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THE GENERALIZED q-OPERATOR ${}_r\Phi_s$ AND ITS APPLICATIONS IN q-IDENTITIES

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Abstract. Based on the basic hypergeometric series ${}_r\phi_s$, we construct a new generalized q-operator ${}_r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{array};q,-c\theta\right)$ and obtain some of its identities. Using these identities, we generalize several well-known q-identities, such as the q-Gauss sum, the q-Chu-Vandermonde sum, and the q-Pffaf-Saalschütz sum.

Keywords: The q-operator, q-Gauss sum, q-Chu-Vandermonde sum, q-Pffaf-Saalschütz sum.

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1. Introduction and Notations

In this paper, we will follow the notations that were used in [5]. We assume that $|q| < 1$. Let a be a complex variable. The q-shifted factorial is defined by [5]

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We adopt the following compact notation for the multiple q-shifted factorial:

$$(a_1,\ldots,a_r;q)_n = (a_1;q)_n\ldots(a_r;q)_n,$$

where n is an integer or ∞.

The basic hypergeometric series ${}_r\phi_s$ is defined by:

$${}_r\phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{array};q,x\right) = \sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k\,(b_1;q)_k\cdots(b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}x^k,$$

where $r,s\in\mathbb{N}$; $a_1,\ldots,a_r,b_1,\ldots,b_s\in\mathbb{C}$; and none of the denominator factors evaluate to zero.

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The convergence conditions for the basic hypergeometric series ${}_r\phi_s$ are as follows [5]:

(a) If $s > r-1$, the series is convergent for all $x \in \mathbb{C}$.
(b) If $s < r-1$, the series is convergent only when $x = 0$.
(c) If $s = r-1$, the series is convergent for $|x| < 1$.

The most important case of the above series is when $r = s+1$,

$$ {}_{s+1}\phi_s\left(\begin{array}{c} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{array}; q, x\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_{s+1};q)_k}{(q;q)_k\,(b_1;q)_k \cdots (b_s;q)_k}\, x^k, \quad |x| < 1. $$

The q-binomial coefficient is given as

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n; \\ 0, & \text{otherwise,} \end{cases} $$

where $n, k \in \mathbb{N}$.

The following identities will be used in this paper [5]:

$$ (a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}. \tag{1} $$

$$ (q/a;q)_k = (-a)^{-k} q^{\binom{k+1}{2}} (aq^{-k};q)_\infty/(a;q)_\infty. \tag{2} $$

Cauchy identity is given by [5]

$$ \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{3} $$

The special case of Cauchy identity was founded by Euler which is given by [5]:

$$ \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} x^k = (x;q)_\infty. \tag{4} $$

The q-Gauss sum is [5]

$$ {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \quad |c/ab| < 1. \tag{5} $$

q-Chu-Vandermonde's sums are [5]

$$ {}_2\phi_1\left(\begin{array}{c} q^{-n}, b \\ c, \end{array}; q, cq^n/b\right) = \frac{(c/b;q)_n}{(c;q)_n}. \tag{6} $$

$$ {}_2\phi_1\left(\begin{array}{c} q^{-n}, b \\ c, \end{array}; q, q\right) = \frac{(c/b;q)_n}{(c;q)_n} b^n. \tag{7} $$

The q-Pfaff-Saalschütz's sum is [5]

$$ {}_3\phi_2\left(\begin{array}{c} a, b, q^{-n} \\ c, abq^{1-n}/c \end{array}; q, q\right) = \frac{(c/a, c/b;q)_n}{(c, c/ab;q)_n}. \tag{8} $$

The operator θ is defined as [7]

$$ \theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \tag{9} $$

Theorem 1.1. [7]. The Leibniz rule for θ. *Let θ be defined as in* (9), *then*

$$ \theta^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{10} $$

The following identities are easy to prove:

Theorem 1.2. [2, 9, 11] *Let θ be defined as in* (9), *then*

$$\theta^k\left\{a^n\right\}=\frac{(q;q)_n}{(q;q)_{n-k}}a^{n-k}q^{\binom{k}{2}-nk+k}. \tag{11}$$

$$\theta^k\left\{(at;q)_\infty\right\}=(-t)^k(at;q)_\infty. \tag{12}$$

$$\theta^k\left\{\frac{(at;q)_\infty}{(av;q)_\infty}\right\}=v^kq^{-\binom{k}{2}}(t/v;q)_k\frac{(at;q)_\infty}{(av/q^k;q)_\infty},\quad |av|<1. \tag{13}$$

In 1997, Chen and Liu [2] defined the q-exponential operator $E(b\theta)$ as follows:

$$E(b\theta)=\sum_{n=0}^{\infty}\frac{(b\theta)^nq^{\binom{n}{2}}}{(q;q)_n}. \tag{14}$$

In 2007, Fang [3] defined the Cauchy operator ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};q,-c\theta\right)$ as follows:

$${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};q,-c\theta\right)=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(-c\theta)^n. \tag{15}$$

Fang [3] proved the following result:

Theorem 1.3. [3]. *Let ${}_1\Phi_0\left(\begin{matrix} b \\ - \end{matrix};q,-c\theta\right)$ be defined as in* (15) *then:*

$${}_3\phi_2\left(\begin{matrix} q^{-n},a,b \\ c,d \end{matrix};q,q\right)=a^n\frac{(c/a,b;q)_n}{(c,d;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n},q^{1-n}/c,d/b \\ q^{1-n}/c,q^{1-n}/b \end{matrix};q,q\right). \tag{16}$$

The series ${}_r\varphi_s$ is defined as follows [1, 8, 10]:

$${}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\,x^n.$$

Note that when $r=s+1$, we have ${}_{s+1}\varphi_s=\ {}_{s+1}\phi_s$.

In 2010, Zhang and Yang [10] construct the finite q-Exponential Operator ${}_2\mathscr{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix};q,d\theta\right]$ with two parameters as follows:

$${}_2\mathscr{E}_1\left[\begin{matrix} q^{-N},w \\ v \end{matrix};q,d\theta\right]=\sum_{n=0}^{N}\frac{(q^{-N},w;q)_n}{(q,v;q)_n}(d\theta)^n.$$

A generalization for the q-Chu-Vandermonde's summation formula (7) has been found by Zhang and Yang [10] as follows:

Theorem 1.4. [10]. *We have*

$$\sum_{m=0}^{n}\sum_{k=0}^{N}\frac{(q^{-n},a;q)_m}{(q,e;q)_m}\frac{(q^{-N},b;q)_k}{(q,c;q)_k}q^{m+(N-n+m)k}\,(c/b)^k$$
$$=a^n\frac{(e/a;q)_n(c/b;q)_N}{(e;q)_n(c;q)_N}\ {}_4\varphi_2\left(\begin{matrix} q^{-N},q^{-n},b,q^{1-n}/e \\ aq^{1-n}/e,bq^{1-N}/c \end{matrix};q,q\right). \tag{17}$$

In 2014, Fang [4] defined the generalized q-operator as follows:

$$ {}_{s+1}\Phi_s\left(\begin{array}{c} a_0,\cdots,a_s \\ b_1,\cdots,b_s \end{array};q,cD_q\right)=\sum_{n=0}^{\infty}\frac{(a_0,\cdots,a_s;q)_n}{(q,b_1,\cdots,b_s;q)_n}\,(cD_q)^n. $$

Fang found a general identity for the operator ${}_{s+1}\Phi_s$ by solving a general q-difference equation and then used this q-difference equation to give a generalizations of Andrews-Askey integral and Askey-Wilson integral.

In 2016, Li and Tan [6] introduced the generalized q-exponential operator $\mathbb{E}\left[\begin{array}{c} v,u \\ w \end{array}\middle|q;t\theta\right]$ with three parameters as follows:

$$ \mathbb{E}\left[\begin{array}{c} v,u \\ w \end{array}\middle|q;t\theta\right]=\sum_{n=0}^{\infty}\frac{(v,u;q)_n}{(q,w;q)_n}(t\theta)^n. $$

Li and Tan [6] obtained a generalization for the q-Chu-Vandermonde summation formula (6) by using the generalized q-exponential operator $\mathbb{E}\left[\begin{array}{c} u,v \\ w \end{array}\middle|q;t\theta\right]$.

The paper is structured as follows. In section 2, we define the general operator ${}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)$. Some operator identities are also given. Section 3 generalizes many well-known q-identities, including the q-Gauss sum, the q-Chu-Vandermonde sum, and the q-Pffaf-Saalschütz sum.

2. The Generalized q-operator ${}_r\Phi_s$ and its Identities

In this section, we define the generalized q-operator ${}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)$, and then find some of its operator identities.

The generalized q-operator ${}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)$ is defined as follows:

$$ {}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\frac{(-c\theta)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. $$

Several previously described operators can be obtained by passing special values to the generalized q-operator ${}_r\Phi_s$, see [2, 3, 6, 10].

In this paper, we'll denote to the fraction $\dfrac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}$ by W_k. Then the generalized q-operator ${}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)$ can be written as follows:

$$ {}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)=\sum_{k=0}^{\infty}W_k\frac{(-c\theta)^k}{(q;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}. \tag{18} $$

Theorem 2.1. *Let* ${}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)$ *be defined as in* (18), *then*

$$ {}_r\Phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{array};q,-c\theta\right)\left\{\frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\right\}=\frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}W_{k+j+i} $$

$$ \times\,(cx)^{k+j+i}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}\left[(-1)^{k+j+i}q^{\binom{k+j+i}{2}}\right]^{s-r}\frac{(u/w,q/at;q)_i}{(q,q/aw;q)_i}\left(\frac{t}{v}\right)^i\frac{(t/v;q)_j}{(q,q)_j} $$

$$\times\frac{(q/ax;q)_{j+i}}{(q/av;q)_{j+i}}, \tag{19}$$

provided that $max\{|av|,|aw|\}<1$.

Proof. From the definition of the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,-c\theta\right)$ and by using Leibniz rule (10), we have

$$\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,-c\theta\right)\left\{\frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q,q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\theta^k\left\{\frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q,q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\theta^j\left\{\frac{(au,at;q)_\infty}{(av,aw;q)_\infty}\right\}\theta^{k-j}\left\{(axq^{-j};q)_\infty\right\}\\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q,q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}\sum_{i=0}^{j}\begin{bmatrix}j\\ i\end{bmatrix}\theta^i\left\{\frac{(au;q)_\infty}{(aw;q)_\infty}\right\}\theta^{j-i}\left\{\frac{(atq^{-i};q)_\infty}{(avq^{-i};q)_\infty}\right\}\\
&\quad\times\theta^{k-j}\left\{(axq^{-j};q)_\infty\right\}\\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q,q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q,q)_k}{(q,q)_j(q,q)_{k-j}}\sum_{i=0}^{j}\frac{(q,q)_j}{(q,q)_i(q,q)_{j-i}}w^iq^{-\binom{i}{2}}\\
&\quad\times(u/w;q)_i\frac{(au;q)_\infty}{(awq^{-i};q)_\infty}\,q^{-\binom{j-i}{2}}(vq^{-i})^{j-i}(t/v;q)_{j-i}\frac{(atq^{-i};q)_\infty}{(avq^{-i}/q^{j-i};q)_\infty}\\
&\quad\times\theta^{k-j}\left\{(axq^{-j};q)_\infty\right\}\qquad\text{(by using (2) and (13))}\\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q,q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q,q)_k}{(q,q)_j(q,q)_{k-j}}\sum_{i=0}^{j}\frac{(q,q)_j}{(q,q)_i(q,q)_{j-i}}w^iq^{-\binom{i}{2}}\\
&\quad\times(u/w;q)_i\frac{(au;q)_\infty}{(-1)^i(aw)^iq^{-\binom{i}{2}-i}(q/aw;q)_i(aw;q)_\infty}\,q^{-\binom{j-i}{2}}(vq^{-i})^{j-i}(t/v;q)_{j-i}\\
&\quad\times\frac{(-1)^i(at)^iq^{-\binom{i}{2}-i}(q/at;q)_i(at;q)_\infty}{(-1)^j(av)^jq^{-\binom{j}{2}-j}(q/av;q)_j(av;q)_\infty}\\
&\quad\times(-xq^{-j})^{k-j}(-1)^j(ax)^iq^{-\binom{j}{2}-j}(q/ax;q)_j(ax;q)_\infty\qquad\text{(by using (2) and (12))}\\
&=\frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}W_k(cx)^k\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}(-1)^jq^{\binom{j}{2}+j}\frac{(u/w,q/at;q)_i}{(q,q/aw;q)_i}\\
&\quad\times\left(\frac{t}{v}\right)^i\frac{(t/v;q)_{j-i}}{(q,q)_{j-i}}\frac{(q/ax;q)_j}{(q/av;q)_j}\frac{q^{-kj}}{(q;q)_{k-j}}\\
&=\frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{j}W_{k+j}(cx)^{k+j}\left[(-1)^{k+j}q^{\binom{k+j}{2}}\right]^{1+s-r}(-1)^jq^{\binom{j}{2}+j}\\
&\quad\times\frac{(u/w,q/at;q)_i}{(q,q/aw;q)_i}\left(\frac{t}{v}\right)^i\frac{(t/v;q)_{j-i}}{(q,q)_{j-i}}\frac{(q/ax;q)_j}{(q/av;q)_j}\frac{q^{-(k+j)j}}{(q;q)_k}
\end{aligned}$$

$$= \frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}W_{k+j+i}(cx)^{k+j+i}\left[(-1)^{k+j+i}q^{\binom{k+j+i}{2}}\right]^{1+s-r}(-1)^{j+i}$$
$$\times q^{\binom{j+i}{2}+j+i}\frac{(u/w,q/at;q)_i}{(q,q/aw;q)_i}\left(\frac{t}{v}\right)^i\frac{(t/v;q)_j}{(q,q)_j}\frac{(q/ax;q)_{j+i}}{(q/av;q)_{j+i}}\frac{q^{-(k+j+i)(j+i)}}{(q;q)_k}$$
$$= \frac{(au,at,ax;q)_\infty}{(av,aw;q)_\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}W_{k+j+i}(cx)^{k+j+i}\frac{(-1)^kq^{\binom{k}{2}}}{(q;q)_k}\left[(-1)^{k+j+i}q^{\binom{k+j+i}{2}}\right]^{s-r}$$
$$\times \frac{(u/w,q/at;q)_i}{(q,q/aw;q)_i}\left(\frac{t}{v}\right)^i\frac{(t/v;q)_j}{(q,q)_j}\frac{(q/ax;q)_{j+i}}{(q/av;q)_{j+i}}.$$

□

- Setting $x = 0$ in (19), we get

Corollary 2.1.

$${}_r\Phi_s\left(\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_s\end{array};q,-c\theta\right)\left\{\frac{(au,at;q)_\infty}{(av,aw;q)_\infty}\right\} = \frac{(au,at;q)_\infty}{(av,aw;q)_\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W_{j+i}\frac{(cq/a)^{j+i}}{(q/av;q)_{j+i}}$$
$$\times\left[(-1)^{j+i}q^{\binom{j+i}{2}}\right]^{1+s-r}\frac{(u/w,q/at;q)_i}{(q,q/aw;q)_i}\left(\frac{t}{v}\right)^i\frac{(t/v;q)_j}{(q,q)_j}, \tag{20}$$

provided that $max\{|av|,|aw|\} < 1$.

- Putting $v = 0$ in (20) yielding

Corollary 2.2.

$${}_r\Phi_s\left(\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_s\end{array};q,-c\theta\right)\left\{\frac{(au,at;q)_\infty}{(aw;q)_\infty}\right\} = \frac{(au,at;q)_\infty}{(aw;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W_{k+j}\frac{(ct)^k}{(q;q)_k}$$
$$\times\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{(u/w,q/at;q)_j}{(q,q/aw;q)_j}(ct)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}q^{kj(s-r)}, \tag{21}$$

Provided that $|aw| < 1$.

- When $u = 0$ in (21), yielding

Corollary 2.3.

$${}_r\Phi_s\left(\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_s\end{array};q,-c\theta\right)\left\{\frac{(at;q)_\infty}{(aw;q)_\infty}\right\} = \frac{(at;q)_\infty}{(aw;q)_\infty}\sum_{k=0}^{\infty}W_k\left(\frac{cq}{a}\right)^k$$
$$\times\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{(t/w;q)_k}{(q,q/aw;q)_k}, \tag{22}$$

Provided that $|aw| < 1$.

Theorem 2.2. *Let* ${}_r\Phi_s\left(\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_s\end{array};q,-c\theta\right)$ *be defined as in* (18) *and* $n \in \mathbb{Z}^+$*, then*

$${}_r\Phi_s\left(\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_s\end{array};q,-c\theta\right)\left\{a^n\frac{(at,q)_\infty}{(av;q)_\infty}\right\} = a^n\frac{(at,q)_\infty}{(av;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W_{k+j}\frac{(cq/a)^k}{(q/av;q)_{k+j}}$$
$$\times\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{(q^{-n},q/at;q)_j}{(q;q)_j}(ct/av)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{1+s-r}\frac{(t/v;q)_k}{(q;q)_k}q^{kj(1+s-r)}, \tag{23}$$

Provided that $|av| < 1$.

Proof. From (18), we have

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-c\theta\right)\left\{a^n\frac{(at,q)_\infty}{(av;q)_\infty}\right\} \\
&=\sum_{k=0}^{\infty}W_k\,\frac{(-c)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\theta^k\left\{a^n\frac{(at,q)_\infty}{(av;q)_\infty}\right\}.
\end{aligned}
$$

By using Leibniz rule (10), we have

$$
\begin{aligned}
&{}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-c\theta\right)\left\{a^n\frac{(at,q)_\infty}{(av;q)_\infty}\right\} \\
&=\sum_{k=0}^{\infty}W_k\,\frac{(-c)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\theta^j\left\{a^n\right\}\theta^{k-j}\left\{\frac{(atq^{-j};q)_\infty}{(avq^{-j};q)_\infty}\right\} \\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}(-1)^ja^{n-j}q^j(q^{-n};q)_j \\
&\quad\times\theta^{k-j}\left\{\frac{(atq^{-j};q)_\infty}{(avq^{-j};q)_\infty}\right\}\qquad\text{(by using (1) and (11))} \\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}(-1)^ja^{n-j}q^j(q^{-n};q)_j \\
&\quad\times q^{-\binom{k-j}{2}}(vq^{-j})^{k-j}(t/v;q)_{k-j}\frac{(atq^{-i};q)_\infty}{(avq^{-i}/q^{j-i};q)_\infty}\qquad\text{(by using (13))} \\
&=\sum_{k=0}^{\infty}W_k\frac{(-c)^k}{(q;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\sum_{j=0}^{k}\frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}(-1)^ja^{n-j}q^j(q^{-n};q)_j \\
&\quad\times q^{-\binom{k-j}{2}}(vq^{-j})^{k-j}(t/v;q)_{k-j}\frac{(-1)^j(at)^jq^{-\binom{j+1}{2}}(q/at;q)_j(at;q)_\infty}{(-1)^k(av)^kq^{-\binom{k+1}{2}}(q/av;q)_k(av;q)_\infty}\qquad\text{(by using (2))} \\
&=a^n\frac{(at;q)_\infty}{(av;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}W_k\frac{(cq/a)^k}{(q/av;q)_k}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{(q^{-n},q/at)_j}{(q;q)_j}\frac{(t/v;q)_{k-j}}{(q;q)_{k-j}}(t/v)^j \\
&=a^n\frac{(at,q)_\infty}{(av;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W_{k+j}\frac{(cq/a)^k}{(q/av;q)_{k+j}}\left[(-1)^kq^{\binom{k}{2}}\right]^{1+s-r}\frac{(q^{-n},q/at;q)_j}{(q;q)_j} \\
&\quad\times(ctq/av)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{1+s-r}\frac{(t/v;q)_k}{(q;q)_k}q^{kj(1+s-r)}.
\end{aligned}
$$

□

3. Applications of the Generalized Operator ${}_r\Phi_s$

In this section, we apply the generalized operator ${}_r\Phi_s$ to find a generalization for some known q-identities such as the q-Gauss sum (5), q-Chu-Vandermonde sum (7), and q-Saalschütz sum (8).

3.1. **Generalization of the q-Gauss Sum.**

By acting the operator ${}_r\Phi_s$ with respect to the parameter c on both sides of the q-Gauss sum (5) and by using equations (23) and (20), we get a generalization of the q-Gauss sum as follows:

Theorem 3.1. (Generalization of the q-Gauss sum). *Given q-Gauss sum* (5)*, then*

$$\sum_{i=0}^{\infty}\frac{(a,b;q)_i}{(q,c;q)_i}(c/ab)^i\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{-i},q^{1-i}/c;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}(-dq^i/cv)^j W_{k+j}$$
$$\times\frac{(-dq/c)^k}{(q/cv;q)_{k+j}}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(q^i/v;q)_k}{(q;q)_k}q^{kj(1+s-r)}$$
$$=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(b,bq/c;q)_j}{(q,abq/c;q)_j}\left(\frac{-dq}{av}\right)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}W_{k+j}\frac{(-dq/c)^k}{(q/cv;q)_{k+j}}$$
$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}q^{kj(1+s-r)}\frac{(1/vb;q)_k}{(q;q)_k},$$

provided that $|c/ab|<1$.

Proof. The q-Gauss sum (5) is

$$\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q,c;q)_k}(c/ab)^k=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty},\quad |c/ab|<1.$$

we can rewrite the above equation as follows:

$$\sum_{k=0}^{\infty}\frac{(a,b;q)_k}{(q;q)_k}(cq^k;q)_\infty\, c^k(1/ab)^k=\frac{(c/a,c/b;q)_\infty}{(c/ab;q)_\infty}.$$

Multiplying both sides of the above equation by $\frac{1}{(cv;q)_\infty}$, we obtain

$$\sum_{i=0}^{\infty}\frac{(a,b;q)_i}{(q;q)_i}\frac{(cq^i;q)_\infty}{(cv;q)_\infty}c^i(1/ab)^i=\frac{(c/a,c/b;q)_\infty}{(cv,c/ab;q)_\infty}.$$

Acting the operator ${}_r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{array};q,-d\theta\right)$ with respect to the parameter c yielding

$$\sum_{i=0}^{\infty}\frac{(a,b;q)_i}{(q;q)_i}(1/ab)^i_r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{array};q,-d\theta\right)\left\{c^i\frac{(cq^i;q)_\infty}{(cv;q)_\infty}\right\}$$
$$={}_r\Phi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{array};q,-d\theta\right)\left\{\frac{(c/a,c/b;q)_\infty}{(cv,c/ab;q)_\infty}\right\}.$$

By using (23) and (20), we get

$$\sum_{i=0}^{\infty}\frac{(a,b;q)_i}{(q;q)_i}(1/ab)^i c^i\frac{(cq^i,q)_\infty}{(cv;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{-i},q/cq^i;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}(dq^i/cv)^j W_{k+j}$$
$$\times\frac{(dq/c)^k}{(q/cv;q)_{k+j}}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\frac{(q^i/v;q)_k}{(q;q)_k}q^{kj(1+s-r)}$$
$$=\frac{(c/a,c/b;q)_\infty}{(cv,c/ab;q)_\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(b,qb/c;q)_j}{(q,qab/c;q)_j}\left(\frac{dq}{cvb}\right)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}W_{k+j}\frac{(dq/c)^k}{(q/cv;q)_{k+j}}$$
$$\times\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}q^{kj(1+s-r)}\frac{(1/vb;q)_k}{(q;q)_k}$$

$$\sum_{i=0}^{\infty}\frac{(a,b;q)_i}{(q,c;q)_i}(c/ab)^i\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(q^{-i},q^{1-i}/c;q)_j}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}(dq^i/cv)^j W_{k+j}$$

$$\times \frac{(dq/c)^k}{(q/cv;q)_{k+j}} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \frac{(q^i/v;q)_k}{(q;q)_k} q^{kj(1+s-r)}$$

$$= \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(b,bq/c;q)_j}{(q,abq/c;q)_j} \left(\frac{dq}{cvb}\right)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} W_{k+j} \frac{(dq/c)^k}{(q/cv;q)_{k+j}}$$

$$\times \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} q^{kj(1+s-r)} \frac{(1/vb;q)_k}{(q;q)_k}.$$

□

3.2. **Generalization of q-Chu-Vandermonde Sum.** By acting the operator ${}_r\Phi_s$ with respect to the parameter c on both sides of q-Chu-Vandermonde sum (7) and by using equations (20) and (22), we obtain the following generalization of q-Chu-Vanermonde sum:

Theorem 3.2. (Generalization of q-Chu-Vandermonde Sum (7)). *Given q-Chu-Vandermonde sum* (7)*, then*

$$\sum_{k=0}^{n}\sum_{i=0}^{\infty} \frac{(q^{-n},b;q)_k}{(q,c;q)_k} q^k \left(\frac{dq}{c}\right)^i \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} W_i \frac{(q^k/v;q)_i}{(q,q/cv;q)_i}$$

$$= \frac{(c/b;q)_n}{(c;q)_n} b^n \sum_{j=0}^{n}\sum_{i=0}^{\infty} \frac{(q^{-n},q^{1-n}/c;q)_j}{(q,bq^{1-n}/c;q)_j} \left(\frac{dq^{n+1}}{bcv}\right)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}$$

$$\times W_{i+j} \frac{(dq/c)^i}{(q/cv;q)_{i+j}} \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(1+s-r)} \frac{(q^n/v;q)_i}{(q;q)_i}, \tag{24}$$

provided that $\max\{|cv|\} < 1$.

Proof. q-Chu-Vandermonde sum (7) is

$$\sum_{k=0}^{n} \frac{(q^{-n},b;q)_k}{(q,c;q)_k} q^k = \frac{(c/b;q)_n}{(c;q)_n} b^n.$$

$$\sum_{k=0}^{n} \frac{(q^{-n},b;q)_k}{(q;q)_k} q^k (cq^k;q)_\infty = \frac{(cq^n,c/b;q)_\infty}{(cq^n/b;q)_\infty} b^n.$$

Multiplying both sides of the above equation by $\dfrac{1}{(cv;q)_\infty}$, we obtain

$$\sum_{k=0}^{n} \frac{(q^{-n},b;q)_k}{(q;q)_k} q^k \frac{(cq^k;q)_\infty}{(cv;q)_\infty} = \frac{(cq^n,c/b;q)_\infty}{(cv,cq^n/b;q)_\infty} b^n.$$

By acting the operator ${}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-d\theta\right)$ on both sides of the above equation with respect to the parameter c we get

$$\sum_{k=0}^{n} \frac{(q^{-n},b;q)_k}{(q;q)_k} q^k {}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-d\theta\right) \left\{\frac{(cq^k;q)_\infty}{(cv;q)_\infty}\right\}$$

$$= {}_r\Phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,-d\theta\right) \left\{\frac{(cq^n,c/b;q)_\infty}{(cv,cq^n/b;q)_\infty}\right\} b^n.$$

Using (20) and (22) yielding

$$\sum_{k=0}^{n} \frac{(q^{-n},b;q)_k}{(q;q)_k} q^k \frac{(cq^k;q)_\infty}{(cv;q)_\infty} \sum_{i=0}^{\infty} \left(\frac{dq}{c}\right)^i \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} W_i \frac{(q^k/v;q)_i}{(q,q/cv;q)_i}$$

$$= \frac{(cq^n, c/b;q)_\infty}{(cv, cq^n/b;q)_\infty} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(q^{-n}, q^{1-n}/c;q)_j}{(q, bq^{1-n}/c;q)_j} \left(\frac{dq^{n+1}}{cv}\right)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}$$
$$\times W_{i+j} \frac{(dq/c)^i}{(q/cv;q)_{i+j}} \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(1+s-r)} \frac{(q^n/v;q)_i}{(q;q)_i} b^n.$$

Thus, we have

$$\sum_{k=0}^{n}\sum_{i=0}^{\infty} \frac{(q^{-n}, b;q)_k}{(q, c;q)_k} q^k \left(\frac{dq}{c}\right)^i \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} W_i \frac{(q^k/v;q)_i}{(q, q/cv;q)_i}$$
$$= \frac{(c/b;q)_n}{(c;q)_n} b^n \sum_{j=0}^{n}\sum_{i=0}^{\infty} \frac{(q^{-n}, q^{1-n}/c;q)_j}{(q, bq^{1-n}/c;q)_j} \left(\frac{dq^{n+1}}{bcv}\right)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r}$$
$$\times W_{i+j} \frac{(dq/c)^i}{(q/cv;q)_{i+j}} \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(1+s-r)} \frac{(q^n/v;q)_i}{(q;q)_i}.$$

□

- Setting $r = 1$, $s = 0$ and $v = 0$ in (24), we get Lemma 5.1 obtained in Fang [3] (equation (16)).
- Setting $r = 2$, $s = 1$, $a_1 = q^{-N}$, $v = 0$ and $d = -\frac{b_1}{a_1 t} q^N$ in (24), then by using (6) and (1) we get Theorem 2.3 obtained in Zhang and Yang [10] (equation (17)).

3.3. **Generalizations of q-Pfaff-Saalschütz's Sum.** By acting the operator ${}_r\Phi_s$ with respect to the parameter a on both sides of q-Pfaff-Saalschütz's sum (8) and by using equations (19) and (21), we get a generalization of q-Pfaff-Saalschütz's sum as follows:

Theorem 3.3. (Generalization of q-Pfaff-Saalschütz's Sum). *Given q-Pfaff-Saalschütz's sum* (8)*, then*

$$\sum_{k=0}^{\infty} \frac{(q^{-n}, a, b;q)_k}{(q, c, abq^{1-n}/c;q)_k} q^k \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty} W_{l+j+i} (gbq^{1-n+k}/c)^{l+j+i} \left[(-1)^{l+j+i} q^{\binom{l+j+i}{2}}\right]^{s-r}$$
$$\times \frac{(-1)^l q^{\binom{l}{2}}}{(q;q)_l} \frac{(q^{-k}, c/a;q)_i}{(q, q^{1-k}/a;q)_i} \left(\frac{q}{cv}\right)^i \frac{(q/cv;q)_j}{(q,q)_j} \frac{(cq^{n-k}/ab;q)_{j+i}}{(q/av;q)_{j+i}}$$
$$= \frac{(c/a, c/b;q)_n}{(c, c/ab;q)_n} \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(bq/cv, gq^n/a;q)_j}{(q, q/av;q)_j}$$
$$\times (gq^{1-n}/c)^j \left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} W_{i+j} \frac{(gq^{1-n}/c)^i}{(q;q)_i} \left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}. \quad (25)$$

Proof. q-Pfaff-Saalschütz's sum (8) is

$$\sum_{k=0}^{\infty} \frac{(a, b, q^{-n};q)_k}{(q, c, abq^{1-n}/c;q)_k} q^k = \frac{(c/a, c/b;q)_n}{(c, c/ab;q)_n}. \quad (26)$$

Let $a \to aq/c$ and $a \to abq/c$ in (2), respectively, we get

$$(c/a;q)_n = (-1)^n (aq/c)^{-n} q^{\binom{n+1}{2}} \frac{(aq^{1-n}/c;q)_\infty}{(aq/c;q)_\infty}.$$
$$(c/ab;q)_n = (-1)^n (abq/c)^{-n} q^{\binom{n+1}{2}} \frac{(abq^{1-n}/c;q)_\infty}{(abq/c;q)_\infty}.$$

Substitute the above equations in (26), we get

$$\sum_{k=0}^{\infty}\frac{(q^{-n},b;q)_k}{(q,c;q)_k}q^k\frac{(a;q)_\infty}{(aq^k;q)_\infty}\frac{(abq^{1-n+k}/c;q)_\infty}{(abq^{1-n}/c;q)_\infty}$$
$$=\frac{(c/b;q)_n}{(c;q)_n}\frac{(-1)^n(aq/c)^{-n}q^{\binom{n+1}{2}}}{(-1)^n(abq/c)^{-n}q^{\binom{n+1}{2}}}\frac{(aq^{1-n}/c;q)_\infty}{(aq/c;q)_\infty}\frac{(abq/c;q)_\infty}{(abq^{1-n}/c;q)_\infty}.$$

Multiplying both sides of the above equation by $\frac{1}{(av;q)_\infty}$, rearrangement the above equation and applying the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,-g\theta\right)$ with respect to the parameter a, we obtain

$$\sum_{k=0}^{\infty}\frac{(q^{-n},b;q)_k}{(q,c;q)_k}q^k{}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,-g\theta\right)\left\{\frac{(a,aq/c,abq^{1-n+k}/c;q)_\infty}{(av,aq^k;q)_\infty}\right\}$$
$$=b^n\frac{(c/b;q)_n}{(c;q)_n}\,{}_r\Phi_s\left(\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,-g\theta\right)\left\{\frac{(abq/c,aq^{1-n}/c;q)_\infty}{(av;q)_\infty}\right\}.$$

By using equations (19) and (21), we obtain

$$\sum_{k=0}^{\infty}\frac{(q^{-n},b;q)_k}{(q,c;q)_k}q^k\frac{(a,aq/c,abq^{1-n+k}/c;q)_\infty}{(av,aq^k;q)_\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}W_{l+j+i}(gbq^{1-n+k}/c)^{l+j+i}$$
$$\times\left[(-1)^{l+j+i}q^{\binom{l+j+i}{2}}\right]^{s-r}\frac{(-1)^lq^{\binom{l}{2}}}{(q;q)_l}\frac{(q^{-k},c/a;q)_i}{(q,q^{1-k}/a;q)_i}\left(\frac{q}{cv}\right)^i\frac{(q/cv;q)_j}{(q,q)_j}$$
$$\times\frac{(q/abq^{1-n+k}/c;q)_{j+i}}{(q/av;q)_{j+i}}$$
$$=b^n\frac{(c/b;q)_n}{(c;q)_n}\frac{(abq/c,aq^{1-n}/c;q)_\infty}{(av;q)_\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(bq/cv,gq^n/a;q)_j}{(q,q/av;q)_j}$$
$$\times(gq^{1-n}/c)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{i+j}\frac{(gq^{1-n}/c)^i}{(q;q)_i}\left[(-1)^iq^{\binom{i}{2}}\right]^{1+s-r}q^{ij(s-r)}$$

$$\sum_{k=0}^{\infty}\frac{(q^{-n},a,b;q)_k}{(q,c,abq^{1-n}/c;q)_k}q^k(aq/c,abq^{1-n}/c;q)_\infty\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}W_{l+j+i}(gbq^{1-n+k}/c)^{l+j+i}$$
$$\times\left[(-1)^{l+j+i}q^{\binom{l+j+i}{2}}\right]^{s-r}\frac{(-1)^lq^{\binom{l}{2}}}{(q;q)_l}\frac{(q^{-k},c/a;q)_i}{(q,q^{1-k}/a;q)_i}\left(\frac{q}{cv}\right)^i\frac{(q/cv;q)_j}{(q,q)_j}$$
$$\times\frac{(cq^{n-k}/ab;q)_{j+i}}{(q/av;q)_{j+i}}$$
$$=b^n(abq/c,aq^{1-n}/c;q)_\infty\frac{(c/b;q)_n}{(c;q)_n}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(bq/cv,gq^n/a;q)_j}{(q,q/av;q)_j}$$
$$\times(gq^{1-n}/c)^j\left[(-1)^jq^{\binom{j}{2}}\right]^{s-r}W_{i+j}\frac{(gq^{1-n}/c)^i}{(q;q)_i}$$

$$\sum_{k=0}^{\infty}\frac{(q^{-n},a,b;q)_k}{(q,c,abq^{1-n}/c;q)_k}q^k(aq/c;q)_\infty(-1)^n(ab/c)^nq^{-\binom{n}{2}}(c/ab;q)_n(abq/c;q)_\infty$$

$$
\times \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty} W_{l+j+i}(gbq^{1-n+k}/c)^{l+j+i}\left[(-1)^{l+j+i}q^{\binom{l+j+i}{2}}\right]^{s-r}\frac{(-1)^l q^{\binom{l}{2}}}{(q;q)_l}\frac{(q^{-k},c/a;q)_i}{(q,q^{1-k}/a;q)_i}
$$

$$
\times \left(\frac{q}{cv}\right)^i \frac{(q/cv;q)_j}{(q,q)_j}\frac{(cq^{n-k}/ab;q)_{j+i}}{(q/av;q)_{j+i}}
$$

$$
= b^n(abq/c;q)_\infty(-1)^n(a/c)^n q^{-\binom{n}{2}}(c/a;q)_n(aq/c;q)_\infty\frac{(c/b;q)_n}{(c;q)_n}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(bq/cv,gq^n/a;q)_j}{(q,q/av;q)_j}
$$

$$
\times (gq^{1-n}/c)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} W_{i+j}\frac{(gq^{1-n}/c)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}
$$

$$
\sum_{k=0}^{\infty}\frac{(q^{-n},a,b;q)_k}{(q,c,abq^{1-n}/c;q)_k}q^k\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty} W_{l+j+i}(gbq^{1-n+k}/c)^{l+j+i}\left[(-1)^{l+j+i}q^{\binom{l+j+i}{2}}\right]^{s-r}
$$

$$
\times \frac{(-1)^l q^{\binom{l}{2}}}{(q;q)_l}\frac{(q^{-k},c/a;q)_i}{(q,q^{1-k}/a;q)_i}\left(\frac{q}{cv}\right)^i\frac{(q/cv;q)_j}{(q,q)_j}\frac{(cq^{n-k}/ab;q)_{j+i}}{(q/av;q)_{j+i}}
$$

$$
= \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(bq/cv,gq^n/a;q)_j}{(q,q/av;q)_j}
$$

$$
\times (gq^{1-n}/c)^j\left[(-1)^j q^{\binom{j}{2}}\right]^{s-r} W_{i+j}\frac{(gq^{1-n}/c)^i}{(q;q)_i}\left[(-1)^i q^{\binom{i}{2}}\right]^{1+s-r} q^{ij(s-r)}.
$$

□

4. Conclusions

(1) Numerous operators can be obtained by using the generalized q-operator ${}_r\Phi_s$ with some special values.
(2) Several well-known q-identities can be generalized by using the generalized q-operator ${}_r\Phi_s$.

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