TWMS J. App. and Eng. Math. V.13, N.3, 2023, pp. 891-900

A GENERALIZATION OF FIXED POINT RESULTS IN COMPLEX–VALUED QUASI–PARTIAL B–METRIC SPACE

P. GAUTAM[1*], S. VERMA[1], P. GUPTA[1], §

ABSTRACT. The aim of this paper is to establish and prove several results on fixed point portrayed in complex valued metric spaces i.e. to bring in a new type of extended b-metric space, which we call complex–valued quasi–partial b–metric space by considering an adequate partial order in the complex field and partial b–metric functions. Within this paper, fixed point results are obtained in complex–valued quasi–partial b–metric space. To substantiate the authenticity of our results and to distinguish them from existing ones, some illustrative examples are also furnished with graphs.

Keywords: complex–valued b–metric space, complex–valued quasi–partial b–metric space, contractive type mappings, fixed point theorems.

AMS Subject Classification: 47H10, 46T99, 54H25.

1. Introduction

Fixed point theorems is a flourishing field in a variety of structures of spaces like the ones developed by Zamfirescu [1], Emmanuele [2] and Suzuki [3] who studied fixed point theorem in metric spaces extensively. Since this area has developed very fast over the past two decades due to huge applications in various fields such as nonlinear analysis, topology and engineering problems, it has attracted considerable attention from researchers.
Following the flow, Bakhtin [4], Mishra, Sachdeva, Banerjee [5], Dubey, Shukla, Dubey [6] and Czerwik [7] worked on fixed point theorems in b–metric spaces. In 2011, Azzam, Fisher, Khan [8] introduced the notion of complex–valued metric spaces which was the starting point for many researchers during last decades in the field of complex–valued space and, in 2013, Karapinar, Erhan, Öztürk [9] introduced the concept of quasi–partial metric space discussing the existence of fixed points using self–map on this space. Further, Gupta and Gautam [10–12] generalized the framework of quasi–partial metric space to the class of quasi–partial b–metric space and proved some fixed point results. Since then,

[1] Department of Mathematics, Kamala Nehru College, University of Delhi, New Delhi 110049, India.
e-mail: pgautam@knc.du.ac.in; ORCID: http://orcid.org/0000-0002-4031-3672.
[*] Corresponding author.
e-mail: swapnil@knc.du.ac.in; ORCID: http://orcid.org/0000-0002-2041-3334.
e-mail: pranjulgupta.mathematics@knc.du.ac.in; ORCID: http://orcid.org/0000-0002-7181-8526.
§ Manuscript received: April 25, 2021; accepted: July 24, 2021.

Mishra, Sánchez, Gautam, Verma have performed [13–17] significant studies in the setting of quasi–partial b–metric space. In 2013, Rao, Swamy, Prasad [18] introduced fixed point theorems in complex–valued b–metric spaces. In recent year, the concept of b-metric space is been extended by Mukheimer [19–21]. Engrossing work has been done by several authors [22–33] refining this research field.
In this paper, we will introduce a new concept of fixed point theorem in the setting of complex–valued quasi–partial b–metric space. Examples are also given based on the final results.
Throughout this paper, we have used symbols like ζ, $\wp$ which belongs to $\mathbb{R}$, where $\mathbb{R}$ denote the set of all real numbers.

2. Basic Definitions and Preliminaries

Let us recall some basic definitions.

Definition 2.1 ([5]). *A b–metric on a non–empty set X is a mapping $d\colon X \times X \to [0, \infty)$ such that for some real number $s \geq 1$ and all $\zeta, \wp, z \in X$, the following conditions are satisfied:*

(i) $d(\zeta, \wp) = 0$, *if and only if* $\zeta = \wp$,
(ii) $d(\zeta, \wp) = d(\wp, \zeta)$,
(iii) $d(\zeta, \wp) \leq s[d(\zeta, z) + d(z, \wp)]$.

Then the pair (X, d) is said to be a b–metric space and the infimum over all reals $s \geq 1$ satisfying (iii) is well–defined which we call coefficient of (X, d) and denoted by $R(X, d)$.

Definition 2.2 ([7]). *A quasi–partial b–metric on a non–empty set X is a function $qp_b : X \times X \to [0, \infty)$ such that for some real number $s \geq 1$ and for all $\zeta, \wp, z \in X$*

(QPb_1) $qp_b(\zeta, \zeta) = qp_b(\zeta, \wp) = qp_b(\wp, \wp) \Rightarrow \zeta = \wp$,
(QPb_2) $qp_b(\zeta, \zeta) \leq qp_b(\zeta, \wp)$,
(QPb_3) $qp_b(\zeta, \zeta) \leq qp_b(\wp, \zeta)$,
(QPb_4) $qp_b(\zeta, \wp) \leq s[qp_b(\zeta, z) + qp_b(z, \wp)] - qp_b(z, z)$.

Then the pair (X, qp_b) is said to be a quasi–partial b–metric space, and the infimum over all reals $s \geq 1$ satisfying (QPb_4) is well-defined which we call coefficient of (X, qp_b) and denoted by $R(X, qp_b)$.
Let qp_b be a quasi–partial b–metric on the set X. Then

$$dqp_b(\zeta, \wp) = qp_b(\zeta, \wp) + qp_b(\wp, \zeta) - qp_b(\zeta, \zeta) - qp_b(\wp, \wp)$$

is a b–metric on X.

Example 2.1. *Let $M = \mathbb{R}$. Define the metric $qp_b(\zeta, \wp) = |\zeta - \wp| + |\zeta| + |\zeta - \wp|^2$ for any $(\zeta, \wp) \in M \times M$ with $s \geq 2$.*
It can be shown that (M, qp_b) is a quasi-partial b-metric space.
In fact, if $qp_b(\zeta, \zeta) = qp_b(\wp, \wp) = qp_b(\zeta, \wp)$
$\Rightarrow \zeta = \wp$ which shows (QPb_1) is true.
Also $qp_b\ (\zeta, \zeta) \leq qp_b(\zeta, \wp)$ which proves (QPb_2).
Now, $qp_b\ (\zeta, \zeta) = |\zeta| \leq |\zeta - \wp| + |\wp| + |\zeta - \wp|^2$
Since,

$$\begin{aligned} |\zeta| - |\wp| &\leq |(|\zeta| - |\wp|)| \\ &\leq |\zeta - \wp| \\ &\leq |\zeta - \wp| + |\zeta - \wp|^2 \end{aligned}$$

which proves (QPb_3). Now we will prove (QPb_4) with $s = 2$, that is
$qp_b\ (\zeta, \wp) \leq 2[qp_b(\zeta, \delta) + qp_b(\delta, \wp)] - qp_b(\delta, \delta)$
In addition, since
$|\zeta - \wp|^2 \leq (|\zeta - \delta| + |\delta - \wp|)^2 \leq 2(|\zeta - \delta|^2 + |\delta - \wp|^2)$
We have $qp_b\ (\zeta, \wp) + qp_b\ (\delta, \delta)$

$$= |\zeta - \wp| + |\zeta| + |\zeta - \wp|^2 + |\delta|$$

$$\leq 2[|\zeta - \delta| + |\delta - \wp| + |\zeta| + |\delta| + |\zeta - \delta|^2 + |\delta - \wp|^2]$$

Rearranging proves (QPb_4).
Hence (X, qp_b) is a Quasi-Partial b-metric space with $s = 2$.

Before giving the definition of complex–valued b–metric space, let us recall an existing partial order in the complex field $\mathbb{C}$.

Definition 2.3 ([19]). *Given $z_1, z_2 \in \mathbb{C}$, we define the partial order $\precsim$ on $\mathbb{C}$ as follows. We settle $z_1 \precsim z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. This means that $z_1 \precsim z_2$ if and only if one of the following conditions holds:*

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(iii) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
(iv) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

If one of the conditions (i), (ii), and (iii) holds, we write $z_1 \lnsim z_2$. On the other hand we write $z_1 \prec z_2$ if the condition (iii) is satisfied. The subset of $\mathbb{C}$ formed by the complex numbers z such that $0 \prec z$ (respectively $0 \precsim z$), will be denoted by $\mathbb{C}^+$ (respectively $\mathbb{C}^{\geq 0}$). Its members will be called positive (resp. non–negative) complex numbers.
Partial order has the following properties:

(i) If $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$,
(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3$, imply $z_1 \prec z_3$,
(iii) If $z \in \mathbb{C}$ $a, b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$.

Remark 2.1. *Having in mind the above partial order on the complex field, the notion of (real–valued) metric space was extended to complex–valued metric space in [18] by considering the function metric d to take values in the set of non–negative complex numbers instead of $[0, \infty)$ and the lesser or equal symbol of triangular inequality replaced by the partial order $\precsim$ of the complex field.*

Remark 2.2. *Similarly, the Definition 2.1 of (real–valued) b–metric space was extended to complex–valued b–metric space in [18]. For the sake of completeness and defining its corresponding coefficient, we include it next.*

Definition 2.4 ([22]). *A complex valued quasi–partial b–metric on a non–empty set X is a mapping $cqp_b\colon X \times X \to \mathbb{C}^{\geq 0}$ such that for some real number $s \geq 1$ and all $\zeta, \wp, z \in X$:*

$(CQPb_1)$ $cqp_b(\zeta, \zeta) = cqp_b(\zeta, \wp) = cqp_b(\wp, \wp) \implies \zeta = \wp$
$(CQPb_2)$ $cqp_b(\zeta, \zeta) \precsim cqp_b(\zeta, \wp)$,
$(CQPb_3)$ $cqp_b(\zeta, \zeta) \precsim cqp_b(\wp, \zeta)$,
$(CQPb_4)$ $cqp_b(\zeta, \wp) \precsim s[cqp_b(\zeta, z) + cqp_b(\wp, z)] - cqp_b(z, z)]$

Then the pair (X, cqp_b) is said to be a complex quasi–partial b–metric space and the infimum over all reals $s \geq 1$ satisfying $(CQPb_4)$ is well–defined which we call the coefficient of (X, cqp_b) and denoted by $R(X, cqp_b)$.

For a complex quasi–partial b–metric space (X, cqp_b), the function $dcqp_b\colon X \times X \to \mathbb{C}^{\geq 0}$

defined by $dcqp_b(\zeta,\wp) = cqp_b(\zeta,\wp) + cqp_b(\wp,\zeta) - cqp_b(\zeta,\zeta) - cqp_b(\wp,\wp)$ *is a complex b–metric on X.*

Examples of complex quasi–partial b–metric spaces are easily obtained taking into account the existence of real quasi–partial b–metric spaces, [10] and the following result which proof is on the other hand immediate having in mind the definition of the partial order considered in the complex field.

Proposition 2.1. *Let* $qp_{b1}, qp_{b2}\colon X\times X\to[0,\infty)$ *be any two functions and let* $cqp_b\colon X\times X\to\mathbb{C}^{\geq 0}$ *be the function defined by*

$$cqp_b(\zeta,\wp) = qp_{b1}(\zeta,\wp) + i\, qp_{b2}(\zeta,\wp),$$

i.e. qp_{b1} *and* qp_{b2} *are the corresponding real and imaginary part functions,* $Re(cqp_b)$ *and* $Im(cqp_b)$*, of* cqp_b*. Then* cqp_b *is a complex quasi–partial b–metric on* X *if and only if* qp_{b1} *and* qp_{b2} *are both real quasi–partial b–metrics on* X*. If they are, then* $R(X,cqp_b) = \max\{R(X,qp_{b1}),\ R(X,qp_{b2})\}$.

Remark 2.3. *The above proposition reminds how limits of sequences (and series) of complex numbers convergence and limits and continuity of functions of a complex variable rely on the equivalent real property of the real and imaginary part of the sequence of function [34–36]. In this line let us mention that even principal values of improper real functions have been recently evaluated by means of an adequate extension to the complex field of the concept Cauchy principal value of contour integrals [37].*

3. Main Result

Definition 4 allows endowing a quasi-partial b-metric space (X, cqp_b) with a topological structure in a standard way, [11].
In order to facilitate reading the proof of our main result, we recall the following lemma which comes from Azzam, Fisher, Khan [8].

Lemma 3.1. *([8]) Let* (X, cqp_b) *be a complex–valued quasi–partial b–metric space. Given a sequence* $\{\rho_n\}$ *in* X*, then* $\{\rho_n\}$ *is a Cauchy sequence and let* $\{\zeta_n\}$ *be a sequence in X. Then* $\{\zeta_n\}$ *is a Cauchy sequence if and only if* $|cqp_b(\zeta_n,\zeta_{n+m})|\to 0$ *as* $n\to\infty$*, where* $m\in\mathbb{N}$.

Theorem 3.1. *Let* (X, cqp_b) *be a complete complex–valued quasi–partial b–metric space and let* $T : X\to X$ *be a self-map. If there exist non–negative real numbers* a, b, c *with* $R(X,cqp_b)(b+c) < 1-a$ *such that*

$$cqp_b(T\zeta,T\wp) \precsim a\, cqp_b(\zeta,T\zeta) + b\, cqp_b(\wp,T\wp) + c\, cqp_b(\zeta,\wp) \tag{1}$$

for all $\zeta,\wp\in X$*, then* T *has a fixed point.*

Proof. Let $\zeta_0\in X$ and $\{\zeta_n\}$ be a sequence in X such that

$$\zeta_n = T\zeta_{n-1} = T^n\zeta_0.$$

Note that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
cqp_b(\zeta_{n+1}, \zeta_{n+2}) &= cqp_b(T\zeta_n, T\zeta_{n+1}) \\
&\precsim a\ cqp_b(\zeta_{n+1}, T\zeta_{n+1}) + b\ cqp_b(\zeta_n, T\zeta_n) \\
&\quad + c\ cqp_b(\zeta_n, \zeta_{n+1}) \\
&= a\ cqp_b(\zeta_{n+1}, \zeta_{n+2}) + b\ cqp_b(\zeta_n, \zeta_{n+1}) \\
&\quad + c\ cqp_b(\zeta_n, \zeta_{n+1}) \\
(1-a)cqp_b(\zeta_{n+1}, \zeta_{n+2}) &\precsim (b+c)cqp_b(\zeta_n, \zeta_{n+1}) \\
cqp_b(\zeta_{n+1}, \zeta_{n+2}) &\precsim \frac{b+c}{1-a} cqp_b(\zeta_n, \zeta_{n+1}).
\end{aligned} \tag{2}$$

Hence if we take $\gamma = \frac{b+c}{1-a}$, $cqp_b(\zeta_{n+1}, \zeta_{n+2}) \precsim \gamma cqp_b(\zeta_n, \zeta_{n+1})$ and by continuing the process, we have

$$cqp_b(\zeta_{n+1}, \zeta_{n+2}) \precsim \gamma^{n+1} cqp_b(\zeta_0, \zeta_1). \tag{3}$$

Representing the coefficient $R(X, cqp_b)$ by just R, it follows that for all $m \in \mathbb{N}$,

$$\begin{aligned}
cqp_b(\zeta_n, \zeta_{n+m}) &\precsim R[cqp_b(\zeta_n, \zeta_{n+1}) + cqp_b(\zeta_{n+1}, \zeta_{n+m})] \\
&\precsim R\ cqp_b(\zeta_n, \zeta_{n+1}) + R^2\ cqp_b(\zeta_{n+1}, \zeta_{n+2}) + R^2 \\
&\quad\ cqp_b(\zeta_{n+2}, \zeta_{n+m}) \\
&\precsim R\ cqp_b(\zeta_n, \zeta_{n+1}) + R^2\ cqp_b(\zeta_{n+1}, \zeta_{n+2}) + R^3 \\
&\quad\ cqp_b(\zeta_{n+2}, \zeta_{n+3}) + \cdots + R^m\ cqp_b(\zeta_{n+m-1}, \zeta_{n+m}) \\
&\precsim R\ \gamma^n cqp_b(\zeta_0, \zeta_1) + R^2\ \gamma^{n+1} cqp_b(\zeta_0, \zeta_1) + R^3\ \gamma^{n+2} \\
&\quad\ cqp_b(\zeta_0, \zeta_1) + \cdots + R^m\ \gamma^{n+m-1} cqp_b(\zeta_0, \zeta_1) \\
&\precsim R\ \gamma^n cqp_b(\zeta_0, \zeta_1)[1 + R\ \gamma + (R\ \gamma)^2 \\
&\quad + \cdots + (R\ \gamma)^{m-1}].
\end{aligned}$$

Now,

$$\begin{aligned}
|cqp_b(\zeta_n, \zeta_{n+m})| &\leq |R\ \gamma^n cqp_b(\zeta_0, \zeta_1)[1 + R\ \gamma + (R\ \gamma)^2 + \cdots + (R\ \gamma)^{m-1}]| \\
&= R\ \gamma^n cqp_b(\zeta_0, \zeta_1)[1 + R\ \gamma + (R\ \gamma)^2 + \cdots + (R\ \gamma)^{m-1}].
\end{aligned} \tag{4}$$

Since $R\gamma < 1$ and $R \geq 1$ then $\gamma < 1$. We have $\lim\limits_{n\to\infty} \gamma^n \to 0$. This implies $|cqp_b(\zeta_n, \zeta_{n+m})| \to 0$ as $n \to \infty$. By Lemma 3.1, $\{\zeta_n\}$ is a Cauchy sequence. Since (X, cqp_b) is complete, $\{\zeta_n\}$ converges to a point $\rho^* \in X$. Let us see that ρ^* is a fixed point of T .
By axiom $(CQPb_4)$ of complex quasi–partial b–metric,

$$\begin{aligned}
cqp_b(\zeta^*, T\zeta^*) &\precsim R[cqp_b(\zeta^*, \zeta_n) + cqp_b(\zeta_n, T\zeta^*)] \\
&= R[cqp_b(\zeta^*, \zeta_n) + cqp_b(T\zeta_{n-1}, T\zeta^*)] \\
&\precsim R[cqp_b(\zeta^*, \zeta_n) + a\ cqp_b(\zeta_{n-1}, T\zeta_{n-1}) + b\ cqp_b(\zeta^*, T\zeta^*) \\
&\quad + c\ cqp_b(\zeta_{n-1}, \zeta^*)] \\
cqp_b(\zeta^*, \zeta^*) &\precsim \frac{R}{1-bR}[cqp_b(\zeta^*, \zeta_n) + a\ cqp_b(\zeta_{n-1}, \zeta_n) + c\ cqp_b(\zeta_{n-1}, \zeta^*)].
\end{aligned} \tag{5}$$

Taking the absolute value on both sides, we get

$$\begin{aligned}|cqp_b(\zeta^*,\zeta^*)| &\leq \frac{R}{1-bR}|cqp_b(\zeta^*,\zeta_n)+a\ cqp_b(\zeta_{n-1},\zeta_n)+c\ cqp_b(\zeta_{n-1},\zeta^*)| \\ &\leq \frac{R}{1-bR}(|cqp_b(\zeta^*,\zeta_n)|+|a\ cqp_b(\zeta_{n-1},\zeta_n)| \\ &\quad +|c\ cqp_b(\zeta_{n-1},\zeta^*)|) \\ &\leq \frac{R}{1-bR}(|cqp_b(\zeta^*,\zeta_n)|+a\ \gamma^{n-1}|cqp_b(\zeta_0,\zeta_1)| \\ &\quad +c|cqp_b(\zeta_{n-1},\zeta^*)|).\end{aligned}$$

Since $\{\zeta_n\}$ converges to ζ^*, taking $n\to\infty$ it follows that $|cqp_b(\zeta^*,\zeta_n)|\to 0$ and $|cqp_b(\zeta_{n-1},\zeta^*)|\to 0$. Hence

$$|cqp_b(\zeta^*,T\zeta^*)|=0.$$

This yields,

$$\zeta^*=T\zeta^*.$$

And consequently, ζ^* is a fixed point of T. □

Example 3.1. *Let $X=\mathbb{R}$. Define complex–valued quasi–partial b–metric space as*

$$cqp_b\{\zeta,\wp\}=|\zeta-\wp|^2+\ i|\zeta-\wp|$$

Here (X,cqp_b) is a complex–valued quasi–partial b–metric space with $R(X,cqp_b)=1$.

We consider the self mapping T as $Tx=kx, 0<k<1$.

Taking $a=b=\frac{1-k}{3}, c=k$.

Here,$R(X,cqp_b)(b+c)=\frac{(2k+1)}{3}<\frac{k+2}{3}=1-a$, since $k<1$.

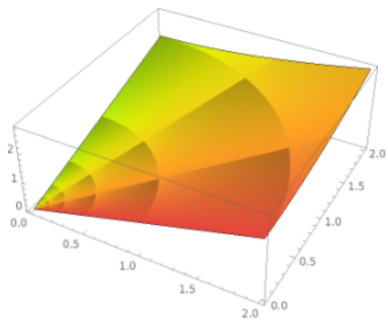

FIGURE 1. T has fixed point.

Case 1: *Let $cqp_b(\zeta,\wp)=(2,3)$ and $k=\frac{1}{4}$. We have $a=b=\frac{3}{12}, c=\frac{1}{4}$ Then,*

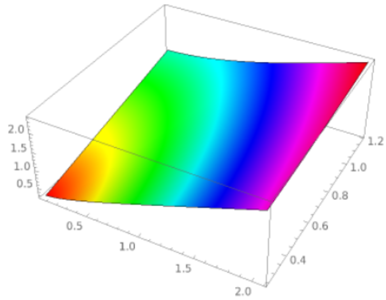

FIGURE 2. T has fixed point.

$$cqp_b(T\zeta, T\wp) \precsim a\ cqp_b(\zeta, T\zeta) + b\ cqp_b(\wp, T\wp) + c\ cqp_b(\zeta, \wp)$$
$$cqp_b(T2, T3) = \frac{1}{16} + i\frac{1}{4} \precsim \frac{3}{12}cqp_b(2, T2) + \frac{3}{12}cqp_b(3, T3) + \frac{1}{4}cqp_b(2, 3)$$

Case 2: *Let* $cqp_b(\zeta, \wp) = (1, 0)$ *and* $k = \frac{1}{3}$. *We have* $a = b = \frac{2}{9}, c = \frac{1}{3}$
Then,

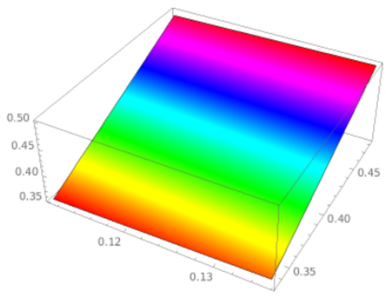

FIGURE 3. T has fixed point.

$$cqp_b(T1, T0) = \frac{1}{9} + i\frac{1}{3} \precsim \frac{2}{9}cqp_b(1, T1) + \frac{2}{9}cqp_b(0, T0) + \frac{1}{3}cqp_b(1, 0)$$

Case 3: *Let* $cqp_b(\zeta, \wp) = (3, 2)$ *and* $k = \frac{1}{5}$. *We have* $a = b = \frac{4}{15}, c = \frac{1}{5}$
Then,

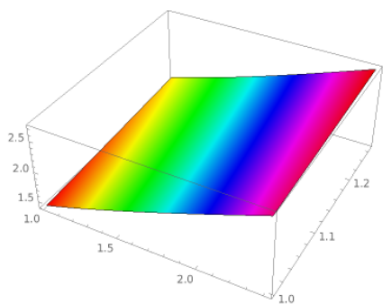

FIGURE 4. T has fixed point.

$$cqp_b(T3, T2) = 1 + i \precsim \frac{4}{15} cqp_b(3, T3) + \frac{4}{15} cqp_b(2, T2) + \frac{1}{5} cqp_b(3, 2)$$

Therefore T has a fixed point.
A graphical representation of the self-–map T is shown in Figure 1,2,3,4.

Corollary 3.1. *Consider (X, cqp_b) be a complete complex–valued quasi–partial b–metric space with the coefficient $R = R(X, cqp_b)$. Let $T: X \to X$ be a mapping (for some fixed n) satisfying that there exist non–negative real numbers a, b, c with $R(X, cqp_b)(b + c) < 1 - a$ such that*

$$cqp_b(T^n\zeta, T^n\wp) \precsim a\ cqp_b(\zeta, T^n\zeta) + b\ cqp_b(\wp, T^n\wp) + c\ cqp_b(\zeta, \wp)$$

for all $\zeta, \wp \in X$, then T has a fixed point in X.

Proof: *Theorem 3.1 provides that T^n has got a fixed point ρ in X.*
From $T^n(\zeta) = \zeta$, it follows that

$$\begin{aligned}
cqp_b(T(\zeta), \zeta) &= cqp_b(T(T^n(\zeta)), \zeta) \\
&= cqp_b(T^n(T(\zeta)), T^n(\zeta)) \\
&\precsim cqp_b(T(\zeta), T^n(T(\zeta))) + b\ cqp_b(\zeta, T^n(\zeta)) + c\ cqp_b(T(\zeta), \zeta) \\
&= a\ cqp_b(T(\zeta), T(T^n(\zeta))) + b\ cqp_b(\zeta, \zeta) + c\ cqp_b(T(\zeta), \zeta) \\
&\precsim a\ cqp_b(T(\zeta), \zeta) + bR\ cqp_b(T(\zeta), \zeta) + cR\ cqp_b(T(\zeta), \zeta) \\
&\precsim (a + bR + cR)\ cqp_b(T(\zeta), \zeta).
\end{aligned}$$

Taking absolute value on both sides, we obtain that

$$|cqp_b(T(\zeta), \zeta)| \leq (a + bR + cR)|cqp_b(T(\zeta), \zeta)|.$$

Taking into account that $R(X, cqp_b)(b + c) < 1 - a$, the above inequality implies that

$$|cqp_b(T(\zeta), \zeta)| = 0,$$

which is

$$T(\zeta) = \zeta = T^n(\zeta).$$

So, T has a fixed point.

The last results and Proposition 1 give as an immediate consequence the corresponding results for real valued functions. For complete and shortness we include just the real version of the last one which in fact comprehends both results and indeed means an extension of the Mishra, Sachdeva, Bernejee [5] result for b–metric spaces who in turn extended the Reich [38] result for complete metric spaces.

4. Conclusions

In this paper, we have extended fixed point theorem in quasi–partial b–metric space to fixed point theorems in complex quasi–partial b–metric space, by using the partial order on complex numbers and by considering the function metric d to take values in the set of non–negative complex numbers instead of $[0, \infty)$ and the lesser or equal symbol of triangular inequality replaced by the partial order $\precsim$ of the complex field. Complex valued quasi–partial b–metric space is considered as a generalization of quasi–partial b–metric space by changing the definition of real valued metric into complex valued metric. This change is expected to bring broad applications of fixed point theorems.

References

[1] Zamfirescu, T., (1972), Fixed point theorems in metric spaces, Archiv der Mathematik, 2, 1, pp. 292–298.

[2] Emmanuele, G., (1981), Fixed point theorems in complete metric spaces, Nonlinear Analysis: Theory, Methods & Applications, 5, 3, pp. 287–292.

[3] Suzuki, T., (2009), A new type of fixed point theorem in metric spaces, Nonlinear Analysis: Theory, Methods & Applications, 71, 11, pp. 5313–5317.

[4] Bakhtin, I. A., (1989), The contraction principle in quasi metric spaces, Funct. Anal., 30, pp. 26–37.

[5] Mishra, P. K., Sachdeva, S., Banerjee, S. K., (2014), Some fixed point theorems in b–metric space, Turkish Journal of Analysis and Number Theory, 2, 1, pp. 19–22.

[6] Dubey, A. K., Shukla, R., Dubey, R. P., (2014), Some fixed point results in b–metric spaces, Asian Journal of Mathematics and Applications, 6, pp. 2307–7743.

[7] Czerwik, S., (1972), Contraction mapping in b–metric spaces, Acta Mathematica et I formatica Universties Ostraviensis, 1, pp. 5–11.

[8] Azzam, A., Fisher, B., Khan, M., (2011), Common fixed point theorems in complex–valued metric spaces, Numerical Functional Analysis and Optimization, 32, 3, pp. 243–253.

[9] Karapinar, E., Erhan, M., Öztürk, A., (2013), Fixed Point theorems on Quasi–partial Metric spaces, Math. Comput. Modell., 57, pp. 2442–2448.

[10] Gupta, A., Gautam, P., (2015), Quasi–partial b–metric spaces and some related fixed point theorems, Fixed Point Theory Appl., 18, doi:10.1186/s13663-015-0260-2.

[11] Gupta, A., Gautam, P., (2016), Topological structure of quasi–partial b–metric spaces, Int. J. Pure Math. Sci., 17, pp. 8–18.

[12] Gupta, A., Gautam, P., (2015), Some coupled fixed point theorems on quasi–partial b–metric spaces, Int. J. Math. Anal., 9, pp. 293–306.

[13] Gautam, P., Verma, S., (2021), Fixed point via implicit contraction mapping on quasi-partial b-metric space, The Journal of Analysis, doi: 10.1007/s41478-021-00309-6.

[14] Gautam, P., Sánchez Ruiz, L. M., Verma, S., (2021), Fixed point of interpolative Rus–Reich–Ćirić contraction mapping on rectangular quasi-partial b-metric space, Symmetry, 13, pp. 32.

[15] Mishra, V. N., Sánchez Ruiz, L. M., Gautam, P., Verma, S., (2020), Interpolative Reich-Rus-Ćirić and Hardy-Rogers contraction on quasi-partial b-metric space, Mathematics, 8, 9, pp. 1598.

[16] Gautam, P., Mishra, V. N., Ali, R., Verma, S., (2021) Interpolative Chatterjea and cyclic Chatterjea contraction on quasi–partial b–metric space, AIMS Mathematics, 6, pp. 1727–1742.

[17] Mishra, L. N., Mishra, V. N., Gautam, P., Negi, K., (2020), Fixed point theorems for cyclic–Ciric–Reich—Rus contraction mapping in Quasi–Partial b–Metric Spaces, Scientific Publications of the State University of Novi Pazar Ser. A: Appl. Math. Inform. and Mech, 12, 1, pp. 47-56.

[18] Rao, K. P. R., Swamy, P. R., Prasad, J. R. A., (2013), Common fixed point theorem in complex–valued b–metric spaces, Bulletin of Mathematics and Statistics Research, 7, 3, pp. 149–161.

[19] Mukheimer, A. A., (2014), Some common fixed point theorems in complex–valued b–metric spaces, Hindawi Publishing Coorporation, The Scientific World Journal, Article ID 587825.

[20] Mukheimer, A., (2014), Common fixed-point theorems for a pair of mappings in complex valued bmetric spaces, Advances in Fixed Point Theory, Vol. 4, No. 3, 344-354.

[21] Mukheimer, A., (2019), Some fixed-point theorems in complex valued b-metric spaces, Italian Journal of Pure and Applied Mathematics, Volume 42, 115–125.

[22] Hasanah, D., (2017), Fixed point theorem in complex–valued b–metric spaces, Cauchy–Jurnal Mathematika Murni Dan Apikasi, 2017, 4, pp. 138–145.

[23] Sitthikul, K., Saejung, S., (2012), Some fixed point theorems in complex–valued metric spaces, Fixed Point Theory and Appl., 2012, 1, pp. 189.

[24] Sintunavarat, W., Kumam, P., (2012), Generalised common fixed point theorems in complex–valued metric spaces and applications, Journal of Inequalities and Applications, doi: 10.1186/1029-242X-2012–84.

[25] Nashine, H. K., Imdad, M., Hasan, M., (2014), Common fixed point theorems under rational contractions in complex–valued metric spaces, Journal of Nonlinear Science and appl., 7, pp. 42–50.

[26] Wu, H. -C., (2017), Coincidence point and common fixed point theorems in the product spaces of quasi-ordered metric spaces, J. Nonlinear Var. Anal., 1, pp. 175–199.

[27] Mlaiki, N. M., (2014), Common fixed points in complex S-metric space, Adv. Fixed Point Theory, 4, no.4, 509-524.

[28] F. Rouzkard, M., (2012), Imdad Some common fixed point theorems on complex valued metric spaces, Comput. Math. Appl., 64 (6), pp. 1866–1874.
[29] Gautam, P., Sánchez Ruiz, L. M., Verma, S., Gupta, G., (2021), Common Fixed point results on generalized weak compatible mapping in quasi-partial b-metric space, Journal of Mathematics, vol. 2021, Article ID 5526801.
[30] Gupta, A., Gautam, P., (2015), A version of coupled fixed point theorems on quasi-partial b-metric spaces, Adv. Fixed Point Theory, 5, pp. 407-419.
[31] Gautam, P., Verma, S., De La Sen, M., Marwaha, P. R., (2021), On $\omega-$ interpolative berinde weak contraction in quasi-partial b-metric spaces, Int. J. Anal. Appl., vol. 19, no. 4, pp. 619-632.
[32] Gautam, P., Verma, S., De La Sen, M., Sundriyal, S., (2021), Fixed point results for $\omega-$ interpolative chatterjea type contraction in quasi-partial b-metric space, Int. J. Anal. Appl., 19 (2), pp. 280-287.
[33] Gupta, A., Gautam, P., (2016), A new common tripled fixed point result in two quasi-partial b-metric spaces, Journal of Advances in Mathematics and Computer Science, pp. 1-20.
[34] Levinson, N., Redheffer, R. M., (1988), Complex Variables, McGraw-Hill: New York, NY, USA.
[35] Brown, J. W., Churchill, R. V., (2013), Complex Variables and Applications, McGraw-Hill: New York, NY, USA.
[36] Sánchez Ruiz, L. M., Legua, M., (2016), Fundamentos de Variable Compleja y Aplicaciones, Universitat Politècnica de València: Valencia, Spain.
[37] Sánchez Ruiz, L. M., Legua, M., (2017), Cauchy Principal Value Contour Integral with Applications, Entropy, 2017, 19, pp. 215, doi:10.3390/e19050215.
[38] Reich, S., (1971), Some remarks concerning contraction mappings, Canad. Math. Bull, 14, 1, pp. 121–124.

Dr. Pragati Gautam is an Associate Professor in the Department of Mathematics, Kamala Nehru College, University of Delhi.

Ms. Swapnil Verma is an alumna of Indian Institute of Technology and she is an Assistant Professor in the Department of Mathematics, Kamala Nehru College, University of Delhi.

Pranjul Gupta is a research scholar in the Department of Mathematics, Kamala Nehru College, University of Delhi. Her field of research is fixed point theory and its applications