TWMS J. App. and Eng. Math. V.13, N.3, 2023, pp. 1083-1095

PAIR DIFFERENCE CORDIAL LABELING OF SOME UNION OF GRAPHS

R. PONRAJ[1*], A. GAYATHRI[2], S. SOMASUNDARAM[2], §

Abstract. Let $G = (V, E)$ be a (p, q) graph.
Define

$$\rho = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm\rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of the union of some graphs like path, cycle, star and bistar graph.

Keywords: Path, star, cycle, bistar, comb, fan.

AMS Subject Classification: 05C78.

1. Introduction

In this paper we consider only finite, undirected and simple graphs.Cordial labeling was introduced by cachit[1]. Subsequently several authors studied cordial related labeling [2,3,8,9]. The notion of pair difference cordial labeling of a graph was introduced in [6] and the pair difference cordial labeling behavior of several graphs like path, cycle, star, triangular snake, alternate triangular snake,quadrilatral snake, alternate quadrilatral snake, butterfly have been investigated in [6,7]. In this paper we investigate the pair difference

[1] Department of Mathematics, Sri Paramakalyani College, Alwarkurichi, 627 412, India.
e-mail: ponrajmaths@gmail.com, ORCID: https://orcid.org/0000-0001-7593-7429.
[*] Corresponding author.
[2] Department of Mathematics, Manonmaniam Sundaranar University, Tirunelvelii 627 012, India.
e-mail: gayugayathria555@gmail.com, ORCID: https://orcid.org/0000-0002-1004-0175.
e-mail: somutvl@gmail.com, ORCID: https://orcid.org/0000-0001-6980-7468.
§ Manuscript received: April 15, 2021; accepted: October 22, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.3

cordial labeling behavior of the union of some graphs like path,cycle,star and bistar graph.

2. Preliminaries

Definition 2.1. *The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup E(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.*

Definition 2.2. *The subdivision graph $S(G)$ of a graph G is obtained by replacing each edge uv by a path uvw.*
a

Definition 2.3. *The graph $F_n = P_n + K_1$ is called the fan graph.*

Definition 2.4. *A wheel is the graph $W_n = C_n + K_1$ where C_n is the cycle $u_1u_2 \cdots u_nu_1$ and $V(K_1) = \{u\}$, u is the central vertex of the wheel.*

Definition 2.5. *The graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy G_2, where G_1 is graph of order n.*

Definition 2.6. *The graph $C_n \odot K_1$ is the called the crown graph.*

Definition 2.7. *The graph $P_n \odot K_1$ is the called the comb graph.*

Definition 2.8. *The bistar $B_{n,n}$ is the graph obtained by joining the apex vertices of two copies of $K_{1,n}$.*

3. Pair difference cordial labeling

Definition 3.1. Let $G = (V, E)$ be a (p, q) graph.
Define

$$\rho = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels.
Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph.

Theorem 3.1. *$K_{1,n} \cup P_n$ is pair difference cordial.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let P_n be the path $z_1z_2 \cdots z_n$.

Define the map $f : V(S(K_{1,n})) \to \{\pm 1, \pm 2, \cdots, \pm n\}$ by

$$\begin{aligned} f(u) &= 2 & \\ f(u_i) &= i & 1 \le i \le n \\ f(z_i) &= -i & 1 \le i \le n-2 \\ f(z_{n-1}) &= -n & \\ f(z_n) &= -n+1. & \end{aligned}$$

Clearly $\Delta_{f_1} = n, \Delta_{f_1^c} = n-1$. Therefore f is a pair difference cordial labeling of $K_{1,n} \cup P_n$. □

Theorem 3.2. *$S(K_{1,n}) \cup P_n$ is pair difference cordial.*

Proof. Let P_n be the path $z_1 z_2 \cdots z_n$. Let $V(S(K_{1,n})) = \{x, x_i, y_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{xx_i, x_i y_i : 1 \le i \le n\}$. Note that $S(K_{1,n}) \cup P_n$ has $3n+1$ vertices and $3n-1$ edges.

Case 1. $n \equiv 0 \pmod 4$.
Define the map $f : V(S(K_{1,n}) \cup P_n) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n}{2}\}$ by

$$\begin{aligned} f(x) &= -\frac{3n}{2} & \\ f(x_i) &= 2i & 1 \le i \le \frac{n}{2} \\ f(x_{\frac{n+2i}{2}}) &= -2i & 1 \le i \le \frac{n}{2} \\ f(y_i) &= 2i-1 & 1 \le i \le \frac{n}{2} \\ f(y_{\frac{n+2i}{2}}) &= -2i+1 & 1 \le i \le \frac{n}{2}. \end{aligned}$$

Assign the labels $(n+1), (n+2)$ respectively to the vertices z_1, z_2 and assign the labels $-(n+1), -(n+2)$ respectively to the vertices z_3, z_4. Next assign the labels $(n+3), (n+4)$ respectively to the vertices z_5, z_6 and assign the labels $-(n+3), -(n+4)$ respectively to the vertices z_7, z_8. Proceeding like this until we reach z_n.

Case 2. $n \equiv 1 \pmod 4$.
Define the map $f : V(S(K_{1,n}) \cup P_n) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+1}{2}\}$ by

$$\begin{aligned} f(x) &= -\frac{3n-1}{2} & \\ f(x_i) &= 2i & 1 \le i \le \frac{n+1}{2} \\ f(x_{\frac{n+2i+1}{2}}) &= -2i & 1 \le i \le \frac{n-1}{2} \\ f(y_i) &= 2i-1 & 1 \le i \le \frac{n+1}{2} \\ f(y_{\frac{n+2i+1}{2}}) &= -2i+1 & 1 \le i \le \frac{n-1}{2}. \end{aligned}$$

Assign the labels $(n+2), (n+3)$ respectively to the vertices z_1, z_2 and assign the labels $-(n), -(n+1)$ respectively to the vertices z_3, z_4. Next assign the labels $(n+4), (n+5)$ respectively to the vertices z_5, z_6 and assign the labels $-(n+2), -(n+3)$ respectively to the vertices z_7, z_8. Proceeding like this until we reach z_{n-1}.Finally assign the label $-\frac{3n+1}{2}$

to the vertex z_n.

Case 3. $n \equiv 2 \pmod 4$.
Assign the labels as in case 1 to the vertices $x, x_i, y_i (1 \leq i \leq n)$ and $z_i, (1 \leq i \leq n-2)$. Lastly assign the labels $-\frac{3n}{2}, \frac{3n}{2}$ respectively to the vertices z_{n-1}, z_n.

Case 4. $n \equiv 3 \pmod 4$.
Assign the labels as in case 1 to the vertices $x, x_i, y_i (1 \leq i \leq n)$ and $z_i, (1 \leq i \leq n-3)$. Finally assign the labels $\frac{3n+1}{2}, -\frac{3n-3}{2}, -\frac{3n-1}{2}$ respectively to the vertices z_{n-2}, z_{n-1}, z_n. The Table 1 given below establish that this vertex labeling f is a pair difference cordial of $S(K_{1,n}) \cup P_n$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{3n}{2}$	$\frac{3n-2}{2}$

Table 1

□

Theorem 3.3. *$K_{1,n} \cup K_{2,n}$ is pair difference cordial if and only if $n \leq 4$.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let $V(K_{2,n}) = \{v, w, v_i : 1 \leq i \leq n\}$ and $E(K_{2,n}) = \{vv_i, wv_i : 1 \leq i \leq n\}$. Obviously $K_{1,n} \cup K_{2,n}$ has $2n+3$ vertices and $3n$ edges.

Case 1. $n = 1, 2, 3, 4$.
The Table 2 and Table 3 shows that $K_{1,n} \cup K_{2,n}$ is pair diffrence cordial for $n = 1, 2, 3, 4$.

n	u	u_1	u_2	u_3	u_4
1	1	2			
2	3	-3	2		
3	4	-3	3	-4	
4	4	3	5	-4	-5

Table 2

n	v	w	v_1	v_2	v_3	v_4
1	-1	1	-2			
2	2	-2	1	-1		
3	2	-2	1	-1	3	
4	2	-2	1	-1	3	-3

Table 3

Case 2. $n \geq 5$.
Suppose f is a pair difference cordial labeling of $K_{1,n} \cup K_{2,n}$. Assume that $f(u) = l_1, f(v) = l_2, f(w) = l_3$ then the maximum value of Δ_{f_1} is attained when $f(u_i) = l_1 - 1, f(u_j) = l_1 + 1, f(v_k) = l_2 - 1, f(v_l) = l_2 + 1, f(v_m) = l_3 - 1, f(v_n) = l_3 + 1$ for some i, j, k, l, m

and n. Therefore $\Delta_{f_1} \leq 2+2+2$. That is $\Delta_{f_1} \leq 6$. This implies $\Delta_{f_1^c} \geq 3n-6$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq 3n-12 > 1$, a contradiction. Therefore $K_{1,n} \cup K_{2,n}$ is not pair difference cordial for $n \geq 5$. □

Theorem 3.4. *$K_{1,n} \cup S(K_{1,n})$ is pair difference cordial if and only if $n \leq 6$.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let $V(S(K_{1,n})) = \{x, x_i, y_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{xx_i, x_iy_i : 1 \leq i \leq n\}$. Clearly $K_{1,n} \cup S(K_{1,n})$ has $3n+2$ vertices and $3n$ edges.

Case 1. $n = 1, 2, 3, 4, 5, 6$.
The Table 4, Table 5 and Table 6 shows that $S(K_{1,n}) \cup K_{1,n}$ is pair diffrence cordial for $n = 1, 2, 3, 4, 5, 6$.

n	u	u_1	u_2	u_3	u_4	u_5	u_6
1	-1	-2	-3				
2	-2	-1	-3				
3	-4	5	-3	-5			
4	-6	5	7	-5	-7		
5	-7	7	8	-5	-6	-8	
6	-9	8	9	10	-7	-8	-10

TABLE 4

n	x	x_1	x_2	x_3	x_4	x_5	x_6
1	2	1	3				
2	4	-4	3				
3	4	2	4	-2			
4	6	2	4	-2	-4		
5	7	2	4	6	-2	-4	
6	7	2	4	6	-2	-4	-6

TABLE 5

n	y_1	y_2	y_3	y_4	y_5	y_6
1	4	-4				
2	2	1				
3	1	3	-1			
4	1	3	-1	-3		
5	1	3	5	-1	-3	
6	1	3	5	-1	-3	-5

TABLE 6

Case 2. $n \geq 7$.
Suppose f is a pair difference cordial labeling of $K_{1,n} \cup S(K_{1,n})$. Assume that $f(u) = l_1, f(x_i) = l_i$ then the maximum value of Δ_{f_1} is attained when $f(u_i) = l_1 - 1, f(u_j) = l_1 + 1$ and $f(y_k) = l_2 - 1, f(y_l) = l_2 + 1$ for some i, j, k and l.Therefore $\Delta_{f_1} \leq 2 + n$. This implies $\Delta_{f_1^c} \geq 2n-2$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq n-4 > 1$, a contradiction. Therefore $K_{1,n} \cup S(K_{1,n})$ is not pair difference cordial for $n \geq 7$. □

Theorem 3.5. *$K_{1,n} \cup K_{1,n}$ is pair difference cordial if and only if $n \leq 4$.*

Proof. Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$ be the vertex set and edge set of first copy . Let the vertex set of second copy of $K_{1,n}$ be $\{x, x_i : 1 \leq i \leq n\}$ and the edge set be $\{xx_i : 1 \leq i \leq n\}$. Note that $K_{1,n} \cup K_{1,n}$ has $2n+2$ vertices and $2n$ edges.

Case 1. $n = 1, 2, 3, 4$.
The Table 7 and Table 8 shows that $K_{1,n} \cup K_{1,n}$ is pair diffrence cordial for $n = 1, 2, 3, 4$.

n	u	u_1	u_2	u_3	u_4
1	3	1	2		
2	1	2	3		
3	2	1	3	4	
4	2	1	3	4	5

TABLE 7

n	x	x_1	x_2	x_3	x_4
1	-3	-1	-2		
2	-1	-2	-3		
3	-1	-2	-3	-4	
4	-2	-1	-3	-4	-5

TABLE 8

Case 2. $n \geq 5$.
Suppose f is a pair difference cordial labeling of $K_{1,n} \cup K_{1,n}$. Assume that $f(u) = l_1, f(x) = l_2$ then the maximum value of Δ_{f_1} is attained when $f(u_i) = l_1 - 1, f(u_j) = l_1 + 1$ and $f(x_r) = l_2 - 1, f(x_s) = l_2 + 1$ for some i, j, r and s. Therefore $\Delta_{f_1} \leq 4$.This implies $\Delta_{f_1^c} \geq 2n - 4$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq 2n - 8 > 1$, a contradiction. Therefore $K_{1,n} \cup K_{1,n}$ is not pair difference cordial for $n \geq 5$. □

Theorem 3.6. *The union of bistar $B_{n,n}$ and the path P_n , $B_{n,n} \cup P_n$ is pair difference cordial if and only if $n \leq 5$.*

Proof. Let P_n be the path $z_1 z_2 \cdots z_n$. Let $V(B_{n,n}) = \{x, y, x_i, y_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{xx_i, yy_i : 1 \leq i \leq n\} \cup \{xy\}$. Obviously $B_{n,n} \cup P_n$ has $3n + 2$ vertices and $3n$ edges.

Case 1. $n = 1$.
First assign the labels $2, -1$ to the vertices x, y respectively. Next assign the labels $1, -2$ to the vertices x_1, y_1 respectively. Now assign the label 1 to the vertex z_1.
Case 2. $n = 2$.

First assign the labels $2, -2$ to the vertices x, y respectively. Next assign the labels $1, 4, -1, -3$ respectively to the vertices x_1, x_2, y_1, y_2. Finally assign the labels $-4, 3$ to the vertices z_1, z_2 respectively.

Case 3. $3 \leq n \leq 5$.
Define the map $f : V(B_{n,n} \cup P_n) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{3n+2}{2} \rfloor\}$ by

$$
\begin{aligned}
f(x) &= 2 \\
f(y) &= -2 \\
f(x_1) &= 1 \\
f(x_2) &= 3 \\
f(x_i) &= f(x_{i-1}) + 1 && 3 \leq i \leq n \\
f(y_i) &= -f(x_i) && 1 \leq i \leq n.
\end{aligned}
$$

Next consider the vertices $z_i, 1 \leq i \leq n$. Table 9 gives the vertex labels to the vertices $z_i, 1 \leq i \leq n$ for $n = 3, 4, 5$.

n	u_1	u_2	u_3	u_4	u_5
3	5	-4	-5		
4	6	7	-6	-7	
5	7	8	-7	-8	-7

TABLE 9

Case 3. $n \geq 6$.
Suppose f is a pair difference cordial labeling of $B_{n,n} \cup P_n$. There are two cases arises.
Subcase 1.n is even.
The maximum number of edges with the label 1 among the vertex labels $1, 2, 3, \cdots, \frac{n}{2}$ respectively to the vertices $z_i, 1 \leq i \leq \frac{n}{2}$ is $\frac{n}{2} - 1$ and the maximum number of edges with the label 1 among the vertex labels $-1, -2, -3, \cdots, -\frac{n}{2}$ to the vertices $z_i, 1 \leq i \leq \frac{n}{2}$ respectively is $\frac{n}{2} - 1$. Also assume that $f(x) = l_1, f(y) = l_2$ then $\Delta_{f_1} = 4$. Therefore $\Delta_{f_1} \leq (\frac{n}{2} - 1) + (\frac{n}{2} - 1) + 4 = n + 2$. This implies $\Delta_{f_1^c} \geq 2n - 2$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq n - 4 > 1$, a contradiction.
Subcase 2.n is odd.
In this case one vertex label is repeated. This vertex labels gives maximum one edge with label 1. Therefore $\Delta_{f_1} \leq (\frac{n-1}{2} - 1) + (\frac{n-1}{2} - 1) + 5 = n + 2$. This implies $\Delta_{f_1^c} \geq 2n - 2$. Hence $\Delta_{f_1^c} - \Delta_{f_1} \geq n - 4 > 1$, a contradiction. Hence $B_{n,n} \cup P_n$ is pair difference cordial if and only if $n \leq 5$. □

Theorem 3.7. *$F_n \cup P_n$ is pair difference cordial for all values of n.*

Proof. Let P_n be the path $z_1 z_2 \cdots z_n$. Let $V(F_n) = \{x, x_i, : 1 \leq i \leq n\}$ and $E(F_n) = \{xx_i, : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\}$. Clearly $F_n \cup P_n$ has $2n + 1$ vertices and $3n - 2$ edges.
First assign the labels $1, 2, 3, \cdots, n$ to the vertices $z_1, z_2, z_3, \cdots, z_n$ respectively and assign the label -1 to the vertex x. Now consider the vertices $x_i, 1 \leq i \leq n$.There are four cases arises.
Case 1. $n \equiv 0 \pmod 4$.
Assign the labels $-1, -2$ to the vertices x_1, x_2 respectively and assign the labels $-4, -3$ respectively to the vertices x_3, x_4. Now assign the labels $-5, -6$ to the vertices x_5, x_6 respectively and assign the labels $-8, -7$ to the vertices x_7, x_8. Proceeding like this until we reach x_n. In this process the vertex x_n get the label $-n + 1$.
Case 2. $n \equiv 1 \pmod 4$.
As in case 1, assign the labels to the vertices $x_i, 1 \leq i \leq n$. Here note that the vertex x_n

get the label $-n$.
Case 3. $n \equiv 2 \pmod 4$.
As in case 1, assign the labels to the vertices $x_i, 1 \le i \le n$. In this method the vertex x_n get the label $-n$.
Case 4. $n \equiv 3 \pmod 4$.
As in case 1, assign the labels to the vertices $x_i, 1 \le i \le n$. Note that the vertices x_{n-1}, x_n get the label $-n+1, -n$.
The Table 10 given below establish that this vertex labeling f is a pair difference cordial of $F_n \cup P_n$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{3n-1}{2}$	$\frac{3n-3}{2}$
n is even	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

TABLE 10

□

Theorem 3.8. *$F_n \cup S(K_{1,n})$ is pair difference cordial for all values of n.*

Proof. Let $V(S(K_{1,n})) = \{x, x_i, y_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{xx_i, x_iy_i : 1 \le i \le n\}$. Let $V(F_n) = \{u, u_i, : 1 \le i \le n\}$ and $E(F_n) = \{uu_i, : 1 \le i \le n\} \cup \{u_iu_{i+1} : 1 \le i \le n-1\}$. Note that $F_n \cup S(K_{1,n})$ has $3n+2$ vertices and $4n-1$ edges. There are two cases arises.
Case 1.n is even.
Define the map $f : V(F_n \cup S(K_{1,n})) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+2}{2}\}$ by

$$
\begin{aligned}
f(u) &= 3n+2 \\
f(x) &= -(3n+2) \\
f(u_i) &= i && 1 \le i \le \frac{n}{2} \\
f(u_{\frac{n+2i}{2}}) &= -f(u_i) && 1 \le i \le \frac{n}{2} \\
f(x_i) &= \frac{n+4i}{2} && 1 \le i \le \frac{n}{2} \\
f(x_{\frac{n+2i}{2}}) &= -f(x_i) && 1 \le i \le \frac{n}{2} \\
f(y_i) &= \frac{n+4i-2}{2} && 1 \le i \le \frac{n}{2} \\
f(y_{\frac{n+2i}{2}}) &= -f(y_i) && 1 \le i \le \frac{n}{2}.
\end{aligned}
$$

Example 3.1. *A pair difference cordial labeling of $F_6 \cup S(K_{1,6})$ is shown in Figure 1.*

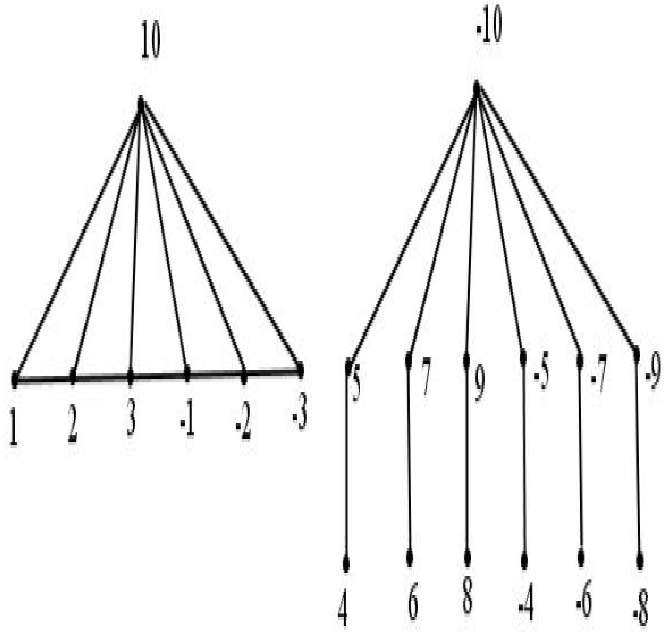

FIGURE 1

Case 2. n is odd.
Define the map $f : V(F_n \cup S(K_{1,n})) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+1}{2}\}$ by

$$f(u) = -(3n+1)$$
$$f(x) = -(3n-1)$$
$$f(u_i) = i \qquad 1 \le i \le \frac{n+1}{2}$$
$$f(u_{\frac{n+2i+1}{2}}) = -f(u_i) \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(x_i) = \frac{n+4i-1}{2} \qquad 1 \le i \le \frac{n+1}{2}$$
$$f(x_{\frac{n+2i+1}{2}}) = -f(x_i) \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(y_i) = \frac{n+4i-3}{2} \qquad 1 \le i \le \frac{n+1}{2}$$
$$f(y_{\frac{n+2i+1}{2}}) = -f(y_i) \qquad 1 \le i \le \frac{n-1}{2}.$$

The Table 11 given below establish that this vertex labeling f is a pair difference cordial of $F_n \cup S(K_{1,n})$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
n is odd	$\frac{4n}{2}$	$\frac{4n-2}{2}$
n is even	$\frac{4n}{2}$	$\frac{4n-2}{2}$

TABLE 11

□

Example 3.2. *A pair difference cordial labeling of* $F_7 \cup S(K_{1,7})$ *is shown in Figure 2.*

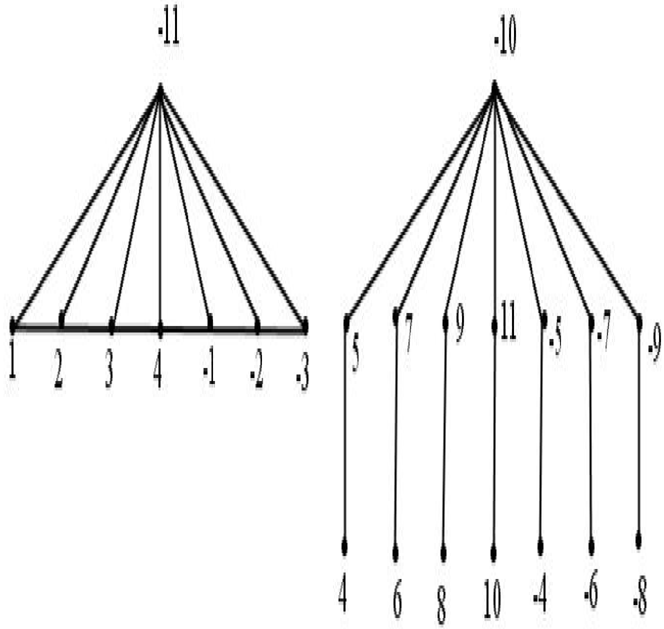

FIGURE 2

Theorem 3.9. $F_n \cup F_n$ *is pair difference cordial for all values of* n.

Proof. Let $V(F_n) = \{x, y, x_i, y_i : 1 \leq i \leq n\}$ and $E(F_n) = \{xx_i, yy_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\}$. Clearly $F_n \cup F_n$ has $2n+2$ vertices and $4n-2$ edges.
Define the map $f : V(F_n \cup F_n) \to \{\pm 1, \pm 2, \cdots, \pm(n+1)\}$ by

$$\begin{aligned} f(x) &= n+1 \\ f(y) &= -(n-1) \\ f(x_i) &= i && 1 \leq i \leq n \\ f(y_i) &= -i && 1 \leq i \leq n-1 \\ f(y_n) &= -(n+1). \end{aligned}$$

Here $\Delta_{f_1} = \Delta_{f_1^c} = 2n-1$. Hence $F_n \cup F_n$ is pair difference cordial for all values of n.

□

Theorem 3.10. $(C_n \odot K_1) \cup (P_n \odot K_1)$ *is pair difference cordial for all values of* n.

Proof. Let $V((C_n \odot K_1) \cup (P_n \odot K_1)) = \{x_i, y_i, u_i, v_i : 1 \leq i \leq n\}$ and $E((C_n \odot K_1) \cup (P_n \odot K_1)) = \{x_i x_{i+1}, x_i u_i, y_i v_i : 1 \leq i \leq n\} \cup \{y_i y_{i+1} : 1 \leq i \leq n-1\}$. Obviously $(C_n \odot K_1) \cup (P_n \odot K_1)$ has $4n$ vertices and $4n-1$ edges.
Define the map $f : V(F_n \cup F_n) \to \{\pm 1, \pm 2, \cdots, \pm(n+1)\}$ by

$$\begin{aligned} f(x_i) &= i && 1 \leq i \leq n \\ f(y_i) &= -i && 1 \leq i \leq n \\ f(u_i) &= 2n-i+1 && 1 \leq i \leq n \\ f(v_i) &= -2n+i-1 && 1 \leq i \leq n. \end{aligned}$$

Since $\Delta_{f_1} = \frac{4n}{2}$ and $\Delta_{f_1^c} = \frac{4n-2}{2}$, f is a pair difference cordial labling.

□

Theorem 3.11. *$W_n \cup S(K_{1,n})$ is pair difference cordial for all values of n.*

Proof. Let $V(S(K_{1,n})) = \{x, x_i, y_i : 1 \le i \le n\}$ and $E(S(K_{1,n})) = \{xx_i, x_iy_i : 1 \le i \le n\}$. Let $V(W_n) = \{x, x_i, : 1 \le i \le n\}$ and $E(W_n) = \{xx_i, : 1 \le i \le n\} \cup \{x_ix_{i+1} : 1 \le i \le n-1\} \cup x_1x_n$. Note that $W_n \cup S(K_{1,n})$ has $3n+2$ vertices and $4n$ edges. There are two cases arises.

Case 1.n is even.

Define the map $f : V(W_n \cup S(K_{1,n})) \to \{\pm1, \pm2, \cdots, \pm\frac{3n+2}{2}\}$ by

$$f(x) = \frac{n+2}{2}$$
$$f(y) = -\frac{n+2}{2}$$
$$f(x_i) = i \qquad 1 \le i \le \frac{n}{2}$$
$$f(x_{\frac{n+2i}{2}}) = -f(x_i) \qquad 1 \le i \le \frac{n}{2}$$
$$f(y_i) = \frac{n+4i}{2} \qquad 1 \le i \le \frac{n}{2}$$
$$f(y_{\frac{n+2i}{2}}) = -f(y_i) \qquad 1 \le i \le \frac{n}{2}$$
$$f(z_i) = \frac{n+4i+2}{2} \qquad 1 \le i \le \frac{n}{2}$$
$$f(y_{\frac{n+2i}{2}}) = -f(z_i) \qquad 1 \le i \le \frac{n}{2}.$$

Case 2.n is odd.

Define the map $f : V(W_n \cup S(K_{1,n})) \to \{\pm1, \pm2, \cdots, \pm\frac{3n+1}{2}\}$ by

$$f(x) = \frac{n+3}{2}$$
$$f(y) = -\frac{n+1}{2}$$
$$f(x_i) = i \qquad 1 \le i \le \frac{n+1}{2}$$
$$f(x_{\frac{n+2i+1}{2}}) = -f(x_i) \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(y_i) = \frac{n+4i+3}{2} \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(y_{\frac{n+2i-1}{2}}) = -f(y_i) \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(z_i) = \frac{n+4i+1}{2} \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(z_{\frac{n+2i-1}{2}}) = -f(z_i) \qquad 1 \le i \le \frac{n-1}{2}$$
$$f(y_n) = -\frac{n+3}{2}$$
$$f(z_n) = -\frac{n+1}{2}.$$

In both cases, these vertex labeling gives that $\Delta_{f_1} = \Delta_{f_1^c} = 2n$. Hence $W_n \cup S(K_{1,n})$ is pair difference cordial.

□

Example 3.3. *A pair difference cordial labeling of* $W_8 \cup S(K_{1,8})$ *is shown in Figure 3.*

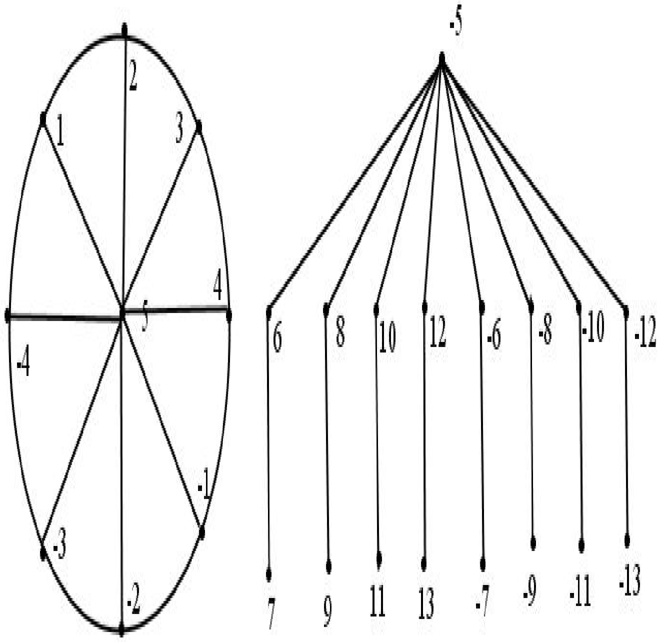

FIGURE 3

4. Conclusions

In this paper, we have studied about the pair difference cordial labeling behavior of union of some graphs. Investigation of the pair difference cordiality of some other special graphs are the open problems.

References

[1] Cahit, I., (1987), Cordial Graphs A weaker version of Graceful and Harmonious graphs, Ars combin., 23, 201–207.
[2] Ebrahim Salehi., Seth Churchman., Tahj Hill., Jim Jordan., (2014), Product Cordial Sets of Trees, Congressus Numerantium., 220, 183–193.
[3] Ebrahim Salehi., (2012), Product Cordial Sets of Long Grids, Ars combin., 107, 339–351.
[4] Gallian, J. A., (2016), A Dynamic survey of graph labeling, The Electronic Journal of Combinatorics., 19.
[5] Harary, F., (1969), Graph theory, Addision wesley, New Delhi.
[6] Ponraj, R., Gayathri, A., and Soma Sundaram, S., (2021), Pair difference cordial labeling of graphs, J.Math. Comp.Sci., Vol.11(3), 2551–2567.
[7] Ponraj, R., Gayathri, A., and Soma Sundaram, S., (2021), Pair difference cordiality of some snake and butterfly graphs, Journal of Algorithm and Computation., 53 issue 1, 149–163.
[8] Seoud, M. A., Salman M. S., (2015), On Difference cordial graphs, Mathematica Aeterna., Vol 5, 189–199.
[9] Seoud, M. A., Salman M. S., (2016), Some results and examples on difference cordial graphs, Turkish Journal of Mathematics., 40, 417–427.

R. Ponraj did his Ph.D in Manonmaniam Sundaranar University, Tirunelveli, India. His research interest in Graph Theory. He is currently an Assistant Professor at Sri Paramakalyani College, Alwarkurichi, India.

A. Gayathri did her M.Phil degree at St. Johns College, Palayamkottai, Tirunelveli, India. She is currently a research scholar, Register number: 20124012092023 in Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli. Her research interest is in Graph Theory. She has published four papers in journals.

S. Somasundaram did his Ph.D at I.I.T, Kanpur, India. He was in the faculty of Mathematics, Manonmaniam Sundaranar University, Tirunelveli. He retired as Professor in June 2020. His research interests include Analysis and Graph Theory.