Mapana – Journal of Sciences
2023, Vol. 22., No. 02, *1-14*
ISSN 0975-3303 | https://doi.org/10.12723/mjs.65.1

Coupling Distance in Graphs

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Abstract:

In this paper, the coupling distance of simple connected graphs is introduced. Different parameters of coupling distance like coupling eccentricity, coupling radius, coupling diameter, coupling center, and coupling periphery are defined. The coupling parameters for different standard graphs are obtained.

Keywords: coupling distance, coupling eccentricity, coupling radius, coupling diameter.

1. Introduction

With the advent of connected networks, graph theory is no more limited only to showing interconnections between entities. Researchers have explored different possibilities with the study of different distance concepts. There are many types of distances in graphs, the eccentric distance being the shortest distance between any two vertices. Distance in graphs by F Buckley and F Harary[1] gives an insight into distance concepts in graphs. The distance concepts like superior distance[5], signal distance[6], detour distance[2], and D-distance[9] are the inspiration for this work.

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In this paper, we introduce coupling distance in graphs and study some of its properties. The coupling distance between any two vertices is the summation of the length of the shortest path between every pair of vertices added to the total number of vertices present on the path. Theorems related to the relation between coupling distance parameters like coupling radius, coupling diameter, coupling center, and coupling periphery are stated and proved. In this paper, only non-trivial, finite, undirected simple, and connected graphs are considered. For undefined terminologies refer to the book Graph Theory by Harary[3].

2. Preliminaries

Definition 2.1: A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, v_3 \dots v_n\}$ called vertices and another set $E = \{e_1, e_2, e_3 \dots e_m\}$ whose elements are called edges, such that each edge $'e_k'$ is associated with a pair of vertices (v_i, v_j).

Definition 2.2 [3]: The order and size of G are given by $|V(G)| = n$ and $|E(G)| = m$ respectively.

Definition 2.3 [3]: The distance $d(u, v)$ between vertices u and v is the minimum number of edges in a $u - v$ path.

Definition 2.4 [4]: The eccentricity of $u \in V(G)$ is $ecc(u) = max\{d(u, v): v \in V(G)\}$. The radius of G is $rad(G) = min\{ecc(u): u \in V(G)\}$ and the diameter of G is $diam(G) = max\{ecc(u): u \in V(G)\}$.

Definition 2.5 [7,8]: The friendship graph F_n can be constructed by joining n copies of the cycle C_3 with a common vertex. F_n has $2n + 1$ vertices and $3n$ edges.

3. Coupling Distance in Graphs

In this section, coupling distance is introduced and its parameters like coupling eccentricity, coupling radius, and coupling diameter are also defined.

Definition 3.1: The coupling distance between any two vertices in a graph is defined

as $\mathfrak{C}_d(u, v) = d(u, v) + \sum_{w \in Pd(u,v)} |w|$,where $d(u, v)$ is the geodesic distance between u and v and $Pd(u, v)$ is the geodesic path between u and v.

Example 3.1: Consider the graph given in Figure-1

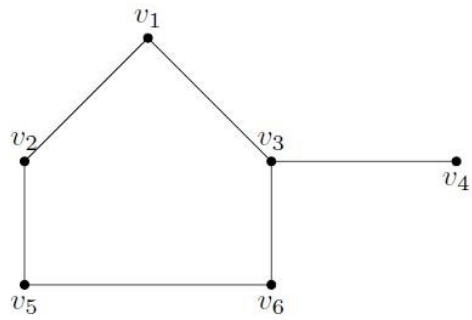


Figure-1

The shortest distance between v_1 and v_2 is 1 and the number of vertices on the path is 2. Therefore, the coupling distance is $1 + 2 = 3 = \mathfrak{C}_d(v_1, v_2)$. There are two paths between v_2 and v_3, the first path is $v_2 - v_1 - v_3$ and the second path is $v_2 - v_5 - v_6 - v_3$. The first path is shorter than the second path. Therefore, the coupling distance between v_2 and v_3 is $2 + 3 = 5 = \mathfrak{C}_d(v_2, v_3)$. Similarly, $\mathfrak{C}_d(v_2, v_4) = 7$. Symbolically, $\mathfrak{C}_d(v_1, v_2) = d(v_1, v_2) + \sum_{w \in Pd(v_1,v_2)} |w| = 1 + 2 = 3$.

$\mathfrak{C}_d(v_2, v_3) = d(v_2, v_3) + \sum_{w \in Pd(v_2,v_3)} |w| = 2 + 3 = 5$.

$\mathfrak{C}_d(v_2, v_4) = d(v_2, v_4) + \sum_{w \in Pd(v_2,v_4)} |w| = 3 + 4 = 7$.

Observation 3.1:

(i) For any graph G, let $u, v \in V(G)$. If $d(u, v) = n$ then $\mathfrak{C}_d(u, v) = 2n + 1$.

(ii) For a connected graph G, $\mathfrak{C}_d(u, v) > d(u, v)$.

(iii) For any graph G, $\mathfrak{C}_d(u, v) \geq 3, \forall\, u, v \in V(G), u \neq v$.

(iv) For any two vertices in a complete graph K_n, $\mathfrak{C}_d(u, v) = 3$.

(v) For any graph G, $\mathfrak{C}_d(u, v) = 0$ if G is disconnected or if $u = v$.

(vi) For any two vertices $u, v \in V(G)$ with $|V(G)| = p$, then $0 \leq d(u, v) < \mathfrak{C}_d(u, v)$.

4. Results on Coupling Distance in Graphs

Observation 4.1: The coupling distance is symmetric, therefore $\mathfrak{C}_d(u,v) = \mathfrak{C}_d(v,u)$.

Theorem 4.2: Let $P: u_1, u_2, \dots, u_n$ be a shortest path in a graph G, then $\mathfrak{C}_d(u_1,u_n) = \mathfrak{C}_d(u_1,u_m) + \mathfrak{C}_d(u_m,u_n)$, where $1 < m < n$.

Proof: Let G be a connected graph. By the definition of coupling distance, we know that $\mathfrak{C}_d(u,v) = d(u,v) + \sum_{w \in Pd(u,v)} |w|$. Let $u_1, u_2, \dots, u_n$ be the vertices of graph G. Let $u_1 - u_m$ and $u_m - u_n$ be the shortest paths in G.

$\mathfrak{C}_d(u_1,u_m) = d(u_1,u_m) + \sum_{z \in Pd(u_1,u_m)} |z| = (m-1) + m = 2m - 1$ and

$\mathfrak{C}_d(u_m,u_n) = d(u_m,u_n) + \sum_{z \in Pd(u_m,u_n)} |z| = (n-m) + \sum_{z \in Pd(u_m,u_n)} |z| = (n-m) + (n-m+1) = 2n - 2m + 1.$

$$\mathfrak{C}_d(u_1,u_m) = \mathfrak{C}_d(u_m,u_n) = 2m - 1 + 2n - 2m + 1 = 2n \qquad (1)$$

$$\mathfrak{C}_d(u_1,u_n) = d(u_1,u_n) + \sum_{z \in Pd(u_1,u_n)} |z| = (n-1) + n = 2n - 1 \qquad (2)$$

From equation (1) and (2), we obtain $\mathfrak{C}_d(u_1,u_n) = \mathfrak{C}_d(u_1,u_m) + \mathfrak{C}_d(u_m,u_n)$.

Theorem 4.3: For any graph G, and for any $u \neq v$, ~~and~~ $\mathfrak{C}_d(u,v) = 3$ if and only if there exists atmost one edge between u and v.

Proof: If $u, v \in V(G)$ and $(u,v) \in E(G)$ then $d(u,v) = 1$. Therefore $\mathfrak{C}_d(u,v) = d(u,v) + \sum_{w \in Pd(u,v)} |w| = 1 + 2 = 3$. Conversely, by the definition of coupling distance we have $d(u,v) + \sum_{w \in Pd(u,v)} |w| = 3$. Every path contains two end points. Hence the value of $\sum_{w \in Pd(u,v)} |w| = 2$ and $d(u,v) = 1$ is the only combination for which $\mathfrak{C}_d(u,v) = 3$. This implies there exists at most one edge between u and v.

5. $\mathfrak{C}$-Eccentricity

Definition 5.1: Let u be the vertex in a connected graph G. The coupling eccentricity or $\mathfrak{C}$-eccentricity of u is the coupling distance to the vertex farthest from u. Thus $\mathfrak{C}e(u) = max\{\mathfrak{C}_d(u, v): v \in V\}$. A $\mathfrak{C}$-eccentric vertex $\mathfrak{C}E(u) = \{v \in V: \mathfrak{C}_d(u, v) = \mathfrak{C}e(u)\}$.

Definition 5.2: The $\mathfrak{C}$-radius (coupling radius) and $\mathfrak{C}$-diameter (coupling diameter) is defined by $\mathfrak{C}rad(G) = min\{\mathfrak{C}e(u): u \in V\}$ and $\mathfrak{C}diam(G) = max\{\mathfrak{C}e(u): u \in V\}$ respectively. u is called a coupling central vertex if $\mathfrak{C}e(u) = \mathfrak{C}rad(G)$. v is called a coupling peripheral vertex if $\mathfrak{C}e(v) = \mathfrak{C}diam(G)$. The coupling center of G $\mathfrak{C}R(G)$ is the set of all coupling central vertices. The coupling periphery of G $\mathfrak{C}P(G)$ is the set of all coupling peripheral vertices. A graph G is said to be coupling self-centered if and only if $\mathfrak{C}rad(G) = \mathfrak{C}diam(G)$.

Table 5.1: From the Figure-1, we tabulate $\mathfrak{C}$-eccentricity and $\mathfrak{C}$-eccentric vertex of v.

Vertex $v \in V(G)$	$\mathfrak{C}$-eccentricity $\mathfrak{C}e(v)$	$\mathfrak{C}$-eccentric vertex $\mathfrak{C}E(v)$
v_1	5	$\{v_4, v_5, v_6\}$
v_2	7	$\{v_4\}$
v_3	5	$\{v_2, v_5\}$
v_4	7	$\{v_2, v_5\}$
v_5	7	$\{v_4\}$
v_6	5	$\{v_1, v_2, v_4\}$

$\mathfrak{C}rad(G) = 5$, $\mathfrak{C}diam(G) = 7$, $\mathfrak{C}R(G) = \{v_1, v_3, v_6\}$ and $\mathfrak{C}P(G) = \{v_2, v_4, v_5\}$.

Observation 5.1:

(i) $ecc(u) < \mathfrak{C}e(u)$ for any vertex $u \in V(G)$.

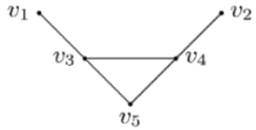
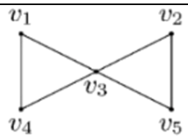
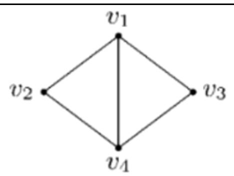
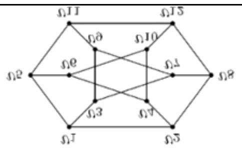
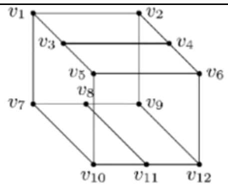
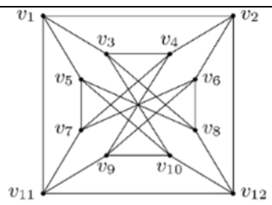
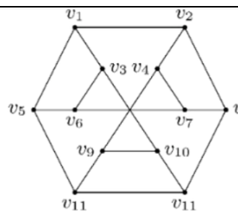
(ii) $rad(G) < \mathfrak{C}rad(G)$.

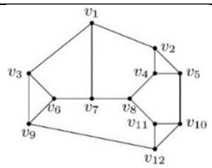
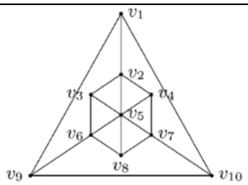
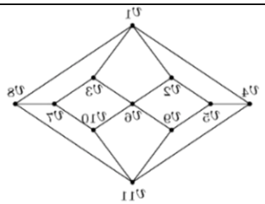
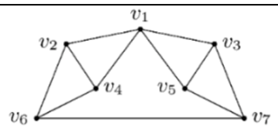
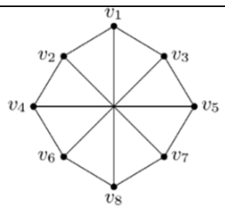
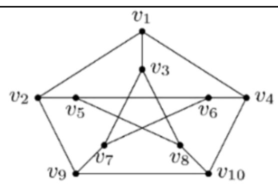
(iii) $diam(G) < \mathfrak{C}diam(G)$.

(iv) The eccentric vertex $E(u)$ is equal to the coupling eccentric vertex $\mathfrak{C}E(u)$ for any graph G.

 ISSN 0975-3303

Table 5.2: The coupling distance parameters of some standard graphs are given in the table below.

Graph	Figure	rad(G)	diam(G)	R(G)	P(G)
Bull Graph		5	7	$\{v_3, v_4, v_5\}$	$\{v_1, v_2\}$
Butterfly Graph		3	5	$\{v_3\}$	$\{v_1, v_2, v_4, v_5\}$
Diamond Graph		3	5	$\{v_1, v_4\}$	$\{v_2, v_3\}$
Durer Graph		7	9	$\{v_1, v_2, v_5, v_8, v_{11}, v_{12}\}$	$\{v_3, v_4, v_6, v_7, v_9, v_{10}\}$
Bidiaskis cube		7	7	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$
Chvatal Graph		5	5	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$
Franklin Graph		7	7	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$

Graph	Figure	rad(G)	diam(G)	R(G)	P(G)
Frucht Graph		7	9	$\{v_1, v_2, v_3, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$	$\{v_4, v_5, v_6, v_9\}$
Golomb Graph		5	7	$\{v_2, v_5, v_6, v_7\}$	$\{v_1, v_3, v_4, v_8, v_9, v_{10}\}$
Herschel Graph		7	9	$\{v_1, v_2, v_3, v_4, v_6, v_8, v_9, v_{10}, v_{11}\}$	$\{v_5, v_7\}$
Moser Spindle Graph		5	5	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
Wagner Graph		5	5	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$
Petersen Graph		5	5	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$	$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$

The following observations are made from Table 5.2.

Observation 5.2:

(i) The regular graph, Bidiakis cube, chvatal graph, franklin graph, moser spindle graph, wagner graph and petersen graph are all coupling self-centered graphs.

(ii) The butterfly graph has a unique center $\mathfrak{C}R(G)$.

(iii) In a chvatal graph $\mathfrak{C}_d(u,v) = \begin{cases} 3, & if\ (u,v) \in E(G) \\ 5, & otherwise \end{cases}$

(iv) The induced subgraph of coupling periphery of golomb graph is a null graph.

(v) For any coupling self-centered graph $\mathfrak{C}R(G) = \mathfrak{C}P(G) = V(G)$.

Theorem 5.1: For any graph $G(V,E)$, $u \in V$, $\mathfrak{C}e(u) = 2\ ecc(u) + 1$.

Proof: The proof follows from the definition of eccentricity of a vertex. The eccentricity $ecc(u)$ of a vertex u is the length of the path between u and the farthest vertex v from u in the graph. Therefore, path length is equal to $ecc(u)$. Now let us consider a vertex u and v if the length of path is 1. They are adjacent and $\mathfrak{C}_d(u,v) = 3$ as there are two vertices on the edge. Similarly, if the path length is 2 there will be three vertices on the path u, w, v where w is the intermediate vertex. Therefore $\mathfrak{C}_d(u,v) = 5$ for a path of length 2. Now for a path of length n there will be $n+1$ vertices in the path. Therefore, the coupling eccentricity for a path of length n is given by

$\mathfrak{C}_d(u,v) = n + n + 1.$

$\mathfrak{C}_d(u,v) = ecc(u) + ecc(u) + 1.$

$\mathfrak{C}_d(u,v) = 2\ ecc(u) + 1.$

$\mathfrak{C}_d(u,v) = \mathfrak{C}e(u).$

Theorem 5.2: If coupling eccentricity of any vertex of a graph G is either r_1 or r_2 then $\mathfrak{C}R(G) = V(G) - \mathfrak{C}P(G)$.

Proof: Let $\mathfrak{C}e(v_n) = r_1$ or $\mathfrak{C}e(v_n) = r_2\ \forall\ v_n \in V(G)$, where $(r_1 < r_2)$ then $\mathfrak{C}rad(G) = r_1$ and $\mathfrak{C}diam(G) = r_2$. Some of the vertices belongs to $\mathfrak{C}R(G)$ say s vertices and some of the vertices belongs to $\mathfrak{C}P(G)$

say t vertices. Hence $s + t = n$ (number of vertices in G). Therefore $\mathfrak{C}R(G) + \mathfrak{C}P(G) = V(G)$. Hence the result.

Theorem 5.3: For any graph G which is not coupling self-centered~~,~~ if $\mathfrak{C}diam(G) = \mathfrak{C}rad(G) + t$ then t is a positive even integer.

Proof: Let $\mathfrak{C}diam(G) = t = 2t_1 + 1$ by Theorem 5.1.

Let $\mathfrak{C}rad(G) = r = 2r_1 + 1$.

$\mathfrak{C}diam(G) - \mathfrak{C}rad(G) = [2(d_1) + 1] - [2(r_1) + 1]$

$\mathfrak{C}diam(G) - \mathfrak{C}rad(G) = 2(d_1 - r_1)$

$\mathfrak{C}diam(G) - \mathfrak{C}rad(G) = t$ (where t is an even positive integer).

Therefore, $\mathfrak{C}diam(G) = \mathfrak{C}rad(G) + r$.

Observation 5.3: The following statements are true for a complete graph K_n.

(i) For any complete graph K_n $\forall\, n > 1$, $\mathfrak{C}rad(G) = \mathfrak{C}diam(G) = 3$.

(ii) K_n is coupling self-centered.

Theorem 5.4: The vertex set $V(G)$ forms the coupling center and coupling periphery of K_n, where $n > 1$.

Proof: For a complete graph K_n, the coupling distance between any two vertices is 3. Since the degree of every vertex $v \in V(G)$ is $n - 1$. Therefore, the coupling eccentricity of every vertex v is constant $\mathfrak{C}e(v) = 3$ and all the vertices adjacent to v are the coupling eccentric vertices of v. Since we have $\deg(v) = n - 1$ all the vertices are eccentric vertices and the coupling eccentricity of every $v \in V(G)$ being constant. The coupling radius and coupling diameter of K_n is same, hence K_n is self-centered. Therefore $\mathfrak{C}R(G) = \mathfrak{C}P(G) = V(G)$.

Observation 5.4: For any Path P_n,

(i) $\mathfrak{C}R(P_n)$ has unique couple center if n is odd.

(ii) $\mathfrak{C}R(P_n)$ contains a pair of coupling central vertices if n is even.

(iii) $\mathfrak{C}P(P_n)$ has only end vertices.

Theorem 5.5: For any path graph P_n, where $n \geq 2$

$$\mathfrak{C}rad(G) = \begin{cases} n, & if \ \ n \, is \, odd \\ n+1, & ifn \ \ is \ even \end{cases}$$

Proof: Case(i): If n is odd. From the Observation-4.4(i) any odd path P_n contains a unique vertex which forms the coupling center of the graph. Then the pendant vertices v_1 and v_n from the coupling eccentric vertices of the unique vertex v_i, which forms the coupling center. The distance $d(v_i, v_1) = d(v_i, v_n) = \frac{n-1}{2}$. Then from the Observation-3.1

Case (i) :if $d(u, v) = n$, then $\mathfrak{C}_d(u, v) = 2n + 1$.

Therefore $\mathfrak{C}_d(v_i, v_1) = \mathfrak{C}_d(v_i, v_n) = 2\left(\frac{n-1}{2}\right) + 1 = n$.

Case (ii): If n is even we know-from the Observation-5.4(ii) that there is a pair of intermediate adjacent vertices which forms the coupling center of P_n. Let v_p and v_q be the intermediate adjacent vertices of the path $\mathfrak{C}R(P_n) = \{v_p, v_q\}$. Then we have $\mathfrak{C}e\left(v_p\right) = \mathfrak{C}e(v_q)$. The coupling eccentric vertices of v_p and v_q are v_n and v_1 respectively. Since distance between the central vertices and the pendant vertices must be same, $d\left(v_p, v_n\right) = d(v_q, v_1)$. Hence $d\left(v_p, v_n\right) = d\left(v_q, v_1\right) = \frac{n}{2}$. But from the observation-3.1(i), if $d(u, v) = n$ then $\mathfrak{C}d(u, v) = 2n + 1$. Therefore if $d\left(v_p, v_n\right) = d\left(v_q, v_1\right) = \frac{n}{2}$ then $\mathfrak{C}d\left(v_p, v_n\right) = \mathfrak{C}d\left(v_q, v_1\right) = 2\left(\frac{n}{2}\right) + 1$. Hence $\mathfrak{C}rad(G) = n + 1$.

Theorem 5.6: For path graph $P_{n,}$ the coupling diameter is $\mathfrak{C}diam(P_n) = 2n - 1$.

Proof: For any path P_n the diameter is the distance between the end vertices of the path. A path is a trial in which vertices are not repeated and has n vertices and $n-1$ edges. Therefore, the distance between the two end vertices v_1 and v_n is given by $d(v_1, v_n) = n - 1$, which is equal to the total number of edges. Now, from the observation-3.1(i) if $d(v_1, v_n) = n - 1$, then $\mathfrak{C}d(v_1, v_n) = 2(n-1) + 1 = 2n - 2 + 1 = 2n - 1$. Hence $\mathfrak{C}diam(P_n) = 2n - 1$.

Observation 5.5: For any wheel graph W_n

(i) W_4 is coupling self-centered graph.

(ii) When $n \geq 5$ there exist a unique coupling center $|\mathfrak{C}R(W_n)| = 1$ and $|\mathfrak{C}P(W_n)| = n - 1$.

Theorem 5.7: For a wheel graph W_n, where $n > 4$ $\mathfrak{C}rad = 3$ and $\mathfrak{C}diam = 5$.

Proof: Let W_n be the wheel graph where $V(W_n) = \{v_1, v_2, v_3, \dots v_c, \dots, v_i\}$. Let v_c be the vertex adjacent to all other vertices of a graph. Therefore $\deg(v_c) = n - 1$ and degree of all other vertices $\deg(v_i) = 3$ where $v_i \in V(W_n) - \{v_c\}$. Since every vertex of $V(W_n) - \{v_c\}$ is incident on the vertex v_c the distance $d(v_c, v_i) = 1$ and $\mathfrak{C}d(v_c, v_i) = 2(1) + 1 = 3$ (From the Observation-3.1(i)). Now consider any two vertices v_1 and v_2 other than v_c then shortest distance between them will be either path P_1 or P_2. P_1 is given by $v_1 - v_2$ if v_1 and v_2 are adjacent. P_2 is given by $v_1 - v_c - v_2$ if v_1 and v_2 are not adjacent.

Case(i): Consider P_1 , $d(v_1, v_2) = 1$ and $\mathfrak{C}d(v_1, v_2) = 2(1) + 1 = 3$ (by Observation- 3.1(i)).

Case(ii): Consider P_2 , $d(v_1, v_2) = 2$ and $\mathfrak{C}d(v_1, v_2) = 2(2) + 1 = 5$(by Observation- 3.1(i)). Now the coupling eccentric values of any vertex $v_i \in V(W_n)$ is given by $\mathfrak{C}e(v_i) = 3$ or 5. Therefore $\mathfrak{C}rad(W_n) = 3$ and $\mathfrak{C}diam(W_n) = 5$.

Theorem 5.8: For a star graph S_n where $n \geq 3$

(i) $|\mathfrak{C}R(S_n)| = 1$.

(ii) $v_i \in \mathfrak{C}P(S_n)$ where v_i belongs to the set of all pendant vertices of S_n and $|\mathfrak{C}P(S_n)| = n - 1$.

Proof: Case(i): Every star graph S_n, where $n \geq 3$ contains a central vertex and $n - 1$ pendant vertices. The degree of the central vertex is $n - 1$ and the degree of pendant vertices is one. Now the distance between the central vertex v_c and pendant vertex v_i given by $d(v_c, v_i) = 1$. Therefore $\mathfrak{C}(v_c, v_i) = 3$. The shortest distance between any two pair of pendant vertices is given by $d(v_1, v_2) = 2$. Therefore $\mathfrak{C}d(v_1, v_2) = 2(2) + 1 = 5$. $\mathfrak{C}e(v_c) = 3$ being the smallest eccentric value of S_n, v_c becomes a vertex with a unique coupling eccentric value. Hence $|\mathfrak{C}R(S_n)| = 1$.

Case(ii): In case(i) we see that $|\mathfrak{C}R(S_n)| = 1$. v_c is the unique vertex with the least eccentric value and all the other remaining pendant vertices $v_i \in V(S_n) - \{v_c\}$ have the same coupling eccentricity $\mathfrak{C}e(v_i) = 5$. Then $\mathfrak{C}diam(S_n) = 5$ and $\mathfrak{C}rad(S_n) = 3$. Therefore, the set of all end vertices of S_n form the coupling periphery of S_n.

Observation 5.6: For any cycle C_n

(i) C_n is coupling self-centered.

(ii) $|\mathfrak{C}R(G)| = |\mathfrak{C}P(G)| = n$ for any cycle C_n.

(iii) $V(C_n) \in \mathfrak{C}R(G), V(C_n) \in \mathfrak{C}P(G), V(C_n) = \mathfrak{C}R(C_n) = \mathfrak{C}P(C_n)$.

Theorem 5.9: For any cycle C_n

$$\mathfrak{C}rad(C_n) = \mathfrak{C}diam(C_n) = \begin{cases} n, & if\ n\ is\ odd \\ n+1, & if\ n\ is\ even \end{cases}$$

Proof: Case(i): Let C_n be a cycle graph where n is odd. For every vertex $v_i \in V(C_n)$ the farthest vertex lies at a distance of $\frac{n-1}{2}$ from it. Therefore $e(v_i) = \frac{n-1}{2}$. Therefore, coupling eccentricity of every vertex of C_n is given by $\mathfrak{C}e(v_i) = 2\left(\frac{n-1}{2}\right) + 1 = n$. Therefore, for an odd cycle which is also self-centered. $\mathfrak{C}rad(C_n) = \mathfrak{C}diam(C_n) = n$.

Case(ii): For an even cycle C_n, where $v_i \in V(C_n)$ the vertex farthest from v_i lies at a distance of $\frac{n}{2}$ from it. Therefore $e(v_i) = \frac{n}{2}$. Now the coupling eccentricity of every vertex of C_n is given by $\mathfrak{C}e(v_i) = 2\left(\frac{n}{2}\right)$+1=n+1. Hence for an even cycle which is also self-centered, $\mathfrak{C}rad(C_n) = \mathfrak{C}diam(C_n) = n + 1$.

Observation 5.7: For a friendship graph F_n

(i) $|\mathfrak{C}R(F_n)| = 1$.

(ii) $|\mathfrak{C}P(F_n)| = n - 1$.

(iii) $v_i \in \mathfrak{C}P(F_n)$ where $\deg(v_i) = 1$ and $v_i \in F_n$.

Theorem 5.10: For any friendship graph F_n , $\mathfrak{C}rad(F_n) = 3$ and $\mathfrak{C}diam(F_n) = 5$.

Proof: We can obtain F_n by joining $'n'$ C_3 cycle graph with a common vertex. The $'n'$ in F_n denotes the number of C_3 cycles in F_n. There are $2n + 1$ vertices and $3n$ edges in F_n. The graph F_1 is copy of cycle C_3. The friendship graph F_2 is obtained by joining two C_3 cycles with a common vertex. Let $v_c \in V(F_n)$ be the vertex common for every cycle C_3 in F_n. Then $\deg(v_c) = 2n$. Since v_c is adjacent to every vertex $v_i \in V(F_n) - v_c$, $d(v_c, v_i) = 1$ and $\mathfrak{C}d(v_c, v_i) = 3$. Let v_i, $v_j \in V(F_n)$ - $\{v_c\}$ be the adjacent vertices. Therefore $d(v_1, v_2) = 1$, $\mathfrak{C}d(v_1, v_2) = 3$, $v_1, v_2 \neq v_c$ and $(v_1, v_2) \in E(F_n)$. The other possibility being $(v_1, v_2) \notin E(F_n)$. Then the distance between v_1 and v_2 is given by the path $v_1 - v_c - v_2$ since v_c is adjacent to all vertices of F_n. Now $d(v_1, v_2) = 2$ and $\mathfrak{C}d(v_1, v_2) = 5$ where $(v_1, v_2) \notin E(F_n)$ and $v_1, v_2 \neq v_c$. Hence, we have $\mathfrak{C}e(v) = 3$ or 5 where $v \in V(F_n)$. Therefore $\mathfrak{C}rad(F_n) = 3$ and $\mathfrak{C}diam(F_n) = 5$.

Acknowledgements

The authors are indebted to the reviewers for their insightful comments and suggestions.

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