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DYNAMICS, CONTROL, STABILITY, DIFFUSION AND SYNCHRONIZATION OF MODIFIED CHAOTIC COLPITTS OSCILLATOR WITH TRIANGULAR WAVE NON-LINEARITY

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Abstract. The purpose of this paper is to introduce a new chaotic oscillator. Although different chaotic systems have been formulated by earlier researchers, only a few chaotic systems exhibit chaotic behaviour. In this work, a new chaotic system with chaotic attractor is introduced for triangular wave non-linearity. It is worth noting that this striking phenomenon rarely occurs in respect of chaotic systems. The system proposed in this paper has been realized with numerical simulation. The results emanating from the numerical simulation indicate the feasibility of the proposed chaotic system. More over, chaos control, stability, diffusion and synchronization of such a system have been dealt with.

Keywords: Chaos, Colpitts oscillator, Lyapunov exponent, Diffusion, Stability, Synchronization, Triangular Wave Non-linearity.

AMS Subject Classification: 34H10, 93C15, 34H15.

1. Introduction

The study of chaotic dynamical systems is drawing the attention of the researchers in the recent times. Research on a chaotic system with chaotic attractor is posing several challenges thereby making the study quite interesting.

A non-linear dynamical system exhibiting complex and unpredictable behavior is called chaotic system [1]. The parameter values are varying with range and the sensitivity depends on initial conditions. These are the remarkable properties [2] of chaotic systems. Sometimes, the chaotic systems are deterministic [3, 4] and they have long-term unpredictable behavior [5-6].

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While chaotic systems are highly sensitive, their sensitivity depends on their initial conditions. The chaotic nature is one of the qualitative [7, 8] properties of a dynamical system [9, 10].

The controlling of the chaotic systems may be accomplished in three ways such as stabilization [11, 12] of unstable periodic motion "contained" in the chaotic set, suppression of chaotic behavior by external forcing like periodic noise, periodic parametric perturbation and algorithm of various automatic control like feedback [13, 14], backstepping [15], sample feedback, time delay feedback, etc.

There exist two ways for the application of controls in a chaotic system. The first one is the change of attractor of the system. The second one is the change in the point position of the phase space for the system which is a constant value in its parameter.

A continuous, repeated and alternating wave production without any input is an oscillator. Converting power supply to an alternating current signal is one of the primary properties of oscillators. The signal of feedback containing a pair of coils and an inductive divider in the server is called Colpitts oscillator [16, 17]. Due to some parametric change and the variation of input, the chaotic nature may occur in Colpitts oscillators.

In this paper, a new chaotic Colpitts oscillator is proposed. It is a modified form of the earlier version of Colpitts oscillators. In section 2, the modified form of Colpitts oscillator [18, 19] is presented with the formulation of the mathematical model. In addition, invariant property, equilibrium point and Lyapunov exponents [20-23] are investigated. In section 3, adaptive backstepping technique [24] is explained for the proposed system. In section 4, a non linear feedback system is established. The control strategy of backstepping is employed to analyze the non linear feedback system in section 5. Finally, the numerical simulation [25-28] is upheld for the hypothetical outcomes.

2. The Mathematical Model of Chaotic Colpitts Oscillator

The depiction of simplified illustrative diagram for modified Colpitts oscillator is undertaken in Figure 1. In addition to Electronic devices, communication systems also have wide usage of the Colpitts oscillator. It is a single-transistor implementation of a sinusoidal oscillator.

The following are the hypotheses for simplifying the extensive simulation of the complete circuit model.

- The base-emitter(B-E) driving point(V-I) characteristic of the R_E with triangular wave function is

$$I_E = f\left(V_{BE}\right) = I_S\left[\frac{2a}{\pi}\sin^{-1}\left(\sin\left(\frac{2\pi}{p}(x_1+x_3)\right)\right)\right]$$

  where I_S is the emitter current (inverse saturation current), a is amplitude and p is period of the B-E junction.
- The state space is schematically represented in Figure 1.

$$\begin{aligned}
R_C C_1 \frac{dV_{C_1}}{dt} &= V_0 - V_{C_1} - V_{C_2} + R_C I_L - R_C f\left(V_{BE}\right) \\
R_C C_2 \frac{dV_{C_2}}{dt} &= V_0 - V_{C_1} - V_{C_2} - R_C I_0 + R_C I_L \\
C_3 \frac{dV_{C_3}}{dt} &= I_L - (1-\alpha) f\left(V_{BE}\right) \\
L\frac{dI_L}{dt} &= -R_b I_L - V_{C_1} - V_{C_2} - V_{C_3}
\end{aligned}$$

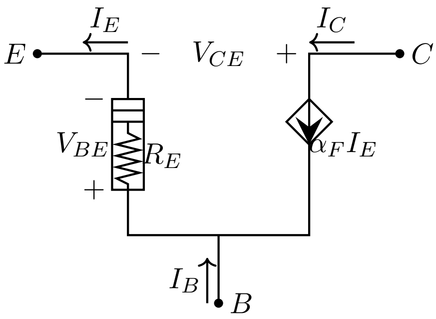


FIGURE 1. The circuit diagram

The following is the proposed new system with Colpitts oscillator:

$$\begin{aligned} \dot{x}_1 &= \sigma_1(-x_1 - x_2) + x_4 - \gamma\phi(x_1, x_3) \\ \dot{x}_2 &= \varepsilon_1\sigma_1(-x_1 - x_2) + \varepsilon_1 x_4 \\ \dot{x}_3 &= \varepsilon_2(x_4 - (1-\alpha)\gamma\phi(x_1, x_3)) \\ \dot{x}_4 &= -x_1 - x_2 - x_3 - \sigma_2 x_4 \end{aligned} \tag{1}$$

where $\phi(x_1, x_3) = \dfrac{2a}{\pi}\sin^{-1}\left(\sin\left(\dfrac{2\pi}{p}(x_1 + x_3)\right)\right)$.

In system (1), the state variables are assumed as x_1, x_2, x_3 and x_4 along with six positive parameters, σ_1, γ, ε_1, ε_2, σ_2 and α. The system (1) is an autonomous system to which a triangular wave expression is associated.

With the modification of coordinates provided by the scheme $(x_1,\ x_2,\ x_3,\ x_4) \mapsto (-x_1,\ -x_2,\ -x_3,\ -x_4)$, the system (1) is found to be invariant.

The mathematical system of the Colpitts oscillator mathematical system when equated to zero gives the equilibrium points of the system as specified below:

$$\begin{aligned} \sigma_1(-x_1 - x_2) + x_4 - \gamma\phi(x_1, x_3) &= 0 \\ \varepsilon_1\sigma_1(-x_1 - x_2) + \varepsilon_1 x_4 &= 0 \\ \varepsilon_2(x_4 - (1-\alpha)\gamma\phi(x_1, x_3)) &= 0 \\ -x_1 - x_2 - x_3 - \sigma_2 x_4 &= 0 \end{aligned} \tag{2}$$

Solving the system (2), it is seen that the new chaotic system (2) has a unique equilibrium at the origin.

The Jocobian matrix of the system (1) at the equilibrium point E is given by

$$J_E = \begin{bmatrix} -\sigma_1 - 4\gamma a/p & -\sigma_1 & -4\gamma a/p & 1 \\ -\varepsilon_1\sigma_1 & -\varepsilon_1\sigma_1 & 0 & \varepsilon_1 \\ -\varepsilon_2(1-\alpha)4\gamma a/p & 0 & -\varepsilon_2(1-\alpha)4\gamma a/p & \varepsilon_2 \\ -1 & -1 & -1 & -\sigma_2 \end{bmatrix} \tag{3}$$

The corresponding characteristic equation of Colpitts oscillator system (1) with respect to E is given by the relation

$$\Delta_1\lambda^4 + \Delta_2\lambda^3 + \Delta_3\lambda^2 + \Delta_4\lambda + \Delta_5 = 0 \tag{4}$$

where
$\Delta_1 = 1$
$\Delta_2 = \dfrac{-4\alpha\varepsilon_2\gamma a + \varepsilon_1\sigma_1 p + 4\varepsilon_2\gamma a + 4\gamma a + \sigma_1 p + \sigma_2 p}{p}$

$$\Delta_3 = \frac{\begin{bmatrix} -4\alpha\varepsilon_1\varepsilon_2\gamma\sigma_1 a - 4\alpha\varepsilon_2\gamma\sigma_1 a - 4\alpha\varepsilon_2\gamma\sigma_2 a + 4\varepsilon_1\varepsilon_2\gamma\sigma_1 a + 4\varepsilon_1\gamma\sigma_1 a \\ +\varepsilon_1\sigma_1\sigma_2 p + \varepsilon_1 p + 4\varepsilon_2\gamma\sigma_1 a + 4\varepsilon_2\gamma\sigma_2 a + \varepsilon_2 p + 4\gamma\sigma_2 a + \sigma_1\sigma_2 p + p \end{bmatrix}}{p}$$

$$\Delta_4 = \frac{\begin{bmatrix} -4\alpha\varepsilon_1\varepsilon_2\gamma\sigma_1\sigma_2 a - 4\alpha\varepsilon_1\varepsilon_2\gamma a - 4\alpha\varepsilon_2\gamma\sigma_1\sigma_2 a + 4\varepsilon_1\varepsilon_2\gamma\sigma_1\sigma_2 a \\ +4\varepsilon_1\varepsilon_2\gamma a + \varepsilon_1\varepsilon_2\sigma_1 p + 4\varepsilon_1\gamma\sigma_1\sigma_2 a + 4\varepsilon_1\gamma a + 4\varepsilon_2\gamma\sigma_1\sigma_2 a + \varepsilon_2\sigma_1 p \end{bmatrix}}{p}$$

$$\Delta_5 = \frac{4\varepsilon_1\varepsilon_2\gamma\sigma_1 a}{p}$$

Applying Routh-Hurwitz stability criterion [29] to the characteristic equation, we conclude that the system is unstable for all values of the parameters at the equilibrium position E.

From the Jacobian matrix (3), among the states x_1, x_2, x_3 and x_4, if x_1 and x_3 are both positive or negative or of opposite signs, it implies "*Hopf bifurcation*". This phenomenon is also known as "*Poincaré–Andronov–Hopf bifurcation*". This bifurcation leads a local birth of "*chaos*" nature in modified Colpitts oscillator (1).

Interestingly, the system (1) is chaotic for the parameters

$$\varepsilon_1 = 1,\ \varepsilon_2 = 20,\ \sigma_1 = 1.49,\ \sigma_2 = 0.872,\ \gamma = 38.00,\ \alpha = \frac{255}{256}$$

Lyapunov exponents may be considered as one of the keys to differentiate between chaotic, hyperchaotic, stable and periodic nature of the systems.

Table 1 gives the details of the chaotic and hyperchaotic nature of the system. For this calculation, the observation time (T) is considered as 500 and the sampling time (Δt) is taken as 0.5. For various initial conditions, the system (1) exhibits chaotic and hyperchaotic nature.

By applying Wolf algorithm [30], the Lyapunov exponents corresponding to the new chaotic system (1) are obtained as follows:

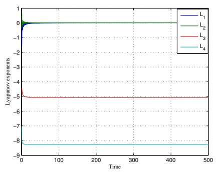
Sl. No.	Parameter, a, p	Initial condition	LEs	Sign of the LEs	Nature
1	$\gamma = 2.2001, a = 1, p = 1$	0.00001, 0.00001, 0.00001, 0.00001	0.009914, 0.010189, −5.086796, −8.273187	[≈ 0, +, −, −]	Chaotic
2	$\gamma = 2.2001, a = 1, p = 2$	0.00001, 0.00001, 0.00001, 0.00001	0.009914, 0.010189, −5.086796, −8.273187	[≈ 0, +, −, −]	Chaotic
3	$\gamma = 2.2001, a = 1, p = 3$	0.00001, 0.00001, 0.00001, 0.00001	0.009914, 0.010189, −5.086796, −8.273187	[≈ 0, +, −, −]	Chaotic
4	$\gamma = 38.000, a = 1, p = 1$	0.00001, 0.00001, 0.00001, 0.00001	0.605403, 0.608392, −3.807690, −51.008494	[+, +, −, −]	Hyperchaotic
5	$\gamma = 38.000, a = 1, p = 2$	0.00001, 0.00001, 0.00001, 0.00001	0.562656, 0.568263, −3.837630, −41.737089	[+, +, −, −]	Hyperchaotic
6	$\gamma = 38.000, a = 1, p = 2$	1.00000, 1.00000, 1.00000, 1.00000	0.562656, 0.568263, −3.837630, −42.690649	[+, +, −, −]	Hyperchaotic
7	$\gamma = 38.000, a = 1, p = 2$	1.50000, 1.50000, 1.50000, 1.50000	0.562656, 0.568263, −3.837630, −42.774085	[+, +, −, −]	Hyperchaotic
8	$\gamma = 38.000, a = 1, p = 2$	2.00000, 2.00000, 2.00000, 2.00000	0.562656, 0.568263, −3.837630, −42.896532	[+, +, −, −]	Hyperchaotic

TABLE 1. LEs of system (1) for observation time (T) = 500, sampling time (Δt) = 0.5, $\varepsilon_1 = 1$, $\varepsilon_2 = 20$, $\sigma_1 = 1.49$, $\sigma_2 = 0.872$, $\alpha = \frac{255}{256}$, $\gamma = 2.2001, 38.00$ with various sampling and observation times using Wolf algorithm

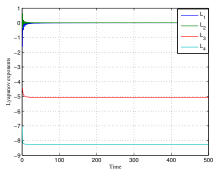
From Table 1, the Lyapunov exponential dimension is calculated. The attractor of the new system is observed to be a strange attractor with fractal dimensions.

Through numerical simulation, the chaotic attractor of the system (1) is obtained as shown in Figure 3.

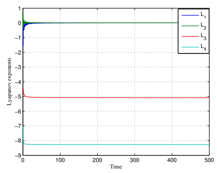
Figure 2 depicts the Lyapunov exponents of the modified Colpitts oscillator and Figure 3 shows the chaotic nature of the modified Colpitts oscillator and Poincaré Map of the modified Colpitts oscillator.



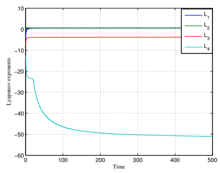
(A) $\gamma = 2.2001$, $a = 1$, $p = 1$



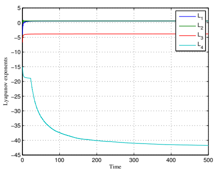
(B) $\gamma = 2.2001$, $a = 1$, $p = 2$



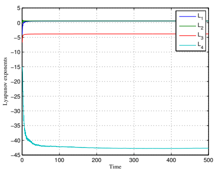
(C) $\gamma = 2.2001$, $a = 1$, $p = 3$



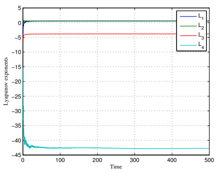
(D) $\gamma = 38$, $a = 1$, $p = 1$



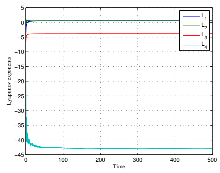
(E) $\gamma = 38$, $a = 1$, $p = 2$



(F) $\gamma = 38$, $a = 1$, $p = 2$

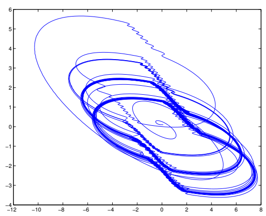


(G) $\gamma = 38$, $a = 1$, $p = 2$

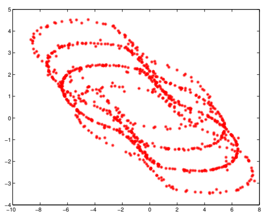


(H) $\gamma = 38$, $a = 1$, $p = 2$

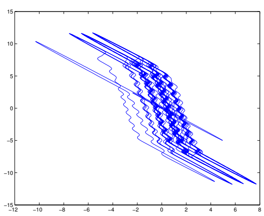
FIGURE 2. Lyapunov exponents of the Modified Colpitts oscillator



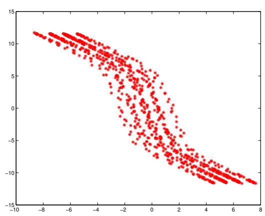
(A) Chaotic nature between x_1 and x_2



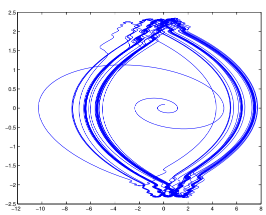
(B) Poincaré Map between x_1 and x_2



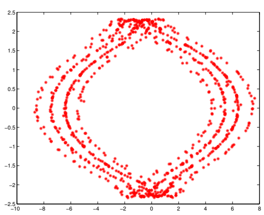
(C) Chaotic nature between x_1 and x_3



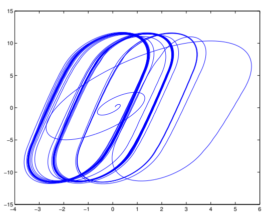
(D) Poincaré Map between x_1 and x_3



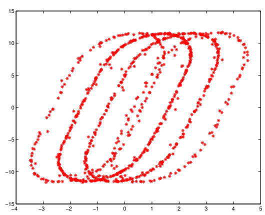
(E) Chaotic nature between x_1 and x_4



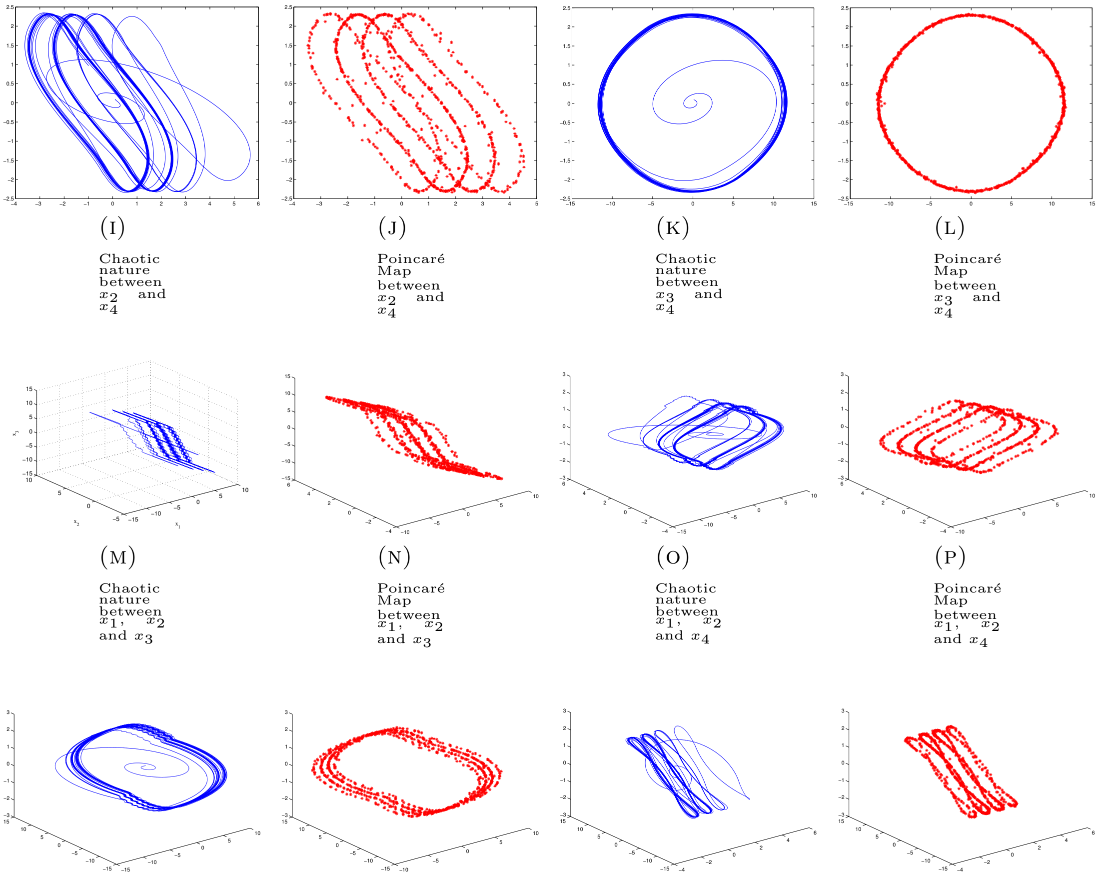
(F) Poincaré Map between x_1 and x_4



(G) Chaotic nature between x_2 and x_3



(H) Poincaré Map between x_2 and x_3



(I) Chaotic nature between x_2 and x_4

(J) Poincaré Map between x_2 and x_4

(K) Chaotic nature between x_3 and x_4

(L) Poincaré Map between x_3 and x_4

(M) Chaotic nature between x_1, x_2 and x_3

(N) Poincaré Map between x_1, x_2 and x_3

(O) Chaotic nature between x_1, x_2 and x_4

(P) Poincaré Map between x_1, x_2 and x_4

(Q) Chaotic nature between x_1, x_3 and x_4

(R) Poincaré Map between x_1, x_3 and x_4

(S) Chaotic nature between x_2, x_3 and x_4

(T) Poincaré Map between x_2, x_3 and x_4

FIGURE 3. Portrait of Colpitts

The study of qualitative properties is one of the utilities of this paradigm. The stability control, limit cycle, periodicity and chaos are some notable qualitative properties. The following theorems bring out the local stability properties of the modified Colpitts oscillator.

Theorem 2.1. *The interior equilibrium point E is locally asymptotically stable in the positive octant.*

Proof. By *divergence criterion* theorem, assume

$$\theta(x_1, x_2, x_3, x_4) = \frac{1}{x_1 x_2 x_3 x_4} \tag{5}$$

where $\theta(x_i, i = 1, 2, 3, 4) > 0$ if $x_i > 0, i = 1, 2, 3, 4$.

Now consider

$$\begin{aligned} p_1 &= \sigma_1(-x_1 - x_2) + x_4 - \gamma\phi(x_1, x_3) \\ p_2 &= \varepsilon_1\sigma_1(-x_1 - x_2) + \varepsilon_1 x_4 \\ p_3 &= \varepsilon_2(x_4 - (1-\alpha)\gamma\phi(x_1, x_3)) \\ p_4 &= -x_1 - x_2 - x_3 - \sigma_2 x_4 \end{aligned} \tag{6}$$

where $\phi(x_1,x_3)=\frac{2a}{\pi}\sin^{-1}\left(\sin\left(\frac{2\pi}{p}(x_1+x_3)\right)\right)$.

Define

$$\nabla=\frac{\partial}{\partial x_1}(p_1\theta)+\frac{\partial}{\partial x_2}(p_2\theta)+\frac{\partial}{\partial x_3}(p_3\theta)+\frac{\partial}{\partial x_4}(p_4\theta) \tag{7}$$

We have to determine ∇ given by Equation (7) along with the trajectories provided by Equations (5) and Equation (6). We obtain

$$\begin{aligned}\nabla=&-\frac{[\sigma_1+\gamma 4a/p]\,x_1x_2x_3x_4+[\sigma_1(-x_1-x_2)+x_4-\gamma\phi(x_1,x_3)]\,x_2x_3x_4}{x_1^2x_2^2x_3^2x_4^2}\\&-\frac{\varepsilon_1\sigma_1x_1x_2x_3x_4+[\varepsilon_1\sigma_1(-x_1-x_2)+\varepsilon_1x_4]x_1x_3x_4}{x_1^2x_2^2x_3^2x_4^2}\\&-\frac{[\varepsilon_2(1-\alpha)\gamma 4a/p]\,x_1x_2x_3x_4+\varepsilon_2\,[x_4-(1-\alpha)\gamma\phi(x_1,x_3)]\,x_1x_2x_4}{x_1^2x_2^2x_3^2x_4^2}\\&-\frac{\sigma_2x_1x_2x_3x_4+(-x_1-x_2-x_3-\sigma_2x_4)x_1x_2x_3}{x_1^2x_2^2x_3^2x_4^2}\end{aligned}$$

which is less than zero.

From *Benedixon-Dulac criterion*, it it clear that the first octant does not contain any limit cycle.

Consequently, the equilibrium provided by E is found to be locally asymptotically stable.

The relation between the limit cycle and closed trajectories exhibits the local asymptotic stability. The following theorem is concerned with the stability under closed trajectory using Bendixson's criteria theorem. □

Theorem 2.2. *There is no closed trajectory for the interior equilibrium point.*

Proof. Define

$$\Psi(x_i,i=1,2,3,4)=\frac{\partial p_1}{\partial x_1}+\ldots+\frac{\partial p_4}{\partial x_4} \tag{8}$$

Find Ψ along with the trajectories associated with Equation (8). It follows that

$$\Psi=-\sigma_1-\gamma 4a/p-\varepsilon_1\sigma_1-\varepsilon_2(1-\alpha)\gamma 4a/p-\sigma_2\neq 0 \tag{9}$$

Hence, by applying *Bendixson's criteria theorem* to Equation (9), it is seen that there is no closed trajectory surrounding the point E.

Hence, limit cycle does not exist emcompassing E.

Therefore, the point E is evidential to be locally asymptotically stable.

In oscillator, exhibiting stable periodic orbit and it corresponds to a special type of solution for a oscillator. The following theorem focuses attention on the nontrivial periodic solution. □

Theorem 2.3. *The modified Colpitts oscillator given by Equation (1) has a nontrivial periodic solution.*

Proof. Define

$$\begin{aligned}\Phi &= \frac{d}{dt}\left(\frac{x_1^2+x_2^2+x_3^2+x_4^2}{2}\right) = x_1\frac{dx_1}{dt}+x_2\frac{dx_2}{dt}+x_3\frac{dx_3}{dt}+x_4\frac{dx_4}{dt} \\ &= x_1\dot{x}_1+x_2\dot{x}_2+x_3\dot{x}_3+x_4\dot{x}_4 = \sum_{i=1}^{4} x_i\frac{dx_i}{dt}\end{aligned} \tag{10}$$

Find Φ from Equation (10) along the trajectories Equation (1). We see that

$$\begin{aligned}\Phi &= x_1[\sigma_1(-x_1-x_2)+x_4-\gamma\phi(x_1,x_3)]+x_2[\varepsilon_1\sigma_1(-x_1-x_2)+\varepsilon_1x_4] \\ &\quad +x_3[\varepsilon_2(x_4-(1-\alpha)\gamma\phi(x_1,x_3))]+x_4[-x_1-x_2-x_3-\sigma_2x_4] \\ &= -\sigma_1x_1^2-\sigma_1x_1x_2+x_1x_4-\gamma x_1\phi(x_1,x_3)-\varepsilon_1\sigma_1x_1x_2-\varepsilon_1\sigma_1x_2^2+\varepsilon_1x_2x_4 \\ &\quad +\varepsilon_2x_3x_4-\varepsilon_2(1-\alpha)x_3\gamma\phi(x_1,x_3)-x_1x_4-x_2x_4-x_3x_4-\sigma_2x_4^2 \\ &= -(\sigma_1x_1^2+\sigma_1x_2^2+\sigma_2x_4^2) \\ &\quad -\sigma_1x_1x_2(1+\varepsilon_1)-(1-\varepsilon_1)x_2x_4-(1-\varepsilon_2)x_3x_4-\gamma\phi(x_1,x_3)[x_1+\varepsilon_2(1-\alpha)x_3] \\ &= -(\nabla_1+\nabla_2)\end{aligned} \tag{11}$$

where $\nabla_1 = \sigma_1x_1^2+\varepsilon_1\sigma_1x_2^2+\sigma_2x_4^2$

$\nabla_2 = \sigma_1(1+\varepsilon_1)x_1x_2+(1-\varepsilon_1)x_2x_4+(1-\varepsilon_2)x_3x_4+\gamma\phi(x_1,x_3)(x_1+\varepsilon_2(1-\alpha)x_3)$

It is observed that $\nabla_1+\nabla_2$ is positive for $x_1^2+x_2^2+x_3^2+x_4^2 < a$ and negative for $x_1^2+x_2^2+x_3^2+x_4^2 > b$, where a, b are positive constants.

This implies that any solution $x_i(t)$ of (1) which starts in the annulus $a < \sum_{i=1}^{4} x_i^2 < b$.

Hence, by *Poincaré-Bendixson* theorem, there exists atleast one periodic solution $x_i(t), i = 1,2,3,4$ of Equation (1) lying in this annulus.

Hence, the modified Colpitts oscillator Equation (1) has a nontrivial periodic solution. □

The study of control refers to the process of influencing the behaviour of an oscillator to achieve a desired goal, primarily through the use of feedback control. The following section describes the backstepping control when the parameter values are unknown.

3. Adaptive Backstepping control of the modified Colpitts oscillator with unknown parameters

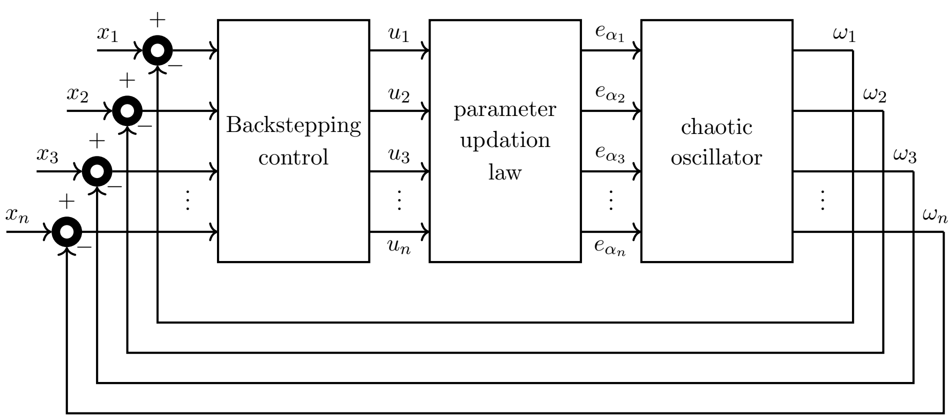


Figure 4. Block diagram for adaptive backstepping control

3.1. **Pseudo Algorithm for Adaptive Synchronization of Chaotic Oscillator.** Suppose the chaotic dynamics is defined by

$$\dot{x} = Ax + f(x) + u, \tag{12}$$

where $x \in \mathbb{R}^n$ is the state of the system, A is the $n \times n$ matrix of the system parameters, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the system and $u \in \mathbb{R}^n$ is the controller of the response system.

Thus, the synchronization error problem is to find a controller u so that

$$\lim_{t\to\infty} \|e(t)\| = 0. \tag{13}$$

In the adaptive synchronization of chaotic systems, it is assumed that the parameters of the master and slave systems are unknown. To fix the notation, suppose that the vector α represents the parameters of master system and also that the vector β represents the parameters of the slave system.

In this case, the synchronizing controller is taken as

$$u(t) = u(x,\ \hat{\alpha},\ \hat{\beta}), \tag{14}$$

where $\hat{\alpha},\ \hat{\beta}$ are estimates of the unknown parameters vectors α and β.

3.1.1. *Design of the parameter Update Law.* A quadratic Lyapunov function can be used for the adaptive synchronization method, viz

$$V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + ... + x_n^2 + e_{\beta_1}^2 + e_{\beta_2}^2 + ... + e_{\beta_n}^2 + e_{\alpha_1}^2 + e_{\alpha_2}^2 + ... + e_{\beta_n}^2) \tag{15}$$

where $\alpha = (\alpha_1,\ \alpha_2, ...\alpha_n)$ and $\beta = (\beta_1,\ \beta_2, ...\beta_n)$.

Then the Lyapunov function V is differentiated along the trajectories of the error dynamics and parameter update law is carefully designed so that $\dot{V}$ is a negative definite function.

3.2. **Adaptive Backstepping control of the modified Colpitts oscillator.** The modified Colpitts oscillator system is given by the dynamics with controllers

$$\begin{aligned}
\dot{x}_1 &= \sigma_1 \left(-x_1 - x_2\right) + x_4 - \gamma\phi \left(x_1, x_3\right) + u_1 \\
\dot{x}_2 &= \varepsilon_1\sigma_1 \left(-x_1 - x_2\right) + \varepsilon_1 x_4 + u_2 \\
\dot{x}_3 &= \varepsilon_2 \left(x_4 - \left(1-\alpha\right)\gamma\phi \left(x_1, x_3\right)\right) + u_3 \\
\dot{x}_4 &= -x_1 - x_2 - x_3 - \sigma_2 x_4 + u_4
\end{aligned} \tag{16}$$

where $\phi\left(x_1, x_3\right) = \dfrac{2a}{\pi} \sin^{-1}\left(\sin\left(\dfrac{2\pi}{p}(x_1 + x_3)\right)\right)$.

In system (16), x_1, x_2, x_3 and x_4 are state variables and $u_1,\ u_2,\ u_3$ and u_4 are adaptive controllers.

The synchronization error is defined as $e_i = y_i - x_i,\ i = 1, 2, 3, 4$.

The unknown parameters are updated by

$$\begin{aligned}
e_{\sigma_1} &= \sigma_1 - \widehat{\sigma}_1(t), \quad e_{\sigma_2} = \sigma_2 - \widehat{\sigma}_2(t) \\
e_{\varepsilon_1} &= \varepsilon_1 - \widehat{\varepsilon}_1(t), \quad e_{\varepsilon_2} = \varepsilon_2 - \widehat{\varepsilon}_2(t) \\
e_{\alpha} &= \alpha - \widehat{\alpha}(t), \qquad e_{\gamma} = \gamma - \widehat{\gamma}(t)
\end{aligned} \tag{17}$$

By differentiating (17) with respect to 't', one obtains

$$\begin{aligned} \dot{e}_{\sigma_1} &= -\dot{\widehat{\sigma}}_1(t), \quad \dot{e}_{\sigma_2} = -\dot{\widehat{\sigma}}_2(t) \\ \dot{e}_{\varepsilon_1} &= -\dot{\widehat{\varepsilon}}_1(t), \quad \dot{e}_{\varepsilon_2} = -\dot{\widehat{\varepsilon}}_2(t) \\ \dot{e}_{\alpha} &= -\dot{\widehat{\alpha}}(t), \quad \dot{e}_{\gamma} = -\dot{\widehat{\gamma}}(t) \end{aligned}$$

At this stage, the state of the system is considered as

$$\dot{x}_1 = \sigma_1 \left(-x_1 - x_2\right) + x_4 - \gamma\phi\left(x_1, x_3\right) + u_1 \tag{18}$$

where x_2 is regarded as virtual controller.

In order to stabilize the system, the suitable Lyapunov function is defined as

$$V_1\left(x_1\right) = \frac{1}{2}x_1^2 + \frac{1}{2}e_{\sigma_1}^2 + \frac{1}{2}e_{\gamma}^2$$

By differentiating V_1 with respect to t,

$$\begin{aligned} \dot{V}_1 &= x_1\dot{x}_1 + e_{\sigma_1}\dot{e}_{\sigma_1} + e_{\gamma}\dot{e}_{\gamma} \\ &= x_1\left[\sigma_1\left(-x_1 - x_2\right) + x_4 - \gamma\phi\left(x_1, x_3\right) + u_1\right] + e_{\sigma_1}\left(-\dot{\widehat{\sigma}}_1\right) + e_{\gamma}\left(-\dot{\widehat{\gamma}}\right) \end{aligned} \tag{19}$$

where x_2 is regarded as virtual controller and is defined as

$$x_2 = \beta_1\left(x_1\right) \text{ and } \beta_1\left(x_1\right) = 0.$$

The controller u_1 is assumed as

$$u_1 = -x_1 + \widehat{\sigma}_1 x_1 - x_4 + \widehat{\gamma}\phi\left(x_1, x_3\right) \tag{20}$$

and the unknown parameters $\widehat{\sigma}_1$ and $\widehat{\gamma}$ are updated by

$$\begin{aligned} \dot{\widehat{\sigma}}_1 &= -x_1^2 + e_{\sigma_1} \\ \dot{\widehat{\gamma}} &= -x_1\phi\left(x_1, x_3\right) + e_{\gamma} \end{aligned} \tag{21}$$

On substitution of (20) and (21) into (19), we get

$$\dot{V}_1 = -x_1^2 - e_{\sigma_1}^2 - e_{\gamma}^2$$

which is found to be a negative definite function.

Hence by Lyapunov stability theory, the system is globally asymptotically stable.

Now define the relation between β_1 and x_2 by

$$\omega_2 = x_2 - \beta_1$$

Consider the subsystem (x_1, ω_2). We have

$$\begin{aligned} \dot{x}_1 &= -e_{\sigma_1}x_1 - \sigma_1\omega_2 - e_{\gamma}\phi\left(x_1, x_3\right) - x_1 \\ \dot{\omega}_2 &= -\varepsilon_1\sigma_1 x_1 - \varepsilon_1\sigma_1\omega_2 + \varepsilon_1 x_4 + u_2 \end{aligned}$$

Define V_2 by the Lyapunov function as

$$V_2 = V_1 + \frac{1}{2}\omega_2^2 + \frac{1}{2}e_{\varepsilon_1}^2$$

On differentiating V_2 with respect to t, we get

$$\dot{V}_2 = x_1\dot{x}_1 + e_{\sigma_1}\left(-\dot{\widehat{\sigma}}_1\right) + e_{\gamma}\left(-\dot{\widehat{\gamma}}\right) + e_{\varepsilon_1}\left(-\dot{\widehat{\varepsilon}}_1\right) + \omega_2\dot{\omega}_2 \tag{22}$$

The controller u_2 is assumed as

$$u_2 = \sigma_1 x_1 + \widehat{\varepsilon}_1\left(\sigma_1 x_1 + \sigma_1\omega_2 - x_4\right) + x_3 - \omega_2 \tag{23}$$

Let x_3 be the virtual controller. It is defined as $x_3 = \beta_2(x_1, \omega_2)$ with the assumption that $\beta_2(x_1, \omega_2) = 0$.

The parameter ε_1 is estimated as $\dot{\hat{\varepsilon}}_1 = -\omega_2(\sigma_1 x_1 + \sigma_1\omega_2 - x_4) + e_{\varepsilon_1}$ (24)

Substituting (23) and (24) into (22), we get

$$\dot{V}_2 = -x_1^2 - e_{\sigma_1}^2 - e_\gamma^2 - w_2^2 - e_{\varepsilon_1}^2$$

which is a negative definite function.
Hence by Lyapunov stability theory, the system is globally asymptotically stable.
The relation between x_3 and β_2 is defined by

$$\omega_3 = x_3 - \beta_2$$

Consider the subsystem $(x_1, \omega_2, \omega_3)$. We have

$$\begin{aligned}
\dot{x}_1 &= -e_{\sigma_1}x_1 - \sigma_1\omega_2 - e_\gamma\phi(x_1, x_3) - x_1\\
\dot{\omega}_2 &= -e_{\varepsilon_1}(\sigma_1 x_1 + \sigma_1\omega_2 - x_4) - \omega_2 + \sigma_1 x_1 + \omega_3\\
\dot{\omega}_3 &= \varepsilon_2(x_4 - (1-\alpha)\gamma\phi(x_1, x_3)) + u_3
\end{aligned}$$

Now consider the Lyapunov function

$$V_3 = V_2 + \frac{1}{2}\omega_3^2 + \frac{1}{2}e_{\varepsilon_2}^2 + \frac{1}{2}e_\alpha^2$$

The derivative of V_3 with respect to t is obtained as

$$\dot{V}_3 = \dot{V}_2 + \omega_3\dot{\omega}_3 + e_{\varepsilon_2}\dot{e}_{\varepsilon_2} + e_\alpha\dot{e}_\alpha \tag{25}$$

where $u_3 = -\omega_2 - \omega_3 + \widehat{\varepsilon}_2\gamma\phi(x_1, x_3) - \varepsilon_2\widehat{\alpha}\gamma\phi(x_1, x_3)$ (26)

Let us denote the virtual controller by x_4. It is defined as $x_4 = \beta_3(x_1, \omega_2, \omega_3)$ and we assume that $\beta_3(x_1, \omega_2, \omega_3) = 0$.
The parameters are estimated as

$$\begin{aligned}
\dot{\widehat{\varepsilon}}_2 &= -\omega_3\gamma\phi(x_1, x_3) + e_{\varepsilon_2}\\
\dot{\widehat{\alpha}} &= \omega_3\varepsilon_2\gamma\phi(x_1, x_3) + e_\alpha
\end{aligned} \tag{27}$$

Substitute (26) and (27) into (25). Then we get

$$\dot{V}_3 = -x_1^2 - e_{\sigma_1}^2 - e_\gamma^2 - w_2^2 - e_{\varepsilon_1}^2 - w_3^2 - e_{\varepsilon_2}^2 - e_\alpha^2$$

which is a negative definite function.
Hence by the theory of Lyapunov, it follows that the system provided by Equation (16) is stable.
Now the relation between x_4 and β_3 is defined by

$$\omega_4 = x_4 - \beta_3$$

Consider the subsystem $(x_1, \omega_2, \omega_3, \omega_4)$ provided by

$$\begin{aligned}
\dot{x}_1 &= -e_{\sigma_1}x_1 - \sigma_1\omega_2 - e_\gamma\phi(x_1, x_3) - x_1\\
\dot{\omega}_2 &= -e_{\varepsilon_1}(\sigma_1 x_1 + \sigma_1\omega_2 - x_4) - \omega_2 + \omega_3 + \sigma_1 x_1\\
\dot{\omega}_3 &= \varepsilon_2\omega_4 - e_{\varepsilon_2}\gamma\phi(x_1, x_3) + e_\alpha\varepsilon_2\gamma\phi(x_1, x_3) - \omega_2 - \omega_3\\
\dot{\omega}_4 &= -x_1 - x_2 - x_3 - \sigma_2\omega_4 + u_4
\end{aligned}$$

Now consider the Lyapunov function

$$V_4 = V_3 + \frac{1}{2}\omega_4^2 + \frac{1}{2}e_{\sigma_2}^2$$

The derivative of V_4 with respect to t is obtained as

$$\dot{V}_4 = \dot{V}_3 + \omega_4\dot{\omega}_4 + e_{\sigma_2}\dot{e}_{\sigma_2} \tag{28}$$

where $u_4 = -\varepsilon_2\omega_3 + x_1 + x_2 + x_3 + \widehat{\sigma}_2\omega_4 - \omega_4$ (29)

By working backward, the parameter is estimated as

$$\dot{\widehat{\sigma}}_2 = e_{\sigma_2} - w_4^2 \tag{30}$$

Substitute (29) and (30) into (28). Then we are led to

$$\dot{V}_4 = -x_1^2 - e_{\sigma_1}^2 - e_{\gamma}^2 - w_2^2 - e_{\varepsilon_1}^2 - w_3^2 - e_{\varepsilon_2}^2 - e_{\alpha}^2 - w_4^2 - e_{\sigma_2}^2$$

which is a negative definite function.
By the stability theory due to Lyapunov, it is seen that the Colpitts oscillator provided by Equation (1) is asymptotically stable.

3.3. **Numerical simulation.** For the numerical simulations, the fourth order Runge-Kutta method is used to solve the system using MATLAB ode45.

The initial value of the drive system (16) are taken as

$$x_1(0) = 1.9124, x_2(0) = 1.3942, x_3(0) = 1.3125 \text{ and } x_4(0) = 1.9873.$$

The initial conditions of the parameters are taken as

$$\begin{aligned} \widehat{\sigma}_1(0) &= 10.9546, & \widehat{\sigma}_2(0) &= 5.9353, \\ \widehat{\alpha}(0) &= 3.8765, & \widehat{\gamma}(0) &= 2.1654, \\ \widehat{\varepsilon}_1(0) &= 7.8762, & \widehat{\varepsilon}_2(0) &= 9.9876 \end{aligned}$$

The adaptive backstepping controllers are updated by

$$\begin{aligned} u_1 &= -x_1 + \widehat{\sigma}_1 x_1 - x_4 + \widehat{\gamma}\phi(x_1, x_3) \\ u_2 &= \sigma_1 x_1 + \widehat{\varepsilon}_1(\sigma_1 x_1 + \sigma_1\omega_2 - x_4) + x_3 - \omega_2 \\ u_3 &= -\omega_2 - \omega_3 + \widehat{\varepsilon}_2\gamma\phi(x_1, x_3) - \varepsilon_2\widehat{\alpha}\gamma\phi(x_1, x_3) \\ u_4 &= -\varepsilon_2\omega_3 + x_1 + x_2 + x_3 + \widehat{\sigma}_2\omega_4 - \omega_4 \end{aligned}$$

The parameter values are updated by

$$\begin{aligned} \dot{\widehat{\sigma}}_1 &= -x_1^2 + e_{\sigma_1} \\ \dot{\widehat{\gamma}} &= -x_1\phi(x_1, x_3) + e_{\gamma} \\ \dot{\widehat{\varepsilon}}_1 &= -\omega_2(\sigma_1 x_1 + \sigma_1\omega_2 - x_4) + e_{\varepsilon_1} \\ \dot{\widehat{\varepsilon}}_2 &= -\omega_3\gamma\phi(x_1, x_3) + e_{\varepsilon_2} \\ \dot{\widehat{\alpha}} &= \omega_3\varepsilon_2\gamma\phi(x_1, x_3) + e_{\alpha} \\ \dot{\widehat{\sigma}}_2 &= e_{\sigma_2} - w_4^2 \end{aligned}$$

Figure 5 depicts the parameter estimation of the modified Colpitts oscillator.
Figure 6 depicts the stability of the modified Colpitts oscillator.

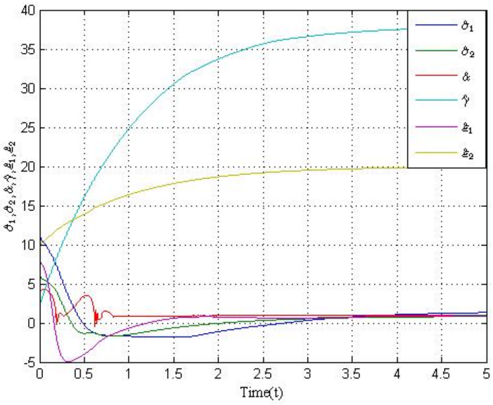


FIGURE 5. The parameter estimation of the modified Colpitts oscillator

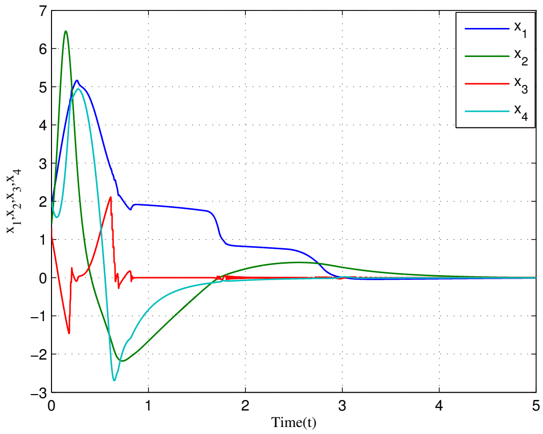


FIGURE 6. The stability of the modified Colpitts oscillator

4. Synchronization of modified chaotic Colpitts oscillator wtih non-linear control

The synchronization of a chaotic system is another way of explaining the sensitivity based on the initial conditions. One has to design *master-slave* or *drive-response* coupling between the two chaotic systems such that the time evolution becomes ideal.

In general, the two dynamic systems involved in the synchronization are called the master and slave systems, respectively. A well-designed controller will make the trajectory of the slave system track and trajectory of the master system, that is, the two systems will be synchronous.

The following sub-section contains the detailed explanation of the synchronization process for the modified Colpitts oscillator using non-linear control.

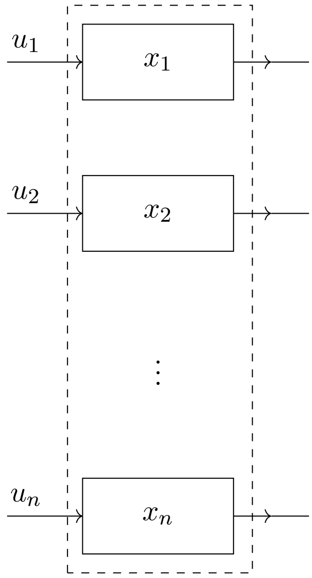


FIGURE 7. Block diagram for non-linear controller

4.1. **Pseudo Algorithm for Synchronization of modified chaotic oscillator using Non-linear Feedback method.** In general, the two dynamics in synchronization are called the master and slave systems respectively. A well designed controller will make the trajectory of slave system track the trajectory of the slave system track the trajectory of the master system.

Consider the chaotic system described by the dynamics

$$\dot{x} = Ax + f(x) \tag{31}$$

where $x \in \mathbb{R}^n$ is the state of the system, A is the $n \times n$ matrix of the system parameters and $f : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the system. The system (31) is considered as the master or drive system.

Consider the slave system with the controller $[u_1,\ u_2,\ u_3, ...u_n]^T$ described by the dynamics

$$\dot{y} = By + f(y) + u \tag{32}$$

where $y \in \mathbb{R}^n$ is the state of the system, B is the $n \times n$ matrix of the system parameters and $g : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear part of the system and $u \in \mathbb{R}^n$ nonlinear part of the slave system. If $A = B$ and $f = g$, then x and y are the states of two identical chaotic systems. If $A \neq B$ or $f \neq g$, then x and y are the states of two different chaotic systems. The chaotic systems (31) and (32) depend not only on state variables but also on time t and the parameters.

The synchronization error is defined as

$$e = y - x. \tag{33}$$

Then the synchronization error dynamics is obtained as

$$\dot{e} = By - Ax + g(y) - f(x) + u. \tag{34}$$

The synchronization error system control a controlled chaotic system with control input $[u_1,\ u_2,\ u_3, ...u_n]$.

Thus, the global synchronization problem is essentially to find a nonlinear feedback controller u so as to stabilize the error dynamics (34) for all initial conditions $e(0) \in \mathbb{R}^n, i.e. \lim_{t\to\infty} \|e(t)\| = 0$ for all initial conditions $e(0) \in \mathbb{R}^n$.

The nonlinear control design uses Lyapunov function methodology for establishing the synchronization of master system (31)and (32). By the Lyapunov function methodology,a candidate Lyapunov function is taken as

$$V(e) = e^T Pe \tag{35}$$

where P is a $n \times n$ positive definite matrix.

Note that $V : \mathbb{R} \to \mathbb{R}$ is a positive definite function by construction. It is assumed that the parameters of master and slave systems are known that the states of both systems (31) and (32) are measurable.

If a controller u can be found such that

$$\dot{V}(e) = -e^T Qe \tag{36}$$

where Q is a positive definite matrix, then $\dot{V}$ is a negative definite function. Hence, by Lyapunov stability theory, the error dynamics (34) is globally exponetially stable and hence the condition $\lim_{t\to\infty} \|e(t)\| = 0$ will be satisfied for all initial conditions $e(0) \in \mathbb{R}^n$. Then the state of the master system (31) and the slave system (32) will be globally exponentially synchronized.

4.2. **Synchronization of modified chaotic Colpitts oscillator using Non-linear Feedback method.** The synchronization of modified Colpitts oscillator is now taken up. The drive-response formalism is utilized. The identical synchronization is elaborated between the modified Colpitts oscillators.

The chaos synchronization basically requires the global asymptotic stability of the error dynamics

$$\text{i.e., } \lim_{t\to\infty} \|e(t)\| = 0.$$

The modified Colpitts oscillator is taken as drive system, which is described by

$$\begin{aligned}
\dot{x}_1 &= \sigma_1 \left(-x_1 - x_2\right) + x_4 - \gamma\phi\left(x_1, x_3\right) \\
\dot{x}_2 &= -\varepsilon_1\sigma_1 x_1 - \varepsilon_1\sigma_1 x_2 + \varepsilon_1 x_4 \\
\dot{x}_3 &= \varepsilon_2 x_4 - \varepsilon_2\left(1-\alpha\right)\gamma\phi\left(x_1, x_3\right) \\
\dot{x}_4 &= -x_1 - x_2 - x_3 - \sigma_2 x_4
\end{aligned} \tag{37}$$

where x_1, x_2, x_3 and x_4 are state variables, σ_1, σ_2, ε_1, ε_2, γ, α are positive parameters and $\phi\left(x_1, x_3\right) = \frac{2a}{\pi}\sin^{-1}\left(\sin\left(\frac{2\pi}{p}(x_1 + x_3)\right)\right)$.

The modified Colpitts oscillator is also taken as the response system which is described by

$$\begin{aligned}
\dot{y}_1 &= \sigma_1 \left(-y_1 - y_2\right) + y_4 - \gamma\phi\left(y_1, y_3\right) + u_1 \\
\dot{y}_2 &= -\varepsilon_1\sigma_1 y_1 - \varepsilon_1\sigma_1 y_2 + \varepsilon_1 y_4 + u_2 \\
\dot{y}_3 &= \varepsilon_2 y_4 - \varepsilon_2\left(1-\alpha\right)\gamma\phi\left(y_1, y_3\right) + u_3 \\
\dot{y}_4 &= -y_1 - y_2 - y_3 - \sigma_2 y_4 + u_4
\end{aligned} \tag{38}$$

where $\phi\left(y_1, y_3\right) = \frac{2a}{\pi}\sin^{-1}\left(\sin\left(\frac{2\pi}{p}(y_1 + y_3)\right)\right)$.

The synchronization error occurring in the system is defined by

$$e_i = y_i - x_i, i = 1, 2, 3, 4 \tag{39}$$

The resulting error dynamics of the system is governed by the set of equations

$$\begin{aligned} \dot{e}_1 &= -\sigma_1 e_1 - \sigma_1 e_2 + e_4 - \gamma\phi(y_1, y_3) + \gamma\phi(x_1, x_3) + u_1 \\ \dot{e}_2 &= -\varepsilon_1\sigma_1 e_1 - \varepsilon_1\sigma_1 e_2 + \varepsilon_1 e_4 + u_2 \\ \dot{e}_3 &= \varepsilon_2 e_4 - \varepsilon_2(1-\alpha)\gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) + u_3 \\ \dot{e}_4 &= -e_1 - e_2 - e_3 - \sigma_2 e_4 + u_4 \end{aligned} \tag{40}$$

where $u = (u_1, u_2, u_3, u_4)^T$ is the non-linear controller to be designed so as to synchronize the states of identically modified Colpitts oscillator.

Now the objective is to find the control law u_i, $i = 1, 2, 3, 4$ for stabilizing the error variable of the system (40) at the origin.

Let the energy source function Lyapunov be chosen as

$$V = \frac{1}{2}\sum_{i=1}^{4} e_i^2 \tag{41}$$

The derivative of (41) with respect to t is provided by

$$\dot{V} = \sum_{i=1}^{4} e_i \dot{e}_i \tag{42}$$

Substituting (39) and (40) into (42) we are led to the relation

$$\begin{aligned} \dot{V} = & e_1(-\sigma_1 e_1 - \sigma_1 e_2 + e_4 - \gamma\phi(y_1, y_3) + \gamma\phi(x_1, x_3) + u_1) \\ & + e_2(-\varepsilon_1\sigma_1 e_1 - \varepsilon_1\sigma_1 e_2 + \varepsilon_1 e_4 + u_2) \\ & + e_3(\varepsilon_2 e_4 - \varepsilon_2(1-\alpha)\gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) + u_3) \\ & + e_4(-e_1 - e_2 - e_3 - \sigma_2 e_4 + u_4) \end{aligned}$$

The controllers are defined by

$$\begin{aligned} u_1 &= \sigma_1 e_2 - e_4 + \gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) \\ u_2 &= \varepsilon_1\sigma_1 e_1 - \varepsilon_1 e_4 \\ u_3 &= \varepsilon_2(1-\alpha)\gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) - \varepsilon_2 e_4 - e_3 \\ u_4 &= e_1 + e_2 + e_3 \end{aligned} \tag{43}$$

Therefore the relation (42) becomes

$$\dot{V} = -\sigma_1 e_1^2 - \varepsilon_1\sigma_1 e_2^2 - e_3^2 - \sigma_2 e_4^2$$

which is a negative definite function.

Thus, by Lyapunov stability theory, the error dynamics provided by (40) is found to be globally asymptotically stable for all initial conditions $e(0) \in R^4$.

Thus, the states of the drive and response system synchronize globally and asymptotically.

4.3. **Numerical simulation.** For the numerical simulations, the fourth order Runge-Kutta method is used to solve the system using MATLAB ode45.

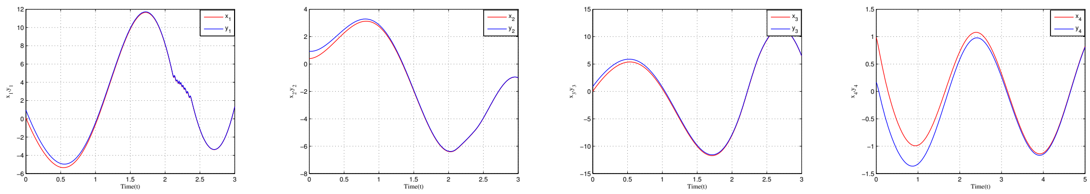
The initial value of the drive system (37) are taken as

$$x_1(0) = 0.09124, x_2(0) = 0.3942, x_3(0) = 0.0125 \text{ and } x_4(0) = 0.9823$$

and inital value of the response system (38) are taken as

$$y_1(0) = 0.9546, y_2(0) = 0.9353, y_3(0) = 0.8765 \text{ and } y_4(0) = 0.1654.$$

The non-linear controllers are updated by (43).



(A) x_1 and y_1 (B) x_2 and y_2 (C) x_3 and y_3 (D) x_4 and y_4

FIGURE 8. Synchronization of the Modified Colpitts oscillator

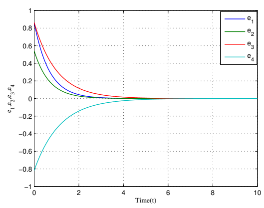


FIGURE 9. Error Dynamics of Chaotic Colspitts oscillator

5. The synchronization of Colpitts oscillator via Backstepping Control

The backstepping technique is a cyclic procedure through a suitable Lyapunov function along with a feedback controller. It leads to the global stability synchronization of the strict feedback chaotic systems. In this section, the backward backstepping method is employed for the proposed system.

In general, the two dynamics in synchronization are called the *master* and *slave* systems respectively. A well designed controller will make the trajectory of slave system track the trajectory of the master system.

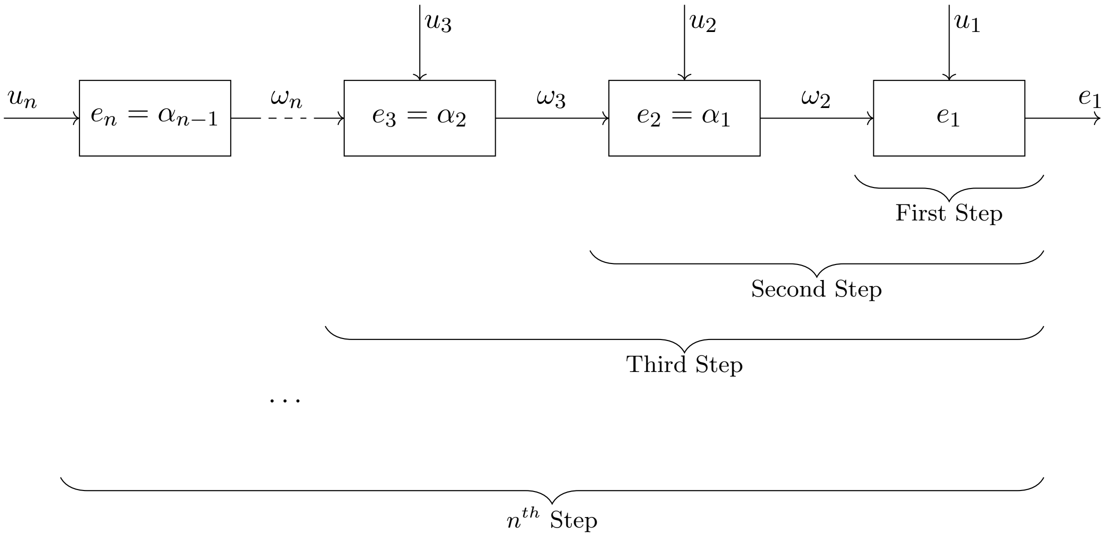


FIGURE 10. Block diagram for n-step backstepping control

Consider the chaotic system described by the dynamics

$$\begin{aligned} \dot{x}_1 &= F_1(x_1,\ x_2...x_n), \\ \dot{x}_2 &= F_2(x_1,\ x_2...x_n), \\ \vdots \quad &\vdots \quad \vdots \\ \dot{x}_n &= F_n(x_1,\ x_2...x_n), \end{aligned} \tag{44}$$

where $x \in \mathbb{R}^n$ is the state of the system. The system (44) is considered as the *master* system.

The *slave* system is a chaotic system with the controller $u = [u_1,\ u_2,\ u_3...u_n]^T$ described by the dynamics

$$\begin{aligned} \dot{y}_1 &= G_1(y_1,\ y_2...y_n) + u_1, \\ \dot{y}_2 &= G_2(y_1,\ y_2...y_n) + u_2, \\ \vdots \quad &\vdots \quad \vdots \\ \dot{y}_n &= G_n(y_1,\ y_2...y_n) + u_n, \end{aligned} \tag{45}$$

where $y \in \mathbb{R}^n$ is the state of the slave system and $F_i,\ G_i(i = 1,\ 2,\ 3...n)$ linear or nonlinear functions with input from systems (44) and (45).

If $F_i = G_i$ for all i, then the system (44) and (45) are called *identical* and otherwise the are *non-identical* chaotic systems.

The synchronization error is defined as

$$e_i = y_i - x_i,\ \text{i=1, 2,}\ \ \text{3...n}. \tag{46}$$

Then the synchronization error dynamics is obtained as,

$$\begin{aligned} \dot{e}_1 &= G_1(y_1,\ y_2...y_n) - F_1(x_1,\ x_2...x_n) + u_1, \\ \dot{e}_2 &= G_2(y_1,\ y_2...y_n) - F_2(x_1,\ x_2...x_n) + u_2, \\ \vdots \quad &\vdots \quad \vdots \\ \dot{e}_n &= G_n(y_1,\ y_2...y_n) - F_n(x_1,\ x_2...x_n) + u_n \end{aligned} \tag{47}$$

The chaos synchronization problem basically requires the global asymptotically stability of the error dynamics (47), i.e.

$$\lim_{t\to\infty} \|e(t)\| = 0 \tag{48}$$

for all initial conditions $e(0) \in \mathbb{R}^n$.

Backstepping design procedure is recursive and guarantee global stability performance of strict-feedback chaotic systems. By using the backstepping design, at the i^{th} step, the i^{th} order subsystem is stabilized with respect to a Lyapunov function V_i, by the virtual control α_i and a control input function u_i.

Consider the global asymptotic stability of the system

$$\dot{e}_1 = G_1(y_1,\ y_2...y_n) - F_1(x_1,\ x_2...x_n) + u_1, \tag{49}$$

where u_1 is control input, which is the function of the error vector e_i, and the state variables $x(t) \in \mathbb{R}^n,\ y(t) \in \mathbb{R}^n$. As long as this feedback stabilizes, the system (49) will converge to zero as $t \to \infty$, where $e_2 = \alpha_1(e_1)$ is regarded as a virtual controller.

For the design of $\alpha_1(e_1)$ to stabilize the subsystem (49), we consider the Lyapunov function defined by

$$V_1(e) = e_1^T P_1 e_1, \tag{50}$$

where P_1 is a positive definite matrix.

Suppose the derivative of V_1 is

$$\dot{V}_1 = -e_1^T Q_1 e_1, \tag{51}$$

where Q_1 is a positive definite matrix.

Then $\dot{V}_1$ is a negative definite function.

Thus by a Lyapunov stability theory, the error dynamics(49) is globally asymptotically stable.

The function $\alpha_1(e_1)$ is an estimative function when e_2 is considered as a controller.

The error between e_2 and $\alpha_1(e_1)$ is

$$w_2 = e_2 - \alpha_1(e_1) \tag{52}$$

Considering $(e_1,\ w_2)$ subsystem given by

$$\begin{aligned} \dot{e}_1 &= G_1(y_1,\ y_2...y_n) - F_1(x_1,\ x_2...x_n) + u_1, \\ \dot{w}_2 &= G_2(y_1,\ y_2...y_n) - F_2(x_1,\ x_2...x_n) - \dot{\alpha}_1(e_1) + u_2. \end{aligned} \tag{53}$$

Let e_3 as a virtual controller in system (53), assume that when

$$e_3 = \alpha_2(e_1,\ w_2). \tag{54}$$

The system (53) is made globally asymptotically stable.

Consider the Lyapunov function defined by

$$V_2(e_2,\ w_2) = V_1(e_1) + w_2^T P_2 w_2 \tag{55}$$

where P_2 is a positive definite matrix.

Suppose the derivative of $V_2(e_1,\ w_2)$ is

$$\dot{V}_2 = -e_1^T Q_1 e_1 - w_2^T Q_2 w_2, \tag{56}$$

where $Q_1,\ Q_2$ are a positive definite matrix.

Then $\dot{V}_2(e_1,\ w_2)$ is a negative definite function.

Thus by a Lyapunov stability theory, the error dynamics (53) is globally asymptotically stable. The virtual controller $e_3 = \alpha_2(e_1,\ w_2)$ and the state feedback input u_2 makes the system (53) asymptotically stable.

For the n^{th} state of the error dynamics, define the error variable w_n as

$$w_n = e_n - \alpha_{n-1}(e_1,\ w_2,\ w_3....w_{n-1}) \tag{57}$$

Considering the $(e_1,\ w_2,\ w_3....w_n)$ subsystem given by

$$\begin{array}{lcl} \dot{e}_1 & = & G_1(y_1,\ y_2...y_n) - F_1(x_1,\ x_2...x_n) + u_1, \\ \dot{w}_2 & = & G_2(y_1,\ y_2...y_n) - F_2(x_1,\ x_2...x_n) - \dot{\alpha}_1(e_1) + u_2 \\ \vdots & \vdots & \vdots \\ \dot{w}_n & = & G_n(y_1,\ y_2...y_n) - F_n(x_1,\ x_2...x_n) - \dot{\alpha}_{n-1}(e_1,\ w_2,\ ...w_{n-1}) + u_2 \end{array} \tag{58}$$

Consider the Lyapunov function defined by

$$V_n(e_2,\ w_2,\ ...w_n) = V_{n-1}(e_1,\ w_2...w_{n-1}) + w_n^T P_n w_n \tag{59}$$

where P_n is a positive definite matrices.

Suppose the derivative of $V_n(e_1,\ w_2,\ w_3....w_n)$ is

$$\dot{V}_n(e_1,\ w_2,\ ...w_n) = -e_1^T Q_1 e_1 - w_2^T Q_2 w_2... - w_n^T Q_n w_n \tag{60}$$

where $Q_1,\ Q_2...Q_n$ are a positive definite matrix.

Then $\dot{V}_2(e_1,\ w_2,\ w_3....w_n)$ is a negative definite function on $\mathbb{R}^n$.

Thus by a Lyapunov stability theory, the error dynamics (58) is globally asymptotically stable. The virtual controller

$$e_n = \alpha_{n-1}(e_1,\ w_2,\ ...w_{n-1}), \tag{61}$$

and the state feedback input u_n makes the system (58) asymptotically stable.

Hence the state of the master and slave systems are globally and asymptotically synchronized.

5.1. **The synchronization of Colpitts oscillator via Backstepping Control.** The error dynamics system is taken as

$$\begin{aligned} \dot{e}_4 &= -e_1 - e_2 - e_3 - \sigma_2 e_4 + u_1 \\ \dot{e}_3 &= \varepsilon_2 e_4 - \varepsilon_2 (1-\alpha)\gamma(\phi(y_1,y_3) - \phi(x_1,x_3)) + u_2 \\ \dot{e}_2 &= -\varepsilon_1\sigma_1 e_1 - \varepsilon_1\sigma_1 e_2 + \varepsilon_1 e_4 + u_3 \\ \dot{e}_1 &= -\sigma_1 e_1 - \sigma_1 e_2 + e_4 - \gamma(\phi(y_1,y_3) - \phi(x_1,x_3)) + u_4 \end{aligned} \tag{62}$$

Now the objective is to find the control laws $u_i (i = 1,2,3,4)$ for stabilizing the error variables of the system (62) at the origin.

First consider the stability of the system

$$\dot{e}_4 = -e_1 - e_2 - e_3 - \sigma_2 e_4 + u_1 \tag{63}$$

where e_3 is considered as virtual controller provided by

$$e_3 = \beta_1(e_4) \text{ and } \beta_1(e_4) = 0$$

The Lyapunov function is defined as

$$V_1 = \frac{1}{2} e_4^2 \tag{64}$$

The derivative of V_1 with respect to t is obtained as

$$\dot{V}_1 = e_4 \dot{e}_4 \tag{65}$$

If $\beta_1 = 0$ and $u_1 = e_1 + e_2$, then we obtain

$$\dot{V}_1 = -\sigma_2 e_4^2 \tag{66}$$

which is a negative definite function.

Hence the system (63) is globally asymptotically stable.

The function $\beta_1(e_4)$ is an estimator when e_3 is considered as virtual controller.

The relation between e_3 and β_1 is defined by

$$\omega_2 = e_3 - \beta_1 = e_3$$

Consider the subsystem (e_4, ω_2) given by

$$\begin{aligned} \dot{e}_4 &= -\omega_2 - \sigma_2 e_4 \\ \dot{\omega}_2 &= \varepsilon_2 e_4 - \varepsilon_2 (1-\alpha)\gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) + u_2 \end{aligned} \tag{67}$$

Let e_2 be a virtual controller in system (67).

Assume that when $e_2 = \beta_2(e_4, \omega_2)$, the system (67) is rendered globally asymptotically stable.

Consider the Lyapunov function defined by

$$V_2 = V_1 + \frac{1}{2}\omega_2^2$$

The derivative of V_2 with respect to t is

$$\dot{V}_2 = e_4\dot{e}_4 + \omega_2\dot{\omega}_2$$

If $\beta_2 = 0$ and $u_2 = -(\varepsilon_2 - 1)e_4 + \varepsilon_2(1-\alpha)\gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) + e_2 - \omega_2$, then we obtain

$$\dot{V}_2 = -\sigma_2 e_4^2 - \omega_2^2$$

which is a negative definite function.

Hence by Lyapunov stability theory, the system is stable.

Let us consider the relation between e_2 and β_2 defined by

$$\omega_3 = e_2 - \beta_2 = e_2$$

Now the subsystem $(e_4, \omega_2, \omega_3)$ is considered as

$$\begin{aligned} \dot{e}_4 &= -\omega_2 - \sigma_2 e_4 \\ \dot{\omega}_2 &= e_4 + \omega_3 - \omega_2 \\ \dot{\omega}_3 &= -\varepsilon_1\sigma_1 e_1 - \varepsilon_1\sigma_1\omega_3 + \varepsilon_1 e_4 + u_3 \end{aligned} \tag{68}$$

Consider the function V_3 due to Lyapunov function defined by

$$V_3 = V_2 + \frac{1}{2}\omega_3^2$$

On differentiating V_3 with respect to t, we get

$$\dot{V}_3 = e_4\dot{e}_4 + \omega_2\dot{\omega}_2 + \omega_3\dot{\omega}_3$$

If $\beta_3 = 0$ and $u_3 = -\omega_2 - \varepsilon_1 e_4$, then we obtain

$$\dot{V}_3 = -\sigma_2 e_4^2 - \omega_2^2 - \varepsilon_1\sigma_1\omega_3^2$$

which is a negative definite function.

Now the relation between e_1 and β_3 is defined as

$$\omega_4 = e_1 - \beta_3 = e_1$$

Let us consider the subsystem $(e_4, \omega_2, \omega_3, \omega_4)$ provided by

$$\begin{aligned} \dot{e}_4 &= -\omega_2 - \sigma_2 e_4 \\ \dot{\omega}_2 &= e_4 + \omega_3 - \omega_2 \\ \dot{\omega}_3 &= -\varepsilon_1\sigma_1\omega_4 - \varepsilon_1\sigma_1\omega_3 - \omega_2 \\ \dot{\omega}_4 &= -\sigma_1\omega_4 - \sigma_1\omega_3 + e_4 - \gamma(\phi(y_1, y_3) - \phi(x_1, x_3)) + u_4 \end{aligned} \tag{69}$$

Consider the Lyapunov function

$$V_4 = V_3 + \frac{1}{2}\omega_4^2$$

The derivative of V_4 with respect to t is

$$\dot{V}_4 = e_4\dot{e}_4 + \omega_2\dot{\omega}_2 + \omega_3\dot{\omega}_3 + \omega_4\dot{\omega}_4$$

If $\beta_4 = 0$ and $u_4 = \varepsilon_1\sigma_1\omega_3 + \sigma_1\omega_3 - e_4 + \gamma\left(\phi\left(y_1, y_3\right) - \phi\left(x_1, x_3\right)\right)$, then we obtain

$$\dot{V}_4 = -\sigma_2 e_4^2 - \omega_2^2 - \varepsilon_1\sigma_1\omega_3^2 - \sigma_1\omega_4^2$$

which is a negative definite function.

Hence by Lyapunov stability theory, the system is stable.

5.2. **Numerical simulation.** For the numerical simulations, the fourth order Runge-Kutta method is used to solve the system using MATLAB ode45.

The initial value of the drive system (37) are taken as

$$x_1(0) = 0.09124, x_2(0) = 0.3942, x_3(0) = 0.0125 \text{ and } x_4(0) = 0.9823$$

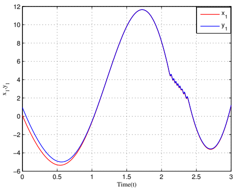
and inital value of the response system (38) are taken as

$$y_1(0) = 0.9546, y_2(0) = 0.9353, y_3(0) = 0.8765 \text{ and } y_4(0) = 0.1654.$$

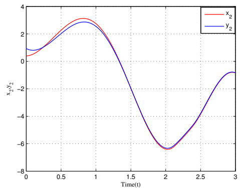
The backstepping controllers of (62) are updated by

$$\begin{aligned} u_1 &= e_1 + e_2 \\ u_2 &= -(\varepsilon_2 - 1)e_4 + \varepsilon_2(1-\alpha)\gamma\left(\phi\left(y_1, y_3\right) - \phi\left(x_1, x_3\right)\right) + e_2 - \omega_2 \\ u_3 &= -\omega_2 - \varepsilon_1 e_4 \\ \text{and } u_4 &= \varepsilon_1\sigma_1\omega_3 + \sigma_1\omega_3 - e_4 + \gamma\left(\phi\left(y_1, y_3\right) - \phi\left(x_1, x_3\right)\right) \end{aligned}$$

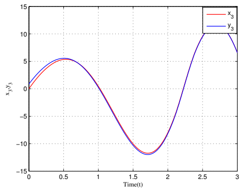
Figure 11 portrays the chaos synchronization of identical drive and response systems provided by Equations (37) and (38), respectively.



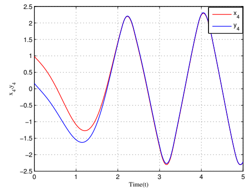
(A) x_1 and y_1



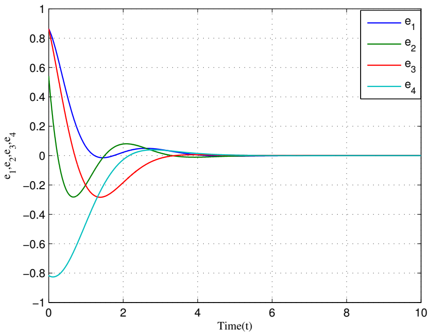
(B) x_2 and y_2



(C) x_3 and y_3



(D) x_4 and y_4



(E) Error Dynamics of modified Colpitts oscillator

FIGURE 11 Synchronization of identical modified Colpitts oscillator, error plot for identical modified Colpitts oscillator

6. Circuit Implementation

In order to verify the dynamical properties of the modified Colpitts oscillator, an operational amplifier circuit is designed in accordance with the equation (1). The circuit is designed by linear resistance and linear capacitors. The allowable voltage range of operational amplifiers leads to the appropriate variables proportional compression transformation to the state variables of the system. According to the circuit diagrams, the corresponding oscillation circuit equation is described as follows

$$\begin{aligned}
\dot{x}_1 &= \sigma_1(-x_1 - x_2) + x_4 - \gamma\phi(x_1, x_3)\\
\dot{x}_2 &= \varepsilon_1\sigma_1(-x_1 - x_2) + \varepsilon_1 x_4\\
\dot{x}_3 &= \varepsilon_2(x_4 - (1-\alpha)\gamma\phi(x_1, x_3))\\
\dot{x}_4 &= -x_1 - x_2 - x_3 - \sigma_2 x_4
\end{aligned}$$

where $\phi(x_1, x_3) = \dfrac{2a}{\pi}\sin^{-1}\left(\sin\left(\dfrac{2\pi}{p}(x_1 + x_3)\right)\right)$ and the parameter values are

$$\sigma_1 = \frac{R_2\,(R_5 + R_8)}{R_5 R_1 C_1 R_3\,(R_6 + R_7)} = \frac{R_{36}\,(R_{31} + R_{32})}{R_{37} R_{31}\,(R_{34} + R_{35})}, \quad \sigma_2 = \frac{R_{64} R_{76} R_{78}}{R_{63} C_4 R_{65} R_{75} R_{77}},$$

$$\varepsilon_1 = \frac{R_{28} R_{37}}{R_{27} C_2 R_{29} R_{36}}, \qquad \varepsilon_2 = \frac{R_{42} R_{46}}{R_{41} C_3 R_{43} R_{45}},$$

$$\gamma = \frac{R_2 R_{20}}{R_1 C_1 R_3 R_{17}} = \frac{R_{58}}{R_{55}}, \qquad \alpha = \frac{R_{46} - R_{45}}{R_{46}}$$

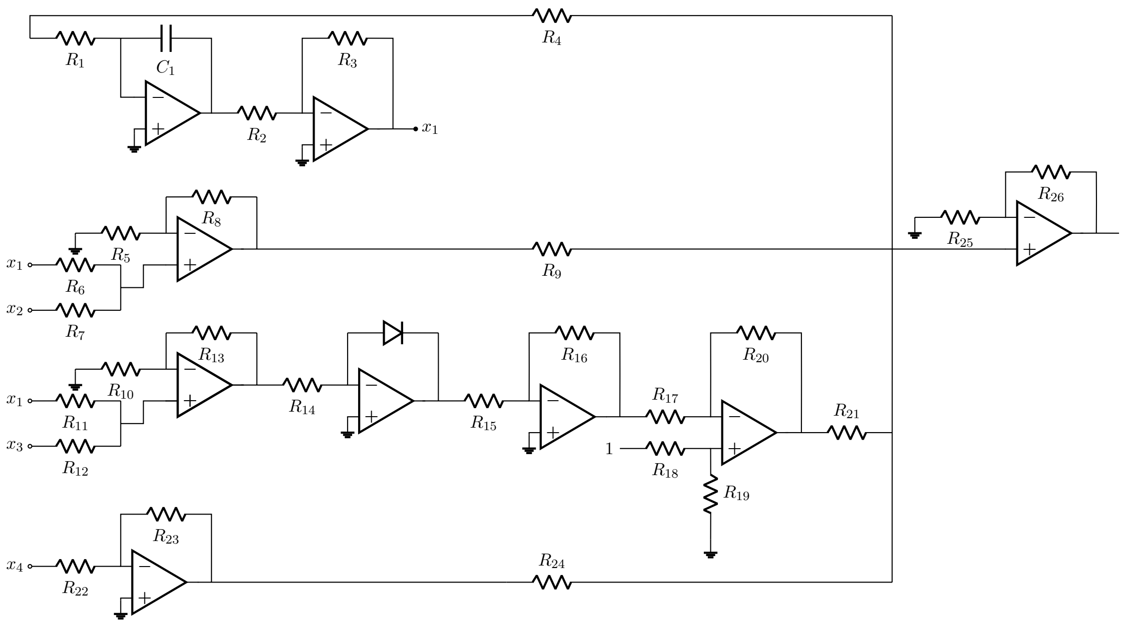


FIGURE 12. Op Amp Circuit diagram of chaotic variable x_1

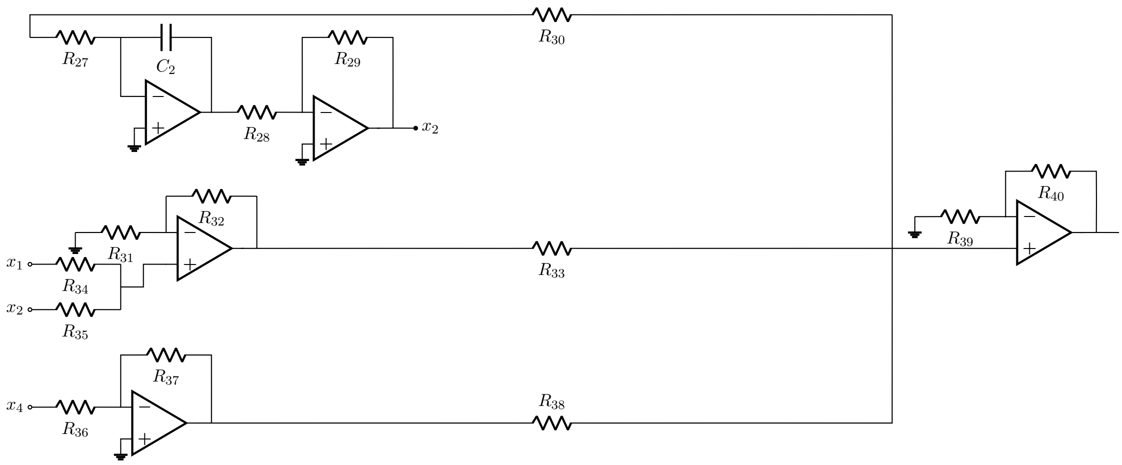


FIGURE 13. Op Amp Circuit diagram of chaotic variable x_2

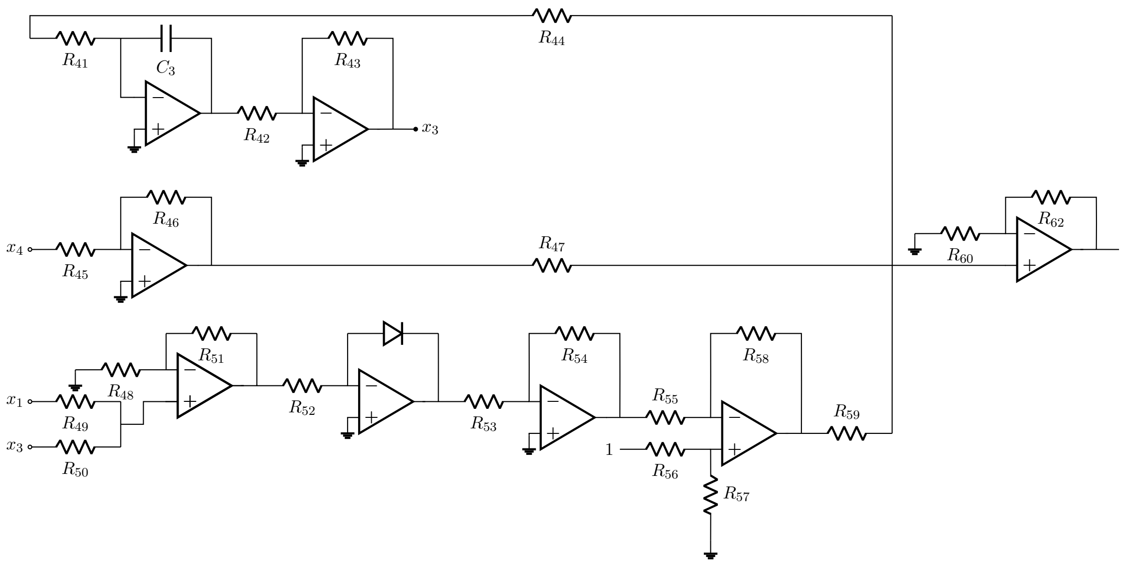


FIGURE 14. Op Amp Circuit diagram of chaotic variable x_3

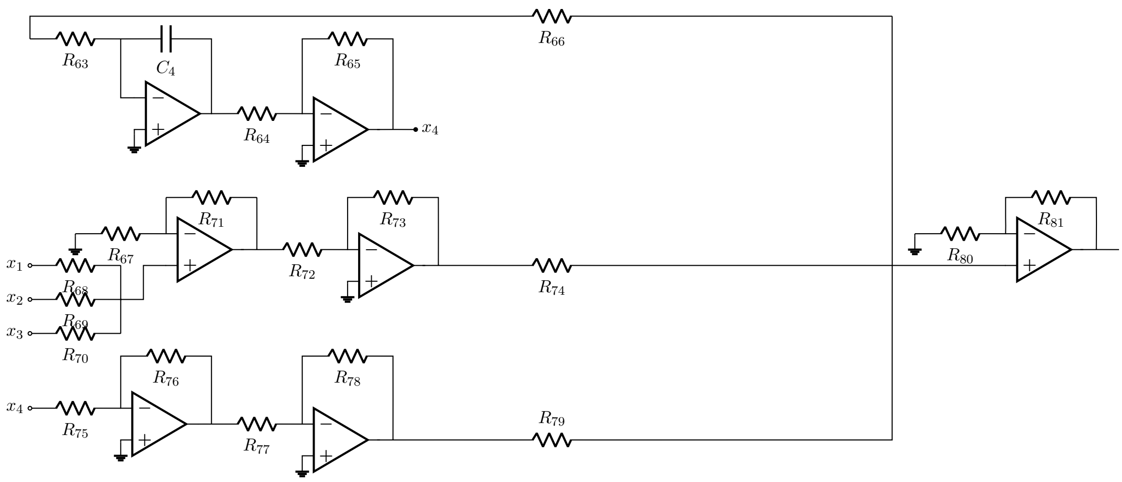


FIGURE 15. Op Amp Circuit diagram of chaotic variable x_4

7. CONCLUSIONS

In this paper, the Colpitts oscillator with triangular wave non-linearity is analyzed. The qualitative properties of the modified Colpitts oscillator is analyzed in this study. It exhibits the chaotic and hyperchaotic nature for some specified initial conditions and parameters. By Wolf method, the Lyapunov exponent's is calculated. For some initial conditions, it exhibits the dissipative nature. The adaptive backstepping control technique is used to control the system. Synchronization, the non-linear and backstepping control are utilized. Numerical simulations support the results. MATLAB is used for numerical simulation.

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