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GRUNDY COLORING OF MIDDLE GRAPH OF WHEEL GRAPH FAMILIES

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Abstract. A Grundy k-coloring of a graph G is a proper k-coloring of vertices in G using colors $\{1, 2, \cdots, k\}$ such that for any two colors i and j, $i < j$, any vertex colored j is adjacent to some vertex colored i. The *First-Fit* or Grundy chromatic number (or simply Grundy number) of a graph G, denoted by $\Gamma(G)$, is the largest integer k, such that there exists a Grundy k-coloring for G. It can be easily seen that $\Gamma(G)$ equals to the maximum number of colors used by the greedy (or *First-Fit*) coloring of G [10]. In this paper, we obtain the Grundy chromatic number of middle graph of graph G, denoted by $M(G)$, where G be a cycle or sunlet graph or star graph or wheel graph or helm graph.

Keywords: Grundy chromatic number, middle graph,cycle, sunlet graph, star graph, wheel graph, helm graph.

AMS Subject Classification: 05C15, 05C75.

1. Introduction

The notion of Grundy colorings was introduced by Patrick Michael Grundy in 1939. Author has dealing with combinatorial games contained ideas that led to the concept of Grundy colorings of graphs. A Grundy coloring [4, 6, 16] of a graph G is a proper vertex coloring of G (whose colors, as usual, are positive integers) having the property that for every two colors i and j with $i \leq j$, every vertex colored j has a neighbour colored i. Consequently, every Grundy coloring is a complete coloring.

The 4-coloring of the tree T_1 of Figure 1 is a Grundy 4-coloring and is therefore a complete 4-coloring as well. However, the complete 3-coloring of T_2 shown in Figure 1 is not a Grundy 3-coloring.

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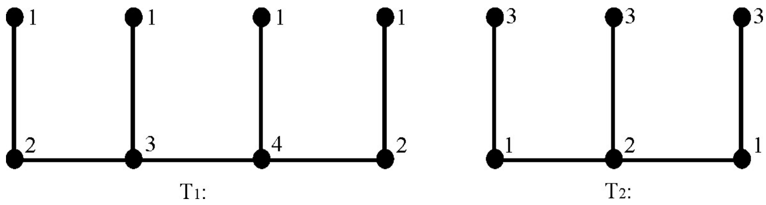

Figure 1. Complete and Grundy colorings

Recall that a greedy coloring [1] c of a graph G is obtained from an ordering ϕ : $v_1, v_2, \cdots, v_n$ of the vertices of G in some manner, by defining $c(v_1) = 1$, and once colors have been assigned to $v_1, v_2, \cdots, v_p$ for some integer p with $1 \leq p \leq n$, $c(v_{p+1})$ is defined as the smallest color not assigned to any neighbour of v_{p+1} belonging to the set $\{v_1, v_2, \cdots, v_p\}$. The coloring c so produced is then a Grundy coloring of G. (i.e.), every greedy coloring is a Grundy coloring. The maximum positive integer k for which a graph G has a Grundy k-coloring is denoted by $\Gamma(G)$ and is called the Grundy chromatic number of G or more simply the Grundy number of G. If the Grundy number of a graph G is k, then in any Grundy k-coloring of G (using the colors $1, 2, \cdots, k$), every vertex v of G colored k must be adjacent to a vertex colored i for each integer i with $1 \leq i \leq k$. Thus $\Delta(G) \geq deg(v) \geq (k-1)$ and so

$$\Gamma(G) \leq \Delta(G) + 1$$

for every graph G. Since every Grundy coloring of a graph G is a proper coloring, it follows that,

$$\chi(G) \leq \Gamma(G).$$

The Grundy number of a graph was perhaps introduced for the first time by Christen and Selkow [2]. In [3], Erdös et., al. proved that the Grundy number of a graph is in fact the same as chromatic number of a graph which was defined and studied independently by Simmons [12]. In [8] the authors studied the Grundy number of hypercubes and determined the exact values. From computational point of view, polynomial time algorithms for determining the Grundy number have been given for trees in [7] and for partial k-trees in [13]. In a manuscript [5] the NP-completeness of determining the Grundy number of general graphs has been proved. Therefore, they gave an affirmative answer to the problem 10.4 posed in the graph coloring problem book [9] which asks whether determining the Grundy chromatic number of graphs is an NP-complete problem.

2. Preliminaries

All graphs we consider are simple and finite. A closed trail whose origin and internal vertices are distinct is called a *cycle*. The $n-$*Sunlet graph*[10] on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n. A *star graph* [14] is a complete bipartite graph in which $n-1$ vertices have degree 1 and a single vertex have degree $(n-1)$. It is denoted by $K_{1,n}$. For any integer $n \geq 4$, the wheel graph W_n[15] is the $n-$vertex graph obtained by joining a vertex u_n to each of the $n-1$ vertices $\{u_1, u_2, \cdots, u_{n-1}\}$ of the cycle graph c_{n-1}. The Helm graph H_n[11] is the graph obtained from a wheel graph W_n by adjoining a pendent edge to each vertex of the $(n-1)-$cycle in W_n.

The middle graph [14] of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is the vertex and the other is an edge incident with it and it is denoted by $M(G)$.

In following section, we obtain the Grundy chromatic number of middle graph of graph G, denoted by $M(G)$, where G be a cycle or sunlet graph or star graph or wheel graph or helm graph.

3. Main Results

Now we consider G be a cycle of order m. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and let $M(G) = V(G) \cup \{u'_i : 1 \leq i \leq m\}$, where $u'_i (1 \leq i \leq m)$ is the vertices formed by subdividing edges of G in cyclic order.

Theorem 3.1. *Let G be a cycle of order $m \geq 3$, then the middle graph of G has a grundy number 4. i.e., $\Gamma(M(G)) = 4$.*

Proof. Define a mapping, $\sigma : V(M(G)) \to \mathbb{N}$ as follows:

Case (i):: *For* $m \equiv 0 \mod 4$.

For $1 \leq i \leq m$,

$$\sigma(u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \mod 4 \\ 3, & \text{if } i \equiv 2 \mod 4 \\ 1, & \text{if } i \equiv 0, 3 \mod 4 \end{cases} \tag{1}$$

$$\sigma(u'_i) = \begin{cases} 1, & \text{if } i \equiv 1 \mod 4 \\ 2, & \text{if } i \equiv 2 \mod 4 \\ 4, & \text{if } i \equiv 3 \mod 4 \\ 3, & \text{if } i \equiv 0 \mod 4 \end{cases} \tag{2}$$

Case (ii):: *For* $m \equiv 1 \mod 4$

The color classes as followed from equation (1) and (2) of order $m-1$. Also $\sigma(u'_m) = 4$ and $\sigma(u_m) = 2$.

Case (iii):: *For* $m \equiv 2 \mod 4$

The color classes as followed from equation (1) and (2) of order $m-2$. Also $\sigma(u'_{m-1}) = \sigma(u'_m) = 2$, $\sigma(u_{m-1}) = 1$ and $\sigma(u_m) = 3$.

Case (iv):: *For* $m \equiv 3 \mod 4$

The color classes as followed from equation (1) and (2) of order $m-3$. Also $\sigma(u'_{m-2}) = 2$, $\sigma(u'_{m-1}) = 3$, $\sigma(u'_{m-1}) = 1$, $\sigma(u_{m-2}) = 1$, $\sigma(u_{m-1}) = 2$ and $\sigma(u_m) = 3$.

We see that all the vertices of $u'_j\,(1 \leq j \leq n)$ have degree 4 and vertices of $u_i\,(1 \leq i \leq m)$ has degree 2.

$$\begin{aligned} \Gamma(G) &\leq \Delta(G) + 1 \\ \Gamma(M(G)) &\leq \Delta(M(G)) + 1 \leq 4 + 1 \\ \Gamma(M(G)) &\leq 5. \end{aligned}$$

Assume $\Gamma(M(G)) = 5$. Then by the construction any vertex of $u'_i\,(1 \leq i \leq m)$ is given the color 5, the in-order to satisfy the grundy coloring, the vertex which is given the color 5 must be adjacent to all other vertices which is given the colors 1, 2, 3 and 4.

If the colors 3 and 4 are given to the vertices of $u'_i\,(1 \leq i \leq m)$ then the colors 1 and 2 must be given to the vertices of $u_i\,(1 \leq i \leq m)$. Since the degree of any vertex in $u_i\,(1 \leq i \leq m)$ is 2. The color 2 which is given to the vertices of $u_i\,(1 \leq i \leq m)$ has only two adjacent vertices one is given color 5 and the other one will be given the color 3 or 4, which is not a grundy number since the color 2 is not satisfied without the color 1. i.e., the color 2 does not has any adjacent vertices with the color 1, which is contradicts. Then $\Gamma(M(G)) < 5 \neq 5$.

Suppose $\Gamma(M(G)) < 4$, i.e., $\Gamma(M(G)) = 3$.
Then it will be a proper coloring of $M(G)$. Since, the grundy number indicates the maximum number of colors used it can not be 3. Hence $\Gamma(M(G)) = 4$. □

Now we consider the middle graph of sunlet graph denoted by $M(G)$. Let $V(G) = \{u_i : 1 \leq i \leq m\} \cup \{u'_i : 1 \leq i \leq m\}$, where u_i is the vertices of cyclic order of G and u'_i is the pendent vertex of u_i. Now we define the vertex set $V(M(G)) = V(G) \cup \{v_j : 1 \leq j \leq n\} \cup \{v'_j : 1 \leq j \leq n\}$, where v_j $(1 \leq j \leq n)$ is the vertices of $u_i u_{i+1}$ $(1 \leq i \leq m-1)$ and $u_m u_1$. Also v'_j $(1 \leq j \leq n)$ is the vertices of $u_i u'_i$ $(1 \leq i \leq m)$.

Theorem 3.2. *Let G be a sunlet graph of order $m \geq 3$, then the middle graph of G has a grundy number 5. i.e., $\Gamma(M(G)) = 5$.*

Proof. Define a mapping, $\sigma : V(M(G)) \to \mathbb{N}$ as follows:
$\sigma(u_i) = \sigma(u'_i) = 2,\ 1 \leq i \leq m$
$\sigma\left(v'_j\right) = 1,\ 1 \leq j \leq n$

$$\sigma(v_j) = \begin{cases} 3, & \text{if } j \equiv 1 \mod 3 \\ 4, & \text{if } j \equiv 2 \mod 3 \\ 5, & \text{if } j \equiv 0 \mod 3 \end{cases}$$

Assume $\Gamma(M(G)) \leq 6$. The middle graph of G exists m cliques of order 4. If we give the color 6 to the vertex v_j and the color 4 to v_{j-1} and 5 to v_{j+1}. Let the color 6 and 5 having the same clique of order 4, the remaining two vertices color 2 to u_i and 1 to u'_i.

- Suppose the vertex v_{j-2} received the color 4, then the color 3 to u_{j-1} and 2 to u'_{j-1} or the color 2 to u_{j-1} and 3 to u'_{j-1}, which is contradiction.
- Suppose the vertex v_{j-2} received the color 3, then the color 4 to u_{j-1} and 2 to u'_{j-1} or the color 2 to u_{j-1} and 4 to u'_{j-1}, which is contradiction.

So, $\Gamma(M(G)) \leq 5$.
Suppose $\Gamma(M(G)) < 5$, i.e., $\Gamma(M(G)) = 4$.
Since there exists a clique of order 4, i.e., $\Gamma(M(G)) \geq 4$. The grundy number indicates the maximum number of colors used it can not be 4. Hence $\Gamma(M(G)) = 5$. □

Consider the middle graph of star graph denoted by $M(G)$. Let $V(G) = \{u_0\} \cup \{u_i : 1 \leq i \leq m\}$. Now we define the vertex set $V(M(G)) = V(G) \cup \{s_i : 1 \leq i \leq m\}$, where s_i is the subdivision vertices of $u_0 u_i$, $(1 \leq i \leq m)$.

Theorem 3.3. *Let G be a star graph of order $m \geq 3$, then the middle graph of G has a grundy number $m+1$. i.e., $\Gamma(M(G)) = m+1$.*

Proof. Define a mapping, $\sigma : V(M(G)) \to \mathbb{N}$ as follows:

$$\begin{aligned} \sigma(u_0) &= m+1; \\ \sigma(s_i) &= i,\ 1 \leq i \leq m; \\ \sigma(u_i) &= 1,\ 2 \leq i \leq m; \\ \sigma(u_1) &= 2. \end{aligned}$$

Let us assume that $\Gamma(M(G)))$ is greater than $m+1$, i.e. $\Gamma(M(G)) = m+2$. Since there exist a clique of order $m+1$. Inequality of grundy numbers as follows,

$$\phi(M(G)) \geq \Gamma(M(G)) \geq \chi(M(G)).$$

Here $m+1 = \phi(M(G)) \geq \Gamma(M(G))$. Hence $\Gamma(M(G)) \geq m+1$. Therefore $\Gamma(M(G)) = m+1$. □

We consider the middle graph of wheel graph denoted by $M(G)$. Let $V(G) = \{u_m\} \cup \{u_i : 1 \leq i \leq m-1\}$, where u_m is the centre vertex of G. and u_i $(1 \leq i \leq m-1)$ is the pendent vertex of G. Now we define the vertex set $V(M(G)) = V(G) \cup \{v_i : 1 \leq i \leq m-1\} \cup \{v_i' : 1 \leq i \leq m\}$, where v_i is the vertices of $u_m u_i$, $(1 \leq i \leq m-1)$. Also v_i' $(1 \leq i \leq m-1)$ is the vertices of $u_i u_{i+1}$, $(1 \leq i \leq m-1)$ and v_m' is the vertex of $u_m u_1$.

Theorem 3.4. *Let G be a wheel graph of order $m \geq 4$, then the middle graph of G has a grundy number $m+2$. i.e., $\Gamma(M(G)) = m+2$.*

Proof. Define a mapping, $\sigma : V(M(G)) \to \mathbb{N}$ as follows:
$\sigma(u_m) = 1$; $\sigma(v_i) = i+3$, $1 \leq i \leq m-1$;
For $1 \leq i \leq m-1$,

$$\sigma(u_i) = \begin{cases} 1, & \text{if } i \equiv 1 \mod 3 \\ 3, & \text{if } i \equiv 2 \mod 3 \\ 2, & \text{if } i \equiv 0 \mod 3 \end{cases}$$

$$\sigma(v_i') = \begin{cases} 2, & \text{if } i \equiv 1 \mod 3 \\ 1, & \text{if } i \equiv 2 \mod 3 \\ 3, & \text{if } i \equiv 0 \mod 3 \end{cases}$$

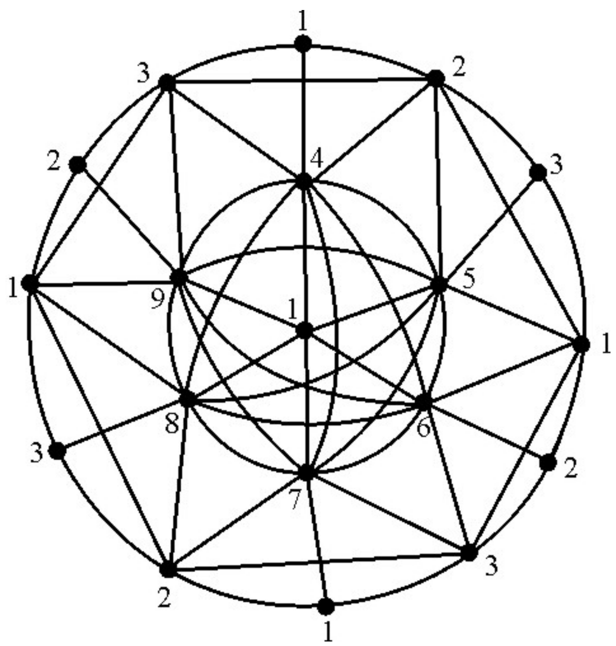

FIGURE 2. $\Gamma(M(W_6)) = 9$.

Let us assume that $\Gamma(M(G)))$ is greater than $m+2$, i.e. $\Gamma(M(G)) = m+3$. Now we assigned to $v_1, v_2, \cdots, v_{m-1}$ for some integer k with $5 \leq k \leq m+3$.

Suppose the color 4 or 3 or 2 or 1 to u_m, then the color $1,2,3$ or $1,2,4$ or $1,3,4$ or $2,3,4$ to u_i, v_i' and v_{i-1}' (where $v_{(i-1=0)}' = v_{m-1}'$) $(1 \leq i \leq m-2)$, received, all with distinct colors. This is the contradiction by grundy number. Thus, we have $\Gamma(M(G)) \leq m+2$. Therefore $\Gamma(M(G)) = m+2$. $\square$

We consider the middle graph of helm graph denoted by $M(G)$. Let $V(G) = \{u_m\} \cup \{u_i : 1 \leq i \leq m-1\} \cup \{s_i : 1 \leq i \leq m-1\}$, where s_i $(1 \leq i \leq m-1)$ is the pendent vertices of u_i $(1 \le i \le m-1)$. Let $V(M(G)) = V(G) \cup \{v_i : 1 \le i \le m-1\} \cup \{v_i' : 1 \le i \le m-1\} \cup \{s_i' : 1 \le i \le m-1\}$, where v_i is the corresponding vertex to the edge $u_m u_i$, v_i' is the corresponding vertex to the edge $u_i u_{i+1}$ and s_i' is the corresponding vertex to the edge $u_i s_i$.

Theorem 3.5. *Let G be a helm graph of order $m \ge 4$, then the middle graph of G has a grundy number $m+3$. i.e., $\Gamma(M(G)) = m+3$.*

Proof. Define a mapping, $\sigma : V(M(G)) \to \mathbb{N}$ as follows:

Case (i):: For $m \equiv 0 \mod 6$

$\sigma(u_m) = 1;\ \sigma(v_i) = i,\ 5 \le i \le m+3;$

For $1 \le i \le m-1$,

$$\sigma(u_i) = \begin{cases} 4, & \text{if } i \equiv 1 \mod 3 \\ 3, & \text{if } i \equiv 2 \mod 3 \\ 2, & \text{if } i \equiv 0 \mod 3 \end{cases}$$

For $1 \le i \le m-2$,

$$\sigma(v_i') = \begin{cases} 1, & \text{if } i \equiv 1 \mod 3 \\ 4, & \text{if } i \equiv 2 \mod 3 \\ 3, & \text{if } i \equiv 0 \mod 3 \end{cases}$$

For $2 \le i \le m-2$,

$$\sigma(s_i') = \begin{cases} 1, & \text{if } i \equiv 0 \mod 3 \\ 2, & \text{if } i \equiv 1,2 \mod 3 \end{cases}$$

For $1 \le i \le m-1$,

$$\sigma(s_i) = \begin{cases} 1, & \text{if } i \equiv 1,2 \mod 3 \\ 2, & \text{if } i \equiv 0 \mod 3 \end{cases}$$

$\sigma(v_m - 1') = 2;\ \sigma(s_1') = 3;\ \sigma(s_{m-1}') = 4;$

Case (ii):: For $m \not\equiv 0 \mod 6$

$\sigma(u_m) = 1;\ \sigma(v_i) = i,\ 5 \le i \le m+3;$

For $1 \le i \le m-1$,

$$\sigma(u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \mod 2 \\ 1, & \text{if } i \equiv 0 \mod 2 \end{cases}$$

For $1 \le i \le m-1$,

$$\sigma(v_i') = \begin{cases} 3, & \text{if } i \equiv 1 \mod 2 \\ 4, & \text{if } i \equiv 0 \mod 2 \end{cases}$$

For $1 \le i \le m-2$,

$$\sigma(s_i') = \begin{cases} 1, & \text{if } i \equiv 1 \mod 2 \\ 2, & \text{if } i \equiv 0 \mod 2 \end{cases}$$

For $1 \le i \le m-1$,

$$\sigma(s_i) = \begin{cases} 1, & \text{if } i \equiv 0 \mod 2 \\ 2, & \text{if } i \equiv 1 \mod 2 \end{cases}$$

Now we claim case(i) and case(ii):

Let us assume that $\Gamma(M(G)))$ is greater than $m+3$, i.e. $\Gamma(M(G)) = m+4$. Now we assigned to $v_1, v_2, \cdots, v_{m-1}$ for some integer k with $6 \le k \le m+4$.

Suppose the color 5, or 4 or 3 or 2 or 1 to u_m, then the color $1,2,3,4$ or $1,2,3,5$ or $1,2,4,5$ or $1,3,4,5$ or $2,3,4,5$ to u_i, v'_i, s'_i and v'_{i-1} $(1 \leq i \leq m-2)$ (where $v'_{(i-1=0)} = v'_{m-1}$), received, all with distinct colors. This is the contradiction by grundy number.

Thus, we have $\Gamma(M(G)) \leq m+3$. Therefore $\Gamma\left(M\left(G\right)\right) = m+3$. $\square$

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