delta}^u_{N-L-1}, \bar{\delta}^y_{N-L-1}\right)$, we are now ready to define the coefficients of the constraint tightenings (57e) and (57f). To this end, we define

$$\begin{aligned} &f_{u,k} = 0,\ f_{\alpha,k} = c_{\alpha,k}^y,\ f_{\sigma,k} = c_{\sigma,k}^y,\ f_{c,k} = \bar{\delta}_k^y, \\ &g_{u,k} = 0,\ g_{\alpha,k} = c_{\alpha,k}^u,\ g_{\sigma,k} = c_{\sigma,k}^u,\ g_{c,k} = \bar{\delta}_k^u, \end{aligned} \tag{62}$$

for $k = 0, \dots, n-1$, and

$$\begin{aligned} f_{u,k+n} &= f_{u,k} + f_{\alpha,k}c_{pe}^{\xi} + f_{\sigma,k}c_{pe}^{\xi}\bar{\delta}^y_{N-1}, \\ f_{\alpha,k+n} &= f_{u,k+n}\Gamma_{uy}c_{\alpha,L-1}^y + c_{\alpha,k+n}^y, \\ f_{\sigma,k+n} &= f_{u,k+n}\Gamma_{uy}c_{\sigma,L-1}^y + c_{\sigma,k+n}^y, \\ f_{c,k+n} &= f_{c,k} + f_{\alpha,k}c_{pe}^{\xi}\xi_{\max} + f_{\sigma,k}\left(\bar{\delta}^y_{N-1}c_{pe}^{\xi}\xi_{\max} + \bar{\delta}^y_n\right), \\ g_{u,k+n} &= g_{u,k} + g_{\alpha,k}c_{pe}^{\xi} + g_{\sigma,k}c_{pe}^{\xi}\bar{\delta}^u_{N-1}, \\ g_{\alpha,k+n} &= g_{u,k+n}\Gamma_{uy}c_{\alpha,L-1}^y + c_{\alpha,k+n}^u, \\ g_{\sigma,k+n} &= g_{u,k+n}\Gamma_{uy}c_{\sigma,L-1}^y + c_{\sigma,k+n}^u, \\ g_{c,k+n} &= g_{c,k} + g_{\alpha,k}c_{pe}^{\xi}\xi_{\max} + g_{\sigma,k}\left(\bar{\delta}^u_{N-1}c_{pe}^{\xi}\xi_{\max} + \bar{\delta}^u_n\right), \end{aligned} \tag{63}$$

for $k = 0, \dots, L-2n-1$. Note that the coefficients are similar to (15) and (16) in the state-feedback case. However, due to the prediction of the prestabilizing input, the coefficients of the input constraint tightening no longer depend on the feedback matrix $\tilde{K}$. This comes at the cost of having to account not only for the output prediction error, but also for the prediction error of the prestabilizing input.

Next, we want to derive upper bounds for these prediction errors, similar to Lemma 2. To this end, we define the undisturbed output at time $t+k$ resulting from an open-loop application of $\overline{v}^*(t)$ as

$$\hat{y}^*_{t+k} := \tilde{C}_K\tilde{A}_K^k\xi_t + \tilde{C}_K\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{B}\overline{v}_i^*(t) + Dv_k^*(t), \tag{64}$$

as well as the undisturbed prestabilizing input at time $t+k$ resulting from an open-loop application of $\overline{v}^*(t)$ as

$$\hat{u}^*_{t+k} := \tilde{K}\tilde{A}_K^k\xi_t + \tilde{K}\sum_{i=0}^{k-1}\tilde{A}_K^{k-1-i}\tilde{B}\overline{v}_i^*(t) + v_k^*(t), \tag{65}$$

where $\xi_t = \begin{bmatrix} u_{[t-n,t-1]}^\top & y_{[t-n,t-1]}^\top \end{bmatrix}^\top$ is the extended state measured at time t. Moreover, we define the trajectories

$$\check{y}^* := H_{L+n}(\hat{y}^d)\alpha^*(t), \tag{66}$$

$$\check{u}^* := H_{L+n}(\hat{u}^d)\alpha^*(t), \tag{67}$$

where $\hat{y}^d$ and $\hat{u}^d$ correspond to the undisturbed data in the Hankel matrices, that is, $\hat{y}_k^d = y_k^d - \delta_k^{d,y}$ and $\hat{u}_k^d = u_k^d - \delta_k^{d,u}$ for $k = 0, \dots, N-1$, with $\delta_k^{d,y} = \tilde{C}_K\sum_{i=0}^{k-1}\tilde{A}_K\tilde{E}\tilde{w}_i^d + \tilde{w}_k^d$ and $\delta_k^{d,u} = \tilde{K}\sum_{i=0}^{k-1}\tilde{A}_K\tilde{E}\tilde{w}_i^d$. Note that, due to the absence of disturbances in the Hankel matrices, $\check{y}^*_{t+k}$ and $\check{u}^*_{t+k}$ can be be formulated analogous to (64) and (65) as trajectories resulting from the open-loop application of $\overline{v}^*(t)$ with the initial condition

$$\check{\xi}_t = \begin{bmatrix} \check{u}^*_{[t-n,t-1]} \\ \check{y}^*_{[t-n,t-1]} \end{bmatrix} = \begin{bmatrix} u_{[t-n,t-1]} - H_n\left(\delta^{d,u}_{[0,N-L-1]}\right)\alpha^*(t) + \sigma^{u*}_{[-n,-1]}(t) \\ y_{[t-n,t-1]} - H_n\left(\delta^{d,y}_{[0,N-L-1]}\right)\alpha^*(t) + \sigma^{y*}_{[-n,-1]}(t) \end{bmatrix}, \tag{68}$$

instead of ξ_t due to (57b) and (57c). With these definitions, we are now ready to state the following lemma.

Lemma 3. *If (57) is feasible at time t, then*

$$\left\|\hat{y}^*_{t+k} - \overline{y}^*_k(t)\right\|_\infty \leq c^y_{\alpha,k}\left\|\alpha^*(t)\right\|_1 + c^y_{\sigma,k}\left\|\sigma^*(t)\right\|_\infty, \tag{69}$$

$$\left\|\hat{u}^*_{t+k} - \bar{u}^*_k(t)\right\|_\infty \leq c^u_{\alpha,k}\left\|\alpha^*(t)\right\|_1 + c^u_{\sigma,k}\left\|\sigma^*(t)\right\|_\infty, \tag{70}$$

hold for all $k = 0, \dots, L-1$.

Proof. We show only (69) and note that showing (70) works analogous. Firstly, note that $\hat{y}^*$ and $\check{y}^*$ are (undisturbed) trajectories of System (52) with the same input sequence, but with different initial conditions. Therefore, the following holds for $k = 0, \dots, L-1$

$$\left\|\hat{y}^*_{t+k} - \check{y}^*_{t+k}\right\|_\infty = \left\|\tilde{C}_K\tilde{A}_K^k\left(\xi_t - \check{\xi}_t\right)\right\|_\infty. \tag{71}$$

$$= \left\|\tilde{C}_K\tilde{A}_K^k \cdot \begin{bmatrix} \sigma^{u*}_{[-n,-1]}(t) - H_n\left(\delta^{d,u}_{[0,N-L-1]}\right)\alpha^*(t) \\ \sigma^{y*}_{[-n,-1]}(t) - H_n\left(\delta^{d,y}_{[0,N-L-1]}\right)\alpha^*(t) \end{bmatrix}\right\|_\infty. \tag{72}$$

$$\leq \rho_k^y\left\|\sigma^*(t)\right\|_\infty + \rho_k^y\left\|\begin{bmatrix} H_n\left(\delta^{d,u}_{[0,N-L-1]}\right) \\ H_n\left(\delta^{d,y}_{[0,N-L-1]}\right) \end{bmatrix}\alpha^*(t)\right\|_\infty. \tag{73}$$

$$\leq \rho_k^y\left\|\sigma^*(t)\right\|_\infty + \rho_k^y\overline{\delta}_{N-L-1}\left\|\alpha^*(t)\right\|_1. \tag{74}$$

Next, we bound

$$\left\|\check{y}^*_{t+k} - \overline{y}^*_k(t)\right\|_\infty = \left\|H_1\left(\hat{y}^d_{[n+k,N-L+k]}\right)\alpha^*(t) - H_1\left(y^d_{[n+k,N-L+k]}\right) + \sigma^{y*}_k(t)\right\|_\infty, \tag{75}$$

$$= \left\|H_1\left(\delta^{d,y}_{[n+k,N-L+k]}\right)\alpha^*(t) + \sigma^{y*}_k(t)\right\|_\infty, \tag{76}$$

$$\leq \overline{\delta}^y_{N-L+k}\,\|\alpha^*(t)\|_1 + \left\|\sigma^{y*}_k(t)\right\|_\infty, \tag{77}$$

for $k = 0, \ldots, L-1$. Finally, plugging (74) and (77) into

$$\left\|\hat{y}^*_{t+k} - \overline{y}^*_k(t)\right\|_\infty \leq \left\|\hat{y}^*_{t+k} - \check{y}^*_{t+k}\right\|_\infty + \left\|\check{y}^*_{t+k} - \overline{y}^*_k(t)\right\|_\infty, \tag{78}$$

yields (69). ■

We, again, close the loop by applying the optimal solution of (57) in an n-step manner, that is, $u_{t+k} = \tilde{K}\xi_{t+k} + \overline{v}^*_k(t)$ for $k = 0, \ldots, n-1$, where $\overline{v}^*_k(t)$ is the optimal solution of (57) for the prediction step k. Note that ξ_{t+k} contains the inputs $u_{[t+k-n,t+k-1]}$ and the measured outputs $y_{[t+k-n,t+k-1]}$. We are now ready to state practical exponential stability, and input as well as output constraint satisfaction of the closed loop. To this end, following the approach by Berberich et al.,[4] we now consider the Lyapunov function

$$V_t := J^*_L(v_{[t-n,t-1]}, y_{[t-n,t-1]}) + \gamma W(\xi_t), \tag{79}$$

for some $\gamma > 0$, where $W(\xi)$ is an IOSS Lyapunov function, which exists due to detectability of (A, C).[33]

Theorem 2. *Suppose that Assumption 2 holds. Then, for any $V_{\mathrm{ROA}} > 0$, there exist $\underline{\lambda}_\alpha$, $\overline{\lambda}_\alpha$, $\underline{\lambda}_\sigma$, $\overline{\lambda}_\sigma$ such that for all λ_α, λ_σ satisfying*

$$\underline{\lambda}_\alpha \leq \lambda_\alpha \leq \overline{\lambda}_\alpha, \quad \underline{\lambda}_\sigma \leq \lambda_\sigma \leq \overline{\lambda}_\sigma, \tag{80}$$

there exist $\bar{\tilde{w}}_{\max}$, $\overline{c}_{pe} > 0$ as well as a continuous, strictly increasing function $\beta : [0, \bar{\tilde{w}}_{\max}] \to [0, V_{\mathrm{ROA}}]$ with $\beta(0) = 0$, such that for all $\tilde{w}_{\max}$ and c_{pe} satisfying

$$\tilde{w}_{\max} \leq \min\left\{\bar{\tilde{w}}_{\max}, \frac{\overline{c}_{pe}}{c_{pe}}\right\}, \tag{81}$$

the following holds for the closed loop resulting from the application of the n-step MPC scheme:

(i) If $V_0 \leq V_{\mathrm{ROA}}$, then OCP (57) is feasible at any time $t \geq 0$.
(ii) For any initial condition satisfying $V_0 \leq V_{\mathrm{ROA}}$ it holds that $y_t \in \mathbb{Y}$ and $u_t \in \mathbb{U}$ for all $t \geq 0$, and V_t converges exponentially to $V_t \leq \beta(\bar{\tilde{w}}_{\max})$.

This result is similar to the state-feedback case (Theorem 1). Also the proof works along the lines of the proof of Theorem 1, where the candidate solution can be chosen analogously. The main difference lies in the modified constraint tightening. For a discussion on the role of the parameters in the above statement, we refer to the discussion below Theorem 1.

Proof. We consider the candidate solutions

$$\overline{v}'(t+n) = \begin{bmatrix} \overline{v}^*_{[0,L-n-1]}(t) \\ \overline{v}'_{[L-2n,L-n-1]}(t+n) \\ 0_n \end{bmatrix}, \tag{82}$$

$$\overline{y}'(t+n) = \begin{bmatrix} y_{[t,t+n-1]} \\ \hat{y}^*_{[t+n,t+L-n-1]} \\ \overline{y}'_{[L-2n,L-n-1]}(t+n) \\ 0_n \end{bmatrix}, \tag{83}$$

and

$$\bar{u}'(t+n) = \begin{bmatrix} u_{[t,t+n-1]} \\ \hat{u}^*_{[t+n,t+L-n-1]} \\ \bar{u}'_{[L-2n,L-n-1]}(t+n) \\ 0_n \end{bmatrix}, \tag{84}$$

where $\bar{v}'_{[L-2n,L-n-1]}(t+n)$ is the input steering the system to the origin in n steps and $\bar{y}'_{[L-2n,L-n-1]}(t+n)$, $\bar{u}'_{[L-2n,L-n-1]}(t+n)$ are the associated output and prestabilizing input. Moreover, we define the candidate solutions

$$\alpha'(t+n) = H^{\dagger}_{v\hat{\xi}} \begin{bmatrix} \bar{v}'(t+n) \\ \xi_t \end{bmatrix}, \tag{85}$$

and

$$\sigma'(t+n) = \begin{bmatrix} H_{L+n}\left(y^d\right) \\ H_{L+n}\left(u^d\right) \end{bmatrix} \alpha'(t+n) - \begin{bmatrix} \bar{y}'(t+n) \\ \bar{u}'(t+n) \end{bmatrix}. \tag{86}$$

Using these candidate solutions, the proof works analogous to the proof of Theorem 1. ■

4.2 | Data-driven estimation of system constants

In the following, we provide data-based procedures to compute (over-approximations of) the coefficients ρ^y_k and ρ^u_k for $k = 0, \dots, L-1$, $\bar{\delta}^y_k$ and $\bar{\delta}^u_k$ for $k = 0, \dots, N-1$, as well as Γ_{uy} an c^{ξ}_{pe}. First, we note that upper bounds $\left\|\tilde{A}^k_K\right\|_\infty \leq \rho_{\tilde{A},k}$ and $\left\|\sum_{i=0}^{k-1} \tilde{A}_K^{k-1-i} \tilde{E}\right\|_\infty w_{\max} \leq \bar{\delta}'_k$ can be over-approximated from data based on Reference 29, analogously to $\rho_{A,k}$ and $\bar{d}_k$ in Section 3.3. As $\|\tilde{K}\|_\infty$ is known, we can directly choose $\rho^u_k = \|\tilde{K}\|_\infty \rho_{\tilde{A},k}$ for $k = 0, \dots, L-1$ and $\bar{\delta}^u_k = \|\tilde{K}\|_\infty \delta'_k$ for $k = 0, \dots, N-1$ to obtain suitable over-approximations of the system constants. In order to obtain similar results for ρ^y_k and $\bar{\delta}^y_k$ we need to bound $\|\tilde{C}_K\|_\infty \leq \rho_{\tilde{C}}$ first. This is done by slightly modifying the approach from[29] as discussed in the following.

We consider the data matrices

$$X := \begin{bmatrix} \xi^d_n & \xi^d_{n+1} & \dots & \xi^d_{N-1} \end{bmatrix},$$

$$Y := \begin{bmatrix} y^d_n & y^d_{n+1} & \dots & y^d_{N-1} \end{bmatrix},$$

$$V := \begin{bmatrix} v^d_n & v^d_{n+1} & \dots & v^d_{N-1} \end{bmatrix}, \tag{87}$$

where $\xi^d_k = \begin{bmatrix} u^{d\top}_{k-n} & \dots & u^{d\top}_{k-1} & y^{d\top}_{k-n} & \dots & y^{d\top}_{k-1} \end{bmatrix}^\top$ for $k = n, \dots, N$. In order to compute a (possibly small) over-approximation of $\rho_{\tilde{C}}$, we solve the optimization problem

$$\min_{\tau,\bar{\sigma}^2} \quad \bar{\sigma}^2, \tag{88a}$$

$$\text{s.t.} \quad P_1(\bar{\sigma}^2) - \tau P_2 \succcurlyeq 0, \tag{88b}$$

$$\bar{\sigma}^2 \geq 0, \tag{88c}$$

$$\tau \geq 0, \tag{88d}$$

where P_1 and P_2 are defined as follows

$$P_1(\overline{\sigma}^2) = \begin{bmatrix} -I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{\sigma}^2 I \end{bmatrix}, \tag{89}$$

$$P_2 = \begin{bmatrix} -\begin{bmatrix} X \\ V \end{bmatrix}^\top & Y^\top \\ 0 & I \end{bmatrix}^\top \cdot \begin{bmatrix} -I^\top & 0 \\ 0 & n\tilde{w}_{\max}^2 N \end{bmatrix} \cdot \begin{bmatrix} -\begin{bmatrix} X \\ V \end{bmatrix}^\top & Y^\top \\ 0 & I \end{bmatrix}. \tag{90}$$

We denote the solution of (88) by $\overline{\sigma}^*$. By equivalence of norms, we can choose $\rho_{\tilde{C}} = \sqrt{n\overline{\sigma}^*}$. For a detailed discussion of the approach we refer to Reference 29. Using this result, we can now choose $\rho_k^y = \rho_{\tilde{C}}\rho_{\tilde{A},k}$ for $k = 0, \dots, L-1$ and $\overline{\delta}_k^y = \rho_{\tilde{C}}\delta_k'$ for $k = 0, \dots, N-1$, which yields suitable over-approximations.

Analogous to Section 3.3 we approximate $c_{pe}^{\xi} \approx \left\| H_{u\xi}^{\dagger} \right\|_1$ with

$$H_{v\xi} = \begin{bmatrix} H_{L+n}\left(v^d\right) \\ H_1\left(\xi_{[0,N-L-n]}^d\right) \end{bmatrix}, \tag{91}$$

and, again, note that the error between $\left\| H_{u\xi}^{\dagger} \right\|_1$ and $\left\| H_{u\hat{\xi}}^{\dagger} \right\|_1$ is small for small disturbances. Berberich et al.[21] show a procedure to compute Γ_{uy} from undisturbed data. In the following, we adapt the procedure in order to approximate Γ_{uy} for disturbed data. To this end, we consider the input/output sequences $\left\{v_k', y_k'\right\}_{k=-n}^{2n-1}$ and $\left\{v_k'', y_k''\right\}_{k=-n}^{n-1}$, introduced above (59). In order to over-approximate the controllability constant, we consider all initial conditions in the polytope $\mathbb{Z} := \mathbb{U}^n \times \mathbb{Y}^n$. We denote the vertices of $\mathbb{Z}$ by $z_{\text{init}}^i = \begin{bmatrix} v_{\text{init}}^{i\top} & y_{\text{init}}^{i\top} \end{bmatrix}^\top$ with $i = 1, \dots, N_v$ and N_v being the number of vertices of $\mathbb{Z}$. First, for all $i = 1, \dots, N_v$, we solve

$$\min_{\substack{\alpha,\sigma, \\ v',y'}} \quad \left\| v_{[0,n-1]}' \right\|_1 + \lambda_\alpha' w_{\max} \|\alpha\|_2^2 + \frac{\lambda_\sigma'}{w_{\max}} \|\sigma\|_2^2, \tag{92a}$$

$$\text{s.t.} \quad \begin{bmatrix} v_{[-n,-1]}' \\ y_{[-n,-1]}' \end{bmatrix} = z_{\text{init}}^i, \tag{92b}$$

$$\begin{bmatrix} v_{[n,2n-1]}' \\ y_{[n,2n-1]}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{92c}$$

$$\begin{bmatrix} v' \\ y' + \sigma \end{bmatrix} = \begin{bmatrix} H_{3n}\left(v^d\right) \\ H_{3n}\left(y^d\right) \end{bmatrix} \alpha. \tag{92d}$$

This yields input sequences $v_{[0,n-1]}'^i$ that (approximately) steer the system from the vertices of $\mathbb{Z}$ to zero in n steps with minimum energy. Note that, due to the disturbances in the data y^d and since that z_{init}^i might not be a trajectory of the undisturbed system (52), a slack as well as a suitable regularization is needed in (92). In order to approximate Γ_{uy}, according to (59), we now need to compute the values $\left\| y_{[0,n-1]}''^i \right\|_\infty$ resulting from the initial condition z_{init}^i after setting $v_{[0,n-1]}'' = 0$. This can be done by solving the optimization problem

$$\min_{\substack{\alpha,\sigma, \\ v'',y''}} \quad \lambda_\alpha'' w_{\max} \|\alpha\|_2^2 + \frac{\lambda_\sigma''}{w_{\max}} \|\sigma\|_2^2, \tag{93a}$$

$$\text{s.t.} \quad \begin{bmatrix} v_{[-n,-1]}'' \\ y_{[-n,-1]}'' \end{bmatrix} = z_{\text{init}}^i, \tag{93b}$$

$$v_{[0,n-1]}'' = 0 \tag{93c}$$

$$\begin{bmatrix} v'' \\ y' + \sigma \end{bmatrix} = \begin{bmatrix} H_{2n}\left(v^d\right) \\ H_{2n}\left(y^d\right) \end{bmatrix} \alpha, \tag{93d}$$

for all $i = 1, \dots, N_v$. Again, we make use of a slack variable to account for the disturbed data and z^i_{init} not being a trajectory of the system. An approximation of Γ_{uy} can now be computed via $\Gamma_{uy} = \max_{i=1,\dots,N_v} \frac{\left\| v'^i_{[0,n-1]} \right\|_1}{\left\| y'^i_{[0,n-1]} \right\|_\infty}$.

5 | NUMERICAL EXAMPLE

As an example, we consider the two mass-spring-system suggested by Wie et al.,[34] with the masses $m_1 = 0.5$ kg, $m_2 = 1$ kg and the spring constant $k = 2\,\text{kg/s}^2$. Discretizing the system with a sampling time of 1 s yields the matrices

$$A = \begin{bmatrix} -0.1799 & 1.1799 & 0.507 & 0.493 \\ 0.59 & 0.41 & 0.2465 & 0.7535 \\ -1.0421 & 1.0421 & -0.1799 & 1.1799 \\ 0.5211 & -0.5211 & 0.59 & 0.41 \end{bmatrix}, \quad B = \begin{bmatrix} 0.7266 \\ 0.1367 \\ 1.014 \\ 0.493 \end{bmatrix}. \tag{94}$$

We assume that full state measurements are available and that $w_{\max} = 10^{-3}$, $u_{\max} = 10$, $x_{\max} = 10$ hold for the constraint sets, where the process disturbance w_k acting on the system, during the data generation and in closed-loop operation, at time k is sampled uniformly from $\mathbb{W}$. As the matrix A has two eigenvalues on the unit disc, the usage of a prestabilizing controller is expected to be advantageous in order to set up the MPC scheme introduced in Section 3. First, we collect apply a PE input sequence of length 50 to the open-loop system and measure the corresponding state sequence. Thereafter, we employ theorem 1 by Berberich et al.[31] to compute a robust linear quadratic regulator for the system based on the available noisy data (using diagonal multipliers to describe the disturbance bound, compare equation (21) in Reference 31). This yields the state-feedback gain

$$K = \begin{bmatrix} 0.4345 & -0.8439 & -0.3665 & -0.6986 \end{bmatrix},$$

which serves as the prestabilizing controller, leading to all eigenvalues of $A_K = A + BK$ being located strictly inside the unit disc.

The goal is to set up the OCP (11) with prediction horizon $L = 12$, a total amount of $N = 50$ data points (of the prestabilized system) in the Hankel matrices, and the parameterization $Q = I_n$, $R = 1$, $\lambda_\alpha = \lambda_\sigma = 100$. To this end, we have to compute the constants (15) and (16) from data. In order to do so, we apply an input sequence of length $N' = 5000$, which is uniformly sampled from $\mathbb{U}$, to the prestabilized system and collect the corresponding N' state measurements. Using these data sequences, we compute $\rho_{A,k}$ for $k = 0, \dots, L-1$ and $\overline{d}_k$ for $k = 0, \dots, N-1$ following the approach mentioned in Section 3.3. Note that the estimation of the system constants also works for a smaller amount of data, at the price of a more conservative overapproximation of these constants. However, for a good approximation of these constants, we need a much larger amount of data than for the prediction via the Hankel matrices (i.e., $N' \gg N$). We now choose an input-state-sequence of length N from the collected data (of total length N'), for which the input sequence is persistently exciting of order $L + n + 1$, to construct the Hankel matrices and approximate the constant c_{pe} as described in Section 3.3. Moreover, we apply the method from Section 3.3, with $\lambda'_\alpha = \lambda'_\sigma = 1$, in order to compute an approximation of the controllability constant Γ. With these approximations of the system constants, we compute the coefficients in (15) and (16). The resulting coefficients as well as the ideal coefficients that would be computable if perfect model knowledge was available, can be found in Figure 1, where the red lines correspond to the ideal coefficients, and the blue lines to the coefficients computed from data. It can be seen that the coefficients computed from data yield over approximations of the "real" coefficients. This is mainly due to the fact that the procedure in Section 3.3 only yields overapproximations of the constants $\rho_{A,k}$ and $\overline{d}_k$. The approximation of c_{pe} by its disturbed counterpart is accurate for the present example,compare the discussion in Section 3.3.

Considering the parameter $b_{c,k}$, it can be seen that for larger k, this coefficient already is close to $u_{\max}$. Even though there is a nonmonotonicity in k, which results from the usage of an n-step MPC scheme and the corresponding recursive definition of the constants in (15) and (16), it is clearly visible that $b_{c,k}$ tends to increase for larger k. Thus, for larger prediction horizons or larger disturbance bounds $w_{\max}$, this parameter would lead to $b_{c,k} > u_{\max}$, which would render (11) infeasible due to (11f). The reason for this conservatism in the constraint tightening lies in the fact that submultiplicativity

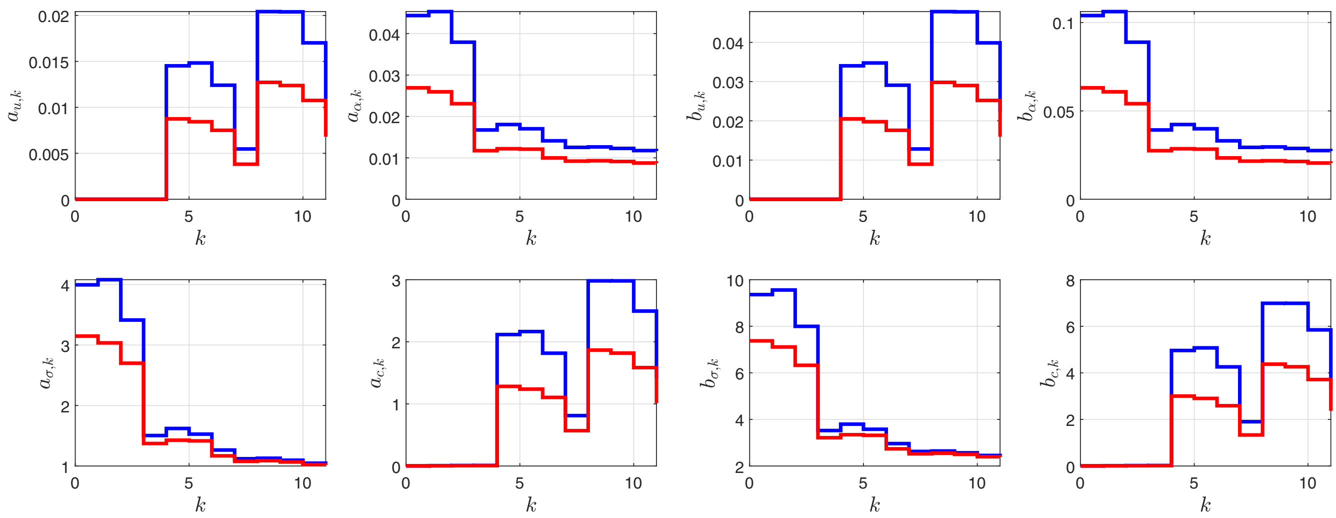

FIGURE 1 Coefficients for the state and input constraint tightening (11e), (11f). The red lines correspond to the ideal coefficients that can be computed if perfect model knowledge is available. The blue lines correspond to the coefficients computed purely from data.

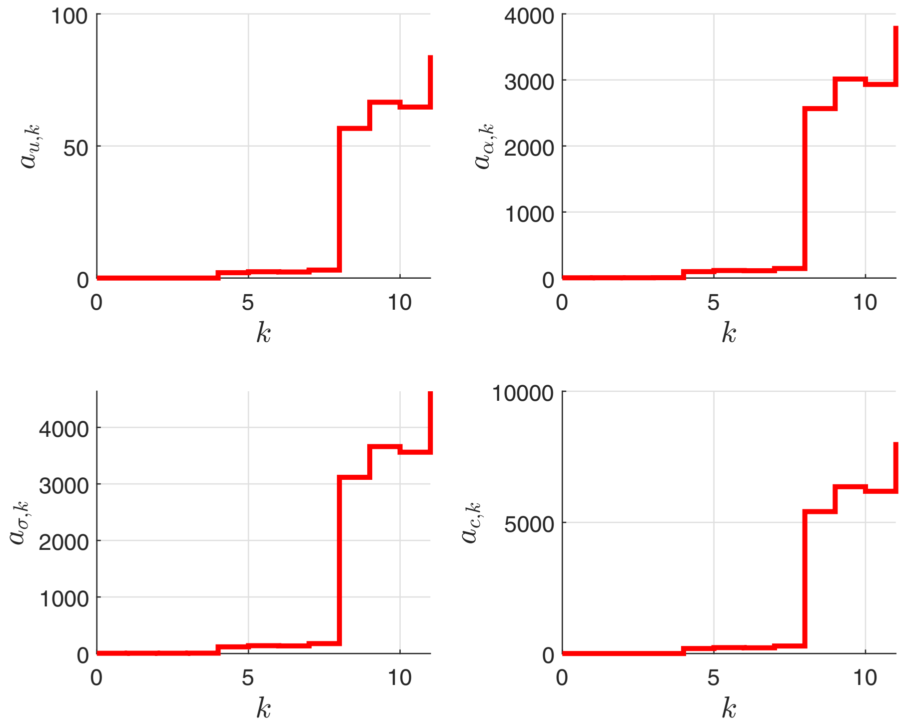

FIGURE 2 Ideal coefficients for the state constraint tightening (11e) without prestabilizing controller

and the triangle inequality were exploited multiple times in the proof of recursive feasibility and constraint satisfaction. Moreover, the estimates for $\rho_{A,k}$ and $\overline{d}_k$ only yield overapproximations of the parameters and for the sake of recursive feasibility, $x_{\max}$ instead of $\|x_t\|_\infty$ is used to define the coefficients (16).

As a motivation for the usage of a prestabilizing controller, the coefficients for $K = 0$ (i.e., without input constraint tightening) are plotted in Figure 2. Note that even for $k = 4$ it holds that $a_{c,k} \approx 233$ which already exceeds $x_{\max}$ and would thus lead to an infeasible OCP even for the smallest possible prediction horizon of the n-step scheme, $L = 4$.

The states and inputs resulting from the n-step scheme in closed loop starting at $x_0 = \begin{bmatrix} 4 & -4 & 0 & 0 \end{bmatrix}^\top$ can be found in Figure 3. It can be seen that the predictive control scheme works as desired, meaning it stabilizes the states at the origin, while satisfying the state and input constraints.

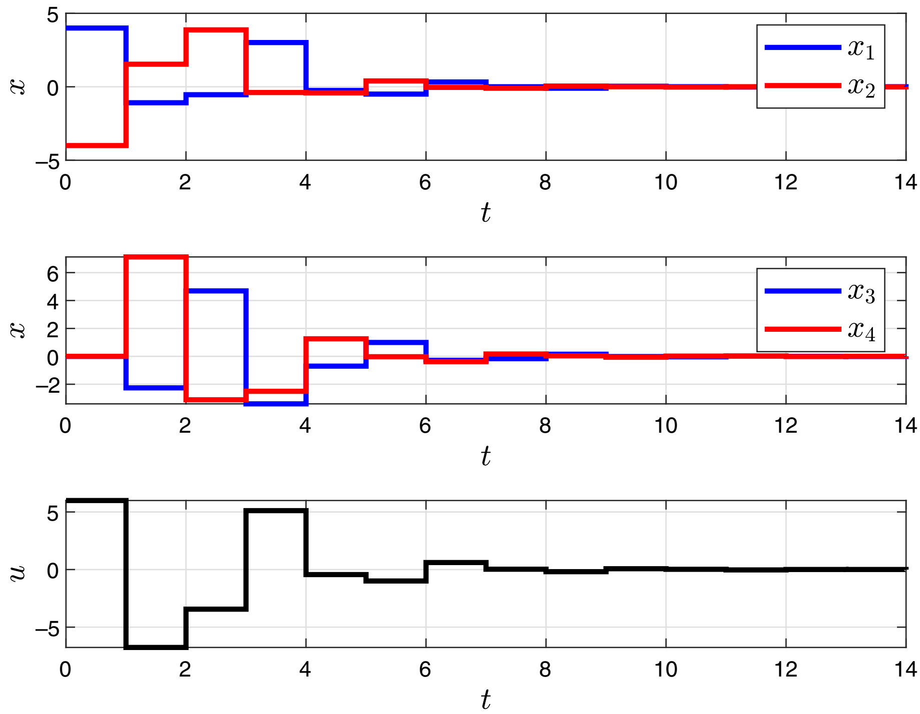

FIGURE 3 Simulation

6 | CONCLUSION

In this paper, we introduced a data-driven predictive control scheme relying on predictions based on a priori measured data, structured in Hankel matrices. The scheme is capable of stabilizing the origin of an LTI system, even in the presence of process disturbances acting on the system state. To this end, a constraint tightening is proposed, which can be set up using only a priori measured data. The presented MPC scheme allows for the usage of a prestabilizing controller and an associated input constraint tightening, which enables the use also for a priori unstable systems. Closed-loop recursive feasibility, practical exponential stability, and constraint satisfaction of the control scheme are shown. Moreover, the MPC, initially introduced for available state measurements, is extended to the case that only output measurements are available. The numerical experiments illustrated the applicability of the proposed approach and underlined the necessity to include a prestabilization and corresponding input constraint tightening in order to design a feasible controller. Interesting issues for future research include the development of less conservative constraint tightenings as well as a data-based technique for obtaining estimates of c_{pe} and the controllability constant Γ from noisy data, which are guaranteed overapproximations of the real system constants.

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CONFLICT OF INTEREST

The authors have declared no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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