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SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS USING FIXED POINT RESULTS IN GENERALIZED METRIC SPACES OF PEROV'S TYPE

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Abstract. In 1964, A. I. Perov generalized the Banach contraction principle introducing, following the work of Đ. Kurepa, a new approach to fixed point problems, by defining generalized metric spaces (also known as vector valued metric spaces), and providing some actual results for the first time. Using the recent approach of coordinate representation for a generalized metric of Jachymski and Klima, we verify in this article some natural properties of generalized metric spaces, already owned by standard metric spaces. Among other results, we show that the theorems of Nemytckii (1936) and Edelstein (1962) are valid in generalized metric spaces, as well. A new application to fractional differential equations is also presented. At the end we state a few open questions for young researchers.

Keywords: Fixed point; vector-valued metric, pseudometric; Perov type; F-contraction; fractional differential equation.

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1. Introduction and preliminaries

Almost 100 years have passed since S. Banach proved his renowned theorem on the existence and uniqueness of a fixed point for any contractive map acting on a complete metric space. Meanwhile, there were thousands of papers by many mathematicians who tried to generalize this famous theorem. These generalizations were basically following two directions: some of them were distorting axioms of metric spaces thus giving raise to a new

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topology, while others were modifying contractive condition, for instance by introducing a new function.

Serbian mathematician Đuro Kurepa was the first to apply a new method in 1933 while working on his doctorate in Paris. He considered as a distance $d(x,y)$ between two points an element of some ordered set, instead just a real number as in the original Fréchet's work. However, he did not try to apply this approach for solving fixed point problems.

A. I. Perov was another mathematician that started a real generalization in this field. Instead of $d(x,y)$ being a number, he considered $d(x,y)$ to be an element of $\mathbb{R}^m$, where m is a natural number greater than 1. He called a generalized metric (later also known as vector valued metric) a map $d : X \times X \to \mathbb{R}^m$, with $m \geq 2$ a natural number. Then, an element $(a_1, \dots, a_m)$ in $\mathbb{R}^m$ could be considered as an $m \times 1$ matrix, and the set $M_{m\times m}$ of non-negative matrices and the product $M_{m\times m} \cdot d(x,y)$, where M is an $m \times m$ matrix, represent a generalization of the product of a scalar with a distance, $\lambda d(x,y)$.

This means that, if we consider the celebrated contractive condition $d(Tx,Ty) \leq \lambda d(x,y)$ in the context of standard metric spaces, it is translated to the condition $d(Tx,Ty) \preceq Md(x,y)$ in the realm of generalized metric spaces, where $\preceq$ is an order relation between m-tuples in $\mathbb{R}^m$. In the first case, the condition that the map $T : X \to X$ has to fulfill to have a fixed point is that $\lambda \in [0,1)$, while in the second case that the spectral radius of the matrix has to be $r(M) < 1$. Therefore, for $m > 1$, it is a generalization of the Banach contraction principle.

As generalized metric spaces (so defined) are the closest to standard metric spaces, we are going to show in this article that some of well-known fixed point theorems can be modified to act in this new framework of vector valued metric spaces. Some results were already obtained by various authors, and some are new.

Let us recall the following. We say that the function $p : X \times X \to [0,+\infty)$ is a pseudometric on X if, for any $x,y,z \in X$, p has the following properties:

P1: $p(x,x) = 0$;
P2: $p(x,y) = p(y,x)$;
P3: $p(x,z) \leq p(x,y) + p(y,z)$.

In order to prove most of the theorems we will need the following recent result (for some useful details see [13]):

Proposition 1.1. [13, Proposition 2.1] *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^m$ be a mapping so that $d = (p_1, \dots, p_m)$, where $p_k : X \times X \to [0,+\infty)$ for $k = 1,2,\dots,m$. Then (X,d) is a generalized metric space (vector valued metric space) if, and only if, $(p_1,\dots,p_m)$ is a separating family of pseudometrics, i.e., for any $x,y \in X$, if $x \neq y$ then $p_i(x,y) > 0$ for some $i \in \{1,2,\dots,m\}$.*

We shall briefly describe main properties of any generalized metric space (X,d).

- A generalized metric space (X,d) is a metrizable space in the sense of [1], [2].
- The generalized metric d is continuous in the sense that $d(x_n,y_n) \to d(x,y)$ whenever $d(x_n,x) \to \theta$ and $d(y_n,y) \to \theta$ as $n \to +\infty$.
- The cone $P = \{(x_1,\dots,x_m) : x_i \geq 0,\ i = 1,\dots,m\}$ in $\mathbb{R}^m$ is normal with the normal constant $K = 1$ under any of equivalent norms on it. Also, it is solid, i.e., $\operatorname{int} P \neq \emptyset$.
- The cone P is regular in the space $\mathbb{R}^m$, meaning that each decreasing (or increasing) sequence in it is convergent.

2. Generalizations of some known results

In what follows, we will assume that (X, d) is a generalized metric space (vector valued metric space) and T is a map of X into itself.

Our first result is a generalization of Bryant Theorem [6].

Theorem 2.1. [6] *If T is a self-map of a complete generalized metric space (X, d) and if, for some positive integer k, T^k is a contraction, then T has a unique fixed point in X.*

Proof. The contraction condition for T^k under generalized metric d yields

$$d\left(T^k x, T^k y\right) \preceq \lambda d\,(x, y) \tag{1}$$

for some $\lambda \in [0, 1)$ and for all $x, y \in X$. Further, by Proposition 1.1 we have

$$p_i\left(T^k x, T^k y\right) \leq \lambda d_i\,(x, y)\,, i = 1, 2, \ldots, m. \tag{2}$$

Since at least one p_{i_0} is a proper metric we get that T^k has a unique fixed point (say z) in X. This means that z is a unique fixed point of T in X. □

In 1969, V. Sehgal [17] proved a fixed point theorem in the case that the iteration k of the mapping $T : X \to X$ depends on a single variable, say x. He also assumed that the mapping is continuous, a hypothesis later relaxed by Gusseman. All this of course resides into the framework of ordinary metric space.

Our result in the context of generalized metric space is the following.

Theorem 2.2. *Let (X, d) be a complete generalized metric space and $T : X \to X$ a mapping satisfying the condition: there exists $\lambda \in [0, 1)$ such that for each $x \in X$, there is a positive integer $k(x)$ such that for all $y \in X$,*

$$d\left(T^{k(x)} x, T^{k(x)} y\right) \preceq \lambda d\,(x, y) \tag{3}$$

holds. Then T has a unique fixed point, say $u \in X$, and $T^n x \to u$ as $n \to +\infty$ for each $x \in X$.

Proof. Using the coordinate notation of Jachymski–Klima [13], (3) becomes

$$p_i\left(T^{k(x)} x, T^{k(x)} y\right) \leq \lambda p_i\,(x, y)\,, i = 1, 2, \ldots, m. \tag{4}$$

From this and [12] the proof follows, as at least for one $i \in \{1, 2, \ldots, m\}$ p_i is a proper metric. □

We shall now consider the famous Jungck theorem [14] on two mappings in the context of generalized metric spaces of Perov's type.

Theorem 2.3. *Let T and I be commuting mappings of a complete generalized metric space (X, d) into itself satisfying the inequality*

$$d\,(Tx, Ty) \preceq \lambda d\,(Ix, Iy) \tag{5}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. If the range of I contains the range of T and if I is continuous, then T and I have a common fixed point.

Proof. Clearly (5) can be written as

$$p_i\,(Tx, Ty) \leq \lambda p_i\,(Ix, Iy)\,,\ i = 1, 2, \ldots, m. \tag{6}$$

For at least one $i \in \{1, 2, \ldots, m\}$ we have that (X, p_i) is a complete metric space. The result then follows by [14]. □

After this it is natural to consider Fisher's [11] theorem with four mappings as a generalization of Jungck's theorem in the context of metric spaces.

Theorem 2.4. [11] *Let S and I be commuting mappings and T and J be commuting mappings of a complete generalized metric space (X, d) into itself satisfying*

$$d\left(Sx, Ty\right) \preceq \lambda d\left(Ix, Jy\right) \tag{7}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. If $SX \subseteq JX$ and $TX \subseteq IX$ for each $x \in X$ and if I and J are continuous, then all S, T, I and J have a unique common fixed point.

Proof. According to Proposition 1.1 it is clear that condition (7) becomes

$$p_i\left(Sx, Ty\right) \leq \lambda p_i\left(Ix, Jy\right), i =, 2, \ldots, m. \tag{8}$$

One should consider that (2), (4), (6) and (8) are inequalities among real numbers, while (1), (3), (5) and (7), are inequalities among vectors in a cone of the space $\mathbb{R}^m$. It is now sufficient, as before, to apply the classical Fisher's theorem to (8) and the proof follows immediately. □

We state also the following theorem from [13].

Proposition 2.1. [13, Proposition 2.2.] *Let (X, d) be a generalized metric space and $p_1, \ldots, p_m$ be pseudometrics associated with d. Let $x_n \in X$ for $n \in \mathbb{N}$ and $x \in X$. The following statements hold:*

1. *$\{x_n\}$ is convergent to x if, and only if, $\lim_{n\to+\infty} p_i\left(x_n, x\right) = 0$ for each $i = 1, 2, \ldots, m$.*
2. *$\{x_n\}$ is a Cauchy sequence if, and only if, $\lim_{n,k\to+\infty} p_i\left(x_n, x_k\right) = 0$ for each $i = 1, 2, \ldots, m$.*

3. Nemytckii and Edelstein theorems in the context of generalized metric spaces

We will now recall a theorem of Nemytckii [16] for compact metric spaces.

Let (X, d) be a metric space and $T : X \to X$. We say that T is contractive if $d\left(Tx, Ty\right) < d\left(x, y\right)$ when $x \neq y$. Any strict contraction map is continuous with respect to the given metric d and can have at most one fixed point. However, it is also possible that the map has no fixed point at all. For example, consider $X = [1, +\infty)$ with $d\left(x, y\right) = |x - y|$ and $Tx = x + \frac{1}{x}$. It is true that $T : X \to X$ and

$$d\left(Tx, Ty\right) = \left|x + \frac{1}{x} - \left(y + \frac{1}{y}\right)\right| = \left(1 - \frac{1}{xy}\right)|x - y| < d\left(x, y\right),$$

if $x \neq y$. We conclude that the map T is contractive but has no fixed point on X.

The above example can be easily adapted to act in a generalized metric space.

Let $X = [1, +\infty)$ and consider the generalized metric $d : X \times X \to \mathbb{R}^2$ (so $m = 2$) defined in the following way:

$$d\left(x, y\right) = \left(|x - y|, |x - y|\right),$$

for each $x, y \in X$.

For the above map T we have:

$$\begin{aligned} d(Tx,Ty) &= (|Tx-Ty|,|Tx-Ty|) \\ &= \left(\left|x+\frac{1}{x}-\left(y+\frac{1}{y}\right)\right|,\left|x+\frac{1}{x}-\left(y+\frac{1}{y}\right)\right|\right) \\ &= \left|\left(1-\frac{1}{xy}\right)|x-y|,\left(1-\frac{1}{xy}\right)|x-y|\right| \\ &\prec ||x-y|,|x-y|| = d(x,y). \end{aligned}$$

We have obtained that the map $T: X \to X$ is contractive in the context of generalized metric space, and clearly has no fixed point at all.

We have the following result for a strict contraction map on a compact generalized metric space, i.e., a generalized metric space in which any sequence has a convergent subsequence.

Theorem 3.1. *Let T be a contractive self-mapping on a compact generalized metric space (X,d). Then T has a unique fixed point.*

Proof. Define a function $F: X \to P$ (P is the cone in $\mathbb{R}^m$) by $F(x) = d(x,Tx)$, $x \in X$. As T and d are continuous mappings, F is also continuous. Indeed, if $x_n \to x$ as $n \to +\infty$, with respect to generalized metric in X then we have

$$F(x_n) = d(x_n, Tx_n) \to d(x,Tx) = F(x),$$

therefore we have proved that F is a continuous mapping.

Further, $F_i(x) = p_i(x,Tx)$ for $i = 1,2,\ldots,m$ and for the particular i for which p_i is a proper metric we obtain that there exists $u \in X$ (because the metric space (X,p_i) is compact) such that

$$F_i(u) = p_i(u,Tu) = \min_{x\in X}\{p_i(x,Tx)\}.$$

Supposing that $u \neq Tu$, and because T is contractive we have

$$F_i(Tu) = p_i(Tu,TTu) < p_i(u,Tu) = F_i(u).$$

This is a contradiction as $F_i(u)$ assumes the minimal value for some $u \in X$. □

We have proved that Nemytckii's theorem from 1936 can be transfered verbatim to generalized metric spaces, also thanks to the cited result from [13].

In 1962, M. Edelstein [10] modified the previous Nemytckii's theorem in the following manner.

Theorem 3.2. [10] *Let (X,d) be a metric space and $T: X \to X$ a strict contraction map. If there exists a point $x_0 \in X$ such that the iterative sequence $\{T^n x_0\}$ contains a convergent subsequence $\{T^{n_i} x_0\}$, then $u = \lim_{i\to+\infty} T^{n_i} x_0$ is the unique fixed point of the strict contraction map, $Tu = u$.*

Our following new result is

Theorem 3.3. *M. Edelstein's theorem remains true in the context of generalized metric spaces.*

Proof. Consider first the sequence $\left\{d\left(T^n x_0, T^{n+1} x_0\right)\right\}$. If the equality $T^{n_0+1}x_0 = T^{n_0}x_0$ is satisfied for some $n_0 \in \mathbb{N}$ then $\{T^n x_0\}$ is a stationary sequence for $n \geq n_0$, therefore $T^{n_0}x_0 = u$. It follows that $Tu = u$.

Suppose now that $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. Then, since the mapping T is contractive, we obtain that $\left\{d\left(T^n x_0, T^{n+1} x_0\right)\right\}$ is a strictly decreasing vector sequence contained in the regular cone P. For this reason it converges. Therefore, $\lim_{i\to+\infty} T^{n_i}x_0 = u$ and because T is a continuous mapping we obtain

$$\lim_{i\to+\infty} T^{n_i+1}x_0 = \lim_{i\to+\infty} TT^{n_i}x_0 = Tu; \text{ likewise } \lim_{i\to+\infty} T^{n_i+2}x_0 = T^2u.$$

It follows that

$$\lim_{i\to+\infty} d\left(T^{n_i}x_0, T^{n_i+1}x_0\right) = d\left(u, Tu\right) \text{ and } \lim_{i\to+\infty} d\left(T^{n_i+1}x_0, T^{n_i+2}x_0\right) = d\left(Tu, T^2u\right).$$

As $\left\{d\left(T^{n_i}x_0, T^{n_i+1}x_0\right)\right\}$ and $\left\{d\left(T^{n_i+1}x_0, T^{n_i+2}x_0\right)\right\}$ are subsequences of a convergent sequence $\left\{d\left(T^{n}x_0, T^{n+1}x_0\right)\right\}$, they both have the same limit. For this reason,

$$d\left(u, Tu\right) = d\left(Tu, T^2u.\right)$$

And also $Tu = u$. If not, we obtain

$$d\left(Tu, T^2u\right) < d\left(u, Tu\right)$$

which is a contradiction. □

Remark 3.1. *The key ingredients to this proof were the continuity of the generalized metric d and the regularity of the cone P in the space $\mathbb{R}^m$.*

4. Applications

Because of coordinate representation of any generalized metric d (Proposition 1.1) for the discussion of applications of this type of spaces we will limit ourselves to the application of Banach Contraction Principle BCP^G. We shall recall its formulation: Let $T : X \to X$ be a self-map of a generalized metric space (X, d) such that there exists $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d\left(Tx, Ty\right) \preceq \lambda \cdot d\left(x, y\right).$$

In explicit coordinate representation the above condition reads as

$$p_i\left(Tx, Ty\right) \leq \lambda \cdot p_i\left(x, y\right), \quad i = 1, 2, \ldots, m.$$

Of course, m is here a given natural number, not smaller than 2. Since among $p_i, i = 1, 2, \ldots, m$ there is at least one proper metric p_i, then the application of generalized metric space reduces to the application of BCP within standard metric space.

To the class of generalized metric space (vector valued metric space) one could add the following other classes of generalized metric space, e.g.:

- generalized partial metric spaces,
- generalized metric-like spaces,
- generalized b-metric spaces,
- generalized partial b-metric spaces,
- generalized b-metric-like spaces.

How do these arise? Take a natural number $m \geq 2$, and one of the following spaces:

- partial metric space [15],
- metric-like space [4],
- b-metric space [8],
- partial b-metric space [18],
- b-metric-like space [3].

If we denote by d the distance function of above spaces and define D by

$$D(x,y) = (\alpha_1 d(x,y), \alpha_2 d(x,y), \ldots, \alpha_m d(x,y)),$$

where $\alpha_i \geq 0, i = 1, 2, \ldots, m$ are such that at least one of them is different from 0, then $D(x,y)$ is called a generalized partial metric (resp. generalized metric-like, generalized b-metric, generalized partial b-metric, and eventually, generalized b-metric-like).

Example 4.1. *Let $X = (C[0,1], \mathbb{R})$ be the set of real continuous functions on $[0,1]$ and $d^{ml}(u,v) = \sup_{t\in[0,1]}(|u(t)| + |v(t)|)$ for all $u, v \in X$. Then $\left(X, d^{ml}\right)$ is an example of metric-like space that is not a partial metric space (and certainly not a metric space).*

Let $m = 3$. Then (X, D), where $D(u,v) = \left(d^{ml}(u,v), \frac{1}{7}d^{ml}(u,v), 4d^{ml}(u,v)\right)$, is an example of generalized metric-like space (with $m = 3$).

In order to show some application of (some kind of) generalized metric space we actually have to proceed in the following way: consider a problem that can be formulated as an existence (and uniqueness) problem for solving certain equality (algebraic, (ordinary or partial) differential, integral, fractional integral, ...), and which can be reformulated as a fixed point problem in some generalized metric space. Then choose some conditions on the given equation that can be reformulated as sufficient conditions for solving the obtained fixed problem.

Definition 4.1. *Let T be a map from a b-metric like space $\left(X, d^{bml}\right)$ (with parameter s) into itself. It is called a generalized (s,q)-Jaggi F-contraction if there exist a strictly increasing map $F : (0, +\infty) \to (-\infty, +\infty)$ and $\tau > 0$ such that for all $x, y \in X$ with $d^{bml}(Tx, Ty) > 0$ and $d^{bml}(x,y) > 0$ the following inequality holds*

$$\tau + F\left(s^q \cdot d^{bml}(Tx, Ty)\right) \leq F\left(M_{bml}^{\alpha,\beta,\gamma}(x,y)\right),$$

where

$$M_{bml}^{\alpha,\beta,\gamma}(x,y) = \alpha \cdot \frac{d^{bml}(x,Tx) \cdot d^{bml}(y,Ty)}{d^{bml}(x,y)} + \beta \cdot d^{bml}(x,y) + \gamma \cdot d^{bml}(x,Ty)$$

with $\alpha + \beta + 2\gamma s < 1$ and $q > 1$.

Now we formulate a positive result. Its proof can be given with the usual approach.

Theorem 4.1. *Let $\left(X, d^{bml}\right)$ be a 0-complete b-metric like space and $T : X \to X$ a (s,q)-Jaggi F-contraction map. Then T has a unique fixed point, say $\sigma \in X$, and if T is a d^{bml}-continuous map then for any $\kappa \in X$ the sequence $T^n\kappa$ converges to σ as $n \to +\infty$.*

With the help of the previous theorem we will show the application of a generalized metric space to the solution of some fractional differential equations.

5. Applications to nonlinear fractional differential equations

Recall the following basic facts from the fractional differential calculus. Given a function $h : [0, +\infty) \to \mathbb{R}$, its Caputo derivative (see [7]) ${}^CD^\beta(h(t))$ of order $\beta > 0$ is defined as

$${}^CD^\beta(h(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} h^{(n)}(s)\, ds \qquad (n-1 < \beta < n, n = [\beta]+1),$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is the gamma function.

The scope of this section is to apply previous results to prove the existence of solutions for nonlinear fractional differential equations of the form

$${}^CD^\beta(\nu(t) + g(t, \nu(t))) = 0 \qquad (0 \leq t \leq 1, \beta < 1) \tag{9}$$

with boundary conditions $\nu(0) = 0 = \nu(1)$, where $\nu \in C([0,1], \mathbb{R})$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The associated Green's function to (9) is given by

$$\mathcal{G}(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if } 0 \le s \le t \le 1 \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \le t \le s \le 1. \end{cases}$$

We shall now state and prove the main result of this section.

Theorem 5.1. *Consider the nonlinear fractional differential equation* (9)*. Let* $\xi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *be a given map and* $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Suppose also that all the following conditions hold true:*

(i) *there exists* $\nu_0 \in C([0,1], \mathbb{R})$ *such that* $\xi\left(\nu_0(t), \int_0^t T\nu_0(t)\right) \ge 0$ *for all* $t \in [0,1]$*, where the map* $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ *is defined as:*

$$T\nu(t) = \int_0^t \mathcal{G}(t,s)\, g(s, \nu(s))\, ds.$$

(ii) *there exists* $\tau > 0$ *such that for all* $\mu, \nu \in X$*,*

$$d^{bml}(T\mu, T\nu) > 0 \text{ and } d^{bml}(\mu,\nu) > 0 \text{ implies } |g(t,a) + g(t,b)| \le \frac{1}{s^{\frac{q}{2}}} M^{\alpha,\beta,\gamma}_{bml}(\mu,\nu)\, e^{-\tau},$$

for all $t \in [0,1]$ *and* $a, b \in \mathbb{R}$ *with* $\nu(a,b) > 0$*, where*

$$M^{\alpha,\beta,\gamma}_{bml}(\mu,\nu) = \alpha \cdot \frac{d^{bml}(\mu, g\mu) \cdot d^{bml}(\nu, g\nu)}{d^{bml}(\mu,\nu)} + \beta \cdot d^{bml}(\mu,\nu) + \gamma \cdot d^{bml}(\nu, g\mu),$$

$\alpha, \beta, \gamma \ge 0$ *with* $\alpha + \beta + 2\gamma s < 1$ *and* $q > 1$.

(iii) *for each* $t \in [0,1]$ *and* $\mu, \nu \in C([0,1], \mathbb{R})$*,* $\xi(\nu(t), \mu(t)) \ge 0$ *implies* $\xi(T\mu(t), T\nu(t)) \ge 0$.

(iv) *for each* $t \in [0,1]$*, if* $\{\nu_n\}$ *is a sequence in* $C([0,1], \mathbb{R})$ *such that* $\nu_n \to \nu$ *in* $C([0,1], \mathbb{R})$ *and* $\xi(\nu_n(t), \nu_{n-1}(t)) \ge 0$ *for all* $n \in \mathbb{N}$ *then* $\xi(\nu_n(t), \nu(t)) \ge 0$ *for all* $n \in \mathbb{N}$.

Then the problem (9) *has at least one solution.*

Proof. Let $X = C([0,1], \mathbb{R})$ be endowed with the b-metric like

$$d_{bml}(\nu,\mu) = \sup_{t\in[0,1]} (|\nu(t)| + |\mu(t)|)^2,$$

for all $\nu, \mu \in X$ (the parameter is $s = 2$). Define a generalized b-metric like on X by

$$D_{bml}(\nu,\mu) =: \underbrace{(d_{bml}(\nu,\mu), \dots, d_{bml}(\nu,\mu))}_{m},$$

where $m \ge 2$ is a given natural number. It is easy to show that $(X, D_{bml}, 2)$ is a 0-complete generalized b-metric like space.

It is obvious that $\nu^* \in X$ is a solution of (9) if, and only if, $\nu^* \in X$ is a solution of the equation $\nu(t) = \int_0^t \mathcal{G}(t,s)\, g(s, \nu(s))\, ds$ for all $t \in [0,1]$. Therefore, the problem (9) can be considered as the problem of finding an element $\nu^* \in X$ that is the fixed point of operator T. To that end, let $\nu, \mu \in X$ be such that $\xi(\nu(t), \mu(t)) \ge 0$ for all $t \in [0,1]$. According to (iii) we have that $\xi(T\nu, T\mu) > 0$. Then using hypotheses (i) and (ii) we obtain the following inequalities

$$\begin{aligned}
|T\nu(t)+T\mu(t)| &= \left|\left|\int_0^t \mathcal{G}(t,s)\, g(s,\nu(s))\, ds\right| + \left|\int_0^t \mathcal{G}(t,s)\, g(s,\mu(s))\, ds\right|\right| \\
&\le \left|\left|\int_0^t \mathcal{G}(t,s)\, |g(s,\nu(s))|\, ds\right| + \left|\int_0^t \mathcal{G}(t,s)\, |g(s,\mu(s))|\, ds\right|\right| \\
&\le \left|\sup_{t\in[0,1]} |g(s,\nu(s))+g(s,\mu(s))| \int_0^t \mathcal{G}(t,s)\, ds\right| \\
&\le \left|\frac{1}{\sqrt{m}s^{\frac{q}{2}}} \sup_{t\in[0,1]} \sqrt{M_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\, e^{-\tau}} \cdot \sup_{t\in[0,1]} \int_0^t \mathcal{G}(t,s)\, ds\right| \\
&\le \frac{1}{\sqrt{m}s^{\frac{q}{2}}} \sup_{t\in[0,1]} \sqrt{M_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\, e^{-\tau}}.
\end{aligned}$$

This means that

$$|T\nu(t)+T\mu(t)|^2 \le \frac{1}{m\cdot s^q} M_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\, e^{-\tau}.$$

Then we have

$$s^q d_{bml}(\nu,\mu) \le M_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\, e^{-\tau}.$$

If we now take $F(\omega)=\ln\omega$ for any $\omega>0$ then F satisfies all conditions of the theorem, and we obtain

$$\ln\left(s^q d_{bml}(\nu,\mu)\right) \le \ln\left(M_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\, e^{-\tau}\right),$$

that is

$$\tau+\ln\left(s^q d_{bml}(\nu,\mu)\right) \le \ln\left(M_{bml}^{\alpha,\beta,\gamma}(\nu,\mu)\right).$$

The above is clearly equivalent to

$$\tau+\ln\left(s^q d_{bml}(\nu,\mu)\right) \le F\left(\alpha\cdot\frac{d^{bml}(\mu,g\mu)\cdot d^{bml}(\nu,g\nu)}{d^{bml}(\mu,\nu)} + \beta\cdot d^{bml}(\mu,\nu) + \gamma\cdot d^{bml}(\nu,g\mu)\right),$$

where $\alpha,\beta,\gamma\ge 0$, $\alpha+\beta+2\gamma s<1$ and $q>1$.

Applying now Theorem 4.1 with $q=2$ we conclude that the map T has a fixed point, which in turn shows that the problem (9) has at least one solution. □

For other applications in this field, see, e.g., [5], [9].

6. Results and Conclusions

We have shown how some theorems from standard metric spaces can be reformulated and proved to be true in generalized (vector valued) metric spaces. We leave to the curious reader to check some other interesting and well-known results from the framework of standard metric spaces and whether they can be transformed to assertions in some kind of generalized spaces.

Those are, for instance:

- Caristi's phenomenon and its consequences;
- Ćirić's generalized contraction;
- Ćirić's quasicontraction;
- Various statements about non-expansive type mappings;
- Greguš fixed point theorem and its generalizations;
- Fixed point of nonself mappings,

and many other statements belonging to the realm of standard metric spaces.

We recall that in this new environment the generalized metric d is a continuous function of two variables, and the cone $P \subset \mathbb{R}^m$ is regular and has a nonempty interior. Moreover, we have a coordinate representation $d = (p_1, \dots, p_m)$ where p_i are pseudometrics (and at least one of them is a proper metric) thanks to the recent result of Jachymski and Klima. This class of cone metric spaces makes them the closest to standard metric spaces and distinguishes them from all other cone spaces. We should also mention that the particular choice of $m = 2$, that is $\mathbb{R}^2$ space does not spoil the generic case results for $m \geq 2$.

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