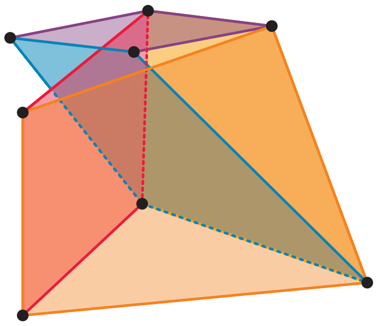
Combinatorial and Computational Methods for the Properties of Homogeneous Polynomials



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Sanat için sanat, matematik için matematik!

L'Art pour l'art, les mathématiques pour les mathématiques!

Kunst um der Kunst willen, Mathematik um der Mathematik willen!

Art for art's sake, mathematics for mathematics' sake!

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Abstract

In this manuscript, we provide foundations of properties of homogeneous polynomials such as the half-plane property, determinantal representability, being weakly determinantal, and having a spectrahedral hyperbolicity cone. One of the motivations for studying those properties comes from the "generalized Lax conjecture" stating that every hyperbolicity cone is spectrahedral. The conjecture has particular importance in convex optimization and has curious connections to other areas. We take a combinatorial approach, contemplating the properties on matroids with a particular focus on operations that preserve these properties. We show that the spectrahedral representability of hyperbolicity cones and being weakly determinantal are minor-closed properties. In addition, they are preserved under passing to the faces of the Newton polytopes of homogeneous polynomials. We present a proved-to-be computationally feasible algorithm to test the half-plane property of matroids and another one for testing being weakly determinantal. Using the computer algebra system Macaulay2 and Julia, we implement these algorithms and conduct tests. We classify matroids on at most 8 elements with respect to the half-plane property and provide our test results on matroids with 9 elements. We provide 14 matroids on 8 elements of rank 4, including the Vámos matroid, that are potential candidates for the search of a counterexample for the conjecture.

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Introduction

A polynomial, all of whose monomials have the same degree, is called *homogeneous*. Properties of homogeneous polynomials with real coefficients have been motivated and studied by different branches of mathematics; differential equations [24, 42, 8], real algebraic geometry [30, 58, 51], convex optimization [29, 5, 54], combinatorics [61, 10, 18], etc. Being hyperbolic is one of these properties. A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is called *hyperbolic* if there exists a direction $e \in \mathbb{R}^n$ such that $h(e)$ does not vanish, and for all $v \in \mathbb{R}^n$ the univariate restriction $h(et - v)$ is real rooted. For every such h, the set of points $v \in \mathbb{R}^n$ for which the univariate restriction $h(et - v)$ has only non-negative roots is called the *hyperbolicity cone* $C_h(e)$ of h. In particular, by [24], such a polynomial h is hyperbolic with respect to every point from the interior of its hyperbolicity cone $C_h(e)$.

Hyperbolicity cones are closed and convex and feasible sets of *hyperbolic programming* that optimizes a linear function on hyperbolicity cones. Another type of optimization, called *semidefinite optimization* (SDP), uses sections of the cone of positive semidefinite matrices as feasible sets. Those sets can be defined as the solution set of linear matrix inequalities and are called *spectrahedral*, or to have a *spectrahedral representation*. While spectrahedral cones can be expressed as hyperbolicity cones of some polynomials, whether every hyperbolicity cone can be defined by linear matrix inequalities is an open question that the *Generalized Lax conjecture* posits. For results supporting the conjecture, we refer to [19, 49, 12, 2].

The studies motivated by the conjecture took a new perspective with the connection between the hyperbolicity and the half-plane property. In combinatorics, being hyperbolic with respect to every point in the positive orthant is called the *half-plane property* (HPP). This property was initially motivated by the theory of electrical networks. An electrical network can be seen as a finite, connected, un-directed graph G whose edges and vertices represent cables and joints, respectively. An application of Kirchoff's matrix tree theorem yields a well-known result in engineering; the spanning tree polynomial of any such graph G has the half-plane property. One can then consider matroids, which generalize the concept of linear independence. In particular, they allow one to consider the half-plane property of objects that are not necessarily constructed from graphs.

A *matroid* M is a finite set $E = [n]$ with a collection $\mathcal{B}$ of its subsets called the *collection of bases* whose elements satisfy the following *basis exchange axiom*:

If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there exists $e' \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$.

Subsets of bases are called *independent*, and as we intuitively expect, the *rank* of a subset $S \subset E$ is the cardinality of the biggest independent set it contains. Also, every basis has the same cardinality. There are several ways to define matroids; once we know one concept, for example, the bases, we can deduce the dependent sets, independent sets, etc. Matroids can be produced from matrices by enumerating their column vectors and considering the independence relation of their columns. They can also be defined from graphs by enumerating the set of their edges and setting the subsets that give rise to spanning trees as bases. On the other hand, there exist some matroids that neither come from matrices nor from graphs. Indeed matroids are broader combinatorial objects.

The basis generating polynomial of a matroid M is

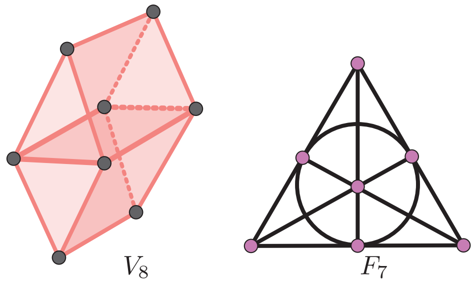
$$h_M := \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i.$$

Such polynomials are homogeneous, *multiaffine* (i.e., every variable has degree at most 1 in it), and in particular, their support elements satisfy the basis exchange axiom. One can then ask whether or when such polynomials are hyperbolic with respect to every point on the positive orthant. A matroid is called to have the half-plane property when its basis generating polynomial has the half-plane property.

In their seminal paper [18] by Choe et al. showed that the support of every homogeneous multiaffine polynomial with the half-plane property is the collection of bases of some matroid. This implies, in particular, that homogeneous multiaffine polynomials with the half-plane property give rise to matroids. On the other hand, not every matroid has the half-plane property. For example, Brändén in [10] showed that the collection of bases of the Fano matroid (F_7) cannot be the support of a polynomial with the half-plane property.

In the context of the generalized Lax conjecture, finding matroids that have the half-plane property (thus hyperbolic) and investigating the spectrahedrality of their hyperbolicity cones gives a way to search for potential counterexamples.

Having a determinantal representation and being weakly determinantal are yet other properties of homogeneous polynomials. A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is called to have a *determinantal representation* if there exists real symmetric positive semi-definite matrices $A_1, \ldots, A_n$ and a non-zero $\lambda \in \mathbb{R}$ such that $h = \lambda \det(x_1 A_1 + \cdots + x_n A_n)$. Moreover, if there exists a power $N \in \mathbb{N}_{>0}$ such that h^N has a determinantal representation, then h is called *weakly determinantal*. Notably, being weakly determinantal implies the half-plane property, and the matrices that give the determinantal representation take a role in the spectrahedral representation of the hyperbolicity cone. On the other hand, not every hyperbolic polynomial is weakly determinantal. Brändén in [11] showed that the basis generating polynomial of the Vámos matroid (V_8) has the half-plane property, but it is not weakly determinantal.



V_8 F_7

In particular, V_8 provides a counterexample for a stronger version of the conjecture. Further such examples can be found in [15, 3].

Moreover, Helton and Vinnikov in [30] gave a criterion for a hyperbolicity cone to be spectrahedral;

The hyperbolicity cone $C_h \supseteq \mathbb{R}_{\geq 0}$ of h is spectrahedral if there exists another hyperbolic polynomial g with $C_h \subseteq C_g$ such that $h \cdot g$ is weakly determinantal.

This brings another facet to the quest to find matroids with the half-plane property that are possible candidates for a counterexample to the generalized Lax conjecture. In particular, one should aim to find more matroids with the half-plane property that are not weakly determinantal.

Furthermore, finding matroids that have the half-plane property and those that do not have it is interesting on its own. It is not easy, in general, to detect the half-plane property of a given matroid. Oxley, for instance, asks for a "practically feasible algorithm for testing whether a matroid is HPP" in his book [50, Problem 15.8.10].

One may then ask the following questions:

- Which operations on polynomials preserve the half-plane property, being weakly determinantal and spectrahedral representability of hyperbolicity cones?
- Which matroids have the half-plane property?
- Are there more matroids that have the half-plane property that are not weakly determinantal?
- Is there a computationally feasible way to test the half-plane property of matroids?

In this manuscript, we first give the necessary background on the mentioned properties and their relations with matroids. We then focus on the first question; the operations on homogeneous polynomials and matroids that preserve those properties.

Especially deletion and contraction operations of matroids have particular importance. Depending on whether a matroid is simple (i.e., has no element that appears in every basis or does not have an element that appears in no basis), their effects on the basis generating polynomial boil down to setting some variables equal to zero or taking derivatives with respect to them. Deletion and contraction operations create new smaller matroids called *minors* from a given matroid. Therefore, minor closedness of the operations of our interest would suggest searching those properties on smaller matroids first, making a classification, and then moving on to the bigger ones.

In [18], Choe et al. showed that the half-plane property of polynomials is closed under taking minors, and in [38], Kummer et al. showed that determinantal representability is minor-closed. We give details about those results while viewing the half-plane property from the perspective of stability (see, for example, [8]). Moreover, in § 2.2, we prove that having a spectrahedral hyperbolicity cone and being weakly determinantal are minor-closed properties. Considering the *polarization* of a homogeneous not necessarily multiaffine polynomial in order to turn it into a multiaffine polynomial in more variables is one of the methods we use. It especially allows us to apply some results on multiaffine polynomials. The mentioned results are part of [39], joint work with Kummer. In summary, we have the following diagram for the properties mentioned above.

$$\begin{tikzcd}
\text{Matroid } M \arrow[d] & M \text{ has the property of interest} \arrow[d, Rightarrow] \\
\text{A minor } M' \text{ of } M & M' \text{ has the property of interest}
\end{tikzcd}$$

We further take a geometric approach and consider the Newton polytopes of matroids called matroid polytopes. We consider the operation of going from the matroid polytope of a given matroid to one of its faces and then going to the matroid corresponding to that face (by [26] faces of matroid polytopes are matroid polytopes of some matroids). We then show the preservation of the properties under this operation for homogeneous polynomials and their Newton polynomials in general.

$$\begin{tikzcd}
\text{Matroid } M \arrow[r] \arrow[d] & \text{Matroid Polytope } P(M) \arrow[d] \\
\text{Matroid } M_F & \text{A Face } F \arrow[l]
\end{tikzcd}$$

Further, in § 3, we discuss the criteria for a matroid to have the half-plane property and to be weakly determinantal. Brändén in[10] gave a criterion for the half-plane property of a matroid M that relies on the non-negativity of the Rayleigh differences

$$\Delta_{ij}(h_M) := \frac{\partial h_M}{\partial x_i} \cdot \frac{\partial h_M}{\partial x_j} - \frac{\partial^2 h_M}{\partial x_i \partial x_j} \cdot h_M$$

of its basis generating polynomial h_M for all indices i, j. In particular, Wagner and Wei in [63] showed that when all proper minors of a matroid have the half-plane property, it is enough to find only one pair of indices for which the Rayleigh difference is non-negative. These criteria create a bridge between the studies on hyperbolic polynomials and the non-negativity of polynomials.

We list several methods one can apply for testing the half-plane property and being weakly determinantal. For the latter, we introduce the SOS-Rayleigh property and show that it is minor closed. A homogeneous multiaffine polynomial is *SOS-Rayleigh* when all of its Rayleigh differences are sums of squares. In particular, by [38], weakly determinantal polynomials are SOS-Rayleigh. Another way to test being weakly determinant is due to [11]; the rank functions of weakly determinantal matroids satisfy the Ingleton inequalities. We implement those criteria and provide an algorithm (Algorithm 1) for testing the half-plane property and another one (Algorithm 2) for testing the SOS-Rayleigh property (in order to use it to disprove being weakly determinantal).

We implement the algorithms using the computer algebra system Macaulay2 [28] and Julia. We use packages “Matroids” by Chen [16, 17] for manipulating matroids in Macaulay2, and “SumsOfSquares” by Cifuentes et. al. [20, 21] for producing symbolic certificates for non-negativity. The Julia package “HomotopyContinuation.jl” by Breiding and Timme [14] is used to compute the critical points in order to disprove the half-plane property.

Using the algorithms, we classify matroids on 8 elements with respect to the half-plane property and also provide our test results on matroids on 9 elements (§ 4). Our classification yields a list of 32 matroids with at most 8 elements that are minor-minimal with respect to not having the half-plane property. These include the ten forbidden minors of rank 3 on 7 elements that were already found in [63], namely the Fano matroid, three of its relaxations, the free extension of $M(K_4)$ by one element and their duals. All other forbidden minors are of rank 4 on 8 elements. We found that 14 matroids of rank 4 on 8 elements, including the Vámos matroid, have the half-plane property but are not weakly determinantal. In particular, they are potential candidates for searching for a counter example for the conjecture. Another particularity is that they have some Rayleigh differences that are non-negative but not a sum of squares of polynomials.

Further, our tests confirm that the Pappus, non-Pappus, and (non-Pappus $\backslash 9)+e$ matroids are the only forbidden minors for the half-plane property that are on 9 elements with rank 3. Among those of rank 4, we provide a list of 4125 matroids that have the half-plane property and a list of 1218 matroids that are forbidden (minimal) minors for the half-plane property.

As future perspectives, our results on the minor closedness of spectrahedral representability suggest the search of spectrahedrality of matroids with the HPP starting from matroids on a small ground set. By using characterizations of matroids from [50] and results supporting the generalized Lax conjecture [12, 2], we conclude that every matroid on at most 5 elements is spectrahedral. There are two matroids on 6 elements whose spectrahedrality is not known. We conclude the script with some open questions.

Chapter 1

Background

1.1 Some Properties of Homogeneous Polynomials

In this section, we provide the necessary background for a better understanding of the main subjects of our study such as Hyperbolic polynomials, Half-plane property, determinantal representability, spectrahedral representability, and their relation to one another. We refer to [24], [54], [9], [63], [30], [59], [11] as main references for more information on this topic.

Hyperbolic Polynomials

A polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ of degree d is called *homogeneous* if all of its monomials have degree d. Another characterization of homogeneous polynomials is the following property:

$$h(\lambda x_1, \dots, \lambda x_n) = \lambda^d h(x_1, \dots, x_n)$$

for all $\lambda \in \mathbb{C}$. Moreover, h is called *multiaffine* if every variable x_i has degree at most one in each monomial. Now, we focus on homogeneous polynomials that are hyperbolic.

Definition 1.1.1. A homogeneous polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ is called *hyperbolic* with respect to $e \in \mathbb{R}^n$ if $h(e) \neq 0$ and for all $v \in \mathbb{R}^n$ the univariate restriction $h(et - v) \in \mathbb{R}[t]$ is real rooted.

In other words, a polynomial h of degree d is hyperbolic with respect to $e \in \mathbb{R}^n$ if lines going through e pierce the real hypersurface defined by h exactly d times counted with multiplicity.

The *hyperbolicity cone* $C_h(e)$ of a hyperbolic polynomial h is the set of points $v \in \mathbb{R}^n$ for which the restriction $h(et - v)$ has only non-negative roots that is

$$C_h(e) := \{v \in \mathbb{R}^n : h(et - v) = 0 \implies t \geq 0\}.$$

Observe that when r is a root of $h(et - v)$, $ar + b$ is a root of $h(et - (av + eb))$ for $a, b \in \mathbb{R}$. Thus if $v \in C_h(e)$, then $cv \in C_h(e)$ for all $c > 0$ such that $C_h(e)$ is a cone.

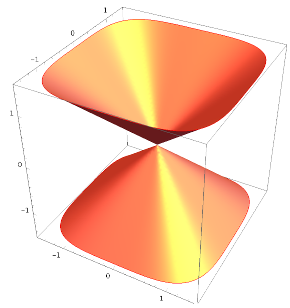


Figure 1.1: The real variety of a non-hyperbolic polynomial of degree 4

Example 1.1.2.
- Consider the polynomial $f(x_1, x_2, x_3) = x_1x_2x_3$ and $e = (1, 1, 1)$. Its univariate restriction $f(t - v_1, t - v_2, t - v_3) = (t - v_1)(t - v_2)(t - v_3)$ is real rooted for all $v \in \mathbb{R}^3$, thus f is hyperbolic with respect to e. Its hyperbolicity cone is the set of points $v \in \mathbb{R}^3$ such that the univariate restriction of f has only non-negative roots. It is immediate to observe that this holds for $v \in \mathbb{R}^3_{\geq 0}$, so $C_f(e) = \mathbb{R}^3_{\geq 0}$.
- Let $X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and consider $h(x_1, x_2, x_3) = \det(X)$. Its univariate restriction $h(v_1 - t, v_2 - t, v_3) = \det \begin{pmatrix} v_1 - t & v_3 \\ v_3 & v_2 - t \end{pmatrix}$ (h is homogeneous, so one can take out the factor -1 and use the restriction to $v - et$) is nothing but the characteristic polynomial of the matrix $V := \begin{pmatrix} v_1 & v_3 \\ v_3 & v_2 \end{pmatrix}$ for all $v \in \mathbb{R}^3$. Therefore, the hyperbolicity cone $C_h(e)$ is the cone of positive semi-definite (PSD) 2×2 matrices, i.e.,
$$C_h(e) = \left\{ v \in \mathbb{R}^3 : V \succeq 0 \right\}.$$

Linear programming optimizes a linear form on some linear slices of the non-negative orthant, and semi-definite programming optimizes a linear form on some slices of the cone of PSD matrices. The examples show that both the non-negative orthant and the cone of PSD matrices are hyperbolicity cones for some polynomials.

Example 1.1.3. Consider the polynomial $h = x_3^4 - x_2^4 - x_1^4$. By considering the shape of its variety $\mathcal{V}_{\mathbb{R}}(h) := \left\{ x \in \mathbb{R}^3 : h(x) = 0 \right\}$ shown in Figure 1.1, one can see that it is not possible to find a line that intersects $\mathcal{V}_{\mathbb{R}}(h)$ at 4 points.

We continue with some propositions that help us understand the hyperbolic polynomials better. First, we show that applying an invertible linear transformation to a hyperbolic polynomial does not harm hyperbolicity.

Proposition 1.1.4. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$, and let $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. Then $h(\mathrm{T}(x))$ is hyperbolic with respect to $\mathrm{T}^{-1}(e)$.*

Proof. Since h is hyperbolic, we have that $h(et - v)$ is real rooted for all $v \in \mathbb{R}^n$. Then, $h(\mathrm{T}(\mathrm{T}^{-1}(e)t - v)) = h(et - \mathrm{T}(v))$ is also real rooted as $\mathrm{T}(v) \in \mathbb{R}^n$. □

Moreover, we can use univariate restrictions to different lines in order to define hyperbolic polynomials.

Proposition 1.1.5. *If $h \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to $e \in \mathbb{R}^n$, then the polynomials, $h(et + v)$, $h(vt - e)$ and $h(vt + e)$ are also real rooted.*

Proof. Since h is hyperbolic, we know that $h(et - v)$ is real rooted. Consider the linear map $\mathrm{T}(x) = -x$. By Proposition 1.1.4, $h(\mathrm{T}(x))$ is hyperbolic with respect to $\mathrm{T}^{-1}(e)$, thus $h(et - \mathrm{T}(v)) = h(et + v)$ is real rooted. Let $a \neq 0$ be a root of $h(vt - e)$. Then, a is also a root of $h(v - \frac{e}{t})$, and $\frac{1}{a}$ is a root of $h(v - et)$ which has to be real as h is hyperbolic with respect to e. Moreover, $a = 0$ cannot be a root of $h(vt - e)$ as $h(-e) \neq 0$ by hyperbolicity. The same argument applies to $h(vt + e)$. □

When we change the line we restrict the polynomial h to, we also adopt the definition of its hyperbolicity cone accordingly. For example,

$$\begin{aligned} C_h(e) &= \{v \in \mathbb{R}^n : h(et - v) = 0 \implies t \geq 0\} \\ &= \{v \in \mathbb{R}^n : h(-et - v) = 0 \implies t \leq 0\} \\ &= \{v \in \mathbb{R}^n : h(et + v) = 0 \implies t \leq 0\}. \end{aligned}$$

Further, an application of Rolle's theorem shows that taking derivatives preserves hyperbolicity.

Proposition 1.1.6. *If $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is hyperbolic with respect to $e \in \mathbb{R}^n$, then $D_e h(x_1, \ldots, x_n) := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} h(x_1, \ldots, x_n)$ is hyperbolic with respect to e.*

Proof. Since h is hyperbolic, $h(et-v)$ has d real roots for all $v \in \mathbb{R}^n$. By Rolle's theorem, the univariate restriction $D_e h(et - v) = \frac{\partial}{\partial t} h(et - v)$ has $d - 1$ real roots that lie in the open intervals between the roots of $h(et - v)$. When h has a root b with multiplicity r, then $D_e h(et - v)$ has b as a root with multiplicity $r - 1$. □

Let us focus on the hyperbolicity cones, their structures, and their properties. By homogeneity, if h is hyperbolic with respect to e, then h is hyperbolic with respect to $-e$. When h is a constant polynomial, it is hyperbolic with respect to every $e \in \mathbb{R}^n$ and the hyperbolicity cone $C_h(e)$ for each e is the whole $\mathbb{R}^n$. When h has degree at least 1, by the definition of the hyperbolicity cone, one observes that $C_h(e) \cap C_h(-e) = \{0\}$, and $C_h(-e) = -C_h(e)$. Note that when h is hyperbolic with respect to e and $e' \in C_h(e)$, h is not necessarily hyperbolic with respect to e'. For example consider the case when $h(e') = 0$. Moreover, when h is hyperbolic with respect to e and e', we do not necessarily have $e \in C_h(e')$ or $e' \in C_h(e)$. Take for instance $e' = -e$. One can observe that if $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ are two polynomials that are hyperbolic with respect to $e \in \mathbb{R}^n$, then $f \cdot g$ is also hyperbolic with respect to e and $C_{fg}(e) = C_f(e) \cap C_g(e)$. The following proposition gives a description of the interior of the hyperbolicity cone.

Proposition 1.1.7. *The hyperbolicity cone $C_h(e)$ of $h \in \mathbb{R}[x_1, \ldots, x_n]$ is the closure of the cone $C_h^{\circ}(e) := \{v \in \mathbb{R}^n : h(et - v) = 0 \implies t > 0\}$, and $C_h^{\circ}(e)$ is the interior of $C_h(e)$.*

Proof. Since $h(et-e)=0$ implies that $t=1>0$ is a root and $h(e)\neq 0$ (without loss of generality we can assume that $h(e)>0$) we have $e\in C_h^\circ(e)$. If $v\in C_h(e)$, then $v+ea\in C_h^\circ(e)$ for all $a>0$. Thus $C_h(e)\subset \overline{C_h^\circ(e)}$. Note that for any family of univariate polynomials whose coefficients continuously vary in v, their roots continuously vary in v (see for example [52, Theorem 1.3.1]). This gives $\overline{C_h^\circ(e)}\subset C_h(e)$.

In order to show the second part, let $v\in C_h^\circ(e)$. Since the roots of the univariate restriction continuously depend on the coefficients, we can find small enough ε such that for each a from the ε neighborhood of v, $h(et-a)$ has non-negative roots. Thus, v is in the interior of $C_h(e)$. Now, let w be from the interior of $C_h(e)$ and assume that $h(et-w)$ has zero as a root. For any $0<\varepsilon<1$, we have that $w+e\varepsilon$ and $w-e\varepsilon$ are in some δ neighborhood of w for a small $\delta>0$ (we can take δ arbitrarily small as it depends on ε). Moreover, $h(et-w+e\varepsilon)$ and $h(et-w-e\varepsilon)$ have roots $-\varepsilon$ and ε respectively. This gives a contradiction with the assumption that w is from the interior of $C_h(e)$. □

Here is another description of the interior of the hyperbolicity cone. It was first mentioned by Gårding in [24].

Proposition 1.1.8 (Proposition 1, in [54])**.** *The cone $C_h^\circ(e)$ is the connected component of*

$$\{x\in\mathbb{R}^n : h(x)\neq 0\}$$

which contains e.

Proof. Let K be the connected component of $\{x\in\mathbb{R}^n : h(x)\neq 0\}$ that contains e. For any $v\in K$, zero is a root of the restriction $h(et-v)$ only if $h(v)=0$. Since $e\in C_h^\circ(e)$ and roots of the univariate restriction continuously depend on the coefficients, $K\subset C_h^\circ(e)$. For the other direction, let $w\in C_h^\circ(e)$. Then $w+be\in C_h^\circ(e)$ for all $b>0$. Therefore,

$$h(aw+(1-a)e+be)=a^d h(w+\frac{(1-a+b)}{a}e)\neq 0$$

for $0<a\leq 1$ and $b>0$ where d is the degree of h. In particular, $h(aw+(1-a)e+te)\in\mathbb{R}[t]$ has positive roots, so that $aw+(1-a)e\in C_h^\circ(e)$ for $0\leq a\leq 1$. Thus, we obtained a path between any point in $C_h^\circ(e)$ and e to show that $C_h^\circ(e)$ is connected. □

Therefore, hyperbolicity cones are semi-algebraic sets, i.e., they can be defined by finite union of sets defined by polynomial inequalities (by [7, Proposition 2.2.4] complements of semi-algebraic sets are semi-algebraic, and by [7, Theorem 2.4.4], connected components of semi-algebraic sets are semi-algebraic).

Below are some more properties of hyperbolicity cones due to Gårding.

Theorem 1.1.9 (Theorem 2, in [24])**.** *Let $h\in\mathbb{R}[x_1,\dots,x_n]$ be hyperbolic with respect to $e\in\mathbb{R}^n$.*

(i) If $e'\in C_h^\circ(e)$, then h is hyperbolic with respect to e' and $C_h^\circ(e)=C_h^\circ(e')$.

(ii) The cones $C_h(e)$ and $C_h^\circ(e)$ are convex.

Proof. (i) : Let $e' \in C_h^\circ(e)$. Since $h(e') = 0$ implies that zero is a root of $h(et + e')$, we have that $h(e') \neq 0$. In order to show that $h(e't + v)$ has only real roots, let $a \in \mathbb{R}_{>0}$ be fixed, $v \in \mathbb{R}^n$ be arbitrary, and consider $h(aie + e't + bv) \in \mathbb{C}[t]$ for all $b \in \mathbb{R}^n_{\geq 0}$. We first prove the following claim.

Claim: For all $b \in \mathbb{R}^n_{\geq 0}$, all roots of $h(aie + e't + bv) \in \mathbb{C}[t]$ have negative imaginary part.

Proof. For $b = 0$, we have that all roots of $h(aie + e't)$ have negative imaginary part, since $e' \in C_h^\circ(e)$ and h is homogeneous. Now, assume that there is a $b > 0$ for which $h(aie + e't + bv) \in \mathbb{C}[t]$ has a non-negative imaginary part. Then, by the continuity of roots in b, there is a $0 < c \leq b$ such that $h(aie + e't + cv) \in \mathbb{C}[t]$ has a real root r. It follows, that ai is a root of $h(xe + e'r + cv) \in \mathbb{C}[x]$. Since $z := e'r + cv$ is real, the existence of such a vector for which $h(xe + z) \in \mathbb{C}[x]$ has a non-real root contradicts the hyperbolicity of h with respect to e, and proves the claim. □

Since all roots of $h(aie + e't + v) \in \mathbb{C}[t]$ have negative imaginary part without depending on the fixed positive value of a, we can consider the limit of $h(aie + e't + v)$ when a is approaching to zero. The continuity of roots in a gives that roots of $h(e't + v) \in \mathbb{C}[t]$ have non-positive imaginary parts. Since the coefficients of $h(e't + v) \in \mathbb{C}[t]$ are real and complex roots come in conjugates, all roots of $h(e't + v)$ are real for an arbitrary $v \in \mathbb{R}^n$ so that h is hyperbolic with respect to e'.

For the equality of the hyperbolicity cones, recall that $C_h^\circ(e)$ is the connected component of $\{x \in \mathbb{R}^n : h(x) \neq 0\}$ that contains e and e', and $C_h^\circ(e')$ is the connected component of $\{x \in \mathbb{R}^n : h(x) \neq 0\}$ that contains e'. Thus $C_h^\circ(e) = C_h^\circ(e')$. One can also observe the equality for their closures, such that $C_h(e) = \overline{C_h^\circ(e)} = \overline{C_h^\circ(e')} = C_h(e')$.

(ii) : Let $v, w \in C_h^\circ(e)$ and $a, b > 0$ with $a + b = 1$. Consider the restriction $h(et - (av + bw)) \in \mathbb{R}[t]$. By (*i*), we may assume that $w = e$. Then the roots of $h(et - (av + be))$ are $ar_i(v) + b > 0$ where $r_i(v)$ are the roots of $h(et - v)$ so that $av + bw \in C_h^\circ(e)$ shows the convexity.

□

We now know that if a homogeneous $h \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to a direction $e \in \mathbb{R}^n$, then it is also hyperbolic with respect to every point from the interior of $C_h(e)$. That does not imply however that we can recover from $C_h^\circ(e)$ every direction for which h is hyperbolic. Recall for example that h is also hyperbolic with respect to every point from $C_h^\circ(-e) = -C_h^\circ(e)$. Moreover, there can be some points e' that are neither in $C_h^\circ(e)$, nor in $-C_h^\circ(e)$ such that h is hyperbolic with respect to e'. In short, the hyperbolicity cone $C_h(e)$ varies with respect to e.

Example 1.1.10. Consider the polynomial $h(x_1, x_2) = x_2^2 - x_1^2$ and $e = (1, 0)$. The univariate restriction $h(t - v_1, v_2) = v_2^2 - (t - v_1)^2$ is real rooted for all $v \in \mathbb{R}^2$ and the defining inequalities of its hyperbolicity cone $C_h(e)$ are $v_1 + v_2 \geq 0$ and $v_1 - v_2 \geq 0$. When we take $e' = (-1, 0)$, we see that h is also hyperbolic with respect to e', and $C_h(e')$ is defined by $-v_1 - v_2 \geq 0$ and $v_2 - v_1 \geq 0$. As illustrated in Figure 1.2, $C_h(e) \cap C_h(e') = \{0\}$.

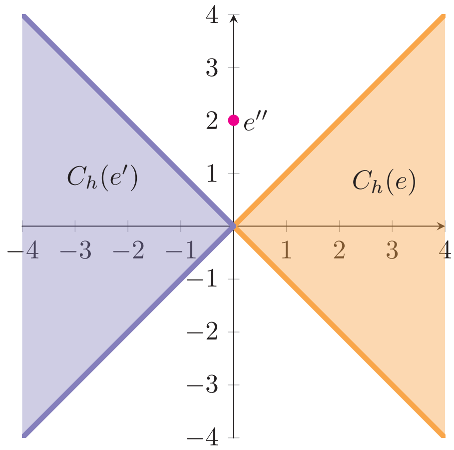


Figure 1.2: Hyperbolicity cones $C_h(e)$ and $C_h(e')$, and the point e''

Now consider $e'' = (0, 2)$. In this case $h(v_1, 2t - v_2) = (2t - v_2)^2 - v_1^2$ is real rooted for all $v \in \mathbb{R}^2$, thus h is hyperbolic with respect to e''. On the other hand, we have that $e'' \notin C_h(e)$ and $e'' \notin C_h(e')$ as illustrated in Figure 1.2. The hyperbolicity cone $C_h(e'')$ is defined by the inequalities $v_2 + v_1 \geq 0$ and $v_2 - v_1 \geq 0$.

Hyperbolicity cones of non-constant polynomials are convex cones that are not the whole $\mathbb{R}^n$. They might however contain a non-trivial linear subspace. In that case we can intersect the hyperbolicity cone with the orthogonal complement of the subspace in order to obtain a regular cone. When a hyperbolic polynomial depends on all the variables (i.e., each variable from its polynomial ring appears at least once after applying arbitrary linear change of coordinates), its hyperbolicity cone contains only the trivial subspace as a subspace so that the cone is regular.

For a hyperbolic polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d, for each $0 \leq m \leq d$, let C_m be the set of $v \in \mathbb{R}^n$ for which zero has multiplicity m as a root of $h(et - v)$. Renegar in [54] shows that the cones C_m give a partition of the boundary of the hyperbolicity cone $C_h(e)$. Moreover, the faces of $C_h(e)$ have the following structure.

Theorem 1.1.11 (Theorem 26 in [54])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$, F be a face of $C_h(e)$ and $x \in F$. The multiplicity of zero as a root of $h(et - x)$ is the minimum of multiplicities of zero of $h(et - y)$ among all $y \in F$ if and only if x is in the relative interior of F.*

Remark 1.1.12. Hyperbolic programming optimizes a linear form on slices of a hyperbolicity cone of some polynomials. It was first introduced by Güler in [29] (see also [5], [54]). As we showed in the Example 1.1.4, the feasible sets of linear programming and semi-definite programming are also feasible sets of hyperbolic programming. The question whether every hyperbolic program can be written as a semi-definite program still remains open as *Generalized Lax Conjecture*. We will give more details on the conjecture and the improvements on it in the following chapters.

Now, let us explore more on the signs of coefficients of univariate real rooted polynomials and their effect on the roots.

Proposition 1.1.13. *Let $h \in \mathbb{R}[t]$ be a univariate real rooted polynomial. Then the following are equivalent:*

(i) For $x \in \mathbb{R}$, $h(x) = 0 \implies x \leq 0$.

(ii) All non-zero coefficients of h have the same sign.

Proof. Assume that (i) is true. Let d be the degree of h. We can factor h as $h = \lambda \prod_{i=1}^{n}(t - r_i)$ where r_i are the roots of h, and $\lambda \in \mathbb{R}$ is a constant. Since $-r_i \geq 0$, the number of sign variation is zero. Now, assume that (ii) is true. Let $\operatorname{var}(h)$ denote the number of sign variations of h, and $\operatorname{pos}(h)$ denote the number of positive roots of h. By Descartes' law of signs (see for example [4, Theorem 2.33]), $\operatorname{var}(h) \geq \operatorname{pos}(h)$. Since $\operatorname{var}(h) = 0$ in our case, we have that $\operatorname{pos}(h) = 0$. □

This gives us another way to express the hyperbolicity cone of a hyperbolic polynomial.

Corollary 1.1.14. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$ with $h(e) > 0$. Then,*

$$C_h(e) = \{v \in \mathbb{R}^n : h(et + v) \in \mathbb{R}[t] \text{ has only non-negative coefficients}\}.$$

The following proposition illustrates the connection between a hyperbolic polynomial whose coefficients have the same sign, and the containment of the non-negative orthant in its hyperbolicity cone.

Proposition 1.1.15. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e = (1, \ldots, 1) \in \mathbb{R}^n$. If h has only non-negative coefficients, then $\mathbb{R}^n_{>0} \subset C_h^\circ(e)$ and $\mathbb{R}^n_{\geq 0} \subset C_h(e)$.*

Proof. Consider the univariate restriction $h(et + v)$ for all $v \in \mathbb{R}^n_{\geq 0}$. Since h has only non-negative coefficients and $h(e) > 0$, $h(et+v)$ also has non-negative coefficients. Thus, by Corollary 1.1.14, $\mathbb{R}^n_{\geq 0} \subset C_h(e)$. For the inclusion of the positive orthant in the interior of $C_h(e)$, let $e' \in \mathbb{R}^n_{>0}$. Zero cannot be a root of $h(et + e')$ as $h(e') > 0$ by the non-negativity of the coefficients of h. This together with the fact that $\mathbb{R}^n_{>0} \subset C_h(e)$ gives that $h(et+e') = 0$ implies $t < 0$. Thus, $\mathbb{R}^n_{>0} \subset C_h^\circ(e)$. □

In particular, a polynomial satisfying the assumptions of the proposition is hyperbolic with respect to every point in the positive orthant.

Moreover, when the hyperbolicity cone of some polynomial does not contain the positive orthant, we can find an invertible linear transformation that translates the cone in a way that it contains the positive orthant.

Lemma 1.1.16. *Let $h(x) \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$ such that $C_h(e)$ does not contain $\mathbb{R}^n_{>0}$. Then, there exists an invertible linear transformation $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(\mathrm{T}(x))$ is hyperbolic with respect to $e' := \mathrm{T}^{-1}(e)$ with $\mathbb{R}^n_{>0} \subset C_{h(\mathrm{T}(x))}(e')$.*

Proof. Since $e \in C_h^\circ(e)$, the interior of the hyperbolicity cone is non-empty, and is the connected component of $\{x \in \mathbb{R}^n : h(e) \neq 0\}$, it is dense in $C_h(e)$. Let $v_1, \ldots, v_n \in C_h^\circ(e)$ be linearly independent vectors. Then, there is an invertible map $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^n$ such that T^{-1} transforms $v_1, \ldots, v_n$ to standard basis vectors. By Proposition 1.1.4, $h(\mathrm{T}(x))$ is hyperbolic with respect to $T^{-1}(e)$, and the image of the map T^{-1} applied on the hyperbolicity cone $C_h(e)$ contains the positive orthant. □

Observe that, by the relation of the roots of a polynomial and the roots of its derivative, the hyperbolicity cone of the derivative of a hyperbolic polynomial contains the hyperbolicity cone of the other. Generally, for any two hyperbolic polynomials, the containment relation of their hyperbolicity cones requires some condition on the roots of their univariate restriction, as shown in the following lemma.

Lemma 1.1.17 (Lemma 3.4 in [39]). *Let $h, g \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. For $v \in \mathbb{R}^n$, let $g_{\min}(v), g_{\max}(v)$ be the smallest and greatest roots of $g(et - v)$ and $h_{\min}(v), h_{\max}(v)$ be the smallest and greatest roots of $h(et - v)$ respectively. Then,*

$$h_{\min}(v) \leq g_{\min}(v) \leq g_{\max}(v) \leq h_{\max}(v)$$

for all $v \in \mathbb{R}^n$ if and only if $C_h \subset C_g$.

Proof. Assume that $C_h \subset C_g$ and let $y := t + h_{\min}(v)$ for $v \in \mathbb{R}^n$. Then, $h(ey - v)$ has its smallest root zero, so that $h(ey - v)$ has only non-negative zeros. Moreover, $-eh_{\min}(v) + v \in C_h$, thus in C_g. Now, consider $g(et - (-eh_{\min}(v) + v)) = g(ey - v)$. By the definition of C_g, $g(ey - v)$ has only non-negative zeros, therefore $h_{\min}(v) \leq g_{\min}(v)$.

For the relation of the maximal roots, let $y' := t + h_{\max}(v)$ for $v \in \mathbb{R}^n$. Then, $h(ey' + v)$ has its maximal root zero, and $eh_{\max}(v) + v \in C_h$ so that $eh_{max}(v) + v \in C_g$. Hence, $g(et + (eh_{\max}(v) + v)) = g(ey' + v)$ has only non-positive roots so that $h_{\max}(v) \geq g_{\max}(v)$.

For the other direction, assume that $h_{\min}(v) \leq g_{\min}(v) \leq g_{\max}(v) \leq f_{\max}(v)$ holds for all $v \in \mathbb{R}^n$. Without loss of generality, assume that $h_{\min}(v) \geq 0$, and let $v \in C_h$. Then, $h(et - v) = 0$ implies that $t \geq 0$. By the assumption and hyperbolicity of g with respect to $e \in \mathbb{R}^n$, $g(et - v) = 0$ implies that $t \geq 0$. Hence $v \in C_g$. □

The Half-Plane Property and Stability

The half-plane property and stability properties of a polynomial are about the half-spaces where the polynomial does not vanish.

Definition 1.1.18. A polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$ said to have the *half-plane property* (HPP) if there exists an open half-plane $H \subset \mathbb{C}$ with $0 \in \partial H$ such that

$$h(x_1, \ldots, x_n) \neq 0$$

whenever $x_1, \ldots, x_n \in H$.

Definition 1.1.19. A polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$ is called *stable* if

$$h(x_1, \ldots, x_n) \neq 0$$

whenever $x_1, \ldots, x_n \in H'$ where $H' := \{x \in \mathbb{C} : \mathrm{Im}(x) > 0\}$.

The following theorem by Hurwitz is an important tool for the theory of stable polynomials that we use several times throughout the text.

Hurwitz's Theorem (Theorem 1.3.8 in [52])**.** *Let $\Omega \subseteq \mathbb{C}^n$ be an open, connected set and $(h_k : k \in \mathbb{N})$ be a sequence of analytic functions that do not vanish on Ω such that it converges uniformly to a function h on every compact subset of Ω. Then h is either non-vanishing on Ω or identically zero.*

Below are some operations that preserve the stability. See also [8] for detailed work by Borcea and Brändén on differential operators that preserve stability.

Lemma 1.1.20 (Lemma 2.4 in [60])**.** *Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ be a stable polynomial.*

(i) Permutation of variables: $h(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is stable where $\sigma : [n] \to [n]$ is a permutation.

(ii) Scaling: For $\lambda \in \mathbb{C}$ and $c \in \mathbb{R}^n_{>0}$, $\lambda h(c_1x_1, \ldots, c_nx_n)$ is stable or identically zero.

(iii) Diagonalization: For $\{i, j\} \subseteq [n]$, $h(x_1, \ldots, x_n)|_{x_i=x_j}$ is stable.

(iv) Specialization: For $z \in \overline{H'}$, $h(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$ is stable or identically zero.

(v) Inversion: If the degree of x_i in h is d_i, then

$$x_i^{d_i} h(x_1, \ldots, x_{i-1}, -x_i^{-1}, x_{i+1}, \ldots, x_n)$$

is stable.

(vi) Differentiation: $\frac{\partial}{\partial x_i} h(x_1, \ldots, x_n)$ is stable or identically zero.

Proof. We only give the provide the proof of $(iv), (v)$ and (vi) as $(i) - (iii)$ are clear.

(iv) When $\mathrm{Im}(z) > 0$, $h(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$ is stable. For the case $\mathrm{Im}(z) = 0$, consider the sequence

$$\big(h(x_1, \ldots, x_{i-1}, z + i2^{-k}, x_{i+1}, \ldots, x_n) : k \in \mathbb{N}\big)$$

and take the limit when k goes to infinity. By Hurwitz's theorem,

$$h(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$$

is stable or identically zero.

(v) If $z \in H'$, then $-z^{-1} \in H'$.

(vi) Without loss of generality, let us assume that $i = 1$. For all $z_2, \dots, z_n \in H'$, the polynomial $\tilde{h}(t) = h(t, z_2, \dots, z_n) \in \mathbb{C}[t]$ has degree d_1 that is the degree of x_1 in h. This follows from the fact that the limit of the sequence $\left(k^{-d_1} h(kx_1, \dots, x_n) : k \geq 1\right)$ when k goes to infinity, is a non-zero polynomial (we may assume that $d_1 \geq 1$ or it falls back to the identically zero case). Now, consider $\tilde{h}(t)$ and $\frac{\partial}{\partial t}\tilde{h}(t)$. By Gauss-Lucas theorem (see for example [52, Theorem 2.1.1]), the zeros of $\frac{\partial}{\partial t}\tilde{h}(t)$ lie in the convex hull of the zeros of $\tilde{h}(t)$. Thus the derivative is stable or identically zero.

□

The last part of the lemma yields the following corollary.

Proposition 1.1.21 (Proposition 2.8 in [18])**.** *Let $h \in \mathbb{C}[x_1, \dots, x_n]$ be stable and let $\lambda_i \in \mathbb{R}_{\geq 0}$. Then, $\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i} h(x_1, \dots, x_n)$ is stable or identically zero.*

Proof. Follows from part (vi) of Lemma 1.1.20. For each $i \in [n]$, $\frac{\partial}{\partial x_i} h(x_1, \dots, x_n)$ is stable, thus it does not vanish when we insert $z_1, \dots, z_n \in H'$ unless the derivative is identically zero. Now it remains to show that their non-negative combinations are either identically zero or non-vanishing. We first need to prove the following claim.

Claim: Given that h is not identically zero, the rational function

$$h^{-1} \left(\sum_{i=1}^n \lambda_i \frac{\partial h}{\partial x_i} \right)$$

has negative imaginary part on $(H')^n$, for $\lambda_i \in \mathbb{R}$, $i \in [n]$ except when it is identically zero.

Proof. First consider the univariate case. If h has degree zero, its derivative is identically zero, thus the claim holds. When h is a stable polynomial of degree $d \geq 1$, it is of the form $h(x) = \lambda \prod_{i=1}^d (x - \alpha_i) \in \mathbb{C}[x]$ with $\lambda \neq 0$ and $\alpha_i \in \mathbb{R}$ (stability in one variable implies real-rootedness). Then, for all $z \in H'$

$$\frac{h'(z)}{h(z)} = \sum_{i=1}^d \frac{1}{z - \alpha_i} \neq 0 \text{ and } \mathrm{Im}(\frac{h'(z)}{h(z)}) < 0.$$

For the multivariate case, consider

$$h(z_1, z_2, \dots, z_{i-1}, x_i, z_{i+1}, \dots, z_n) \in \mathbb{C}[x_i]$$

for all $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n \in H'$ for $i \in [n]$. In this case $h^{-1} \frac{\partial h}{\partial x_i}$ has non-positive imaginary part on $(H')^n$ for all $i \in [n]$, therefore

$$h^{-1}(z_1, \dots, z_n) \left(\sum_{i=1}^n \lambda_i \frac{\partial h}{\partial x_i}(z_1, \dots, z_n) \right)$$

also has non-positive imaginary part for all $z_1, \dots, z_n \in H'$ for $\lambda_i \in \mathbb{R}_{\geq 0}$. By the open mapping theorem, the image of $h^{-1} \left(\sum_{i=1}^n \lambda_i \frac{\partial h}{\partial x_i} \right)$ on $(H')^n$ is either open in $\mathbb{C}$ or $h^{-1}\left(\sum_{i=1}^{n} \lambda_i \frac{\partial h}{\partial x_i}\right)$ is a real constant. But it cannot be a non-zero constant because of the degree difference. Therefore, $h^{-1}\left(\sum_{i=1}^{n} \lambda_i \frac{\partial h}{\partial x_i}\right)$ has only negative imaginary part on $(H')^n$ or it is identically zero. □

Since $h^{-1}\left(\sum_{i=1}^{n} \lambda_i \frac{\partial h}{\partial x_i}\right)$ either has a negative imaginary part on $(H')^n$ or it is identically zero, and since h is non-vanishing on $(H')^n$, $\sum_{i=1}^{n} \lambda_i \frac{\partial h}{\partial x_i}$ is either non-vanishing on $(H')^n$ or it is identically zero. □

For homogeneous polynomials the notion of stability and the half-plane property coincide. This implies that for homogeneous polynomials stability is not about a specific half-plane, but about the existence of a half-plane for which the polynomial has the required property.

Proposition 1.1.22. *A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is stable if and only if h has the half-plane property.*

Proof. The if direction is clear. For the only if direction, assume that h has the half-plane property for a half-plane $H \subset \mathbb{C}$, and that there exists $x_1, \ldots x_n \in \mathbb{C}$ with $\mathrm{Im}(x_i) > 0$ such that $h(x_1, \ldots, x_n) = 0$. Since h is homogeneous, we can find a $c \in \mathbb{C}$ such that $c^d h(x_1, \ldots, x_n) = h(cx_1, \ldots, cx_n) = 0$ where $cx_i \in H$. This contradicts with the half-plane property. □

The fact that in one variable, stability implies real rootedness hints at the connection between stable polynomials and hyperbolic polynomials.

Proposition 1.1.23. *A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ has the half-plane property if and only if it is hyperbolic with respect to every $e \in \mathbb{R}^n_{>0}$.*

Proof. Let h have the half-plane property, and thus be stable. Since stability in one variable implies real-rootedness, for any $e \in \mathbb{R}^n_{>0}$, the univariate restriction $h(et - v)$ is real rooted for every $v \in \mathbb{R}^n$. The stability of $h(et - v)$ follows from the fact that

$$(H')^n = \{et - v : v \in \mathbb{R}^n,\, e \in \mathbb{R}^n_{>0} \text{ and } t \in H'\}\,.$$

Assume that $h(e) = 0$ for some $e \in \mathbb{R}^n_{>0}$. Then $i^d h(e) = h(e_1 i, \ldots, e_n i) = 0$ which contradicts the stability. For the other direction, let h be hyperbolic with respect to every point in the positive orthant. Then for any $e \in \mathbb{R}^n_{>0}$, $h(et - v)$ is real rooted for every $v \in \mathbb{R}^n$. In particular for $t = i$, $h(et - v) = h(e_1 i - v_1, \ldots, e_n i - v_n) \neq 0$ for all $e \in \mathbb{R}^n_{>0}$ and $v \in \mathbb{R}^n$ so that $h(x_1, \ldots, x_n) \neq 0$ for $x_1, \ldots, x_n \in H'$ shows the half-plane property. □

By Lemma 1.1.16, we can always find a linear transformation to bring a hyperbolic polynomial and its hyperbolicity cone in a form that it is stable. Therefore for further studies on properties of hyperbolic polynomials and their hyperbolicity cones, we can restrict our focus on hyperbolic polynomials that are stable. Below are some examples of stable polynomials.

Example 1.1.24.

- Elementary symmetric polynomials

$$E_{k,n}(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \ldots x_{i_k}$$

  are stable since they are hyperbolic with respect to $e = (1, \ldots, 1)$ and their hyperbolicity cone contains $\mathbb{R}^n_{>0}$ [18, Theorem 9.1].

- For $n \times n$ positive semi-definite (PSD) matrices $A_1, \ldots A_n$ and an $n \times n$ real symmetric matrix B,

$$h(x_1, \ldots, x_n) = \det(A_1 x_1 + \cdots + A_n x_n + B)$$

  is stable [9, Proposition 2.4]

- The basis generating polynomial of the Vámos matroid has the half-plane property [63] (see §1.2 for the definition of the basis-generating polynomial of a matroid).

When there is zero sign variation among the coefficients of a hyperbolic polynomial, we obtain an easier characterization of stability.

Corollary 1.1.25. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial whose coefficients have the same sign. Then, h is stable if it is hyperbolic with respect to $e = (1, \ldots, 1)$.*

Proof. By Proposition 1.1.15, h is hyperbolic with respect to every $e \in \mathbb{R}^n_{>0}$. □

Note that the condition on the sign variations of coefficients is essential for the previous corollary.

Further, the coefficients of stable polynomials do not have sign variation.

Proposition 1.1.26 (Theorem 6.1 in [18])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a stable polynomial of degree d. Then all non-zero coefficients of h have the same sign.*

Proof. We apply induction on n. For $n = 1$, the claim holds as $h(x_1) = a_d x_1^d$ for some $a_d \in \mathbb{R}$. For $n \geq 2$ assume that the claim holds for $n - 1$. Let $d_n \leq d$ be the degree of x_n in h. One can write h as

$$h(x_1, \ldots, x_n) = \sum_{k=0}^{d_n} x_n^k h_k(x_1, \ldots, x_{n-1})$$

where $h_k(x_1, \ldots, x_{n-1}) := \frac{1}{k!} \frac{\partial^k}{x_n^k} h(x_1, \ldots, x_n)|_{x_n=0}$. In particular, ever coefficient of h corresponds to exactly one coefficient of h_k. Since each h_k is a homogeneous polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ and by Lemma 1.1.20 taking derivatives and setting some variables equal to zero preserves stability, h_k is stable. Therefore, by the induction hypothesis all non-zero coefficients of each h_k have the same sign. Now we need to show that all h_k have the same sign.

Note that by Proposition 1.1.15, h is hyperbolic with respect to every point in the positive orthant. In particular, $h(e) \neq 0$ for $e \in \mathbb{R}^n_{>0}$. Now consider $h(e't + v)$ for $e' = (0, \ldots, 0, 1) \in \mathbb{R}^n$ and $v = (1, \ldots, 1, 0) \in \mathbb{R}^n$. We have

$$h(e't + v) = h(1, \ldots, 1, t) = \sum_{k=0}^{d_n} t^k h_k(1, \ldots, 1).$$

This polynomial has only non-positive zeros, since for $t > 0$, $e'' := (1, \ldots, 1, t) \in \mathbb{R}^n_{>0}$ is in the interior of the hyperbolicity cone of h, thus $h(e'') = h(e't+v) \neq 0$. Moreover, since h is stable, by Lemma 1.1.20, $h(1, \ldots, 1, t)$ is stable, hence real-rooted. By Proposition 1.1.13, for a univariate real rooted polynomial, having only non-positive zeros implies that there is no sign variation among its non-zero coefficients. Therefore, all $h_k(1, \ldots, 1)$ have the same sign for $0 \leq k \leq d_n$. □

The following theorem illustrates the connection between the stability of polynomials that are not necessarily homogeneous and hyperbolicity of their homogenization.

Theorem 1.1.27 (Proposition 1.1 in [8])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial of degree d, and $\hat{h} \in \mathbb{R}[x_1, \ldots, x_{n+1}]$ be the unique homogeneous polynomial of degree d such that $\hat{h}(x_1, \ldots, x_n, 1) = h(x_1, \ldots, x_n)$. Then, h is stable if and only if $\hat{h}$ is hyperbolic with respect to every $e \in \mathbb{R}^{n+1}_{\geq 0}$ with $e_{n+1} = 0$ and $(e_1, \ldots, e_n) \in \mathbb{R}^n_{>0}$.*

Proof. Let h be stable of degree d and

$$\hat{h}(x_1, \ldots, x_{n+1}) := x^d_{n+1} h(x_1 x^{-1}_{n+1}, \ldots, x_n x^{-1}_{n+1})$$

be its homogenization. Consider the limit h_{hom} of the sequence

$$\left(k^{-d} h(kx_1, \ldots, kx_n) : k > 0\right)$$

when k goes to infinity. This limit is the homogeneous degree d part of h. By Hurwitz's theorem, h_{hom} is stable, and by Proposition 1.1.23, it is hyperbolic with respect to every $e \in \mathbb{R}^n_{>0}$. In particular, $h_{hom}(e) \neq 0$ for all $e \in \mathbb{R}^n$. Hence, for all $e \in \mathbb{R}^{n+1}_{\geq 0}$ with $e_{n+1} = 0$ and $(e_1, \ldots, e_n) \in \mathbb{R}^n_{>0}$, $\hat{h}(e_1, \ldots, e_n, 0) = h_{hom}(e_1, \ldots, e_n) \neq 0$. Now consider the univariate restriction $\hat{h}(et + v) = \hat{h}(e_1 t + v_1, \ldots, e_n t + v_n, v_{n+1})$ for all $v \in \mathbb{R}^{n+1}$. When $v_{n+1} = 0$, the restriction is just $h_{hom}(e_1 t + v_1, \ldots, e_n t + v_n)$, which by the hyperbolicity of h_{hom}, is real rooted for all $(v_1, \ldots, v_n) \in \mathbb{R}^n$. When $v_{n+1} \neq 0$,

$$\hat{h}(e_1 t + v_1, \ldots, e_n t + v_n, v_{n+1}) = v^d_{n+1} h((e_1 t + v_1) v^{-1}_{n+1}, \ldots, (e_n t + v_n) v^{-1}_{n+1})$$

is real rooted since $h(e_1 t + v_1, \ldots, e_n t + v_n)$ is real rooted by the stability of h. Hence, $\hat{h}$ is hyperbolic with respect to the desired vectors.

Now assume that $\hat{h}$ is hyperbolic with respect to all $e \in \mathbb{R}^{n+1}_{\geq 0}$ with $e_{n+1} = 0$ and $(e_1, \ldots, e_n) \in \mathbb{R}^n_{>0}$. Then, $\hat{h}(e_1, \ldots, e_n, 0) \neq 0$ for all $(e_1, \ldots, e_n) \in \mathbb{R}^n_{>0}$. In particular,

$$\hat{h}(e_1 t + v_1, \ldots, e_n t + v_n, 1) = h(e_1 t + v_1, \ldots, e_n t + v_n)$$

is real rooted for all $(v_1, \ldots, v_n) \in \mathbb{R}^n$ and $(e_1, \ldots, e_n) \in \mathbb{R}^n_{>0}$ so that $h(x_1, \ldots, x_n)$ does not vanish for all $x_1, \ldots, x_n \in H'$. Thus, h is stable. □

One can further consider *the polarization* of a homogeneous polynomial in order to turn it into multiaffine symmetric polynomial. A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d can be written as

$$h(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_d = 1}^{n} a_{i_1 \ldots i_d} x_{i_1} \ldots x_{i_d}$$

with symmetric coefficients $a_{i_1 \dots i_d}$. Then, its polarization is the unique polynomial

$$\tilde{h}(x_{11}, \dots, x_{1d}, \dots, x_{n1}, \dots, x_{nd})$$

that is multiaffine and symmetric in $x_{i1}, \dots, x_{id}$ with

$$\tilde{h}(\underbrace{x_1, \dots, x_1}_{d \text{ times}}, \dots, \underbrace{x_n, \dots, x_n}_{d \text{ times}}) = h(x_1, \dots, x_n).$$

The following theorem is one of the *Grace-Walsh-Szegö theorems*. It is a useful tool for relating polynomials with their polarizations.

Walsh's Coincidence Theorem (Theorem 3.4.1b in [52])**.** *Let $h \in \mathbb{C}[x_1, \dots, x_n]$ be symmetric multiaffine polynomial. Then every open half-plane $H \subseteq \mathbb{C}$ containing points $z_1, \dots, z_n$ contains at least one $y \in H$ such that*

$$h(z_1, \dots, z_n) = h(y, \dots, y).$$

As a consequence, we obtain the stability of polarization of a stable polynomial as below.

Corollary 1.1.28. *Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a stable polynomial of degree d. Then, there exists a stable multiaffine polynomial*

$$\tilde{h} \in \mathbb{R}[x_{11}, \dots, x_{1d}, \dots, x_{n1}, \dots, x_{nd}]$$

such that

$$h(x_1, \dots, x_n) = \tilde{h}(\underbrace{x_1, \dots, x_1}_{d \text{ times}}, \dots, \underbrace{x_n, \dots, x_n}_{d \text{ times}}).$$

Proof. Assume that h is stable and consider its polarization

$$\tilde{h}(x_{11}, \dots, x_{1d}, \dots, x_{n1}, \dots, x_{nd}).$$

Since $\tilde{h}$ is multiaffine, it can be written as sum of the product of some elementary symmetric polynomials $E_i := E_{k,d}(x_{i1}, \dots, x_{id}) \in \mathbb{R}[x_{i1}, \dots, x_{id}]$ of degree k for $1 \leq i \leq n$ and for some $0 \leq k \leq d$, thus it is symmetric in each block $x_{i1} \dots x_{id}$ of variables for $1 \leq i \leq n$. Let $H' \subseteq \mathbb{C}$ be the upper half-plane and assume that there are $z_{11}, \dots, z_{1d}, \dots, z_{n1}, \dots, z_{nd} \in H'$ such that

$$\tilde{h}(z_{11}, \dots, z_{1d}, \dots, z_{n1}, \dots z_{nd}) = 0.$$

Then, after applying Walsh's coincidence theorem on each block of variables, we obtain the existence of $y_1, \dots, y_n \in H'$ with

$$\tilde{h}(\underbrace{y_1, \dots, y_1}_{d \text{ times}}, \dots, \underbrace{y_n, \dots, y_n}_{d \text{ times}}) = 0.$$

Since $\tilde{h}(\underbrace{y_1, \dots, y_1}_{d \text{ times}}, \dots, \underbrace{y_n, \dots, y_n}_{d \text{ times}}) = h(\underbrace{y_1, \dots, y_n}_{n \text{ times}})$, this contradicts with the stability of h. □

Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial. The following diagram summarizes the relation of the properties we have seen so far.

$$h \text{ is stable} \quad \Longleftrightarrow \quad h \text{ has HPP} \quad \Longleftrightarrow \quad \begin{array}{c} h \text{ is hyperbolic} \\ \text{with resp. to all } e \in \mathbb{R}^n_{\geq 0} \end{array}$$

Therefore, throughout the text, we use the terms half-plane property, stable and hyperbolic interchangeably.

Determinantal Representability

In this section we focus on polynomials that can be expressed as the determinant of a linear pencil of positive semi-definite matrices. First let us recall the definition of positive semi-definiteness and recall their properties.

Definition 1.1.29. A real symmetric matrix A of size $n \times n$ is called *positive semi-definite* (PSD) if for all $x \in \mathbb{R}^n$, $x^T Ax \geq 0$, and it is indicated by $A \succeq 0$. When we have $x^T Ax > 0$ for all non-zero $x \in \mathbb{R}^n$, A is called *positive definite* (PD), and indicated as $A \succ 0$.

As one can see from the definition, those matrices are closely related to non-negativity of quadratic forms. Throughout the text, $\mathrm{Sym}^n_{\mathbb{R}}$ presents the set of real symmetric matrices of size $n \times n$. One can more generally define semi-definiteness and definiteness on complex Hermitian matrices A and complex vectors x, but throughout the text we are interested in the real case. There are several equivalent ways to define (semi)definiteness as below.

Proposition 1.1.30. *Let $A \in \mathrm{Sym}^n_{\mathbb{R}}$ be a real symmetric matrix. The following are equivalent.*

(i) A is positive semi-definite, i.e., $A \succeq 0$.

(ii) For all $x \in \mathbb{R}^n$, $x^T Ax \geq 0$.

(iii) All $2^n - 1$ principal minors of A are non-negative.

(iv) All eigenvalues of A are non-negative

(v) There exists a factorization $A = BB^T$ where B is a real $n \times r$ matrix, r is the rank of A.

Proposition 1.1.31. *Let $A \in \mathrm{Sym}^n_{\mathbb{R}}$ be a real symmetric matrix. The following are equivalent.*

(i) A is positive definite, i.e., $A \succ 0$.

(ii) For all non-zero $x \in \mathbb{R}^n$, $x^T Ax > 0$.

(iii) All leading principal minors of A are positive.

(iv) All eigenvalues of A are positive

(v) There exists a factorization $A = BB^T$ where B is a real non-singular, square matrix.

Now, let us give the definition of determinantal representability of a homogeneous polynomial.

Definition 1.1.32. A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is called to have a *determinantal representation* if there exists PSD matrices $A_1, \ldots, A_n$ in $\mathrm{Sym}^d_{\mathbb{R}}$ such that $h = \lambda \det(A_1 x_1 + \cdots + A_n x_n)$ for some $\lambda \in \mathbb{R}$.

If there exists a positive integer $r \in \mathbb{N}$ such that h^r has a determinantal representation, then h is called to have a weak determinantal representation.

Remark 1.1.33. The determinantal representability of a polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d can alternatively be defined with real symmetric matrices (not necessarily PSD) $A_1, \ldots, A_n$ in $\mathrm{Sym}^d_{\mathbb{R}}$ for which

$$h = \lambda \det(A_1 x_1 + \cdots + A_n x_n)$$

for some $\lambda \in \mathbb{R}$, and there exists an $e \in \mathbb{R}^n$ such that $e_1 A_1 + \cdots + e_n A_n \succ 0$.

We are interested in polynomials with the half-plane property that have a determinantal representation. The following proposition shows that if a polynomial with a determinantal representation as given in Remark 1.1.33 is stable, the existence of a vector $e \in \mathbb{R}^n$ with $\sum_{i=0}^n A_i e_i$ is equivalent to the condition that A_i's in the representation are positive semi-definite. Hence, Definition 1.1.32 is just an adapted version.

Proposition 1.1.34. *Let $A_1, \ldots, A_n \in \mathrm{Sym}^d_{\mathbb{R}}$ be real symmetric matrices and $e = (1, \ldots, 1) \in \mathbb{R}^n$. If the polynomial $h(x_1, \ldots, x_n) = \det(x_1 A_1 + \cdots + x_n A_n)$ has the half-plane property then the following are equivalent:*

(i) $A(e) := e_1 A_1 + \cdots + e_n A_n = \sum_{i=1}^n A_i \succ 0$.

(ii) $A_1, \ldots, A_n$ *are positive semi-definite.*

Proof. Assume that $A(e) \succ 0$. Then, $h(e) > 0$. Since h is stable, by Proposition 1.1.23, h is hyperbolic with respect to every $e' \in \mathbb{R}^n_{>0}$. Moreover, the univariate restriction $h(e' - et)$ for all $e' \in \mathbb{R}^n_{>0}$ has only positive roots. By the continuity of roots in e', we have that $h(e'' - et)$ for standard basis vectors e'' has only non-negative roots. Since $A(e) \succ 0$, there exists an invertible real matrix $M \in \mathbb{R}^{d \times d}$ such that $M^{-1} A(e) M = D$ where D is diagonal whose entries are eigenvalues λ_i of $A(e)$ (thus they are all positive). The polynomial

$$\begin{aligned} h(e'' - et) =& \frac{1}{\det(M)^2} \det(M^{-1} A(e'') M - Dt) \\ =& \frac{1}{\det(M)^2 \det(D)} \det(B - It) \end{aligned}$$

where B is obtained by scaling the ith row of $M^{-1} A(e'') M$ by $1/\lambda_i$ for $1 \leq i \leq n$ has only non-negative roots. Similar matrices have same eigenvalues and the eigenvalues of B are positive scalings of the eigenvalues of $A(e'')$. Then, matrices $A_i = A(e'')$ for some standard basis vector e'' for $1 \leq i \leq n$ have only non-negative eigenvalues, hence are positive semi-definite.

Now, assume that $A_1, \ldots A_n$ are positive semi-definite. Then, their positive linear combinations are also positive semi-definite. Since h is hyperbolic with respect to every $e' \in \mathbb{R}^n_{>0}$, we especially have that $h(e) = \det(\sum_{i=1}^n A_i) \neq 0$. Hence, $A(e)$ does not have zero as an eigenvalue so that it is positive definite. $\square$

When a homogeneous polynomial with a determinantal representation is hyperbolic, we can express its hyperbolicity cone in terms of semi-definiteness of the pencil of its representing matrices.

Corollary 1.1.35. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial with a determinantal representation*

$$h(x_1, \ldots, x_n) = \det(A(v)) = \det(A_1 x_1 + \cdots + A_n x_n)$$

with PSD matrices $A_1, \ldots A_n$. If h is hyperbolic with respect to $e = (1, \ldots, 1) \in \mathbb{R}^n$, then

$$\begin{aligned} C_h(e) &= \{v \in \mathbb{R}^n : A(v) \succeq 0\} \\ C_h^\circ(e) &= \{v \in \mathbb{R}^n : A(v) \succ 0\}. \end{aligned}$$

Proof. Since h is hyperbolic with respect to $e = (1, \ldots, 1)$, we have that $h(e) = \det(\sum_{i=1}^n A_i) \neq 0$. By the positive semi-definiteness of A_i, it follows that $A(e) = \sum_{i=1}^n A_i$ is positive definite and there exists an invertible matrix $M \in \mathbb{R}^{d \times d}$ such that $M^{-1}A(e)M = D$, where D is the diagonal matrix whose entries are eigenvalues of $A(e)$ (they are all positive). Then, for all $v \in \mathbb{R}^n$, the roots of the univariate restriction

$$h(v - et) = \det(A(v) - tA(e)) = \frac{1}{\det(M)^2} \det(M^{-1}A(v)M - Dt)$$

are scalings of the roots of the characteristic polynomial of $A(v)$ by the eigenvalues of D. Thus, the restriction has only non-negative roots if and only if $A(v)$ has non-negative eigenvalues, and it has only positive roots if and only if $A(v)$ has only positive eigenvalues for all $v \in \mathbb{R}^n$. $\square$

Example 1.1.36. The polynomial

$$h(x_1, x_2, x_3) = 2x_1^2 x_2 + 3x_1^2 x_3 + 2x_1 x_2^2 + 10x_1 x_2 x_3 + 6x_1 x_3^2 + 5x_2^2 x_3 + 6x_2 x_3^2$$

is the determinant of a linear pencil of PSD matrices since

$$\begin{aligned} h(x_1, x_2, x_3) &= \det\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} x_3\right) \\ &= \det\begin{pmatrix} x_1 + x_2 & 0 & x_2 \\ 0 & x_1 + x_2 + 2x_3 & x_1 + 2x_3 \\ x_2 & x_1 + 2x_3 & x_1 + x_2 + 5x_3 \end{pmatrix}. \end{aligned}$$

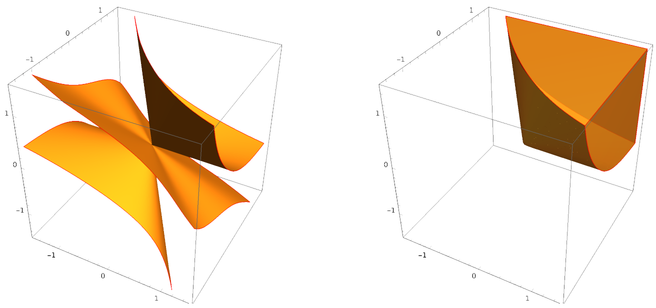
Moreover, h is hyperbolic with respect to $e = (1, 1, 1)$ since $h(1, 1, 1) \neq 0$ and $h(et - v)$ has real roots for all $v \in \mathbb{R}^3$ because real symmetric matrices have real eigenvalues. The hyperbolicity cone of $C_h(e)$ is

$$C_h(e) = \left\{v \in \mathbb{R}^3 : \begin{pmatrix} v_1 + v_2 & 0 & v_2 \\ 0 & v_1 + v_2 + 2v_3 & v_1 + 2v_3 \\ v_2 & v_1 + 2v_3 & v_1 + v_2 + 5v_3 \end{pmatrix} \succeq 0\right\}$$

depicted in Figure 1.3.

The following proposition illustrates the relation between determinantal representability and stability.

Proposition 1.1.37. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. If h has a determinantal representation, then it has the half-plane property.*



(a) Variety $\mathcal{V}_{\mathbb{R}}(h)$ of h (b) Hyperbolicity cone $C_h(e)$ of h

Figure 1.3: Hyperbolicity cone of a polynomial with a determinantal representation and its variety.

Proof. Let $d > 0$ be the degree of h and assume that h has a determinantal representation. Then, there exists PSD matrices $A_1, \ldots, A_n$ of size $d \times d$ such that $h = \det(A_1x_1 + \cdots + A_nx_n)$. By the degree argument, at least one of the A_i's has rank d. As $\sum_{i=1}^n A_i$ is PSD and has rank d, it is PD. Thus, $h(e) > 0$ for $e = (1, \ldots, 1)$.

Now, consider the univariate restriction $h(v - et)$ for all $v \in \mathbb{R}^n$. Without loss of generality, we may assume that $\sum_{i=1}^n A_i = I$. Then, $h(v - et) = \det(A(v) - It)$ is the characteristic polynomial of the real symmetric matrix $A(v)$, thus is real rooted. Therefore, h is hyperbolic with respect to e. Note that if the coefficients of h have the same sign, then by Corollary 1.1.25 we are done. For the general case, note that by Corollary 1.1.35, $C_h(e)$ is the set of points $v \in \mathbb{R}^n$ for which $A(v)$ is PSD. Since each A_i is PSD, for the standard basis vectors $e' \in \mathbb{R}^n_{\geq 0}$, $A(e')$ are also PSD. Thus, they are in $C_h(e)$ and the univariate restriction $h(e' - et)$ has only non-negative roots. Since the roots of the univariate restriction $h(v - et)$ continuously depend on the coefficients and $e \in C_h^\circ(e)$, we have that $v \in \mathbb{R}^n_{>0}$ are in $C_h^\circ(e)$. Thus, h is stable. □

One asks whether every homogeneous polynomial with the half-plane property has a determinantal representation. Peter Lax, in 1958 in [42], conjectured a positive answer for this question for polynomials in 3 variables. Later, Lewis, Parillo, and Romana in [43] pointed out that the results of Helton and Vinnikov in [30] prove the Lax conjecture.

Theorem 1.1.38 (Helton-Vinnikov [30])**.** *A homogeneous polynomial $h \in \mathbb{R}[x_1, x_2, x_3]$ of degree d is hyperbolic with respect to $e \in \mathbb{R}^3$ with $h(e) = 1$ if and only if there exist $d \times d$ real symmetric matrices A_1, A_2, A_3 with $A_1e_1 + A_2e_2 + A_3e_3 = I$ such that*

$$h(x_1, x_2, x_3) = \det(A_1x_1 + A_2x_2 + A_3x_3).$$

Note that, by Lemma 1.1.16, we can apply a transformation on the hyperbolicity cone of h and adjust the statement of the theorem for stable polynomials.

In 2007, Helton and Vinnikov in [30] conjectured that every real zero polynomial in n variables has a determinantal representation. Note that real zero polynomials in homogeneous setting correspond to hyperbolic polynomials, and rigidly convex sets in the homogeneous setting correspond to hyperbolicity cones. Brändén in [11] gave a counter-example to their conjecture and showed that the basis generating polynomial h of the Vámos matroid does not have a weak determinantal representation (h is stable due to Wagner and Wei in [63]). This example also shows that the answer is still negative when we restate the conjecture with weak determinantal representation.

Example 1.1.39. The basis generating polynomial of the Vámos matroid has the half-plane property and it does not have a weak determinantal representation ([11, Theorem 3.3]). We define the Vámos matroid in § 1.2.

Brändén uses the properties of the rank functions of matroids and counts parameters to prove that this polynomial gives a counter-example. In the following sections, we will discuss matroids and their relation with the half-plane property and determinantal representability. After introducing spectrahedral cones, we give more details about other versions of the conjecture.

Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. Below, we illustrate the relation of determinantal representability to stability.

$$\begin{tikzcd}
& \text{h has a determinantal representation} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \\
\text{h is weakly determinantal} \arrow[r, Rightarrow] & \text{h has the HPP} \arrow[r, Leftrightarrow] & \text{h is hyperbolic with resp. to all $e \in \mathbb{R}^n_{\geq 0}$}
\end{tikzcd}$$

Spectrahedral Representability

This section is about spectrahedral representability of convex sets. We will illustrate how this property is closely related to determinantal representability of polynomials and also to hyperbolicity cones.

A convex set is called spectrahedral if it can be represented as the solution set of a system of linear matrix inequalities.

Definition 1.1.40. A set $S \subseteq \mathbb{R}^n$ is called a *spectrahedron* if it has the form

$$S = \{v \in \mathbb{R}^n : A_0 + A_1 v_1 + \cdots + A_n v_n \succeq 0\}$$

for some given real symmetric matrices $A_0, A_1, \ldots, A_n$ of size $d \times d$.

A convex cone $C \subseteq \mathbb{R}^n$ is called *spectrahedral* if there exists real symmetric matrices $A_1, \ldots, A_n$ of size $d \times d$ such that

$$C = \{v \in \mathbb{R}^n : A_1 v_1 + \cdots + A_n v_n \succeq 0\}\,.$$

In other words, spectrahedra are sections of the cone of PSD matrices. Moreover, polyhedral cones are spectrahedra defined by diagonal matrices. Below are some examples of spectrahedra and spectrahedral cones.

Example 1.1.41.

- A tetrahedron T can be defined as
$$T = \left\{ v \in \mathbb{R}^3 : \begin{pmatrix} v_1+v_2-v_3 & 0 & 0 & 0 \\ 0 & v_1-v_2+v_3 & 0 & 0 \\ 0 & 0 & -v_1+v_2+v_3 & 0 \\ 0 & 0 & 0 & -v_1-v_2-v_3+2 \end{pmatrix} \succeq 0 \right\}.$$
The positive semi-definiteness of this matrix is given by non-negativity of the diagonal entries, thus it is defined by linear inequalities. Polytopes are specialization of spectrahedra.

- The spectrahedron S defined as
$$S = \left\{ v \in \mathbb{R}^3 : \begin{pmatrix} 1 & v_1 & v_2 \\ v_1 & 1 & v_3 \\ v_2 & v_3 & 1 \end{pmatrix} \succeq 0 \right\}$$
is called *Samosa* depicted in Figure 1.4a.

- A slice of the spectrahedron S', defined as
$$S' = \left\{ v \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & 0 \\ y & 0 & z+1 \end{pmatrix} \succeq 0 \right\}$$
is depicted in Figure 1.4b

- The spectrahedral cone C defined as
$$C = \left\{ v \in \mathbb{R}^3 : \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0 \right\}$$
is the cone of PSD 2×2 matrices.

- Consider the spectrahedral cone C' defined as
$$C' = \left\{ v \in \mathbb{R}^6 : \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \succeq 0 \right\}$$
where $X := \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ and $X' := \begin{pmatrix} x_4 & x_5 \\ x_5 & x_6 \end{pmatrix}$. It is the direct sum of two cone of PSD 2×2 matrices in $\mathbb{R}^6$.

The spectrahedral cones have the following properties.

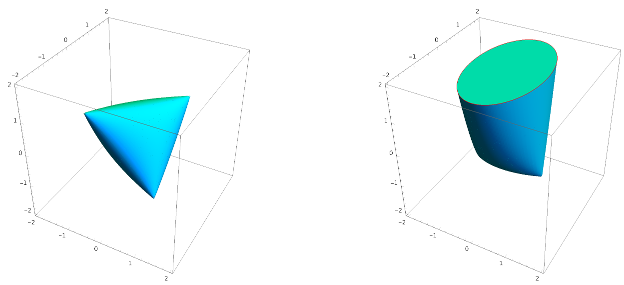
Proposition 1.1.42. *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral cone given by real symmetric matrices $A_1, \dots, A_n$. Then,*

(i) S is closed and convex.

(ii) If there exists $w \in \mathbb{R}^n$ such that $A(w) = \sum_{i=1}^n A_i w_i \succ 0$, then $S^\circ := \{v \in \mathbb{R}^n : A(v) \succ 0\}$ is the interior of S.

(iii) Intersection of two spectrahedral cones is spectrahedral.

Proof. (i) Since a matrix is positive semi-definite if and only if all its principal minors are non-negative, S is defined by finite number of polynomial inequalities, thus it is a closed semi-algebraic set. In order to show the convexity, let $v, w \in S$ and $a, b > 0$ with $a + b = 1$. Since $A(V) \succeq 0$ and $A(w) \succeq 0$, $aA(v) + bA(w) \succeq 0$, therefore $av + bw \in S$.



(a) Samosa (b) A slice of a spectrahedron

Figure 1.4: Examples of spectrahedra

(ii) Let $v \in S^\circ$. Then the characteristic polynomial $\det(A(v) - tI)$ of $A(v)$ has only positive roots, and by the definition of S, $v \in S$. Since the roots continuously depend on coefficients, there exists a neighborhood of v such that for every point w from the neighborhood the characteristic polynomial of $A(w)$ has only positive roots. Thus v cannot be at the boundary so that v is in the interior of S.

For the other inclusion, let v' be in the interior of S. By the definition of S, the characteristic polynomial of $A(v')$ has non-negative roots. Since it is in the interior, there exists a neighborhood of v' that is also in the interior of S. Roots of univariate polynomials continuously depend on their coefficients, and by the assumption, there exists a w in the interior of S for which $A(w) \succ 0$. Since w is in some neighborhood of v', we have $A(v') \succ 0$, thus $v' \in S^\circ$.

(iii) Let S_1 and S_2 be spectrahedral cones defined by the pencils $B(v) \succeq 0$ and $B'(v) \succeq 0$ respectively where $B_1, \ldots, B_n, B'_1, \ldots, B'_n$ are real symmetric matrices. Their intersection $S_1 \cap S_2$ is then defined by the pencil

$$\begin{pmatrix} B(v) & 0 \\ 0 & B'(v) \end{pmatrix} \succeq 0.$$

□

The following proposition shows that, translations of the linear pencil $A(v) \succeq 0$ by a real symmetric matrix A_0 do not change a spectrahedron that has a non-empty interior.

Proposition 1.1.43. *Let S be a spectrahedral cone with a non-empty interior defined as $S := \{v \in \mathbb{R}^n : A_0 + A(v) \succeq 0\}$ and $S' := \{v \in \mathbb{R}^n : A(v) \succeq 0\}$ for some real symmetric matrices $A_0, \ldots, A_n$. Then $S = S'$.*

Proof. Since $0 \in S$, $A_0 \succeq 0$. Then, for every point $v \in S'$, we have that $A(v) + A_0 \succeq 0$ as $A(v) \succeq 0$ by the definition of S'. Thus $v \in S$ and $S' \subseteq S$. Now let $w \in S$ and assume that $A(w)$ has at least one negative eigenvalue. Since S is a convex cone, we can find a big enough $\lambda > 0$ such that $A_0 + \lambda A(w) \in S$ has at least one negative eigenvalue. This contradicts with the definition of S. Thus $A(w) \succeq 0$ for all $w \in S$ so that $w \in S'$ shows the other inclusion.

□

We continue with an example of a special type of spectrahedra called Gram spectrahedra. They especially have an important role on non-negativity of polynomials.

Example 1.1.44. A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $2d$ is a sum of squares (thus non-negative) if there exists a real symmetric PSD matrix G, called *Gram matrix*, of size $\binom{n+d}{d} \times \binom{n+d}{d}$ such that $h = m^T G m$ where m is the vector of all monomials of degree d. The *Gram spectrahedron* of h is the set of all real symmetric PSD matrices G' that give a sum of square representation of h, that is

$$S_G(h) := \left\{ G' \in \mathrm{Sym}_{\mathbb{R}}^{\binom{n+d}{d}} : G' \succeq 0,\, m^T G' m = h \right\}.$$

By Corollary 1.1.35 we have that, when a homogeneous polynomial has a determinantal representation, it is hyperbolic and its hyperbolicity cone is defined by linear matrix inequalities. Further, given a spectrahedral cone defined by a pencil $A_1 + \cdots + A_n \succeq 0$ of some real symmetric matrices $A_1, \ldots, A_n$, consider the polynomial $h := \det(\sum_{i=1}^n A_i x_i)$. The following proposition shows that one can always find some matrices that define a spectrahedron S which makes it a hyperbolicity cone for some polynomial h.

Proposition 1.1.45. *Let $S \subseteq \mathbb{R}^n$ be a spectrahedral cone with a non-empty interior. Then, there exists $d \geq 0$ and real symmetric matrices $B_1, \ldots, B_n \in \mathrm{Sym}_{\mathbb{R}}^d$ such that*

$$S = \{v \in \mathbb{R}^n : B_1 v_1 + \cdots + B_n v_n \succeq 0\}$$

and there exists $w \in \mathbb{R}^n$ for which $B(w) \succ 0$.

Proof. Since S is spectrahedral, there exists real symmetric matrices $A_1, \ldots, A_n$ of some size $m \geq 0$ such that S is defined by the pencil $A(v) \succeq 0$. Then there exists a $d \geq 0$ that is the maximal rank of $A(v)$ for all $v \in S$. Let $w \in S$ with $\mathrm{rk}(A(w)) = d$. Since $A(w) \succeq 0$, there exists an orthogonal matrix M such that $M^T A(w) M =: D$ is a diagonal matrix whose diagonal entries are the eigenvalues of $A(w)$. As $A(w)$ has rank d, without loss of generality, we may assume that all the non-zero entries of D are in the upper left $d \times d$ diagonal block. Matrices A_i are real symmetric, and $(M^T A_i M)^T = M^T A_i M$ for all $1 \leq i \leq n$. Therefore, $M^T A_i M$ are real symmetric and can be written of the form

$$M^T A_i M = \begin{pmatrix} B_i & C_i \\ C_i^T & E_i \end{pmatrix}$$

where B_i are real symmetric matrices of size $d \times d$, and E_i are real symmetric matrices of size $(m-d) \times (m-d)$ for all $1 \leq i \leq n$. Since the $w \in \mathbb{R}^n$ with $A(w) \succeq 0$ is in the interior of S, there exists an $\varepsilon > 0$ neighborhood of w that is also in S. In particular, for all $-\varepsilon < \lambda < \varepsilon$, $A(w) + \lambda A_i \succeq 0$ holds for all $1 \leq i \leq n$. Therefore, for all λ from the range above, we have

$$D + \lambda M^T A_i M \succeq 0$$

for all $1 \leq i \leq n$. By the structure of D, we have $-E_i \succeq 0$ and $E_i \succeq 0$, thus $E_i = 0$ for all $1 \leq i \leq n$. Consider the Laplace expansion formula (see [31, Formula 0.3.1]) for computing $\det(D + \lambda M^T A_i M)$ for all $1 \leq i \leq n$. From the corresponding indices, the sign of $\det(C)^2$ in the expansion is $(-1)^{m\times(m+1)/2}$, thus it is always negative. By the non-negativity of $\det(D + \lambda M^T A_i M)$, we conclude that $C_i = 0$ for all $1 \leq i \leq n$. Therefore,

$$S = \{v \in \mathbb{R}^n : B_1 v_1 + \cdots + B_n v_n \succeq 0\}.$$

Note that for the chosen $w \in S$ above, the entries of the upper diagonal $d \times d$ block are non-zero, and as $A(w) \succeq 0$, they are positive. Hence, we have $B(w) \succ 0$. □

Hence, every spectrahedral cone is a hyperbolicity cone for some polynomials h. One can then wonder whether all hyperbolicity cones are spectrahedral or when they are spectrahedral. The theorem below follows from [30] of Helton and Vinnikov and describes the necessary and sufficient conditions for a hyperbolicity cone to be spectrahedral.

Theorem 1.1.46 (Theorem 2.1 in [59]). *Let $C_h \subseteq \mathbb{R}^n$ be a hyperbolicity cone of some stable polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$. The cone C_h is spectrahedral if and only if there exists a hyperbolic polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ with $C_h \subseteq C_g$ such that $h \cdot g$ has a determinantal representation.*

The hyperbolicity cone of the product of two hyperbolic polynomials is the intersection of their hyperbolicity cones. Thus, the condition about the inclusion of cones in the theorem ensures that the determinantal representation for the product $h.g$ gives exactly the linear matrix inequalities of the hyperbolicity cone of C_h. Notice also that the polynomial h itself does not need to have a weak determinantal representation for its hyperbolicity cone to be spectrahedral.

The *Generalized Lax Conjecture* claims that every hyperbolicity cone satisfies the conditions of the theorem above.

Conjecture 1.1.47 (Generalized Lax Conjecture). *Every hyperbolicity cone is spectrahedral.*

There has been extensive work on the conjecture, providing positive results for some special cases. Below is a summary of the developments on Conjecture 1.1.47.

- Homogeneous cones, i.e., convex cones whose automorphism group acts transitively on their interior, are spectrahedral [19, Chua in 2003].
- Hyperbolicity cones of quadratic polynomials are spectrahedral [49, Netzer and Thom in 2012].
- Hyperbolicity cones of elementary symmetric polynomial are spectrahedral [12, Brändén in 2014].
- Hyperbolicity cones of multivariate matching polynomials are spectrahedral [2, Amini in 2019].

For more positive results on weaker versions of the conjecture, we refer to [36], [48], [56].

In the following chapters we show that spectrahedral representability is preserved under certain operations.

Remark 1.1.48. Recall that semi-definite programming optimizes linear functions on the slices of spectrahedra. Therefore, on the programming point of view, the generalized Lax conjecture posits that every hyperbolic program can be written as a semi-definite program. Further, the criterion for hyperbolicity cones to be spectrahedral implies that the smallest size of the matrices that give the LMIs for a hyperbolicity cone can be bigger than the degree of the polynomial. So even if the conjecture is true, the computational advantages of it depend on the sizes of the matrices.

One can further ask whether the spectrahedrality of the hyperbolicity cone of a polynomial with a determinantal representation is preserved when we take its derivative. The following result by Saunderson provides a positive answer.

Theorem 1.1.49 (Theorem 1 in [56])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial with a determinantal representation. Then the hyperbolicity cone of $D_e h(x_1, \ldots, x_n)$ for any direction $e \in \mathbb{R}^n$ is spectrahedral.*

Later, Kummer in [37, Corollary 5.13] gave an analogous positive answer for higher derivatives.

Remark 1.1.50. By Corollary 1.1.28, it is sufficient to prove Conjecture 1.1.47 for homogeneous multiaffine polynomials.

Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. The following diagram summarizes the relation between the properties.

$$
\begin{tikzcd}
\text{h has the HPP} \arrow[rr, Rightarrow, "/" marking, shift right] \arrow[dr, Rightarrow, "/" marking, shift right] & & \begin{array}{c}\text{h has a determinantal}\\ \text{representation}\end{array} \arrow[ll, Rightarrow, shift right] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
& \begin{array}{c}\text{h is weakly}\\ \text{determinantal}\end{array} \arrow[ul, Rightarrow, shift right] \arrow[r, Rightarrow] & \begin{array}{c}\text{h is hyperbolic}\\ \text{with resp. to all } e \in \mathbb{R}^n_{\geq 0}\\ \text{and } C_h \text{ is spectrahedral}\end{array} \arrow[d, Leftrightarrow] \\
& & \begin{array}{c}\exists \text{ a hyperbolic } g\\ \text{with } C_h \subseteq C_g\\ \text{such that}\\ h.g \text{ is weakly determinantal}\end{array}
\end{tikzcd}
$$

1.2 Matroids

In Example 1.1.39, we mentioned that the basis generating polynomial of the Vámos matroid has the half-plane property but is not weakly determinantal. In this section, we will define the terms such as matroid, basis generating polynomial etc., that might be unknown to the reader. We introduce the basics of the matroid theory and explain their connection to the properties we have seen so far.

Basic Definitions

Matroids are generalizations of the concept of linear independence we are familiar with from linear algebra. They provide a way to keep independence, like combinatorial data, on different objects. There are several equivalent definitions of matroids; however, we give only the ones related to our interests.

Definition 1.2.1. A *matroid* M is a finite set $E = [n] := \{1, \ldots, n\}$ and a collection $\mathcal{I}$ of subsets called *independent sets* that satisfy the following axioms:

- $\emptyset \in \mathcal{I}$.
- If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$.
- If $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Those axioms give us a way to define linear independence on objects that are different than finite set of vectors such as matrices. The maximal independent sets are called *bases*, and one can also define a matroid in terms of its collection of bases as follows.

Definition 1.2.2. A matroid M is a finite set $E = [n] := \{1, \ldots, n\}$ (called *ground set*) with a collection of subsets $\mathcal{B}$ called *collection of bases* that satisfy the following property:

- If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \setminus B_2$, then there exists $e' \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$.

The property is also called the *basis exchange axiom*. It follows from axioms of independence that every element in the collection of bases has the same cardinality. One can also observe that independent sets are subsets of bases.

The basis generating polynomial $h_M \in \mathbb{R}[x_1, \ldots, x_n]$ of a matroid M on $E = [n]$ is

$$h_M = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i.$$

In particular, basis generating polynomials are homogeneous and multiaffine.

The rank $\mathrm{rk}(S)$ of an element $S \subseteq E$ of a matroid M is the cardinality of the maximal independent set contained in S. The rank $\mathrm{rk}(M)$ of the matroid M is the cardinality of its bases elements. One can then identify a matroid by a function called rank function, defined from the power set of E to non-negative numbers, which encodes the rank of every element of M.

Definition 1.2.3. A matroid M is a ground set $E = [n]$ with a function $\mathrm{rk}_M : 2^E \to \mathbb{N}$ called the *rank function* of M satisfying

- If $S \subseteq E$, then $0 \leq \mathrm{rk}_M(S) \leq |S|$.
- If $S \subseteq S' \subseteq E$, then $\mathrm{rk}_M(S) \leq \mathrm{rk}_M(S')$.
- If S and S' are subsets of E, then

$$\mathrm{rk}_M(S \cup S') + \mathrm{rk}_M(S \cap S') \leq \mathrm{rk}_M(S) + \mathrm{rk}_M(S').$$

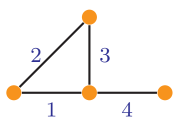
We denote the rank function of M with $\mathrm{rk}_M(\cdot)$ when there is ambiguity, otherwise we denote it by $\mathrm{rk}(\cdot)$.

Example 1.2.4.
- Matrices are finite set of column vectors and those vectors have some linear independence relation among each other. Consider the matrix

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

  We now numerate its columns and see it solely as a finite set $E = [4]$. A subset of E is independent if the corresponding columns are linearly independent. For example $\{1, 4\}$ is independent and the matroid M has the collection of bases $\{\{2, 3, 4\}, \{2, 1, 4\}, \{1, 3, 4\}\}$. Thus, its basis generating polynomial is $h_M = x_2x_3x_4 + x_2x_1x_4 + x_1x_3x_4$. Matroids defined from matrices over some field are called linear matroids.

- Consider the following graph



  with 4 vertices. We enumerate its edges with $E = [4]$ and define the collection $\mathcal{B}$ of its bases as the set of subsets $S \subseteq [4]$ such that the corresponding edges give a spanning tree, i.e., they give a shortest path visiting every vertex. One can check that this collection satisfies the basis exchange axiom, thus we obtain a matroid M'. Moreover, M' has the basis generating polynomial $h_M = x_2x_3x_4 + x_2x_1x_4 + x_1x_3x_4$. A matroid which is defined from a graph is called a *graphical matroid*.

Those two examples indeed define the same matroid, so that the combinatorial relation they have is the same. They both have rank 3, and they have the same collection of bases.

Some other components of matroids are circuits, flats and hyperplanes. A matroid can further be defined axiomatically in terms of its collection of circuits, flats, hyperplanes respectively. We only give their definitions in terms of their effect on the rank of a subset. For more information on matroids and their different definitions, we refer to [50].

Definition 1.2.5. Let M be a matroid on a ground set $E = [n]$.

- Circuits are minimal dependent sets of a matroid. A subset $C \subseteq E$ is called a *circuit* if for any $e \in C$, $C \setminus \{e\}$ is independent.
- The *closure* of a subset $S \subseteq E$ is defined as

  $$\mathrm{cl}(S) := \{e \in E : \mathrm{rk}(S \cup \{e\}) = \mathrm{rk}(S)\}.$$

  Flats correspond to closed sets, i.e., they are equal to their closure. In linear algebra terminology, they are all elements contained in a subspace. A subset $F \subseteq E$ is called a *flat* of M if for any $e \in E \setminus F$, $\mathrm{rk}(F \cup \{e\}) = \mathrm{rk}(F) + 1$. Flats of rank $r - 1$ are called *hyperplanes*. The set of flats of a matroid M has a lattice structure ordered with inclusion.

The following propositions demonstrate some properties of the closure operator that we will use in a sequel.

Proposition 1.2.6 (Lemma 1.4.2 in [50])**.** *Let M be a matroid on a ground set $E = [n]$. For every $S \subseteq E$, S and its closure have the same rank, i.e.,*

$$\mathrm{rk}(S) = \mathrm{rk}(\mathrm{cl}(S)).$$

Proof. Let $\mathcal{I}$ be the collection of independent sets of M and $I \in \mathcal{I}$ be a maximal independent set contained in S. For each $x \in \mathrm{cl}(S) \setminus S$, we have $\mathrm{rk}(I \cup \{x\}) \leq \mathrm{rk}(S \cup \{x\})$. Since $x \in \mathrm{cl}(S)$, by the definition of the closure, $\mathrm{rk}(S \cup \{x\}) = \mathrm{rk}(S)$. Since the rank of S is the size of I and I is independent, $\mathrm{rk}(S \cup \{x\}) = |I| = \mathrm{rk}(I) \leq \mathrm{rk}(I \cup \{x\})$. Then $\mathrm{rk}(S \cup \{x\}) = \mathrm{rk}(I \cup \{x\}) = \mathrm{rk}(I) < |I \cup \{x\}|$ such that $I \cup \{x\}$ is dependent. This implies that I is a maximal independent set of $\mathrm{cl}(S)$, thus $\mathrm{rk}(S) = \mathrm{rk}(\mathrm{cl}(S))$. □

Proposition 1.2.7 (Lemma 1.4.3 in [50])**.** *Let M be a matroid on a ground set $E = [n]$. The closure operator of M has the following properties:*

(i) If $S \subseteq E$, then $S \subseteq \mathrm{cl}(S)$.

(ii) If $S \subseteq S' \subseteq E$, then $\mathrm{cl}(S) \subseteq \mathrm{cl}(S')$.

(iii) If $S \subseteq E$, then $\mathrm{cl}(\mathrm{cl}(S)) = \mathrm{cl}(S)$.

Proof. Let $\mathcal{I}$ be the collection of independent sets of M.

(i) It is clear from the definition of the closure of S.

(ii) Assume that $S \subseteq S'$ and let $x \in \mathrm{cl}(S) \setminus \{x\}$. Let $I \in \mathcal{I}$ be a maximal independent set contained in S. By the definition of closure, $\mathrm{rk}(S \cup \{x\}) = \mathrm{rk}(S)$. Thus, I is also a maximal independent set of $S \cup \{x\}$. Then $S' \cup \{x\}$ contains a maximal independent set $I' \in \mathcal{I}$ such that $I \subseteq I'$ and $x \notin I'$. In particular, I' is a maximal independent set in S', thus $\mathrm{rk}(S' \cup \{x\}) = |I'| = \mathrm{rk}(S')$ so that $x \in \mathrm{cl}(S')$. Therefore, $\mathrm{cl}(S) \subseteq \mathrm{cl}(S')$.

(iii) By (i), $\mathrm{cl}(S) \subseteq \mathrm{cl}(\mathrm{cl}(S))$. For the other inclusion, let $x \in \mathrm{cl}(\mathrm{cl}(S))$. Then by the definition of the closure and by Proposition 1.2.6,

$$\mathrm{rk}(\mathrm{cl}(S) \cup \{x\}) = \mathrm{rk}(\mathrm{cl}(S)) = \mathrm{rk}(S).$$

Since by the properties of the rank function

$$\mathrm{rk}(S) \leq \mathrm{rk}(S \cup \{x\}) \leq \mathrm{rk}(\mathrm{cl}(S) \cup \{x\})$$

and we have equality between the fist part and the last part, we obtain that $\mathrm{rk}(S \cup \{x\}) = \mathrm{rk}(S)$. Thus, $x \in \mathrm{cl}(S)$.

□

The following lemma illustrates the relation between the rank of the intersection of closures of independent sets of a matroid, and the independence of their union.

Lemma 1.2.8. *Let $I, J \subseteq E$ be two independent sets of a matroid M on E. Then*

$$I \cup J \textit{ is independent} \iff \mathrm{rk}(\mathrm{cl}(I) \cap \mathrm{cl}(J)) = 0.$$

Proof. In order to prove the lemma, we first need to prove the following claim.
Claim:

$$\mathrm{rk}(I \cup J) = \mathrm{rk}(I \cup \mathrm{cl}(J)).$$

Proof. $\mathrm{rk}(I \cup J) \leq \mathrm{rk}(I \cup \mathrm{cl}(J))$ since $I \cup J \subseteq I \cup \mathrm{cl}(J)$. For the other direction, by Proposition 1.2.6 we have $\mathrm{rk}(\mathrm{cl}(I \cup J)) = \mathrm{rk}(I \cup J)$. Moreover,

$$\mathrm{rk}(I \cup J) = \mathrm{rk}(\mathrm{cl}(I \cup J)) \geq \mathrm{rk}(I \cup \mathrm{cl}(J))$$

since $I \subset \mathrm{cl}(I \cup J)$ and $\mathrm{cl}(J) \subset \mathrm{cl}(I \cup J)$. Thus, $\mathrm{rk}(I \cup J) \geq \mathrm{rk}(I \cup \mathrm{cl}(J))$. □

Since the rank of an independent set equals to its cardinality,

$$I \cup J \,\text{independent} \iff \mathrm{rk}(I \cup J) = \mathrm{rk}(I) + \mathrm{rk}(J).$$

By the claim we have $\mathrm{rk}(I \cup J) = \mathrm{rk}(\mathrm{cl}(I) \cup J) = \mathrm{rk}(\mathrm{cl}(I) \cup \mathrm{cl}(J))$ so that

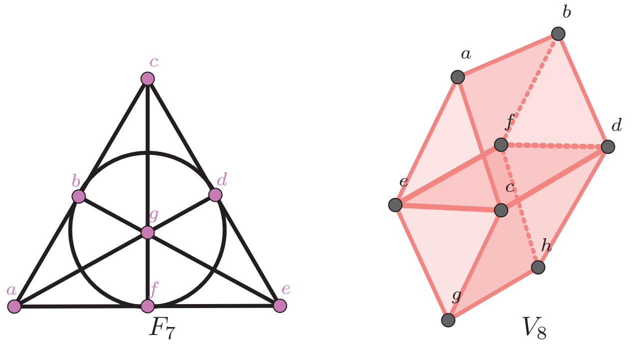
$$I \cup J \,\text{independent} \iff \mathrm{rk}(\mathrm{cl}(I) \cup \mathrm{cl}(J)) = \mathrm{rk}(I) + \mathrm{rk}(J).$$

Then by the last property of the rank function,

$$\begin{aligned} & I \cup J \,\text{independent} \\ \iff & \mathrm{rk}(\mathrm{cl}(I)) + \mathrm{rk}(\mathrm{cl}(J)) - \mathrm{rk}(\mathrm{cl}(I) \cap \mathrm{cl}(J)) = \mathrm{rk}(\mathrm{cl}(I)) + \mathrm{rk}(\mathrm{cl}(J)) \\ \iff & \mathrm{rk}(\mathrm{cl}(I) \cap \mathrm{cl}(J)) = 0. \end{aligned}$$

□

Matroids in general do not need to be produced by a graph or a matrix. They can be constructed by projective geometries or they can sometimes be not representable over any field, i.e., there is no matrix in any field that has the same combinatorial structure. Below are some examples of special matroids.



(a) Fano Plane (b) Vámos Cube

Figure 1.5: Fano matroid F_7 and Vámos matroid V_8.

Example 1.2.9. A special class of matroids is the uniform matroid. A uniform matroid $U_{r,n}$ is a matroid on n elements of rank r such that its collection of bases is all subsets of $[n]$ of size r. For example $U_{2,5}$ has the collection of bases $\mathcal{B} = \{S \subset [5] : |S| = 2\}$. Basis generating polynomial of $U_{r,n}$ is the elementary symmetric polynomial $E_{r,n}$.

Example 1.2.10.

- The Fano matroid F_7 is a matroid defined on the projective Fano plane depicted in Figure 1.5a on its points (we identify the ground sets given by n letters with $[n]$). A 3-element subset B of $E = \{a, \ldots, g\}$ is a basis if the corresponding points do not lie on the same line on the Fano plane. For example $\{a, c, d\}$ is a basis of F_7.

- The Vámos matroid V_8 is a rank 4 matroid on $E = \{a, \ldots, h\}$. It is defined on the vertices of the Vámos cube depicted in Figure 1.5b . A 4 element subset B of E is not a basis if they do not all lie on one of the painted quadrilaterals. It is not representable over any field. Unlike a graph or a projective geometry, Vámos cube is only a picture to illustrate its non-bases. For example $\{a, b, c, d\}$ is a non-bases.

- The matroid P_8 has ground set $E = [8]$ and it has rank 4. It is represented by the matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}$$
over $\mathbb{Q}$. The circuit hyperplanes of P_8 are $\{1, 2, 3, 8\}$, $\{1, 2, 4, 7\}$, $\{1, 3, 4, 6\}$, $\{2, 3, 4, 5\}$, $\{1, 4, 5, 8\}$, $\{2, 3, 6, 7\}$, $\{1, 5, 6, 7\}$, $\{2, 5, 6, 8\}$, $\{3, 5, 7, 8\}$ and $\{4, 6, 7, 8\}$. We denote by P_8', P_8'' and P_8''' the matroids obtained by relaxing the first, the first and the second, resp. all three of the following circuit hyperplanes: $\{1, 4, 5, 8\}$, $\{2, 3, 6, 7\}$ and $\{4, 6, 7, 8\}$.

Some Operations on Matroids

In this section we introduce some operations on matroids such as deleting or contracting an element, taking a direct sum of two matroids, taking duals.

Definition 1.2.11. Let M be a matroid on a ground set $E = [n]$ of rank r with the collection of bases $\mathcal{B}$. The dual matroid M^* of M is a matroid on E of rank $n-r$ with the collection of bases $\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$. The dual M^* is indeed a matroid, see for example [50, Theorem 2.1.1 in].

Hence the basis generating polynomial h_{M^*} of M^* is

$$h_{M^*} = \sum_{B \in \mathcal{B}^*} \prod_{i \in B} x_i = x_1 \ldots x_n h_M(x_1^{-1}, \ldots, x_n^{-1}).$$

A circuit C of a matroid M of rank r that is also a hyperplane is called a circuit hyperplane. A matroid M' obtained by adding a circuit hyperplane to the set of bases of M is called a relaxation of M. It corresponds to deleting a relation that creates some dependencies.

Definition 1.2.12. Let M be a matroid on $E = [n]$ of rank r with the collection of bases $\mathcal{B}$ and a circuit hyperplane C. The matroid M' on E with the collection of bases $\mathcal{B}' = \mathcal{B} \cup C$ is called a *relaxation* of M.

The following theorem shows that a relaxation of a matroid is indeed a matroid.

Proposition 1.2.13. *Let M be a matroid on $E = [n]$ with the collection of bases $\mathcal{B}$ and a circuit hyperplane C. Then $\mathcal{B}' = \mathcal{B} \cup C$ is the set of bases of a matroid M'.*

Proof. Since $\mathcal{B}$ is non-empty, $\mathcal{B}'$ is also non-empty. Let $B_1, B_2 \in \mathcal{B}'$ and $e \in B_1 \setminus B_2$. If $B_1, B_2 \in \mathcal{B}$, then there exists an element $e' \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{e'\} \in \mathcal{B}$ thus it is also in $\mathcal{B}'$. So for the non-trivial case, assume that $B_1 := C, B_2 \in \mathcal{B}'$ and $e \in B_1 \setminus B_2$. Since C is both a circuit and a hyperplane, $B_1 \setminus \{e\}$ is independent, and $\operatorname{rk}(M) - 1 = |B_1 \setminus \{e\}| < |B_2|$. Then, by the third property of the independent sets, there exists $e' \in B_2 \setminus (B_1 \setminus \{e\})$ such that $(B_1 \setminus \{e\}) \cup \{e'\}$ is an independent set of M and it is contained in a maximal independent set. Since, $|\,(B_1 \setminus \{e\}) \cup \{e'\}\,| = |B_1| = |B_2|$, we have that $(B_1 \setminus \{e\}) \cup \{e'\}$ is the maximal independent set containing itself so that it is a basis of M. In the case $\mathcal{B}_1 := C$ and $\mathcal{B}_2 := C$, we know that $C \setminus \{e\}$ for $e \in C$ is an independent set of M as C is a circuit hyperplane. By the axioms of independent sets, there exists a basis $B \in \mathcal{B}$, with $C \setminus \{e\} \subset B$. Thus $\mathcal{B}'$ satisfies the basis exchange axiom in all cases.

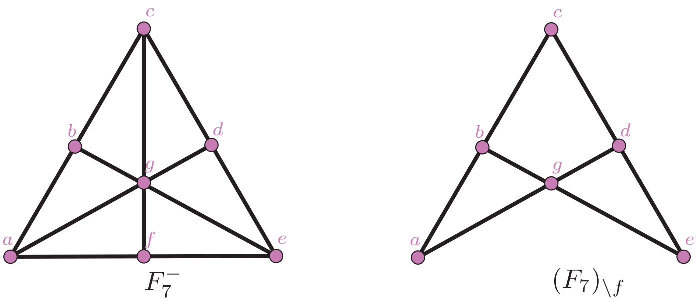
□

One can further delete an element from the ground set of a matroid, depending on the effects this operation has on the collection of the bases of M or M^*, the operations are called deletion and contraction respectively.

Definition 1.2.14. Let M be a matroid on $E = [n]$ of rank r with the collection of bases $\mathcal{B}$ and $e \in E$.

- The matroid $M_{\setminus e}$ on $E \setminus \{e\}$ with the collection of bases consisting of the elements of the set
$$\{B \setminus \{e\} : B \in \mathcal{B}\}$$
that have maximal cardinality is called the *deletion* $M_{\setminus e}$ of M.

  For a subset $S \subset M$, the deletion $M_{\setminus (E \setminus S)}$ of $E \setminus S$ is also called the *restriction* of M to S, and is denoted as $M|_S$.



(a) Relaxation F_7^- of F_7. (b) Deletion $(F_7)_{\setminus f}$ of F_7.

Figure 1.6: Relaxation of the circuit hyperplane $C = \{b, f, d\}$ of F_7 and deletion of the element f from F_7.

- The dual matroid $(M^*_{\setminus e})^*$ of $M^*_{\setminus e}$ is called the *contraction* $M_{/e}$ of M. Its collection of bases consists of the maximal elements of the set

$$\{B' \subseteq E \setminus \{e\} : \text{there is a basis } B \in \mathcal{B} \text{ with } B' \subset B\}.$$

Note that one can apply deletion and contraction operations for a subset $T \subseteq E$ by repeating the operation element-wise for elements of T. An element e of the ground set E of a matroid M is called a *loop* if it is not contained in any basis of M, and it is called a *co-loop* if it is contained in every basis of M. A matroid without loops or co-loops is called *simple*. We can describe the basis generating polynomials of deletions and contractions of a matroid in terms of its basis generating polynomial.

Proposition 1.2.15. *Let M be a matroid on $E = [n]$ of rank r with the collection of bases $\mathcal{B}$.*

(i) If M is simple, then for $e \in E$,

$$h_{M\setminus e} = h_M|_{x_e=0} \text{ and } h_{M/e} = \frac{\partial}{\partial x_e} h_M.$$

(ii) If $e \in E$ is a co-loop of M, then $h_{M\setminus e} = \frac{\partial}{\partial x_e} h_M$.

(iii) If $e \in E$ is a loop of M, then $h_{M/e} = h_M$.

Proof. It follows from the definitions. □

For a polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ of degree r, the polynomial that is the sum of its degree r terms is denoted by $h^{\#}$, and the polynomial that is the sum of lowest degree terms is denoted by $h_{\#}$. In other words,

$$h^{\#} := \lim_{k\to\infty} k^{-r} h(kx_1, \ldots, kx_n) \text{ and } h_{\#} := \lim_{k\to 0} k^{-l} h(kx_1, \ldots, kx_n)$$

where l is the degree of the lowest degree monomial of h. We will refer to them as *leading form* and *initial form* respectively. Then, one can express the basis generating polynomials of deletions and contractions of a simple matroid M as

$$h_{M\setminus e} = h_M|_{x_e=0} = (h_M|_{x_e=1})^{\#} \text{ and } h_{M/e} = \frac{\partial}{dx_e} h_M = (h_M|_{x_e=1})_{\#}.$$

Another operation on matroids is taking minors by applying deletion and contraction operators.

Definition 1.2.16. A *minor* M' of a matroid M on $E = [n]$ is a matroid obtained by some deletion and/or contraction operations for a $T \subseteq E$.

In general, we can express deletion and/or contraction of a set of elements in terms of the leading form and initial form as follows.

Lemma 1.2.17 (Lemma 2.7 in [39])**.** *Let M be a matroid on a ground set E. For any $S \subset E$ we have:*

(i) $h_{M/S} = c \cdot (h_M|_{x_i=1 \text{ for } i\in S})_{\#}$ *for some constant c.*

(ii) $h_{M\setminus S} = c' \cdot (h_M|_{x_i=1 \text{ for } i\in S})^{\#}$ *for some constant c'.*

Proof. By Definition 1.2.14, a subset $I \subset E \setminus S$ is a basis of $M_{/S}$ if and only if there is a basis B of M such that $B \setminus I$ is a basis of $M_{\setminus(E\setminus S)}$, i.e., a maximal independent subset of S. Therefore, the support of $h_{M/S}$ agrees with the support of $f := (h_M|_{x_i=1 \text{ for } i\in S})_{\#}$. Moreover, it also follows from the definition that the coefficient of each monomial in f is the number of bases of $M_{\setminus(E\setminus S)}$.

By the same argument we obtain that the support of $h_{M\setminus S}$ agrees with the support of $g := (h_M|_{x_i=1 \text{ for } i\in S})^{\#}$ and that the coefficient of every monomial in g is the number of bases of $M_{/(E\setminus S)}$. □

A minor M' of a matroid M is called a *forbidden minor* for a property if M' does not have the property and all of its proper minors have the property.

Example 1.2.18. A matroid M is a *sixth root of unity* matroid ($\sqrt[6]{1}$-matroid) if it can be represented by a matrix A with complex entries such that all minors of A lie in the multiplicative group of complex sixth root of unity. Choe *et. al.* in [18, Theorem 8.9] showed that the class of $\sqrt[6]{1}$-matroids and the class of complex uni-modular matroids, i.e., they can be represented by a complex matrix all of whose minors have modulus 1, are equal.

One can also characterize sixth root of unity matroids in terms of forbidden minors.

- A matroid M is a $\sqrt[6]{1}$-matroid, if and only if it has no minors isomorphic to $U_{2,5}, U_{3,5}$, F_7, F_7^*, F_7^-, $(F_7^-)^*$ or P_8 ([18, Theorem 8.15], see also [25]).

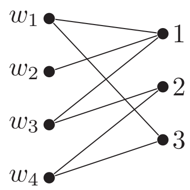
In particular, one then only needs to check whether a given matroid has one of the forbidden minors in order to determine whether the matroid is a $\sqrt[6]{1}$-matroid.

Example 1.2.19. Another class of matroids is transversal matroids. Let S be a finite set, and $\mathcal{D} = (D_1, \ldots, D_n)$ be a finite family of subsets of S, i.e., $D_i \subseteq S$ for all $i \in S$, whose members are not necessarily distinct. Let $J = [n]$ and consider the bipartite graph G' with the vertex set $S \cup J$ and the edge set $\{xj : x \in S,\, j \in J \text{ and } x \in D_j\}$. A *matching* in a graph is a set of edges of G'

such that no two edges have a common endpoint. A subset X of S is called a *partial transversal* of $\mathcal{D}$ if there is a matching in G' in which every edge has one end point in X.

Let $\mathcal{D}$ be a family of subsets of a finite set S. By Theorem 1.6.2 in [50], S with the set of partial transversals of $\mathcal{D}$ as its collection of independent sets defines a matroid M. Matroids defined this way are called *transversal*.

- Let $S = \{w_1, \ldots, w_4\}$ and $\mathcal{D} = (D_1, D_2, D_3)$ where $D_1 = \{w_1, w_2, w_3\}$, $D_2 = \{w_3, w_4\}$ and $D_3 = \{w_1, w_4\}$. Then, we have the following bipartite graph.



  Some of the matchings of maximal size of this graph are $\{w_1 3, w_2 1, w_3 2\}$, $\{w_2 1, w_3 2, w_4 3\}$ and $\{w_1 1, w_3 2, w_4 3\}$ such that, the corresponding partial transversals

$$\{w_1, w_2, w_3\}, \{w_2, w_3, w_4\}, \text{ and } \{w_1, w_3, w_4\}$$

  appear in the collection of bases of the constructed transversal matroid on S.

Moreover, a transversal matroid M on S defined from a bipartite graph G' is called *nice* if there is a collection $\{\lambda_e\}_{e \in E}$ of non-negative edge weights such that

$$c(B; \lambda) := \sum_{\substack{\text{matchings } m \\ V(m) \cap S = B}} \prod_{i \in m} \lambda_i$$

has the same non-zero value for all the bases B of M where E is the edge set of G' and $V(m)$ is the vertex set of the matching m. In particular, the basis generating polynomial of a nice transversal matroid can be written in terms of the matching polynomial of the corresponding bipartite graph. See §10 of [18] for more details.

The direct sum of two matroids is an operation that creates a new matroid from the given ones.

Definition 1.2.20. The direct sum $M \oplus N$ of matroids M and N on disjoint ground sets E and E', with collection of bases $\mathcal{B}$ and $\mathcal{B}'$ respectively is a matroid on $E \cup E'$ such that its collection of bases is

$$\{B \cup B' : B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}.$$

Matroids that cannot be expressed as a direct sum of two other matroids are called *connected*.

Extension and co-extension operations on matroids are two different ways to construct a bigger matroid from a given matroid. For the proofs of propositions 1.2.23, 1.2.24, and for more information on these operations, we refer to [50, §7].

Definition 1.2.21. If a matroid M is obtained from a matroid N by deleting an element e from the ground set of N, then N is called a *single-element extension* of M. If N^* is an extension of the dual M^* of M, then N is called a *co-extension* of M.

In order to learn more about the structure of the matroid that is an extension or a co-extension, we need to consider modular pairs.

Definition 1.2.22. A pair (F_1, F_2) of flats F_1, F_2 of a matroid M is called a *modular pair* if the following holds:

$$\mathrm{rk}(F_1) + \mathrm{rk}(F_2) = \mathrm{rk}(F_1 \cap F_2) + \mathrm{rk}(F_1 \cup F_2).$$

The proposition below gives some insights about the relation between the flats of a matroid and the flats of its extension.

Proposition 1.2.23 (Lemma 7.2.2 in [50]). *Let N be an extension of M by e and let $\mathcal{S}$ be the set of flats F of M such that $F \cup \{e\}$ is a flat of N with $\mathrm{rk}_N(F \cup \{e\}) = \mathrm{rk}_M(F)$. Then $\mathcal{S}$ has the following properties:*

- *If $F \in \mathcal{S}$ and $F^{'}$ is a flat of M with $F \subseteq F^{'}$, then $F^{'} \in \mathcal{S}$.*
- *If $F_1, F_2 \in \mathcal{S}$ and (F_1, F_2) is a modular pair, then $F_1 \cap F_2 \in \mathcal{S}$.*

A set $\mathcal{S}$ of flats of M is called a *modular cut* if it satisfies the conditions above.

Moreover, a modular cut of a matroid gives a unique extension of it.

Proposition 1.2.24 (Lemma 7.2.3[50]). *Let $\mathcal{S}$ be a modular cut of a matroid M on a set E. Then there is a unique extension N of M with e such that $\mathcal{S}$ consists of all flats of M for which $F \cup e$ is a flat of N with $\mathrm{rk}_N(F \cup \{e\}) = \mathrm{rk}_M(F)$. Moreover, for all $L \subseteq E$,*

- $\mathrm{rk}_N(L) = \mathrm{rk}_M(L)$
- $\mathrm{rk}_N(L \cup \{e\}) = \begin{cases} \mathrm{rk}_M(L) & \text{if } \mathrm{cl}_M(L) \in \mathcal{S} \\ \mathrm{rk}_M(L) + 1 & \text{if } \mathrm{cl}_M(L) \notin \mathcal{S} \end{cases}$

Then, we obtain the structure of the flats of the extension of M that corresponds to a modular cut S in terms of the flats of M as follows.

Lemma 1.2.25. *Let $\mathcal{F}$ be the set of flats of a matroid M. Let $\mathcal{S}$ be a modular cut such that $\mathcal{F} \setminus \mathcal{S} \neq \emptyset$. Let N be the one element extension of M by the element e corresponding to the modular cut S. Then,*

(i) For $F \in \mathcal{S}$, $F \cup \{e\}$ is a flat of N having the same rank as F.

(ii) For $F \in \mathcal{F} \setminus \mathcal{S}$, if there is not any $F^{'} \in \mathcal{S}$ with

$$F^{'} \supset F \text{ and } \mathrm{rk}_M(F^{'}) = \mathrm{rk}_M(F) + 1,$$

then $F \cup \{e\}$ is a flat of N of rank $\mathrm{rk}_M(F) + 1$.

(iii) For $F \in \mathcal{F} \setminus \mathcal{S}$, if there exists $F^{'} \in \mathcal{S}$ with

$$F^{'} \supset F \text{ and } \mathrm{rk}_M(F^{'}) = \mathrm{rk}(F) + 1,$$

then $F \cup \{e\}$ is not a flat of N.

(iv) For $F \notin \mathcal{S}$, F is a flat of N with $\mathrm{rk}_M(F)$.

Proof. (i) : It follows from the definition of the extension with respect to the modular cut.

(ii) : Let $F \in \mathcal{F} \setminus \mathcal{S}$ such that there is no $F^{'} \in \mathcal{S}$ with $F^{'} \supset F$ and $\mathrm{rk}_M(F^{'}) = \mathrm{rk}(F) + 1$. Since $F = \mathrm{cl}_M(F)$ is not in $\mathcal{S}$, by Proposition 1.2.24

$$\mathrm{rk}_N(F \cup \{e\}) = \mathrm{rk}_M(F) + 1.$$

Now, we need to show that $F \cup \{e\}$ is a flat of N. Assume that $\mathrm{cl}_N(F \cup \{e\}) \neq F \cup \{e\}$. Then there exists a flat $F^{'}$ of M of $\mathrm{rk}_M(F) + 1$ with $F^{'} \supset F$ such that $\mathrm{cl}_N(F \cup \{e\}) = F^{'} \cup \{e\}$. Therefore,

$$\mathrm{rk}_N(F^{'} \cup \{e\}) = \mathrm{rk}_N(\mathrm{cl}_N(F \cup \{e\})) = \mathrm{rk}_M(F) + 1$$

so that $F^{'} \in \mathcal{S}$. This contradicts with the assumption that there is not an $F^{'} \supset F$ in $\mathcal{S}$ that has rank $\mathrm{rk}_M(F) + 1$. Thus $F \cup \{e\}$ is a flat of N with rank $\mathrm{rk}(F) + 1$.

(iii) : Let $F \in \mathcal{F} \setminus \mathcal{S}$, and $F^{'} \in \mathcal{S}$ be a flat of M with $\mathrm{rk}_M(F^{'}) = \mathrm{rk}_M(F) + 1$ such that $F^{'} \supset F$. Assume that $F \cup \{e\}$ is a flat of N. Then, for any $a \in E \setminus F$,

$$\mathrm{rk}_N(F \cup \{e\} \cup \{a\}) = \mathrm{rk}_N(F \cup \{e\}) + 1 = \mathrm{rk}_M(F) + 2. \tag{1.1}$$

Without loss of generality, we may assume that $\mathrm{cl}_M(F \cup \{a\}) = F^{'}$.

Claim: $\mathrm{rk}_N(\mathrm{cl}_N(F \cup \{e\} \cup \{a\})) = \mathrm{rk}_N(\mathrm{cl}_M(F \cup \{a\}) \cup \{e\})$.

Proof. Since $F \cup \{e\} \cup \{a\} \subseteq \{e\} \cup \mathrm{cl}_N(F \cup \{a\}) = \{e\} \cup \mathrm{cl}_M(F \cup \{a\})$, by the properties of closure, $\mathrm{cl}_N(F \cup \{e\} \cup \{a\}) \subseteq \mathrm{cl}_N(\{e\} \cup \mathrm{cl}_M(F \cup \{a\}))$, thus

$$\mathrm{rk}_N(F \cup \{e\} \cup \{a\}) \leq \mathrm{rk}_N(\{e\} \cup \mathrm{cl}_M(F \cup \{a\})).$$

For the other direction, note that $\mathrm{cl}_N(F \cup \{e\} \cup \{a\}) \supseteq \{e\} \cup \mathrm{cl}_N(F \cup \{a\}) = \{e\} \cup \mathrm{cl}_M(F \cup \{a\})$. Then, by the properties of closure

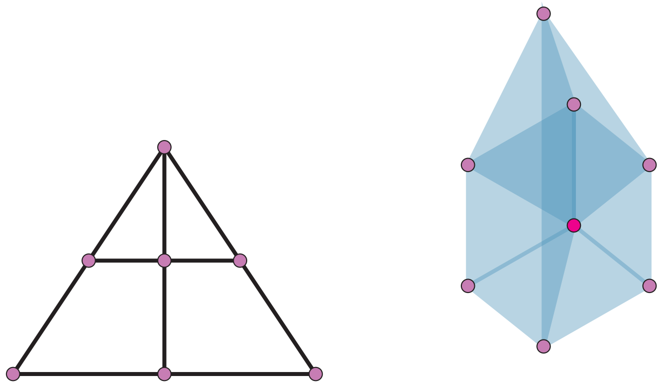
$$\mathrm{rk}_N(F \cup \{e\} \cup \{a\}) \geq \mathrm{rk}_N(\{e\} \cup \mathrm{cl}_M(F \cup \{a\})).$$

□

However,

$$\mathrm{rk}_N(F^{'} \cup e) = \mathrm{rk}_M(F^{'}) = \mathrm{rk}_M(F) + 1$$

contradicts with 1.1.



(a) P_7 (b) $\mathrm{CoExt}(P_7)$

Figure 1.7: The matroids P_7 and $\mathrm{CoExt}(P_7)$.

(iv) : By the properties of the rank function on N, we have that if a flat $F \notin \mathcal{S}$, then for any $a \in (E \cup e) \setminus F$, $\mathrm{rk}_N(F \cup \{a\}) = \mathrm{rk}_M(F) + 1$. Thus, F is a flat of N having rank $\mathrm{rk}_M(F)$.

$\square$

Example 1.2.26. When $\mathcal{S} = \{E\}$, the extension N has the flat $E \cup \{e\}$ of rank $\mathrm{rk}_M(M)$, and flats $F \cup \{e\}$ of rank $\mathrm{rk}_M(F) + 1$ where F is a flat of M with $\mathrm{rk}_M(F) \leq \mathrm{rk}_M(M) - 2$. An extension produced that way is called a free extension of M.

The Half-Plane Property of Matroids

In this section, we connect the properties we have seen in §1.1 with matroids we defined in §1.2.

Recall that the basis generating polynomial

$$h_M = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i$$

of a matroid M on n elements with the collection of bases $\mathcal{B}$ is a homogeneous multiaffine polynomial. Therefore, h_M might have the half-plane property, or it can be weakly determinantal. A matroid is called to have the half-plane property and to be weakly determinantal if its basis generating polynomial has the respective property. Moreover, a matroid M is called hyperbolic if h_M is hyperbolic with respect to the vector $e = (1, \ldots, 1)$, and it is called *spectrahedral* if the hyperbolicity cone $C_{h_M}(e)$ is spectrahedral.

Since the coefficients of the basis generating polynomials have the same sign, we have a more straightforward characterization of their half-plane property.

Corollary 1.2.27. *A matroid M on $E = [n]$ has the half-plane property if and only if it is hyperbolic with respect to $e = (1, \ldots, 1) \in \mathbb{R}^n$.*

Proof. It follows from Corollary 1.1.25 as the basis of the generating polynomial h_M of M has no sign variation on its coefficients. $\square$

Moreover, some operations on matroids preserve the half-plane property. Below we specify some of them.

Proposition 1.2.28 (Proposition 4.2 in [18])**.** *Let M be a matroid on n elements of rank r with the half-plane property. Then M^* also has the half-plane property.*

Proof. Let $h_M \in \mathbb{R}[x_1, \dots, x_n]$ be the basis generating polynomial of M. The basis generating polynomial of M^* is

$$h_{M^*} = (-1)^r x_1 \dots x_n h_M(-x_1^{-1}, \dots, -x_n^{-1}).$$

Since inversion $-x_i^{-1}$ of x_i stays on the same half-plane and scaling preserves stability, h_{M^*} is stable. □

The following proposition shows that the half-plane property is closed under taking minors.

Proposition 1.2.29. *Let M be a matroid on $E = [n]$ with the half-plane property and $e \in E$. Then $M_{\setminus e}$ and $M_{/e}$ also have the half-plane property.*

Proof. First assume that e is not a loop or co-loop. Then, by Proposition 1.2.15

$$h_{M\setminus e} = h_M|_{x_e=0} \text{ and } h_{M/e} = \frac{\partial}{\partial x_e} h_M.$$

Therefore stability of $h_{M\setminus e}$ and $h_{M/e}$ follows from Lemma 1.1.20. When e is a co-loop of M, $h_{M\setminus e} = \frac{\partial}{\partial x_e} h_M$, thus it is stable and when $e \in E$ is a loop of M, $h_{M/e} = h_M$ is stable. □

Corollary 1.2.30. *The half-plane property is minor-closed.*

Proof. It follows from the fact that minors of matroids are obtained by some deletion and contraction operations. □

Recall that the basis generating polynomial of a deletion or contraction of a matroid can be expressed in terms of setting a variable equal to one and taking upper degree or lower degree components.

Proposition 1.2.31. *Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous multiaffine polynomial of degree d with the half-plane property. Then for $e \in [n]$, $(h|_{x_e=1})^{\#}$ and $(h|_{x_e=1})_{\#}$ also have the half-plane property.*

Proof. Since h is homogeneous and multiaffine, when h does not have x_e as a factor with x_e appearing in at least one term of h, we have

$$(h|_{x_e=1})^{\#} := \lim_{k\to\infty} k^{-d} h(kx_1, \dots, kx_n)|_{x_e=1} = h|_{x_e=0} \text{ and}$$

$$(h|_{x_e=1})_{\#} := \lim_{k\to 0} k^{-d+1} h(kx_1, \dots, kx_n)|_{x_e=1} = \frac{\partial}{\partial x_e} h.$$

When h does not have x_e as a factor with x_e not appearing in any of the terms of h, both expressions are just h itself.

On the other hand, when h has x_e as a factor we have

$$(h|_{x_e=1})^{\#} := \lim_{k\to\infty} k^{-d} h(kx_1,\ldots,kx_n)|_{x_e=1} = \frac{\partial}{\partial x_e} h \text{ and}$$

$$(h|_{x_e=1})_{\#} := \lim_{k\to 0} k^{-d+1} h(kx_1,\ldots,kx_n)|_{x_e=1} = \frac{\partial}{\partial x_e} h.$$

Therefore, by Lemma 1.1.20 and by the assumption they all have the half-plane property. □

Moreover, taking the direct sum of two matroids preserve the half-plane property.

Proposition 1.2.32 (Proposition 4.5 in [18]). *Let M, N be two matroids on disjoint ground sets that have the half-plane property. Then $M \oplus N$ also has the half-plane property.*

Proof. It follows from the fact that $h_{M\oplus N} = h_M \cdot h_N$. □

For more operations preserving the half-plane property, we refer to [18].

We have seen some operations on matroids that preserve the half-plane property. One can then wonder which matroids have the half-plane property and whether every hyperbolic matroid is spectrahedral. The basis generating polynomials of matroids with the half-plane property give rise to examples of homogeneous, multiaffine polynomials with the half-plane property. Choe *et al.* with the following theorem showed that, on the other hand, the supports of homogeneous multiaffine polynomials with the half-plane property give rise to matroids by making up the collection of bases for some matroids M.

Theorem 1.2.33 (Theorem 7.1 in [18]). *Let $h = \sum_{S\subset[n]} \alpha_S \left(\prod_{i\in S} x_i\right)$ in $\mathbb{R}[x_1,\ldots,x_n]$ be a homogeneous multiaffine polynomial of degree d with the half-plane property. Then the support*

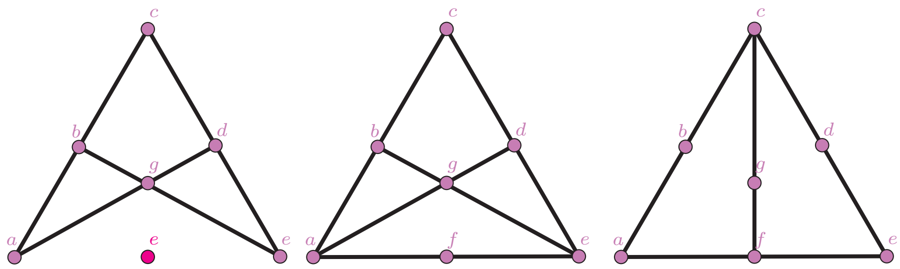
$$\mathrm{Supp}(h) := \{S \subset E : \alpha_S \neq 0\}$$

of h is the set of bases $\mathcal{B}$ for some matroid M of rank d.

By Remark 1.1.50, it is enough to prove the generalized Lax conjecture for homogeneous multiaffine polynomials. While basis generating polynomials of matroids are in the class of multiaffine polynomials, there are multiaffine polynomials that are not a basis generating polynomial for some matroids, for example, those having coefficients different than 1. By Proposition 1.1.26, non-zero coefficients of a homogeneous multiaffine polynomial with the half-plane property have the same sign. Therefore, we can obtain homogeneous multiaffine stable polynomials that do not have all 1 coefficients by applying coordinate changes on the basis generating polynomials of some matroids.

Moreover, Theorem 1.2.33 implies that the support of homogeneous multiaffine polynomials with the half-plane property satisfies the basis exchange axiom. For more information on the structure of the support of stable polynomials, we refer to [18], and [10].

There have been several developments regarding the classification of matroids with respect to the half-plane property. Below, we list some positive evidence for the conjecture from matroid theory.



(a) $M(K_4) + e$ (b) F_7^{--} (c) F_7^{-3}

Figure 1.8: Some matroids on 7 elements that are forbidden minors for the half-plane property.

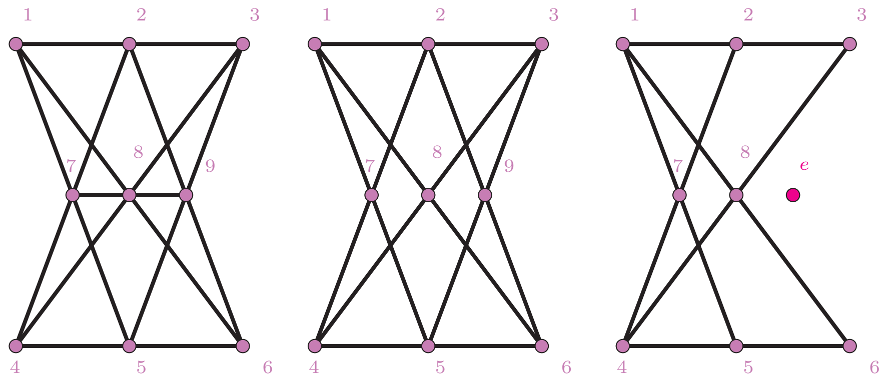
- Uniform matroids have the half-plane property [18, Theorem 9.2], and are spectrahedral [12].
- Sixth root of unity matroids have the half-plane property and are spectrahedral [18].
- Vámos matroid has the half-plane property [63], it is not weakly determinantal [11], and a specialized version of its basis generating polynomial is spectrahedral [35].
- Graphical matroids have the half-plane property and their hyperbolicity cones are spectrahedral [2].

In addition, the following classes of matroids known to have the half-plane property.

- Matroids of rank or co-rank 2 have the half-plane property [18, Corollary 5.5].
- All matroids on at most 6 elements have the half-plane property [18, Proposition 10.4].
- Nice transversal matroids have the half-plane property [18, Corollary 10.3].

On the other hand, not all matroids have the half-plane property.

- Matroids $F_7, F_7^-, F_7^{--}, F^{-3}, M(K_4) + e$ and their duals do not have the half-plane property [18].
- Pappus, non-Pappus and (non-Pappus\9)+e matroids do not have the half-plane property [18].
- No projective geometry has the half-plane property [13].
- Matroids P_8 and P_8'' do not have the half-plane property [18].



(a) Pappus (b) non-Pappus (c) (non-Pappus\9)+e

Figure 1.9: Some matroids on 9 elements that are forbidden minors for the half-plane property.

Remark 1.2.34. There is a weaker version of the half-plane property called the weak half-plane property. A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ has *the weak half-plane property* if there exist another homogeneous polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ with the half-plane property such that h and g have the same support. In other words, such polynomials only need to adjust their coefficients in order to have the half-plane property.

Besides the matroids that have the half-plane property, another class of matroids is known to have the weak half-plane property:

- All matroids that are representable over $\mathbb{C}$ have the weak half-plane property [18].
- Transversal matroids have the weak half-plane property [18].

One may ask whether the collection of bases of any matroid is the support of some homogeneous multiaffine polynomial with the half-plane property. Brändén in [10] showed that the Fano matroid F_7 gives a negative answer to this question as it does not have the weak half-plane property. More generally, there is a class of matroids that are known to not have the weak half-plane property:

- No projective geometry has the weak half-plane property [13].

For more details on the weak half-plane property, see for example [18], [10] and [13]. Throughout the text, we will only focus on the half-plane property.

In this manuscript, we are interested in the following questions:

Question 1. *Can we classify all matroids on at most* 8 *elements in terms of the half-plane property?*

Question 2. *Are there more matroids like the Vámos matroid so they have the half-plane property, but they are not weakly determinantal?*

Question 3. *Is there a computationally feasible algorithm to detect the half-plane property of a matroid?*

Question 4. *Which operations on matroids preserve determinantal representability and/or spectrahedral representability?*

Chapter 2

Operations Preserving Determinantal and Spectrahedral Representability of Matroids

This chapter focuses on operations on homogeneous multiaffine polynomials that preserve their determinantal representability and the spectrahedral representability of their hyperbolicity cone. We are especially interested in operations that concern the minors of a matroid. For more information, we refer to [11], [38], [39].

2.1 Determinantal Representability of Matroids

This section focuses on matroids with a determinantal representation, investigates their structure and further considers operations on matroids that preserve their determinantal representability.

A matroid M on a ground set $E = [n]$ has a determinantal representation if its basis generating polynomial h_M has one, i.e., if there exists real symmetric PSD matrices $A_1, \dots, A_n$ such that $h_M = \lambda \cdot \det(A_1x_1 + \dots + A_nx_n)$ for some $\lambda \in \mathbb{R}$.

By Proposition 1.1.37, we know that if a matroid M has a determinantal representation, then it has the half-plane property. In particular, M is hyperbolic with respect to every point in the positive orthant.

We first define the hyperbolic rank function of a polynomial which will be of use for further results.

Definition 2.1.1. Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. The *hyperbolic rank function* $\mathrm{r}_h : 2^{[n]} \to \mathbb{Z}$ of h is defined for all $S \subseteq [n]$ as

$$\mathrm{r}_h(S) = \deg\left(h(e + t\sum_{i\in S}\delta_i)\right)$$

where δ_i is the i-th standard basis vector.

Let M be a matroid on the ground set $E = [n]$, and $A = [a_1, \dots, a_n]$ be a matrix with real entries whose columns a_i have the same independence relation as the independent sets of M. Then by definition, for all $S \subseteq E$, the rank $\mathrm{rk}_M(S)$ of S in M is

$$\mathrm{rk}_M(S) = \dim(\mathrm{span}_{\mathbb{R}}\{a_i : i \in S\}).$$

Given PSD rank at most one matrices, one can construct a determinantal polynomial whose hyperbolic rank is the rank function of a linear matroid.

Lemma 2.1.2 (cf. p. 5 in [11]). *Let $A_1 = a_1 a_1^T, \ldots, A_n = a_n a_n^T$ be PSD matrices in $\mathrm{Sym}_{\mathbb{R}}^d$ of rank at most one, and let V_i be the column space of A_i for $i \in [n]$. Then*

$$\mathrm{rk}_M(S) = \dim(\sum_{i \in S} V_i) = \deg(\det(I + t \sum_{i \in S} A_i))$$

for all $S \subseteq [n]$, where rk_M is the rank function of the matroid M defined by the matrix $A = [a_1, \ldots, a_n]$ and I is the identity matrix of size d.

Proof. Let $S \subseteq [n]$ be a subset and let $B := (I, A)$ be the matrix constructed by concatenating the identity matrix and A, and D be the diagonal matrix defined as

$$D := \begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix}.$$

Then we have $I + t \sum_{i \in S} A_i = BDB^T$ so that $\det(I + t \sum_{i \in S} A_i) = \det(BDB^T)$. By Cauchy-Binet theorem [31, Formula 0.8.7],

$$\det(BDB^T) = \sum_{S \in \binom{[n+d]}{d}} |B([d], S)|^2 t^{|S \cap \{d+1, \ldots, d+n\}|}$$

where $B([d], S)$ is the $d \times d$ minor of B with columns indexed by S. Therefore, the degree of the polynomial on the right-hand side is the cardinality of the maximal linearly independent subset of S, that is $rk_M(S) = \dim(\sum_{i \in S} V_i)$. $\square$

The following lemma shows, that the hyperbolic rank function of a homogeneous multiaffine polynomial with a determinantal representation is the rank function of a matroid that is representable over $\mathbb{R}$. Further, the matrices giving the determinantal representation have rank at most 1.

Lemma 2.1.3 (cf. Section 3 in [11]). *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous, multiaffine polynomial with a determinantal representation with PSD matrices $A_1, \ldots, A_n \in \mathrm{Sym}_{\mathbb{R}}^d$, i.e., $h = \lambda \cdot \det(A_1 x_1 + \cdots + A_n x_n)$ with non-zero $\lambda \in \mathbb{R}$. Then, the hyperbolic rank function $\mathrm{r}_h(\cdot)$ is the rank function of a matroid M that is representable over $\mathbb{R}$, and the degree of x_i in h is the rank of A_i.*

Proof. Let $A := \sum_{i=1}^n A_i$. Since h has a determinantal representation, it is stable, thus, it is hyperbolic with respect to every point in the positive orthant. Therefore, A is positive definite. Let $V \in \mathbb{R}^{d \times d}$ be an invertible matrix such that $V^T A V$ is the identity matrix I. For $S \subseteq [n]$ and $e = (1, \ldots, 1) \in \mathbb{R}^n$,

$$\begin{aligned} \mathrm{r}_h(S) &= \deg(h(e + t \sum_{i \in S} \delta_i)) = \deg(\lambda \cdot \det(A + t \sum_{i \in S} A_i)) \\ &= \deg(\frac{\lambda}{\det(V)^2} \det(I + t \sum_{i \in S} V^T A_i V)) \end{aligned}$$

where δ_i is the i-th standard basis vector. Let W_i be the column space of $V^T A_i V$. By Lemma 2.1.2, we have $\mathrm{r}_h(S) = \dim(\sum_{i\in S} W_i)$. Since by stability all non-zero coefficients of h have the same sign (Proposition 1.1.26), the degree of x_i in h is $\mathrm{r}_h(\{i\}) = \dim(W_i)$ that is the rank of A_i. As h is multiaffine, A_i are of rank at most 1. One can then easily check that r_h satisfies the properties for being a rank function of a matroid M (see Definition 1.2.3 for the properties). The rank function $\mathrm{r}_h = \mathrm{rk}_M$ is defined in terms of the linear independence relation between the columns of the real matrix $V^T A_i V$, hence M is representable over $\mathbb{R}$. $\square$

Moreover, for any matroid that is representable over $\mathbb{R}$, there is a homogeneous stable polynomial with a determinantal representation whose hyperbolic rank function is the rank function of the matroid.

Lemma 2.1.4 (cf. Section 3 in [11])**.** *Let M be a matroid on the ground set $E = [n]$ that is representable over $\mathbb{R}$, and rk_M be its rank function. Then there is a homogeneous, multiaffine stable polynomial h with a determinantal representation such that its hyperbolic rank function is $\mathrm{r}_h = \mathrm{rk}_M$.*

Proof. Let $A = [a_1, \dots, a_n] \in \mathbb{R}^{d\times n}$ be the matrix that defines M, and V_i be the column space of a_i for $i \in [n]$. Then, for all $S \subseteq E$,

$$\mathrm{rk}_M(S) = \dim\left(\sum_{i\in S} V_i\right).$$

Let $A_i = a_i^T a_i$ for $i \in [n]$. Note that by construction, each A_i is a real symmetric, PSD matrix. Then by Lemma 2.1.3, the polynomial

$$h = \det\left(A_1 x_1 + \cdots + A_n x_n\right)$$

has its hyperbolic rank function as rk_M.

$\square$

Remark 2.1.5. Note that there is an analog of the results in Lemma 2.1.3 and Lemma 2.1.4 for homogeneous, not necessarily multiaffine, polynomials with determinantal representability, relating the rank of A_i to the degree of x_i in the polynomial. In this case, one considers rank functions that define polymatroids. Polymatroids are the generalization of matroids. Their rank function criteria differ from the rank function of matroids by the criterion that the rank of a finite set cannot exceed its cardinality. For more information on polymatroids and the extension of the results, we refer to [11].

Recall that a matroid M is representable over a field if there exists a matrix with entries in the field that defines M. A matroid that is representable over every field is called *regular*. By Theorem 6.6.3 in [50], a matroid is regular if and only if it is representable over $\mathbb{R}$ with a matrix all of whose square sub-determinants are $-1, 1$ or 0.

Let M be a matroid on $E = [n]$ with a determinantal representation given by PSD real symmetric matrices $A_1, \dots A_n \in \mathbb{R}^{d\times d}$ such that $h_M(x_1, \dots, x_n) = \lambda \det(A_1 x_1 + \cdots + A_n x_n)$, for some non-zero $\lambda \in \mathbb{R}$. By Lemma 2.1.3, we may assume that A_i have rank one. Then the A_i have the form $A_i = a_i a_i^T$ with

$a_i \in \mathbb{R}^{d\times 1}$ for $i \in [n]$. Let $A := [a_1, \ldots, a_n]$ so that $h_M = \det(AXA^T)$ where X is the diagonal matrix with x_i, $i \in [n]$. By the Cauchy-Binet theorem [31, Formula 0.8.7],

$$h_M(x_1, \ldots, x_n) = \sum_{S\in\binom{[n]}{d}} |A([d], S)|^2 \prod_{i\in S} x_i = \sum_{B\in\mathcal{B}} \prod_{i\in B} x_i \tag{2.1}$$

where $A([d], S)$ is $d \times d$ minor of A with columns indexed by S and $\mathcal{B}$ is the collection of bases of M. In particular, all minors of A need to be $-1, 1$ or 0. As mentioned above, by Theorem 6.6.3 in [50], such a representation in $\mathbb{R}$ implies regularity. Thus, if a matroid has a determinantal representation, then it is representable over every field.

Conversely, if a matroid M on $E = [n]$ is regular, then there is a real matrix A all of whose minors are $-1, 0$ or 1. Let $a_1, \ldots, a_n$ be the columns of A and let $A_i = a_i a_i^T$. By construction, A_i are real symmetric PSD matrices. Consider the polynomial h defined the same way as h_M in 2.1. By Lemma 2.1.4, the hyperbolic rank function of h is the same as rk_M and all non-zero coefficients of h are 1. Therefore, h is the basis generating polynomial of M. Then, M has a determinantal representation.

Corollary 2.1.6. *A matroid has a determinantal representation if and only if it is regular.*

Example 2.1.7.

- The Fano matroid F_7 is representable only over fields of characteristic two (see [50, Proposition 6.8.7]). By Example 11.1 in [18], it does not have the half-plane property. Since by Proposition 1.1.37 having a determinantal representation implies the half-plane property, F_7 does not have a determinantal representation.
- Consider the uniform matroid $U_{2,4}$. By Theorem 9.1 in [18], it has the half-plane property. On the other hand, by Theorem 6.6.6 in [50] it is one of the forbidden minors for being regular. Therefore, it does not have a determinantal representation (see also Example 2.1.14).
- A graphical matroid $M(G)$, constructed from a graph G, can be represented over $\mathbb{R}$ by the incidence matrix A_G of $D(G)$ with entries in $\{-1, 0, 1\}$, where $D(G)$ is the directed graph obtained by assigning an arbitrary direction to each edge of G. The matrix A_G indeed gives a representation of $M(G)$ over any field (see [50, Lemma 5.1.3]). Therefore, graphical matroids are regular (see also [50, Proposition 5.1.2]) and they have a determinantal representation.

A Criterion for Determinantal Representability

We consider the preservation of determinantal representability under certain operations. Especially a criterion for determinantal representability plays an essential role in obtaining desired results. The content of this section highly relies on the paper [38] by Kummer, Plaumann, and Vinzant.

In order to characterize determinantal representability of homogeneous multiaffine polynomials, we first need to define their Rayleigh differences.

Definition 2.1.8. Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial, and let $i, j \in [n]$. The *Rayleigh difference* $\Delta_{ij}h$ of (i, j) is defined as

$$\Delta_{ij}h = \frac{\partial h}{\partial x_i} \cdot \frac{\partial h}{\partial x_j} - \frac{\partial^2 h}{\partial x_i \partial x_j} \cdot h.$$

The lemma below shows the structure of the Rayleigh differences of a homogeneous polynomial that is the product of two other polynomials. Wagner and Wei first used this identity in [63].

Lemma 2.1.9. *Let $h = f \cdot g$ where $f, g \in \mathbb{R}[x_1, \ldots, x_n]$ and $i, j \in [n]$. Then,*

$$\Delta_{ij}h = f^2 \Delta_{ij} g + g^2 \Delta_{ij} f.$$

Proof. By the definition of the Rayleigh difference Δ_{ij} and the properties of derivatives, we have

$$\begin{aligned}
\Delta_{ij}h &= \frac{\partial}{\partial x_i}(f \cdot g) \cdot \frac{\partial}{\partial x_j}(f \cdot g) - (f \cdot g) \cdot \frac{\partial^2}{\partial x_i \partial x_j}(f \cdot g) \\
&= \left(\frac{\partial f}{\partial x_i} \cdot g + \frac{\partial g}{\partial x_i} \cdot f \right) \cdot \left(\frac{\partial f}{\partial x_j} \cdot g + \frac{\partial g}{\partial x_j} \cdot f \right) \\
&- (f \cdot g) \cdot \left(g \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} + \frac{\partial g}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} + f \cdot \frac{\partial^2 g}{\partial x_i \partial x_j} \right) \\
&= f^2 \cdot \left(\frac{\partial g}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} - g \frac{\partial^2 g}{\partial x_i \partial x_j} \cdot \right) + g^2 \cdot \left(\frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \frac{\partial^2 f}{\partial x_i \partial x_j} \right).
\end{aligned}$$

Therefore, $\Delta_{ij}h = f^2 \Delta_{ij} g + g^2 \Delta_{ij} f$. □

The following result by Kummer, Plaumann, and Vinzant characterizes the determinantal representability of a homogeneous polynomial via the squareness of its Rayleigh differences.

Theorem 2.1.10 (Corollary 5.7 in [38])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous, multiaffine and stable polynomial. Then the following are equivalent:*

- *h has a determinantal representation.*
- *$\Delta_{ij}h$ is a square for all $i, j \in [i]$.*

Using this criterion, one obtains that having a determinantal representation is minor-closed.

Theorem 2.1.11 (Corollary 5.8 in [38])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous, multiaffine, stable polynomial and $1 \leq k \leq n$. If h has a determinantal representation, then so do $\frac{\partial h}{\partial x_k}$ and $h|_{x_k=0}$.*

Proof. Let $1 \leq k, i, j \leq n$, $p := \frac{\partial h}{\partial x_k}$ and $q = h|_{x_k=0}$. One can write h as $h = x_k p + q$. Then

$$\Delta_{ij}h = \left(x_k \frac{\partial p}{\partial x_i} + \frac{\partial q}{\partial x_i} \right) \cdot \left(x_k \frac{\partial p}{\partial x_j} + \frac{\partial q}{\partial x_j} \right) - \left(x_k \frac{\partial^2 p}{\partial x_i \partial x_j} + \frac{\partial^2 q}{\partial x_i \partial x_j} \right) \cdot h.$$

After inserting $x_k p + q$ for h and expanding the expression, we obtain

$$\Delta_{ij}h = x_k^2 \Delta_{ij}p + x_k g + \Delta_{ij}q$$

where $g := \frac{\partial p}{\partial x_i}\frac{\partial q}{\partial x_j} + \frac{\partial q}{\partial x_i}\frac{\partial p}{\partial x_j} - \left(\frac{\partial^2 q}{\partial x_i \partial x_j}p + \frac{\partial^2 p}{\partial x_i \partial x_j}q\right)$ does not depend on x_k. Since h has a determinantal representation, $\Delta_{ij}h$ is a square, therefore $\Delta_{ij}p$ and $\Delta_{ij}q$ are squares as well. This shows that p and q have a determinantal representation. □

Corollary 2.1.12. *Determinantal representability is minor-closed.*

Proof. It follows from the fact that a minor of a matroid is obtained by deletion and contraction operations. By Proposition 1.2.15, a simple matroid obtained by deleting or contracting $e \in E$ has the basis generating polynomial $h_M|_{x_e=0}$ or $\frac{\partial}{\partial x_e}h_M$ respectively. Moreover, if $e \in E$ is a co-loop of M, the deletion $M \setminus e$ has the basis generating polynomial $\frac{\partial}{\partial x_e}h_M$ and when $e \in E$ is a loop of M, the contraction $M \setminus e$ has the basis generating polynomial h_M. Therefore, they are covered by Theorem 2.1.11. □

Recall that for homogeneous, multiaffine polynomials, taking derivative with respect to a variable or setting a variable equal to zero can be expressed as setting that variable equal to one and taking the sum of the lower degree terms or the sum of the upper degree terms respectively.

Corollary 2.1.13. *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous multiaffine polynomial of degree d with a determinantal representation. Then for $e \in [n]$, $(h|_{x_e=1})^{\#}$ and $(h|_{x_e=1})_{\#}$ also have a determinantal representation.*

Proof. Since h is homogeneous and multiaffine, when x_e appears in h and it is not a factor of h we have

$$(h|_{x_e=1})^{\#} := \lim_{k\to\infty} k^{-d}h(kx_1, \ldots, kx_n)|_{x_e=1} = h|_{x_e=0} \text{ and}$$

$$(h|_{x_e=1})_{\#} := \lim_{k\to 0} k^{-d+1}h(kx_1, \ldots, kx_n)|_{x_e=1} = \frac{\partial}{\partial x_e}h$$

for $e \in [n]$. On the other hand, when x_e is a factor of h we have

$$(h|_{x_e=1})^{\#} = \lim_{k\to\infty} k^{-d-1}h(kx_1, \ldots, kx_n)|_{x_e=1} = \frac{\partial}{\partial x_e}h = (h|_{x_e=1})_{\#}$$

and when x_e does not appear in h we have

$$(h|_{x_e=1})_{\#} = \lim_{k\to 0} k^{-d}h(kx_1, \ldots, kx_n)|_{x_e=1} = h = (h|_{x_e=1})^{\#}.$$

Therefore, by Theorem 2.1.11, they have a determinantal representation. □

Here is an example of the usage of the criterion given in Theorem 2.1.10.

Example 2.1.14. Consider the uniform matroid $U_{2,4}$. Its basis generating polynomial $h_{U_{2,4}}$ is the elementary symmetric polynomial $E_{2,4}$. As we mentioned in the previous chapter, this polynomial is stable. However, it does not have a determinantal representation since the Rayleigh difference $\Delta_{12}E_{2,4} = x_3^2 + x_3x_4 + x_4^2$ is not a square. In general, consider the elementary symmetric polynomial $E_{r,n} \in \mathbb{R}[x_1, \ldots, x_n]$ of degree r. For all $1 \leq i < j \leq n$ we have, $\Delta_{ij}E_{1,n} = 1$, $\Delta_{ij}E_{n-1,n} = (x_1 \ldots x_n / x_i x_j)^2$ and $\Delta_{ij}E_{n,n} = 0$. For other values of r, $E_{r,n}$ does not have a determinantal representation (see for example [55, Theorem 1.3]).

One can further consider polynomials with a determinantal representation that can be written as a product of two or more polynomials. The following lemma gives an analog of the criterion for this case.

Lemma 2.1.15 (Lemma 5.6 in [38])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ such that the degree of x_i and x_j are at most 1 in h for some $i, j \in [n]$. If $h = f \cdot g$ with $f, g \in \mathbb{R}[x_1, \ldots, x_n]$, then $\Delta_{ij}h$ is a square if and only if $\Delta_{ij}f$ and $\Delta_{ij}g$ are squares.*

Proof. Assume that $\Delta_{ij}h$ is a square. Since x_i and x_j have degree at most 1 in h (i.e., h is affine in x_i and x_j), f and g are also affine in x_i and x_j. Therefore, either $\frac{\partial f}{\partial x_i} = 0$ or $\frac{\partial g}{\partial x_i} = 0$ (same holds for j) so that either $\Delta_{ij}f = 0$ or $\Delta_{ij}g = 0$. Since $\Delta_{ij}h = f^2\Delta_{ij}g + g^2\Delta_{ij}f$ by Lemma 2.1.9 and $\Delta_{ij}h$ is a square, we conclude that each of $\Delta_{ij}f$ and $\Delta_{ij}g$ is either zero or a square.

For the other direction, assume that $\Delta_{ij}f$ and $\Delta_{ij}g$ are squares. By the degree argument above, we have again that either $\Delta_{ij}f = 0$ or $\Delta_{ij}g = 0$. Since $\Delta_{ij}h = f^2\Delta_{ij}g + g^2\Delta_{ij}f$, h is a square. □

We then immediately obtain that a homogeneous multiaffine polynomial that is the product of two or more polynomials has a determinantal representation if and only if the factors have a determinantal representation.

Corollary 2.1.16 (Corollary 5.9 in [38])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be homogeneous and multiaffine. If $h = f \cdot g$ with $f, g \in \mathbb{R}[x_1, \ldots, x_n]$, then h has a determinantal representation if and only if f and g have a determinantal representation.*

Proof. It follows from Theorem 2.1.10 and Lemma 2.1.15. □

The example below illustrates how Corollary 2.1.16 fails for non-multiaffine polynomials.

Example 2.1.17. Consider the polynomial

$$h = (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + 2x_3x_4)^2.$$

It has a determinantal representation with

$$h = \det \begin{pmatrix} x_1 + x_2 + 2x_4 & x_4 & 0 & -x_2 - x_4 \\ x_4 & x_2 + x_3 + x_4 & x_2 + x_4 & 0 \\ 0 & x_2 + x_4 & x_1 + x_2 + 2x_4 & x_4 \\ -x_2 - x_4 & 0 & x_4 & x_2 + x_3 + x_4 \end{pmatrix}.$$

However, its factor $g := x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + 2x_3x_4$ does not have a determinantal representation as $\Delta_{3,4} = x_1^2 + x_2^2$ is not a square.

Rayleigh differences of homogeneous multiaffine polynomials and their properties, such as non-negativity, will appear in the following sections. We first introduce a term for multiaffine polynomials, all of whose Rayleigh differences are sum of squares.

Definition 2.1.18. A multiaffine polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is called *SOS-Rayleigh* if for all $1 \leq i, j \leq n$, the Rayleigh difference $\Delta_{ij}h$ is sum of squares of polynomials. A matroid is called *SOS-Rayleigh* if its basis generating polynomial is SOS-Rayleigh.

In particular, weakly determinantal polynomials are SOS-Rayleigh.

Theorem 2.1.19 (Corollary 4.3 in [38]). *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be weakly determinantal. Then for all $1 \leq i, j \leq n$, the Rayleigh difference $\Delta_{ij}h$ is a sum of squares.*

Below, we show that being SOS-Rayleigh is preserved under taking derivatives and setting some variables equal to zero.

Theorem 2.1.20 (Theorem 1.12 in [39]). *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a multiaffine stable polynomial and $k \in [n]$. If h is SOS-Rayleigh, then $\frac{\partial h}{\partial x_k}$ and $h|_{x_k=0}$ are SOS-Rayleigh as well.*

Proof. Let $g = \frac{\partial h}{\partial x_k}$ and $f = h|_{x_k=0}$. One calculates

$$\Delta_{ij}h = x_k^2 \cdot \Delta_{ij}g + x_k \cdot p + \Delta_{ij}f$$

for some $p \in \mathbb{R}[x_1, \ldots, x_n]$ that does not depend on x_k. By assumption $\Delta_{ij}h$ is a sum of squares so that it is of the form $\Delta_{ij}h = \sum_{l=1}^{m} s_l^2$ for some $s_l \in \mathbb{R}[x_1, \ldots, x_n]$. Comparing degrees on both sides we obtain that $s_l = a_l x_k + b_l$ for some polynomials $a_l, b_l \in \mathbb{R}[x_1, \ldots, x_n]$ that do not depend on x_k. It follows that $\Delta_{ij}g = \sum_{l=1}^{m} a_l^2$ and $\Delta_{ij}f = \sum_{l=1}^{m} b_l^2$. □

Corollary 2.1.21. *Being SOS-Rayleigh is minor closed.*

Furthermore, being SOS-Rayleigh is closed under taking direct sums.

Proposition 2.1.22. *Let M, N be two matroids on disjoint ground sets that are SOS-Rayleigh. Then $M \oplus N$ is also SOS-Rayleigh.*

Proof. Since $h_{M \oplus N} = h_M \cdot h_N$, it follows from the identity

$$\Delta_{ij}h_{M\oplus N} = h_N^2 \Delta_{ij}h_M + h_M^2 \Delta_{ij}h_N$$

given in Lemma 2.1.9. □

2.2 Spectrahedral Representability of Matroids

In this section, we present our main result on operations preserving spectrahedral representability. Most of the results presented here are a part of [39] joint work with Mario Kummer.

Our main objects are homogeneous multiaffine polynomials with the half-plane property whose hyperbolicity cones are spectrahedral, i.e., they can be defined by linear matrix inequalities. Recall that Theorem 1.1.46 gives a criterion for a hyperbolicity cone to be spectrahedral. A homogeneous multiaffine stable polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ has a spectrahedral hyperbolicity cone C_h if and only if there exists another hyperbolic polynomial $g \in \mathbb{R}[x_1, \dots, x_n]$ with $C_h \subset C_g$ such that $h \cdot g$ has a determinantal representation. One observes that this criterion has two components: the determinantal representability of $h \cdot g$ and the inclusion of the hyperbolicity cones.

The following lemmas focus on the structure of hyperbolicity cones with an inclusion relation.

Lemma 2.2.1 (Lemma 3.5 in [39]). *Let $(h_i)_{i\in\mathbb{N}}, (g_i)_{i\in\mathbb{N}}$ two sequences of polynomials in $\mathbb{R}[x_1, \dots, x_n]$ that are hyperbolic with respect to $e \in \mathbb{R}^n$. Assume that we have $h = \lim_{i\to\infty} h_i$ and $g = \lim_{i\to\infty} g_i$ for polynomials g, h that are hyperbolic with respect to e. If $C_{h_i}(e) \subset C_{g_i}(e)$ for all $i \in \mathbb{N}$, then $C_h(e) \subset C_g(e)$.*

Proof. Since for all $v \in \mathbb{R}^n$ and $i \in \mathbb{N}$, the roots of $h_i(et - v)$ and $g_i(et - v)$ continuously depend on coefficients, and by Lemma 1.1.17, the minimal root of $h_i(et - v)$ is smaller than the minimal root of $g_i(et - v)$, their limits have the same property. Therefore $C_h \subseteq C_g$. □

Below, we consider the structure of the hyperbolicity cone of a polynomial h, which does not depend on all of the variables in its polynomial ring (i.e., its hyperbolicity cone contains a linear subspace). In particular, one can construct a new polynomial with the unused variables and exploit the relation between the hyperbolicity cones when the constructed polynomial is hyperbolic.

Lemma 2.2.2 (Lemma 3.6 in [39]). *Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial that does not depend on x_k and let $h' := x_k^m h$, $m \geq 0$. If h' (and thus h) is hyperbolic with respect to $e \in \mathbb{R}^n$, then*

$$C_h(e) = \{v \in \mathbb{R}^n : \exists \lambda \in \mathbb{R} : v + \lambda\delta_k \in C_{h'}(e)\}$$

where δ_k is the k-th standard basis vector.

Proof. Taking the product of hyperbolic polynomials corresponds to intersecting their hyperbolicity cones. As the hyperbolicity cone of the polynomial x_k^m is the half-space defined by the inequality $x_k \geq 0$, $C_{h'} = C_{x_k^m}(e) \cap C_h(e) \subseteq C_h(e)$. Since h does not depend on x_k, for every $v \in C_{h'}(e)$ and $\lambda \in \mathbb{R}$, $v + \lambda\delta_k \in C_h$.

For the other direction, let $v \in C_h$. Then $h'(et - (v + v_k\delta_k)) = at^m h(et - v)$ has only non-negative roots for some constant $a \in \mathbb{R}$ so that $v + v_k\delta_k \in C_{h'}$. □

Corollary 2.2.3 (Corollary 3.7 in [39]). *Let $h_1, h_2 \in \mathbb{R}[x_1, \dots, x_n]$ be homogeneous polynomials that do not depend on x_k and let $h_1' := x_k^{m_1} h_1$ and $h_2' := x_k^{m_2} h_2$. Assume that h_1', h_2' (and thus h_1, h_2) are hyperbolic with respect to $e \in \mathbb{R}^n$. If $C_{h_1'}(e) \subseteq C_{h_2'}(e)$, then $C_{h_1}(e) \subseteq C_{h_2}(e)$.*

Proof. Follows from Lemma 2.2.2. □

We are especially interested in deletion and contraction operations. In particular, Corollary 2.1.13 motivates the exploration of the initial form and leading form of polynomials and their effect on the inclusion relation of the hyperbolicity cones.

Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$ that does not have x_k as a factor for $k \in [n]$, and consider $(h|_{x_k=1})^{\#}$. In this case, we have $(h|_{x_k=1})^{\#} = h|_{x_k=0}$. Then, the hyperbolicity cone $C_{(h|_{x_k=1})^{\#}}(e)$ is nothing but the intersection of $C_h(e)$ with the hyperplane $x_k = 0$. Therefore, applying the upper sharp operation on two polynomials $h, g \in \mathbb{R}[x_1, \ldots, x_n]$ that are hyperbolic with respect to $e \in \mathbb{R}^n$ with $C_h(e) \subset C_g(e)$, will not affect the inclusion relation.

The proposition below concerns the lower sharp operation.

Proposition 2.2.4 (Corollary 3.8 in [39]). *Let $h_1, h_2 \in \mathbb{R}[x_1, \ldots, x_n]$ be stable and $C_{h_1}(e) \subset C_{h_2}(e)$. Let $h_1' = (h_1|_{x_k=1})_{\#}$ and $h_2' = (h_2|_{x_k=1})_{\#}$. Then $C_{h_1'}(e) \subset C_{h_2'}(e)$.*

Proof. Let d_i be the degree of h_i, and r_i, be the smallest degree of a monomial of $h_i|_{x_k=1}$ for $i = 1, 2$. For all $\gamma > 0$ we denote

$$h_{i,\gamma} = \gamma^{-r_i} h_i(\gamma x_1, \ldots, \gamma x_{k-1}, x_k, \gamma x_{k+1}, \ldots, \gamma x_n).$$

Since $(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \mapsto (\gamma x_1, \ldots, \gamma x_{k-1}, x_k, \gamma x_{k+1}, \ldots, \gamma x_n)$ is an invertible linear map, we have $C_{h_{1,\gamma}}(e) \subset C_{h_{2,\gamma}}(e)$ for all $\gamma > 0$. Since h_i is homogeneous, the limit $\lim_{\gamma \to 0} h_{i,\gamma}$ kills all the monomials, except those that are divisible by $x_k^{d_i - r_i}$. Thus, we have

$$\lim_{\gamma \to 0} h_{i,\gamma} = x_k^{d_i - r_i} \, (h_i|_{x_k=1})_{\#} \, .$$

By Lemma 2.2.1 and Corollary 2.2.3, we obtain $C_{h_1'}(e) \subset C_{h_2'}(e)$. □

The following theorems are our main results on the preservation of spectrahedral representability under deletion and contraction operations on matroids.

Theorem 2.2.5 (Theorem 3.9 in [39]). *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous, multiaffine and stable polynomial and $e = (1, \ldots, 1)$ the all-ones vector. If $\frac{\partial h}{\partial x_k} \neq 0$ and the hyperbolicity cone $C_h(e)$ is spectrahedral, then the hyperbolicity cone of $\frac{\partial h}{\partial x_k}$ is also spectrahedral.*

Proof. Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$ and assume that $C_h(e)$ is spectrahedral. Then, by Theorem 1.1.46, there exists a stable polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$ such that $C_h(e) \subseteq C_g(e)$ and $p := h \cdot g$ has a determinantal representation. We can assume that $k = 1$ and that

$$0 \neq p = \det(x_1 A_1 + \ldots + x_n A_n)$$

where the A_i are positive semidefinite matrices of rank $r_i \geq 1$ for $i \in [n]$. Please note that p does not need to be multiaffine. We write each A_i as the sum of rank 1 matrices:

$$A_i = A_{i1} + \ldots + A_{ir_i},$$

where A_{ij} are positive semidefinite matrices of rank 1, $i \in [n]$, $j \in [r_i]$. Now consider the polarization of p, and define a new polynomial

$$\tilde{p} = \det(x_{11} A_{11} + x_{12} A_{12} + \cdots + x_{1r_1} A_{1r_1} + \cdots + x_{nr_n} A_{nr_n})$$

in $\sum_{i=1}^{n} r_i$ variables. The degree of each variable x_{ij} in $\tilde{p}$ is the rank of A_{ij} so that $\tilde{p}$ is multiaffine. Moreover, since the rank of A_i is r_i by Lemma 2.1.3 (and Remark 2.1.5), there are monomials of $\tilde{p}$ that are divisible by $x_{i1}\cdots x_{ir_i}$. By Theorem 2.1.11 the polynomial

$$\frac{\partial^{r_1}\tilde{p}}{\partial x_{11}\cdots\partial x_{1r_1}} = (\tilde{p}|_{x_{1j}=1 \text{ for } j\in[r_1]})_{\#}$$

has a determinantal representation. The same is thus true for the polynomial $\hat{p}$ obtained from $\frac{\partial^{r_1}\tilde{p}}{\partial x_{11}\cdots\partial x_{1r_1}}$ by setting $x_{ir_1} = \cdots = x_{ir_k} = x_i$ for all $i \in [n]$. By construction we have that

$$\hat{p} = (p|_{x_1=1})_{\#} = (h|_{x_1=1}\cdot g|_{x_1=1})_{\#} = (h|_{x_1=1})_{\#}\cdot(g|_{x_1=1})_{\#}.$$

Since h is multiaffine, we have $(h|_{x_1=1})_{\#} = \frac{\partial h}{\partial x_1}$. Thus in order to prove the claim it remains to show that the hyperbolicity cone of $(h|_{x_1=1})_{\#}$ is contained in the hyperbolicity cone of $(g|_{x_1=1})_{\#}$. This follows from Proposition 2.2.4. □

Note that the proof of Theorem 2.2.5 crucially relies on the assumption that h is multiaffine and it is not known whether one can drop this assumption.

Kummer in [37] proved that if h (not necessarily multiaffine) has a determinantal representation, then the hyperbolicity cone of any iterated derivative in direction e of h is spectrahedral.

Lemma 2.2.6 (Lemma 3.11 in [39]). *Let $h \in \mathbb{R}[x_1,\ldots,x_n]$ be a homogeneous, multiaffine and stable polynomial and $e = (1,\ldots,1)$ the all-ones vector. If $h|_{x_k=0} \neq 0$ and the hyperbolicity cone $C_h(e)$ is spectrahedral, then the hyperbolicity cone of $h|_{x_k=0}$ is also spectrahedral.*

Proof. The hyperbolicity cone of $h|_{x_k=0}$ is the intersection of $C_h(e)$ with the hyperplane $x_k = 0$ and thus spectrahedral. □

Corollary 2.2.7 (Corollary 3.12 in [39]). *The class of spectrahedral matroids is minor-closed.*

Proof. If a matroid M is spectrahedral, then the same is true for its contractions by Theorem 2.2.5 and for its deletions by Lemma 2.2.6. □

Further, taking direct sums preserves spectrahedral representability.

Proposition 2.2.8. *Let M, N be two matroids on disjoint ground sets that are spectrahedral. Then $M \oplus N$ is also spectrahedral.*

Proof. Since $h_{M\oplus N} = h_M \cdot h_N$, the hyperbolicity cone of products of two polynomials is the intersection of their hyperbolicity cones, and that M and N have disjoint ground sets, the spectrahedral representation of the hyperbolicity cone of $h_{M\oplus N}$ comes from the matrix constructed by the block matrices coming from the spectrahedral representation of h_M and the spectrahedral representation of h_N. □

One of the consequences of Theorem 2.2.5 and Lemma 2.2.6 is that being weakly determinantal is preserved under deletion and contraction operations on matroids.

Corollary 2.2.9 (Corollary 3.13 in [39]). *Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be multiaffine, stable and weakly determinantal. Then $\frac{\partial h}{\partial x_k}$ and $h|_{x_k=0}$ are also weakly determinantal.*

Proof. For $\frac{\partial h}{\partial x_k}$ we can apply the proof of Theorem 2.2.5 to the case $g = h^r$ for some $r \geq 0$. The claim for $h|_{x_k=0}$ follows directly from the definition. □

Corollary 2.2.10. *The class of weakly determinantal matroids is minor-closed.*

Moreover, being weakly determinantal is closed under taking direct sums.

Proposition 2.2.11. *Let M, N be two matroids on disjoint ground sets that are weakly determinantal. Then $M \oplus N$ is also weakly determinantal.*

Proof. Since M and N are weakly determinantal, there are powers r, s such that h_M^r and h_N^s have a determinantal representation. In particular, h_M^{rs} and h_N^{rs} have a determinantal representation. As M and N have two disjoint ground sets and $h_{M\oplus N} = h_M \cdot h_N$, the $r\dot{s}$-th power of the product has a determinantal representation with the matrix constructed by block matrices coming from the determinantal representation of h_M^{rs} and h_N^{rs}. Therefore, it is weakly determinantal. □

2.3 Matroid Polytopes

In this section, we take a geometric approach and observe the effect of going to the faces of the matroid polytope of a matroid M on its basis generating polynomials.

The matroid polytope of a matroid is the Newton polytope of its basis generating polynomial.

Definition 2.3.1. The *matroid polytope* $P(M)$ of a matroid M on a non-empty ground set $E = [n]$ with its non-empty collection of bases $\mathcal{B}$ is defined as

$$P(M) = \operatorname{conv}\left\{\sum_{i\in B} \delta_i : B \in \mathcal{B}\right\}$$

where $\delta_i \in \mathbb{R}^n$ denotes the i-th unit vector.

One can further give the description of the matroid polytope of a matroid M in terms of the half-spaces defined by its flats as follows:

$$P(M) = \left\{x \in r \cdot \Delta_n : \sum_{i\in S} x_i \leq \operatorname{rk}(S) \text{ for all flats } S \subseteq E\right\} \tag{2.2}$$

where $r \cdot \Delta_n \subset \mathbb{R}^n$ denotes r-fold dilation of the standard n-simplex $\Delta_n \subset \mathbb{R}^n$ (see for example [46, §4.4]).

Below is an example of a matroid polytope.

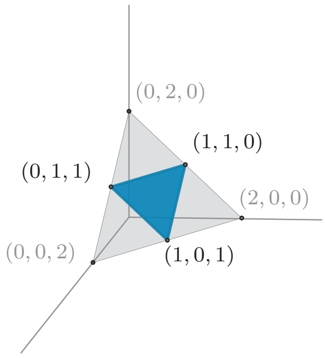


Figure 2.1: Matroid polytope $P(U_{2,3})$

Example 2.3.2. Consider the uniform matroid $U_{2,3}$. Its matroid polytope has vertices $(1,1,0),(1,0,1)$ and $(0,1,1)$. We can also express it in terms of linear inequalities coming from its flats and from the defining inequalities of $2\cdot\Delta_3$. In this case, its flats are $\{x_1,x_2,x_3\}$ of rank 2, and $\{x_1\},\{x_2\}$ and $\{x_3\}$ of rank 1. Then we have the inequalities

$$x_1+x_2+x_3\leq 2,\ x_1\leq 1,\ \ x_2\leq 1,\ \ \text{and } x_3\leq 1$$

that come from the flats. The matroid polytope $P(U_{2,3})$ is depicted in Figure 2.1.

Lemma 2.3.3. *Let M be a matroid on $E=[n]$ of rank r with the collection of bases $\mathcal{B}$. Then the set of vertices of its matroid polytope $P(M)$ is*

$$V_{P(M)} := \left\{\sum_{i\in B}\delta_i : B\in\mathcal{B}\right\}$$

where $\delta_i\in\mathbb{R}^n$ denotes the i-th unit vector.

Proof. First, note that the definition of a matroid polytope concerns only matroids on non-empty ground sets with at least one basis. Therefore, matroid polytopes are non-empty. Let V be the set of vertices of $P(M)$. Since $P(M)$ is the convex hull of the elements of $V_{P(M)}$, we have $V\subseteq V_{P(M)}$ (see [64, Proposition 2.2]). Assume that there is a $v\in V_{P(M)}\setminus V$ and let $B\in\mathcal{B}$ be its defining basis such that $v=\sum_{i\in B}\delta_i$. Since v is not a vertex and $v\in P(M)=\mathrm{conv}(V)$, by Cathéodory's theorem (see [57, Theorem 1.1.4]) there is a minimal $k\in\mathbb{N}$ such that $v=\sum_{i=1}^k\lambda_iv_i$ where $v_i\in V$, $\lambda_i>0$ and $\sum_{i=1}^k\lambda_i=1$. On the other hand, by construction every $v_i\in V$ (and also v) has its coordinates in $\{0,1\}^n$ and has exactly r non-zero entries. Thus for each $i\in[k]$, we have $\lambda_i=0$ or $\lambda_i=1$ and there cannot be more than one non-zero λ_i. Then, $v=v_i$ for some $v_i\in V$ so that $V_{P(M)}\subseteq V$. $\square$

The faces of a matroid polytope are matroid polytopes of some matroids.

Proposition 2.3.4 (Theorem 4.1 in [26])**.** *Let M be a matroid on $E=[n]$ of rank r, $P(M)$ be its matroid polytope. Then, every face F of $P(M)$ is a matroid polytope $P(M_F)$ for some matroid M_F, and M_F is uniquely determined by F.*

Proof. It is sufficient to prove the claim for facets F of $P(M)$. Since F is a facet, the set of its vertices V_F is a subset of the set of vertices $V_{M(P)}$ of $P(M)$. By Lemma 2.3.3, $V_{P(M)} = \left\{\sum_{i\in B} \delta_i : B \in \mathcal{B}\right\}$ where $\mathcal{B}$ is the collection of bases of M. Let $\mathcal{B}_F$ be the collection of $B \in \mathcal{B}$ that corresponds to a vertex of F. In other words, $v_j \in V_F$ are $\sum_{i\in B} \delta_i$ for some $B \in \mathcal{B}_F \subseteq \mathcal{B}$ where δ_i are standard basis vectors. We need to show that the elements of $\mathcal{B}_F \subseteq \mathcal{B}$ satisfy the basis exchange axiom.

By the description in 2.2, a facet F of $P(M)$ there is either a flat $S \subseteq E$ of M such that F is the intersection of $P(M)$ with the hyperplane

$$H_S := \left\{ x \in \mathbb{R}^n : \sum_{i\in S} x_i = \mathrm{rk}(S) \right\},$$

or it is the intersection of $P(M)$ with a coordinate hyperplane $x_i = 0$.

For the first case, let $B_1, B_2 \in \mathcal{B}_F$ be two distinct bases of M and $e \in B_1 \setminus B_2$. In the first case, there is a non-empty flat S of M such that all $v_j \in V_F$ satisfy the equation of H_S. Then $S = \mathrm{cl}(B_1 \cap S) = \mathrm{cl}(B_2 \cap S)$ and $|B_1 \cap S| = |B_2 \cap S| = \mathrm{rk}(S)$. If $\mathrm{rk}(S) = r$, then $F = P(M)$ is not a facet. For $0 < \mathrm{rk}(S) < r$, let $I_1 := B_1 \setminus (B_1 \cap S)$ and $I_2 := B_2 \setminus (B_2 \cap S)$. If $e \in B_1 \setminus I_1$, then $|B_2 \setminus I_2| > |\,(B_1 \setminus I_1) \setminus \{e\}\,|$ therefore, by the properties of independent sets, there exists $e' \in (B_2 \setminus I_2) \setminus ((B_1 \setminus I_1) \setminus \{e\})$ such that $I_3 := ((B_1 \setminus I_1) \setminus \{e\}) \cup \{e'\} \subset S$ is independent. Since I_1 is not a subset of S and S is a flat,

$$\mathrm{rk}(S \cup I_1) = \mathrm{rk}(S) + |I_1| = \mathrm{rk}(\mathrm{cl}(I_3)) + |I_1| = \mathrm{rk}(I_3) + |I_1| = r$$

by the properties of the closure. Therefore $I_1 \cup I_3$ is a basis satisfying the equation of H_F so that $(B_1 \setminus \{e\}) \cup \{e'\} = I_1 \cup I_3 \in \mathcal{B}_{\mathcal{F}}$. If $e \in I_1$, then $|I_2| > |I_1 \setminus \{e\}\,|$ and again by the properties of independent sets there exists $e' \in I_2 \setminus (I_1 \setminus \{e\})$ such that $I_4 := \{e'\} \cup (I_1 \setminus \{e\})$ is independent. By the similar arguments on the definition of flat and the properties of closure, $(B_1 \setminus \{e\}) \cup \{e'\} = I_1 \cup I_4 \in \mathcal{B}_F$. Then, F is the matroid polytope of the matroid M_F defined with the collection of bases $\mathcal{B}_F$.

In the second case, F is the intersection of $P(M)$ with the hyperplane

$$H := \{x \in \mathbb{R}^n : x_i = 0\}.$$

In particular, F consists of all points in $r \cdot \Delta_n \subset \mathbb{R}^n$ that satisfy $\sum_{i\in S} x_i = \mathrm{rk}(S)$ where $S = E \setminus \{i\}$. Then the vertices in V_F correspond to $B \in \mathcal{B}$ with $B \cap \{i\} = \emptyset$. If there are no such $B \in \mathcal{B}$, then $\mathcal{B}_F$ consists of the maximal independent sets I contained in $E \setminus \{i\}$. In both possibilities, $\mathcal{B}_F$ is the collection of the bases of the matroid $M_{\setminus i}$ obtained by the deletion of the element i in M. Since the vertices of F are encoded by the elements of $\mathcal{B}_F$, F is the matroid polytope of $M_F = M_{\setminus i}$. □

One can further describe the basis generating polynomial of the matroid that corresponds to a face of a matroid polytope of a matroid M in terms of initial form and leading form of h_M.

Proposition 2.3.5 (Corollary 4.5 in [39]). *Let M be a matroid on $E = [n]$ and F be a facet of $P(M)$. Then is a subset $S \subset E$ and a constant c such that*

$$h_{M_F} = c \cdot (h_M|_{x_i=1 \text{ for } i\in E\setminus S})^{\#} \cdot (h_M|_{x_i=1 \text{ for } i\in S})_{\#}.$$

Proof. Let F be a facet of $P(M)$ defined as the intersection $P(M) \cap H_S$ for some non-empty flat $S \subseteq E$ of M, where H_S is the set of all points in $\mathbb{R}^n$ satisfying $\sum_{i \in S} x_i = \mathrm{rk}(S)$. Then the set of vertices of F is $V_F = \left\{\sum_{i\in B} e_i : B \in \mathcal{B} \text{ and } |B \cap S| = \mathrm{rk}(S)\right\}$. As the bases of corresponding matroid M_F are encoded by the vertices of F (by Proposition 2.3.4 and by Lemma 2.3.3), we have

$$h_{M_F} = \sum_{\substack{B\in\mathcal{B} \text{ and}\\ |B\cap S| = \mathrm{rk}(S)}} \prod_{i\in B} x_i.$$

Let $p := \left(h_M|_{x_i=1 \text{ for } i\in E\setminus S}\right)^{\#}$ and $q := (h_M|_{x_i=1 \text{ for } i\in S})_{\#}$. Then the monomials of p and q are of the form

$$\prod_{i\in B\cap S} x_i \text{ and } \prod_{i\in B\cap(E\setminus S)} x_i$$

for $B \in \mathcal{B}$ with $|B \cap S| = \mathrm{rk}(S)$ respectively. Note that for all $B \in \mathcal{B}$ with $|B \cap S| = \mathrm{rk}(S)$, S is the closure of the independent set $B \cap S$. Let $B_1, B_2 \in \mathcal{B}$ be bases with $B_1 \cap S \neq B_2 \cap S$ and $|B_1 \cap S| = |B_2 \cap S|$. Then by Lemma 1.2.8, for any independent set I of rank $\mathrm{rk}(M) - \mathrm{rk}(S)$,

$$\begin{aligned} & I \cup (B_1 \cap S) \text{ is independent} \\ \iff & \mathrm{rk}(\mathrm{cl}(B_1 \cap S) \cap \mathrm{cl}(I)) = \mathrm{rk}(S \cap \mathrm{cl}(I)) = \mathrm{rk}(\mathrm{cl}(B_2 \cap S) \cap \mathrm{cl}(I)) = 0 \\ \iff & (B_2 \cap S) \cup I \text{ is independent.} \end{aligned}$$

Therefore, product of a monomial of p and a monomial of q appears as a monomial of h_{M_F}. Letting $A := \{s \subset S : \mathrm{rk}(s) = |s| = \mathrm{rk}(S)\}$, we conclude that each monomial of q has coefficient $|A|$. Similarly, the coefficient of each monomial of p is $|A'|$ where $A' := \{s \subset E \setminus S : \mathrm{rk}(s) = |s| = r(M) - \mathrm{rk}(S)\}$.

If F is the intersection of $r \cdot \Delta_n$ with the coordinate hyperplane $x_i = 0$, then as mentioned in the last part of the proof of Proposition 2.3.4, F is the matroid polytope of $M_F = M_{\setminus i}$. By Lemma 1.2.17, for $S = E \setminus \{i\}$, $h_F = c \cdot (h_M|_{x_i=1\, i\in E\setminus S})^{\#} = c' \cdot (h_M|_{x_i=1\, i\in E\setminus S})^{\#} (h_M|_{x_i=1,\, i\in S})_{\#}$. □

In addition, for a face F of a matroid polytope $P(M)$, one can express the corresponding matroid M_F in terms of the direct sum of some deletion and contraction operations.

Corollary 2.3.6 (Theorem 2 in [27])**.** *Let M be a matroid on $E = [n]$ and F be a face of $P(M)$. Then there is $S \subset E$ such that*

$$M_F = M_{\setminus(E\setminus S)} \oplus M_{/S}.$$

Proof. By Proposition 2.3.5, there is $S \subseteq E$ such that

$$h_{M_F} = c \cdot (h_M|_{x_i=1 \text{ for } i\in E\setminus S})^{\#} \cdot (h_M|_{x_i=1 \text{ for } i\in S})_{\#}.$$

By Lemma 1.2.17, the components of the product are the basis generating polynomials of $M_{\setminus(E\setminus S)}$ and $M_{/S}$ respectfully. Then by the definition of the direct sum of matroids, $M_F = M_{\setminus(E\setminus S)} \oplus M_{/S}$. □

Given a matroid, we consider its matroid polytope, and its faces. We then switch from the faces to the matroids they uniquely determine. We are especially interested in properties that are preserved under this operation. The diagram below illustrates the process.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & P(M) \arrow[d] \\ M_F & F \arrow[l] \end{tikzcd}$$

Corollary 2.3.6 boils this operation down to the direct sum of some deletion and contraction operations. Hence we obtain the following desired result on the preservation of properties we are interested in.

Corollary 2.3.7 (Corollary 4.4 in [39])**.** *Let M be a matroid and F a face of its matroid polytope $P(M)$. Let P be a property that is minor-closed and preserved under taking the direct sum of matroids. If M has property P, then M_F has property P.*

This applies to the half-plane property, being weakly determinantal, spectrahedral and SOS-Rayleigh.

Proof. It suffices to show the claim for facets F of the matroid polytope. It then follows from the description $M_F = M_{\backslash(E\backslash S)} \oplus M_{/S}$ given in Corollary 2.3.6. By Corollary 1.2.30, half-plane property is minor closed. Similarly, the minor-closeness of being weakly determinantal, being SOS-Rayleigh, and spectrahedral representability follow from Corollary 2.2.10, Corollary 2.1.21 and Corollary 2.2.7 respectively.

By Proposition 1.2.32, taking direct sums preserves the half-plane property. Moreover, being weakly determinantal, being spectrahedral and being SOS-Rayleigh are also closed under taking direct sums by Proposition 2.2.11, Proposition 2.2.8 and Proposition 2.1.22 respectively. □

Example 2.3.8. Let M be the matroid with basis generating polynomial

$$h_M = x_1x_2 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4.$$

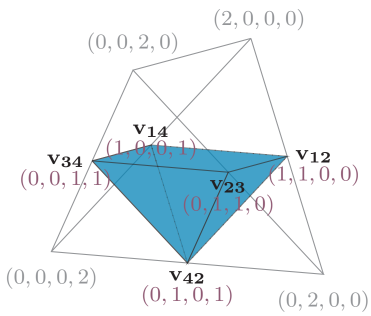
Its matroid polytope $P(M)$ is depicted in Figure 2.2a. The nonempty proper flats of M are $S_1 = \{2\}$, $S_2 = \{4\}$ and $S_3 = \{1, 3\}$. Thus $P(M)$ has the following representation in terms of inequalities:

$$P(M) = \left\{ x \in 2\Delta_4 : \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

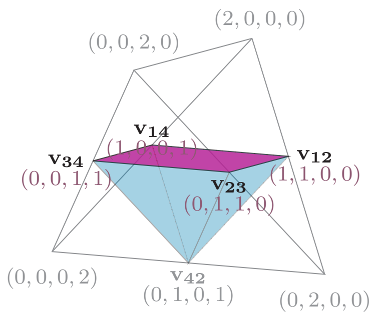
Consider for instance the flat $\{1, 3\}$, the hyperplane $H = \left\{x \in \mathbb{R}^4 : x_1 + x_3 = 1\right\}$ and the face $F = P(M) \cap H$. The matroid M_F is the matroid whose bases are encoded by the vertices of the face F that is illustrated in Figure 2.2b. Its basis generating polynomial is

$$h_{M_F} = x_1x_2 + x_2x_3 + x_1x_4 + x_3x_4 = (x_1 + x_3)(x_2 + x_4).$$

which is consistent with Proposition 2.3.5.



(a) $P(M)$



(b) $P(M_F)$

Figure 2.2: Matroid polytopes $P(M)$ and $P(M_F)$

Newton Polytopes of Stable Polynomials

We further extend our results on matroid polytopes and their faces to the Newton polytope $\mathrm{Newt}(h)$ of any homogeneous stable polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ and its faces F. In this section, by the support of h, we mean the set of all $\alpha \in \mathbb{Z}^n$ such that the monomial $x^\alpha := x^{\alpha_1} \cdots x^{\alpha_n}$ has a non-zero coefficient in h. Then the Newton polytope $\mathrm{Newt}(h)$ is nothing but the convex hull in $\mathbb{R}^n$ of the support of h. If h has the support $\mathcal{S} \subset \mathbb{Z}^n$, it has the form $h = \sum_{\alpha \in \mathcal{S}} c_\alpha x^\alpha$ where c_α are non-zero coefficients in the suitable field ($\mathbb{R}$ in the setting above). Restricting $\mathcal{S}$ to $F \cap \mathcal{S}$, we denote the restriction polynomial by $h_F := \sum_{\beta \in F \cap \mathcal{S}} c_\beta x^\beta$. The following proposition shows that stability is preserved when we apply the operation described above.

Proposition 2.3.9 (Proposition 2.6 in [39])**.** *Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ be stable and F be a face of* $\mathrm{Newt}(h)$*. Then the restriction h_F is stable.*

Proof. Let $h = \sum_{\alpha \in \mathcal{S}} c_\alpha x^\alpha$ with support $\mathcal{S} \subset \mathbb{Z}^n_{\geq 0}$ for non-zero $c_\alpha \in \mathbb{C}$. Consider the hyperplane that defines F. Namely, let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $\langle a, x\rangle \geq b$ for all $x \in \mathrm{Newt}(h)$ with equality exactly when $x \in F$. By Lemma 1.1.20, polynomials in the sequence

$$\left(h_\varepsilon := \varepsilon^{-b} \cdot h(\varepsilon^{a_1} x_1, \ldots, \varepsilon^{a_n} x_n) = \sum_{\alpha \in S} \varepsilon^{\langle a, \alpha \rangle - b} c_\alpha x^\alpha : \varepsilon > 0 \right)$$

are stable. Then by Hurwitz's theorem, the limit $h_F = \lim_{\varepsilon \to 0} h_\varepsilon$ is stable. □

Moreover, for homogeneous multiaffine polynomials, going from the Newton polytope to its faces preserves the determinantal representability of the corresponding polynomial.

Lemma 2.3.10 (Lemma 4.7 in [39])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous and multiaffine polynomial. Let F be a face of* $\mathrm{Newt}(h)$*. If h has a determinantal representation, then h_F has a determinantal representation.*

Proof. Without loss of generality, we can assume that F is a facet of $P :=$ $\mathrm{Newt}(h)$. Since h is multiaffine, by Lemma 2.1.3 we can write

$$h = \det(x_1 A_1 + \ldots + x_n A_n)$$

for some positive semi-definite matrices of rank at most 1, i.e., we have $A_i = a_i^T \cdot a_i$ for some $a_i \in \mathbb{R}^d$ for $i \in [n]$. Let M be the matroid represented by the matrix $V = [a_1, \ldots, a_n]$. Then we have $P = P(M)$. Further, by Corollary 2.3.6 there is a subset S of the ground set of M such that F consists of all $y \in d \cdot \Delta_n$ that satisfy $\sum_{i \in S} y_i = \mathrm{rk}(S) =: r$. We write $A := \sum_{i=1}^n x_i A_i = VXV^T$ where X is a diagonal matrix with diagonal entries x_i for $0 \leq i \leq n$. Considering $\tilde{A} := W^T AW$ for some invertible matrix W, we may assume that the variables x_i for $i \in S$ appear in the upper left $r \times r$ block R_1 of $\tilde{A}$. Let R_2 be the lower right $(d-r) \times (d-r)$ block matrix. We apply Laplace expansion (see [31, Formula 0.3.1]) to the determinant of $\tilde{A}$ and observe that the sum of the terms with maximal degree in x_i for $i \in S$ is the determinant of the matrix

$$\begin{pmatrix} R_1|_{x_j=0,\, j \notin S} & 0 \\ 0 & R_2 \end{pmatrix}$$

therefore, it gives the determinantal representation of h_F. □

Lemma 2.3.11 (Lemma 4.8 in [39])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be homogeneous. Let F be a face of* Newt(h)*. If h has a determinantal representation, then h_F has a determinantal representation.*

Proof. As in the proof of Theorem 2.2.5, by considering the polarization of h, we can find a multiaffine homogeneous polynomial $p \in \mathbb{R}[x_{11}, \ldots, x_{1r_1}, \ldots, x_{nr_n}]$ which has a determinantal representation such that $h = p|_{x_{ij}=x_i}$. In particular, the polytope Newt(h) is the image of Newt(p) under a linear map. There is a face F' of Newt(p) such that $h_F = (p_{F'})|_{x_{ij}=x_i}$. Thus the claim follows from Lemma 2.3.10. □

Corollary 2.3.12 (Corollary 4.9 in [39])**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous and stable polynomial. Let F be a face of* Newt(h)*. If h is weakly determinantal or spectrahedral, then h_F is weakly determinantal or spectrahedral .*

Proof. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $\langle a, p \rangle \geq b$ for all $y \in$ Newt(h) with equality exactly when $y \in F$. If f is spectrahedral, then by Theorem 1.1.46 there is a stable polynomial g whose hyperbolicity cone contains the one of h such that $h \cdot g$ has a determinantal representation. Let F' and F'' be the faces of Newt(g) and Newt$(h \cdot g)$ where $\langle a, - \rangle$ attains its minimum. Then we have $(h \cdot g)_{F''} = h_F \cdot g_{F'}$ and this polynomial has a determinantal representation by Lemma 2.3.11. Writing h_F and $g_{F'}$ as limits as in the proof of Proposition 2.3.9, it follows from Lemma 2.2.1 that the hyperbolicity cone of $g_{F'}$ contains the one of h_F. Thus we conclude that h_F is spectrahedral. If h is weakly determinantal, then we can proceed in the same way by letting $g = h^r$ for some $r \geq 0$. □

Chapter 3

Testing the Properties of Matroids: An Algorithm

In the previous chapter, we focused on operations on matroids that preserve the operations of our interest. In this chapter, we will take further steps to answer the questions posed at the end of §1.2 that are motivated by Conjecture 1.1.47. We list some criteria for having the half-plane property and focus on finding ways to implement them together with the ones for determinantal representability. Further, we present an algorithm that posits to be computationally feasible and efficient throughout our tests. The methods listed in the section were used for conducting tests on matroids on 8 and 9 elements that are part of [39].

In particular, the results on the minor closedness of determinantal representability, half-plane property, being SOS-Rayleigh, and being weakly determinantal have exciting consequences. If a matroid has one of the above properties, all its minors have the property. Therefore, classifying matroids on small elements with respect to the desired property provides us the following: if a given matroid M has the property, then all of its minors have the property, and if it does not have the property, then no matroid having M as a minor has the property. Especially in terms of not possessing the property, it helps us drive conclusions for bigger matroids. Therefore, being able to test those properties in practice helps us come up with potential counter-example candidates for the Generalized lax conjecture. In that context, finding some matroids that have the half-plane property but are not weakly determinantal is primarily of interest.

The Half-Plane Property

In §2.1, Rayleigh differences appeared to be a helpful tool for determining determinantal representability and being weakly determinantal. Brändén, in 2007 showed through the following characterization of homogeneous multiaffine stable polynomials that they also provide a criterion for the half-plane property.

Theorem 3.0.1 (Theorem 5.6 in [10]). *Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ be a multiaffine polynomial. The following are equivalent:*

(i) h is stable,

(ii) for all $1 \leq i, j \leq n$*, the* Rayleigh difference

$$\Delta_{ij}(h) := \frac{\partial h}{\partial x_i} \cdot \frac{\partial h}{\partial x_j} - \frac{\partial^2 h}{\partial x_i \partial x_j} \cdot h$$

is nonnegative on $\mathbb{R}^n$*.*

Consider a simple matroid M on n elements with the half-plane property. Since its basis generating polynomial h_M is multiaffine, for any $i, j \in [n]$ we can express h_M as

$$h_M = x_i p + x_j q + x_i x_j f + g.$$

where $p := \frac{\partial h_M}{\partial x_i}|_{x_j=0}$, $q := \frac{\partial h_M}{\partial x_j}|_{x_i=0}$, $f := \frac{\partial^2 h_M}{\partial x_i \partial x_j}$ and $g := h_M|_{\{x_i=0,\, x_j=0\}}$. After plugging in this expression in the Rayleigh difference $\Delta_{ij} h_M$, we obtain

$$\begin{aligned}\Delta_{ij} h_M &= (p + x_j f) \cdot (q + x_i f) - f \cdot (x_i p + x_j q + x_i x_j f + g) \\ &= pq - gf = \frac{\partial h_M}{\partial x_i}|_{x_j=0} \cdot \frac{\partial h_M}{\partial x_j}|_{x_i=0} - \frac{\partial^2 h_M}{\partial x_i \partial x_j} \cdot h_M|_{\{x_i=0,\, x_j=0\}}.\end{aligned}$$

Recall that the minors of M are matroids obtained as the direct sum of deletion and contraction of some elements of M. By Proposition 1.2.15, these operations correspond to setting some variables of the basis generating polynomial zero and taking derivatives with respect to some variables, respectively. Then we can write the Rayleigh difference $\Delta_{ij} h_M$ as

$$\Delta_{ij}(h_M) = h_{M_{/i\setminus j}} \cdot h_{M_{/j\setminus i}} - h_{M_{/\{i,j\}}} \cdot h_{M_{\setminus k\{i,j\}}}.$$

The following theorem by Wagner and Wei proved that when all proper minors of a matroid M have the half-plane property, it suffices to find *one* Rayleigh difference which is non-negative.

Theorem 3.0.2 (Theorem 3 in [63])**.** *Let M be a matroid on $E = [n]$ all of whose proper minors have the half-plane property. If there exist distinct indices $i, j \in [n]$ such that $\Delta_{ij} h_M(x) \geq 0$ for all $x \in \mathbb{R}^n$, then M has the half-plane property.*

Proof. Assume that every minor M' of M has the half-plane property, and that for the pair of distinct indices $i, j \in [n]$, $\Delta_{ij} h_M$ is non-negative on $\mathbb{R}^n$. Note that for any $k \in [n]$, $M_{/k}$ and $M_{\setminus k}$ are minors of M, therefore, by assumption, they have the half-plane property. Moreover, for any $k \in [n]$ (we may assume that $k \in [n] \setminus \{i, j\}$), we have

$$h_M = x_k h_{M_{/k}} + h_{M_{\setminus k}}$$

such that by the same calculation as in the proof of Theorem 2.1.11, the Rayleigh differences of h_M are of the form

$$\Delta_{rs} h_M = x_k^2 \Delta_{rs} h_{M_{/k}} + x_k g' + \Delta_{rs} h_{M_{\setminus k}} \tag{3.1}$$

where $r, s \in [n] \setminus \{k\}$, and

$$g' := h_{M_{/\{k,r\}}} h_{M_{\setminus k/s}} + h_{M_{/\{k,s\}}} h_{M_{\setminus k/r}} - \left(h_{M_{\setminus k/\{r,s\}}} h_{M_{/k}} + h_{M_{/\{r,s,k\}}} h_{M_{\setminus k}}\right).$$

Since h_M is multiaffine, the degree of x_k in (3.1) is at most 2. By expanding the discriminant formula $(g')^2 - 4\left(\Delta_{rs}h_{M/k}\right)\cdot\left(\Delta_{rs}h_{M\setminus k}\right)$, one can observe that it is symmetric under the permutation of indices $\{r,s,k\}$ (see [63, Proposition 1] for the explicit expansion). We will use this symmetry for passing to the non-negativity of other Rayleigh differences.

Note that each coefficient of (3.1) is some combination of minors of h_M, which by assumption are non-negative on $\mathbb{R}^n$ and by the hypothesis, $\Delta_{ij}h_M$ is non-negative on $\mathbb{R}^n$. Let $k \in [n]\setminus\{i,j\}$ and fix real values for x_l for $l \in [n]\setminus\{i,j,k\}$. By symmetry, the discriminant of $\Delta_{ij}h_M$ as a quadratic polynomial in x_k is equivalent to the discriminant of $\Delta_{i,k}h_M$ as a quadratic form in x_j. Moreover, the sign of the first and the last coefficient of the second quadratic polynomial do not change, and if any of these two coefficients is zero, then the middle coefficient is also zero (by the construction of g' in (3.1)). Therefore, $\Delta_{i,k}h_M$ is non-negative on $\mathbb{R}^n$. By applying this process repetitively, we obtain that for indices $s,r \in [n]\setminus\{i,j\}$, $\Delta_{s,k}h_M$ is non-negative on $\mathbb{R}^n$ as well. □

One of the methods to certify the non-negativity of a polynomial is to find a way to write it as a sum of squares of polynomials. While polynomials with an SOS representation are non-negative on $\mathbb{R}^n$, not every non-negative polynomial is a SOS of polynomials. Still, trying to find one pair of indices for which the corresponding Rayleigh difference has an SOS representation is one way to test and, if applicable, certify the half-plane property of a matroid. We continue listing some criteria that we apply in computational tests we conduct.

Criterion 1. *By Theorem 3.0.2, if all proper minors of a matroid M have the half-plane property and there exists a Rayleigh difference of h_M that is a sum of squares of polynomials, then M has the half-plane property.*

Note in particular that the non-zero Rayleigh differences of the basis generating polynomial of a matroid of rank r are homogeneous and have degree $2r-2$.

Theorem 3.0.3 (Theorem 3.39 in [6])**.** *Let $h \in \mathbb{R}[x_1,\dots,x_n]$ be a homogeneous polynomial of degree $2d$, and m be the vector of all monomials of degree d. Then h is a sum of squares of polynomials if and only if there exists a PSD matrix $G \in \mathrm{Sym}_{\mathbb{R}}^{\binom{n+d}{d}\times\binom{n+d}{d}}$ such that*

$$h(x_1,\dots,x_n) = m^T G m.$$

Recall that a matrix defined as in the theorem is called a Gram matrix. Moreover the condition $h(x_1,\dots,x_n) = m^T G m$, $G \succeq 0$ is a semi-definite program. In particular, the set of possible Gram matrices is given by the intersection of the PSD cone with an affine subspace; letting G be a matrix of entries g_{ij} as variables,

$$h(x_1,\dots,x_n) = m^T G m$$

defines a system of equations, and finding a PSD matrix satisfying those equations defines an SDP program.

Several computer algebra systems implement SOS decomposition algorithms using various SDP solvers (CSDP, SDPA, MOSEK, DSDP, etc.). For example, the Macaulay2 [28] package "SumsOfSquares" by Cifuentes, Kahle, Parrilo, and Peyrl [20, 21] uses interior point methods for producing SOS certificates. For more information on methods for solving SDP and producing non-negativity certificates, we refer to [6]. Numerical algorithms first attempt to find an approximate solution with a Gram matrix with approximate (floating point) entries and, if successful, try to round them to rational numbers. In particular, the existence of a Gram matrix with rational entries certifies the non-negativity of a polynomial with its SOS decomposition.

Theorem 3.0.4 (Theorem 3.43 in [6])**.** *A polynomial $h \in \mathbb{Q}[x_1, \dots, x_n]$ is a sum of squares of polynomials in $\mathbb{Q}[x_1, \dots, x_n]$ if and only if there exists a PSD gram matrix G with rational entries such that describes h.*

Since the basis generating polynomial of a matroid of rank r on n elements is multiaffine, we consider the vector of multiaffine monomials of degree r. Further, since the Rayleigh differences have degree $2r-2$, the size of the Gram matrices we want to find is $\binom{n}{r-1} \times \binom{n}{r-1}$.

Another way to test the half-plane property of a given matroid M on $E = [n]$ is to use the minor closedness of the half-plane property.

The first criterion that needs to be applied for disproving the half-plane property is the following.

Criterion 2. *By Corollary 1.2.30, if M has one of the forbidden minors for the half-plane property as one of its minors, then it does not have the half-plane property.*

In addition, minor closedness can also be used for proving the half-plane property.

Criterion 3. *By Corollary 1.2.30, if there exists another matroid M' on $E' = [m], m > n$ with the half-plane property such that M is a minor of M', then M has the half-plane property as well.*

However, Criterion 3 is only efficient when the structure of the matroid is known and related to matroids with specific properties. For example, Amini and Brändén in [3] describe a way to construct non-representable matroids with the weak half-plane property from d-uniform hyper-graphs. Those matroids have similar properties to the Vámos matroid. In particular, they are good candidates to use as matroids on a large number of elements in the minor search test.

Remark 3.0.5. There are algorithms for testing the half-plane property using quantifier elimination methods that are infeasible in practice. For more information, see [18, §2.6], [53].

When it is not possible to certify the non-negativity of Rayleigh differences of a given matroid that does not have any known forbidden minors as a minor, one can use other methods to disprove the property.

By Proposition 1.1.23, a matroid M on $E = [n]$ with the half-plane property is hyperbolic with respect to $e = (1, \dots, 1) \in \mathbb{R}^n$ and the hyperbolicity cone of h_M contains the non-negative orthant. Moreover, the univariate restrictions $h_M(et - v)$ for $v \in \mathbb{R}^n$ are real rooted. This leads to the following criterion.

Criterion 4. *By Proposition 1.1.23 and by the fact that zeros of a univariate polynomial continuously depend on its coefficients, if there exists* $e', v \in \mathbb{R}^n_{\geq 0}$ *for which* $h_M(e't - v)$ *has some complex zeros, then* M *does not have the half-plane property.*

In practice, checking the number of real roots of $h_M(e't-v)$ for $e, v \in \{0,1\}^n$ provides a quick way to disprove the stability of a matroid.

Moreover, one can use the criterion given in Theorem 3.0.2 for disproving the half-plane property.

Criterion 5. *By Theorem 3.0.2, if there are some points in* $\mathbb{R}^n$ *for which one of the Rayleigh differences of* h_M *takes a negative value, then* M *does not have the half-plane property.*

To search for such points, we dehomogenize h_M and compute its critical points using the Julia package "HomotopyContinuation.jl" by Breiding, and Timme [14]. We later insert the critical points in the Rayleigh differences and check whether the value is negative.

After applying the criteria given above and conducting tests on matroids on 8 elements, we chose the most efficient methods and implemented them as the algorithm described in Algorithm 1. Our test results on 9 elements support the efficiency of the algorithm (see § 4 for the test results). We first check whether a given matroid has a forbidden minor as its minor. If it passes this criterion, then we proceed with the algorithm. The algorithm first uses the computer algebra system Macaulay2 for sums of squares tests on the Rayleigh differences, and in case a matroid fails the tests, it switches to Julia for finding negative points to disprove the half-plane property. The function *solveSOS* in the algorithm is a built-in function in the Macaulay2 package "SumsOfSquares". It attempts to find a PSD Gram matrix with rational entries to produce the sum of squares representation of the Rayleigh difference.

Being SOS-Rayleigh and Weak Determinantal Representability

Recall that a matroid is called SOS-Rayleigh if all Rayleigh differences of h_M are sums of squares of polynomials. While, for matroids all of whose proper minors have the HPP, it is enough to find an SOS representation for one Rayleigh difference of h_M to prove that it has the half-plane property, one needs to go through all the Rayleigh differences and certify that they are sums of squares.

Criterion 6. *By definition, if* $\Delta_{ij} h_M$ *is a sum of squares of polynomials for all* $i, j \in [n]$*, then* M *is SOS-Rayleigh.*

In application, we use the Macaulay2 package "SumsOfSquares" for finding SOS representations of the Rayleigh differences.

Further, a matroid without the half-plane property cannot be SOS-Rayleigh.

Criterion 7. *By Theorem 3.0.2, if* M *does not have the half-plane property, then it is not SOS-Rayleigh.*

Algorithm 1: An algorithm for testing the HPP of a matroid

Data: A matroid M on $E = [n]$ with the collection of bases $\mathcal{B}$, all of whose proper minors have the HPP

Result: Whether M has the HPP

```
E ← {1, ..., n};
𝓑 ← {B : B is a Basis of M};
R ← ℝ[x_1, ..., x_n];
h_M ← ∑_{B∈𝓑} ∏_{i∈B} x_i ;
J ← {(i,j) : 1 ≤ i,j ≤ n, i ≠ j} ;
for (i,j) ∈ J do
    Δ_ij ← (∂h_M/∂x_i)(∂h_M/∂x_j) − (∂²h_M/∂x_i∂x_j) h_M;
    Sols ← solveSOS(Δ_ij) ;                         /* SDP for SOS decomposition */
    if status(Sols) == "SDP solved primal-dual feasible" then
        if ring(GramMatrix(Sols)) == ℚ then
            /* Δ_ij is a SOS                                                  */
            return "M has the HPP"
        end
    end
end
J' ← {(i,j) : 2 ≤ i,j ≤ n, i ≠ j} ;
for (i,j) ∈ J' do
    Δ_ij ← (∂h_M/∂x_i)(∂h_M/∂x_j) − (∂²h_M/∂x_i∂x_j) h_M;
    dehom = Δ_ij|_{x_1←1};
    /* Critical points are found using ''Homotopy Continuation''           */
    CritPts ← {p ∈ ℝ^{n−3} : p is a critical point of dehom};
    E' ← E \ {1, i, j};
    for p ∈ CritPts do
        Subst = dehom|_{{x_{E'_i} ← p_i for i∈[n−3]}};
        if Subst < 0 then
            return "M does not have the HPP"
        end
    end
end
return "Undetected"
```

When there is no Gram matrix with rational entries found that gives the SOS decomposition of a Rayleigh difference, we can try to disprove the property. In order to show that a polynomial is not a sum of squares, one needs to prove that its Gram spectrahedron is empty.

Lemma 3.0.6 (Lemma 5.14 in [39])**.** *Let $h \in \mathbb{R}[x_1, \ldots x_n]$ be a non-zero homogeneous polynomial of degree $2d-2$, and S_G be its Gram spectrahedron defined by the pencil*

$$G_0 + \lambda_1 G_1 + \ldots \lambda_k G_k \succeq 0$$

for some real symmetric matrices $G_1, \ldots, G_k \in \mathbb{R}^{\binom{n}{d-1} \times \binom{n}{d-1}}$. If there exists a positive definite matrix $A \in \mathbb{R}^{\binom{n}{d-1}}$ such that $\operatorname{tr}(AG_i) = 0$ for all $i = 0, \ldots, k$, then S_G is empty.

Proof. Assume that $S_G \neq \emptyset$. Then there exists $\lambda \in \mathbb{R}^k$ such that

$$G = G_0 + \lambda_1 G_1 + \ldots \lambda_k G_k$$

is positive semidefinite with $\operatorname{tr}(AG) = 0$. Since A is positive definite, there exist an invertible matrix $B \in \mathbb{R}^{\binom{n}{d-1} \times \binom{n}{d-1}}$ such that $BB^t = A$. Since G is positive semi-definite and non-zero,

$$0 = \operatorname{tr}(AG) = \operatorname{tr}(BB^t G) = \operatorname{tr}(B^t G B) > 0$$

gives a contradiction. □

In application, a certificate of not being SOS is produced by solving a semi-definite program. The condition of the trace of products of matrices and positive definiteness provide equations for defining the SDP.

Criterion 8. *If there exists a Rayleigh difference of h_M that is certified for not being a sum of squares, then M is not SOS-Rayleigh.*

Implementing the criteria above, we give an algorithm described in 2 for determining the SOS-Rayleigh property of a matroid using the computer algebra system Macaulay2.

By Theorem 2.1.19, being weakly determinantal implies being SOS-Rayleigh. Therefore, one can use negative test results on the SOS-Rayleigh property in order to conclude that a matroid does not have a weak determinantal representation.

Criterion 9. *By Theorem 2.1.19, if a matroid is not SOS-Rayleigh, then it is not weakly determinantal.*

Another method for disproving the weak half-plane property was used by Brändén in [11] for giving a counter-example for a stronger version of the generalized Lax Conjecture. Using the following characterization, he showed that the Vámos matroid V_8 is not weakly determinantal, although it has the half-plane property (see [63]).

Algorithm 2: An algorithm for testing the SOS-Rayleigh property of a matroid

Data: A matroid M on $E = [n]$ with the collection of bases $\mathcal{B}$ with the HPP
Result: Whether M is SOS-Rayleigh

$\mathrm{E} \leftarrow \{1, \dots, \mathrm{n}\}$;
$\mathcal{B} \leftarrow \{\mathrm{B} : \mathrm{B}$ is a Basis of $\mathrm{M}\}$;
$\mathrm{r} \leftarrow \mathrm{rk(M)}$;
$\mathrm{h_M} \leftarrow \sum_{\mathrm{B}\in\mathcal{B}} \prod_{\mathrm{i}\in\mathrm{B}} \mathrm{x_i}$;
$\mathrm{J} \leftarrow \{(\mathrm{i},\mathrm{j}) : 1 \leq \mathrm{i},\mathrm{j} \leq \mathrm{n}, \mathrm{i} \neq \mathrm{j}\}$;
for $(\mathrm{i},\mathrm{j}) \in \mathrm{J}$ **do**
  $\Delta_{\mathrm{ij}} \leftarrow \frac{\partial \mathrm{h_M}}{\partial \mathrm{x_i}} \frac{\partial \mathrm{h_M}}{\partial \mathrm{x_j}} - \frac{\partial^2 \mathrm{h_M}}{\partial \mathrm{x_i} \partial \mathrm{x_j}} \mathrm{h_M}$;
  $\mathrm{Sols} \leftarrow \mathrm{solveSOS}(\Delta_{\mathrm{ij}})$; /* SDP for SOS decomposition */
  if status(Sols) == *"SDP solved primal-dual feasible"* **then**
    if ring(GramMatrix(Sols)) == $\mathbb{Q}$ **then**
      /* Δ_{ij} is a SOS */
    end
  else
    /* Produce a non-SOS certificate */
    $\mathrm{m} \leftarrow \binom{\mathrm{n}}{\mathrm{r}-1}$;
    $\mathrm{k} \leftarrow \mathrm{m(m+1)/2}$;
    $\mathrm{S} \leftarrow \mathbb{R}[\mathrm{a_1}, \dots, \mathrm{a_k}][\mathrm{x_1}, \dots, \mathrm{x_n}]$;
    $\mathrm{Inds} \leftarrow \{\mathrm{T} \subset \mathrm{E} \setminus \{\mathrm{i},\mathrm{j}\} : |\mathrm{T}| = \mathrm{r} - 1\}$;
    $\mathrm{G} \leftarrow \mathrm{GenericMatrix} \in \mathrm{Sym}^{\mathrm{m}}_{\mathrm{S}}$;
    /* Initial Gram matrix with entries $a_1, \dots, a_k$ */
    $\mathrm{M} \leftarrow \left(\prod_{\mathrm{i}\in\mathrm{T}} \mathrm{x_i}\right)_{\mathrm{T}\in\mathrm{Inds}} \in \mathrm{S}^{1\times\mathrm{m}}$;
    /* Vector of multiaffine monomials */
    $\mathrm{Solve}(\mathrm{MGM}^T - \Delta_{ij})$; /* Obtain the relations */
    $\mathrm{G} \leftarrow \mathrm{G}|_{\mathrm{solution(Solve)}}$; /* Insert the relations */
    $\mathrm{NZInd} \leftarrow \{\mathrm{i} \in [\mathrm{k}] : \mathrm{a_i}$ appears in $\mathrm{G}\}$;
    /* $G_0 + \sum_{i\in\mathrm{NZInd}} G_i a_i$ is the linear pencil defining the Gram spectrahedron */
    $\mathrm{Gs} \leftarrow \left\{ \mathrm{G}|_{\left\{ \substack{a_i \leftarrow 1 \\ a_j \leftarrow 0 \text{ for all } j\in\mathrm{NZInd}\setminus\{i\}} \right\}} \in \mathbb{R}^{\mathrm{m}\times\mathrm{m}} : \mathrm{i} \in \mathrm{NZInd} \cup \{0\} \right\}$;
    $\mathrm{b} \leftarrow \mathbf{0} \in \mathbb{R}^{|GS|}$;
    $\mathrm{C} \leftarrow \mathbf{0} \in \mathbb{R}^{m\times m}$;
    $\mathrm{Sdp} \leftarrow \mathrm{SDP(C, Gs, b)}$;
    if status(Sdp) == *"SDP solved primal-dual feasible"* **then**
      if the solution matrix is PD **then**
        /* A non-SOS certificate is produced */
        return *"M is not SOS-Rayleigh"*
      else
        return *"Undetected"*
      end
    else
      return *"Undetected"*
    end
  end
end
return *"M is SOS-Rayleigh"*

Theorem 3.0.7 (Brändén [11]). *Let M be a weakly determinantal matroid on the ground set $[n]$. Then its rank function* rk *satisfies the* Ingleton inequalities*:*

$$\begin{aligned} &\mathrm{rk}(P_1 \cup P_2) + \mathrm{rk}(P_1 \cup P_3) + \mathrm{rk}(P_1 \cup P_4) + \mathrm{rk}(P_2 \cup P_3) + \mathrm{rk}(P_2 \cup P_4) \\ \geq\, &\mathrm{rk}(P_1) + \mathrm{rk}(P_2) + \mathrm{rk}(P_1 \cup P_2 \cup P_3) + \mathrm{rk}(P_1 \cup P_2 \cup P_4) + \mathrm{rk}(P_3 \cup P_4) \end{aligned}$$

for all $P_1, P_2, P_3, P_4 \subset [n]$.

Criterion 10. *Matroids that violate the Ingleton inequalities are not weakly determinantal.*

For the Vámos matroid, there exist disjoint subsets $P_1, P_2, P_3, P_4 \subset [8]$ of size 2 that violate this inequality. Paving matroids with a similar property are called Vámos-like and defined as follows.

Definition 3.0.8. A sparse paving matroid M on a ground set E of rank r is called *Vámos-like* if there exist pairwise disjoint sets P_1, P_2, P_3, P_4, K where $|P_1| = |P_2| = |P_3| = |P_4| = 2$ and $|K| = r - 4$ such that $K \cup P_i \cup P_j$ for $i < j, (i, j) \neq (3, 4)$ are non-bases, while $K \cup P_3 \cup P_4$ is a basis.

A matroid M of rank r is a *sparse paving matroid* if each size-r subset of its ground set is either a basis or a circuit-hyperplane.

In [47] it was shown that Vámos-like matroids violate the Ingleton inequalities.

Criterion 11. *Vámos like matroids are not weakly determinantal.*

Chapter 4

Test Results on Matroids on 8 and 9 Elements

In this chapter, by applying Algorithm 1, we classify all matroids on a ground set E with $|E| \leq 8$ that have the half-plane property. Moreover, we present our test results on the half-plane property of matroids on 9 elements. Since matroids with the half-plane property on at most 7 elements were classified in [18], we start with considering matroids on 8 elements. The test results appear in the pre-print [39]. Source code for the tests can be found at `https://github.com/busrasert/HPP-of-Matroids`.

4.1 Matroids on 8 Elements

We start by narrowing down the set of matroids that have to be considered. For this, we use the results on the half-plane property matroids and preservation of the half-plane property. Note that by [18, Prop. 4.2], having the half-plane property is closed under taking the dual of the matroid. Thus it is enough to consider matroids on 8 elements with rank at most four.

Further, the half-plane property is preserved by direct sums, adjoining loops, or parallel elements [18, §4]. Consequently, it is enough to consider simple connected matroids on 8 elements of rank at most four. Moreover, by [18, Cor. 5.5], all rank-1 and rank-2 matroids have the half-plane property. Thus we only need to consider matroids on eight elements of rank three and four.

We use the Macaulay2 [28] package "Matroids" by Chen [16, 17] where a list of all matroids on at most 8 elements is implemented: The command `allMatroids(8)` yields a list of all matroids on 8 elements. In the following, we will denote by $\mathcal{M}_i$ the i-th element of this list (counting starts from zero). Code files for the tests are available at: `https://github.com/busrasert/HPP-of-Matroids`.

We obtain the number of non-isomorphic matroids on 8 elements with rank 3 or 4 is 1265. While 685 of them are simple, 659 of them are simple and connected. Since the half-plane property is minor-closed, we continue with excluding the forbidden minors for the half-plane property. Choe et. al. in [18] showed that Fano matroid F_7, Non-Fano matroid F_7^-, F_7^{--}, F_7^{-3}, $M(K_4)+e$ and thus their duals do not have the half-plane property. Therefore, any

Matroids	**Number of pairs $(\mathbf{e}, \mathbf{v})$ for which $\mathbf{h_M(et - v)}$ has non-real roots**	**One of the pairs $(\mathbf{e}, \mathbf{v})$**
$\mathcal{M}_{435}$	4	$((1,1,1,1,1,1,0,1),(1,1,0,0,0,0,1,1))$
$\mathcal{M}_{437}$	8	$((0,0,0,0,1,1,1,1),(1,1,1,1,1,1,0,1))$
$\mathcal{M}_{439}$	4	$((1,0,1,1,1,1,1,1),(1,1,1,1,0,0,0,0))$
$\mathcal{M}_{443}$	2	$((1,1,1,1,0,1,1,1),(0,0,0,0,1,1,1,1))$
$\mathcal{M}_{450}$	4	$((0,1,1,1,1,1,1,1),(1,1,0,0,1,1,0,0))$
$\mathcal{M}_{455}$	2	$((1,1,0,1,0,1,0,1),(0,1,1,0,1,0,1,1))$
$\mathcal{M}_{460}$	2	$((0,0,1,1,1,1,1,1),(1,1,0,0,0,0,1,1))$
$\mathcal{M}_{461}$	12	$((1,1,1,1,0,1,1,1),(0,0,1,1,1,1,0,0))$
$\mathcal{M}_{465}$	10	$((0,1,1,1,1,1,1,1),(1,1,0,0,0,0,1,1))$
$\mathcal{M}_{466}$	38	$((0,0,1,1,1,1,0,1),(1,1,0,0,0,0,1,1))$
$\mathcal{M}_{467}$	62	$((1,1,0,1,1,0,1,1),(0,0,1,1,1,1,0,0))$
$\mathcal{M}_{548}$	6	$((1,1,1,1,0,0,0,1),(0,0,0,0,1,1,1,1))$
$\mathcal{M}_{549}$	4	$((1,1,1,1,0,0,1,1),(1,1,1,1,0,0,1,1))$
$\mathcal{M}_{570}$	78	$((1,1,0,0,1,1,1,1),(0,0,1,1,1,1,0,0))$
$\mathcal{M}_{575}$	158	$((1,1,1,1,1,1,0,1),(0,0,0,1,1,0,1,1))$

Table 4.1: 15 matroids and a sample of directions for which they fail the hyperbolicity test.

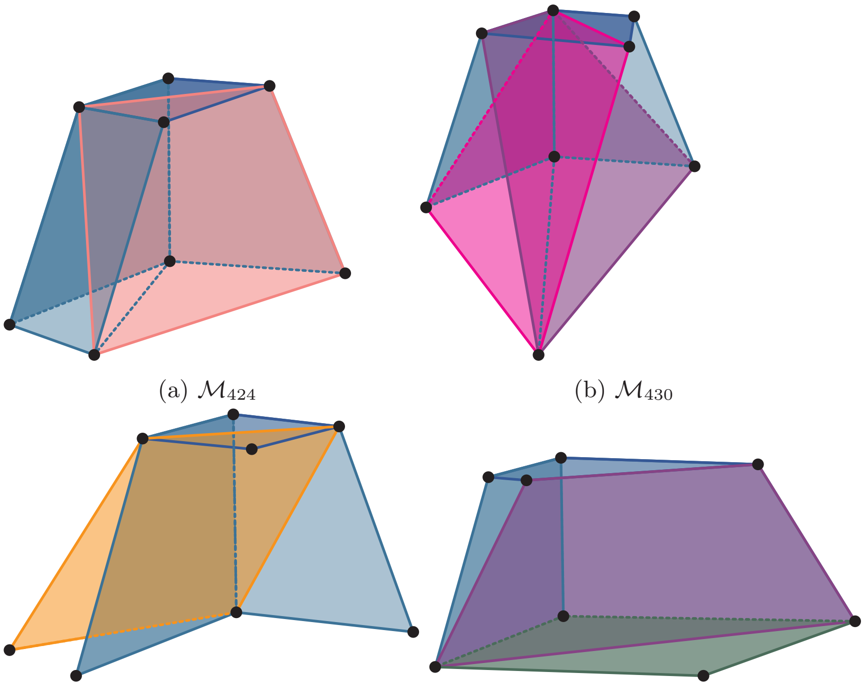
matroid that has one of those matroids as a minor does not have the half-plane property. After excluding such matroids, we are left with 309 simple and connected matroids, all of whose proper minors have the half-plane property.

As described in Algorithm 1, we run a sum of squares test using the Macaulay2 package "SumsOfSquares" by Cifuentes, Kahle, Parrilo, and Peyrl [20, 21] on all Rayleigh differences of the basis generating polynomials of all these 309 matroids. We obtain that for 287 matroids from this list, there exists some indices i, j such that $\Delta_{ij}(h_M)$ is a sum of squares. In particular, for each of the 287 matroids, there were some indices i, j for which the SDP solver could find a Gram matrix with *rational* entries. This gives a *symbolic* certificate for being a sum of squares, and thus by Theorem 3.0.2 it proves that the corresponding matroid has the half-plane property.

For the remaining 22 matroids, we do not know whether they have the half-plane property: the fact that for those 22 matroids, there are no i, j for which the Rayleigh difference is a sum of squares does not imply that these matroids do not have the half-plane property. In particular, there are nonnegative polynomials that are not a sum of squares. On the other hand, we apply some methods listed in the previous chapter in order to eliminate those that do not have the half-plane property. Since all these 22 matroids have rank 4, we conclude that among the matroids on eight elements with rank 3 there are no new forbidden minors for the half-plane property.

We first apply Criterion 4 and test whether there is any matroid M among the 22 matroids, for which $h_M(et - v)$ has some complex roots for some $e, v \in \{0,1\}^8$. We obtain that 15 of the 22 matroids do not pass this test, and the remaining 7 matroids require more tests to prove or disprove that they have the half-plane property. For the list of these 15 matroids whose basis generating polynomials fail this test and for the corresponding points e, v, we refer to Table 4.1.

For each of the remaining 7 matroids M, we found a point $x \in \mathbb{R}^6$ for which $\Delta_{67}(h_M)(x) < 0$, which confirms that they do not have the half-plane property by using the Julia package "HomotopyContinuation.jl" by Breiding and Timme [14]. More precisely, we computed all critical points of the Rayleigh difference and plugged in nearby rational points. The list of points where the Rayleigh



(a) $\mathcal{M}_{424}$ (b) $\mathcal{M}_{430}$

(c) $\mathcal{M}_{431}$ (d) $\mathcal{M}_{436}$

Figure 4.1: Examples of Matroids on 8 elements that are forbidden minors for the HPP.

Matroids	$\mathbf{x} \in \mathbb{R}^{\mathbf{6}}$ **s.t.** $\mathbf{\Delta_{6,7}(h_M)(x) < 0}$
$\mathcal{M}_{424}$	$(4, 30, 1, 7, -32, -4)$
$\mathcal{M}_{430}$	$(80, 19, -31, -31, -17, -4)$
$\mathcal{M}_{431}$	$(60, 27, -90, -22, 27, 5)$
$\mathcal{M}_{436}$	$(40, 309, -40, -306, 9, 73)$
$\mathcal{M}_{462}$	$(20, 55, -11, -4, -52, -19)$
$\mathcal{M}_{463}$	$(30, 399, -111, -10, -368, -28)$
$\mathcal{M}_{550}$	$(50, -25, 94, -45, -142, 66)$

Table 4.2: Seven matroids and points $x \in \mathbb{R}^6$ at which one of their Rayleigh differences is negative.

difference is negative is given in Table 4.2.

We therefore obtain the following result.

Theorem 4.1.1 (Theorem 5.2 in [39]). *There are exactly* 22 *matroids on* 8 *elements all of whose minors have the half-plane property but which do not have the half-plane property themselves. These* 22 *matroids are the matroids* $\mathcal{M}_k$ *for* k *in the following list:*

$$\{424, 430, 431, 435, 436, 437, 439, 443, 450, 455, 460, 461, 462, 463, 465, 466, 467, 548, 549, 550, 570, 575\}.$$

All of them are sparse paving matroids of rank 4. *Furthermore, all of these matroids are self-dual except for the following dual pairs:* $\mathcal{M}_{424} = \mathcal{M}^*_{436}$, $\mathcal{M}_{430} = \mathcal{M}^*_{548}$, $\mathcal{M}_{450} = \mathcal{M}^*_{549}$ *and* $\mathcal{M}_{455} = \mathcal{M}^*_{550}$.

Below, give examples of those 22 matroids and some details about their characteristics.

Example 4.1.2. The matroid $\mathcal{M}_{430}$ is the co-extension $\mathrm{CoExt}(P_7)$ of the ternary 3-spike P_7. Both are depicted in Figure 1.7. Further minors of $\mathcal{M}_{430}$ include P_6, Q_6, R_6, $\mathrm{CoExt}(R_6)$ and $\mathrm{Ext}(Q_6)$ which all have the half-plane property. Its dual is the matroid $\mathcal{M}_{548}$.

Example 4.1.3. Recall that Choe et. al. in [18] showed that P_8, P_8' and P_8'' do not have the half-plane property. In our list, these are the matroids $\mathcal{M}_{575}$, $\mathcal{M}_{570}$ and $\mathcal{M}_{467}$. Furthermore, the matroid $\mathcal{M}_{466}$ is the relaxation P_8''', which does not have the half-plane property as well. In [45, Prop. 4] the matroid P_8''' is denoted by P_3, and it is shown that it is not representable over any field. Further note that in [50] the matroid P_8'' is denoted by $P_8^{=}$. It is an excluded minor for being representable over the field of 4 elements.

Example 4.1.4. The matroid $\mathcal{M}_{431}$ has only four circuit hyperplanes:

$$\{1,2,3,4\}, \{1,2,5,6\}, \{1,3,5,7\} \text{ and } \{2,3,5,8\}.$$

Example 4.1.5. Matroid Betsy Ross (B_{11}) is a matroid on 11 elements of rank 3 defined by the relations depicted in Figure 4.1. It has $\mathcal{M}_{430}$ as a minor, therefore it does not have the half-plane property.

Example 4.1.6. The *Extended Ternary Golay code* is the matroid represented by the following generating matrix of the Extended Ternary Golay code over the field $\mathbb{F}_3$

$$\begin{pmatrix} 1&0&0&0&0&0&1&1&1&2&2&0\\ 0&1&0&0&0&0&1&1&2&1&0&2\\ 0&0&1&0&0&0&1&2&1&0&1&2\\ 0&0&0&1&0&0&1&2&0&1&2&1\\ 0&0&0&0&1&0&1&0&2&2&1&1\\ 0&0&0&0&0&1&0&1&1&1&1&1 \end{pmatrix}.$$

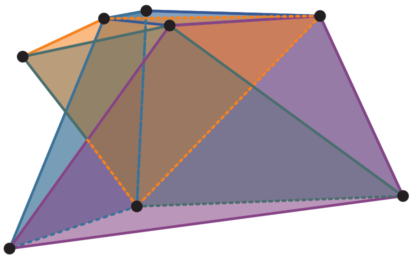
The supports of the codewords form a Steiner System $S(5,6,12)$, i.e., a 12 element set with a collection $\mathcal{D}$ of 6 element subsets (called blocks) such that every 5 element subset of S is contained in exactly one block. Here, a ground set E with 12 elements with the collection $\mathcal{D}$ of hyperplanes define a matroid. It does not have the half-plane property since it has P_8 as a minor.

By minor closedness of the half-plane property, any matroid having one of the listed 22 matroids as a minor does not have the half-plane property.

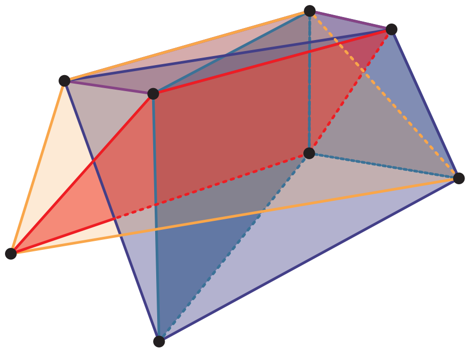
Additionally, co-extensions

$$\mathrm{CoExt}(P_8), \mathrm{CoExt}(J), \mathrm{CoExt}(T_8), \mathrm{CoExt}(S_8),$$
$$\mathrm{CoExt}(N1), \mathrm{CoExt}(N2), \mathrm{CoExt}(AG(2,3)),$$

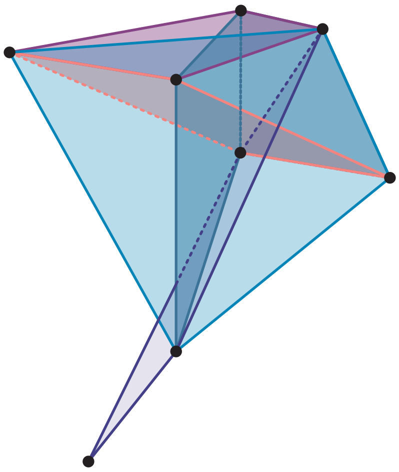
extensions $\mathrm{Ext}(P_8)$, $\mathrm{Ext}(T_8)$, $\mathrm{Ext}(J)$, $\mathrm{Ext}(N1)$ and the matroid $AG(3,3)$ have one of the 22 matroids as a minor. These all do not have the half-plane property.



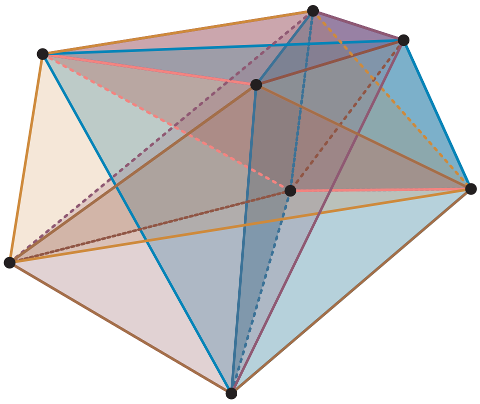
(a) $\mathcal{M}_{462}$



(b) $\mathcal{M}_{465}$



(c) $\mathcal{M}_{549}$



(d) $\mathcal{M}_{575}$

Figure 4.2: Some more examples of the forbidden minors for the HPP.

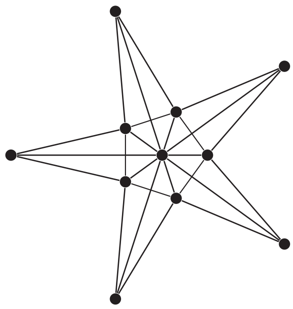


Figure 4.3: Matroid Betsy Ross (B_{11})

Example 4.1.7. Pappus, non-Pappus matroids, and a free extension (Non-Pappus\9) $+e$ do not have the half-plane property (see [18]). The authors of [18] suspect that they are minor minimal. By Theorem 4.1.1, there are no minor-minimal matroids of rank 3 on 8 elements, and none of the forbidden minors in 7 elements of rank 3 is a minor of any of the 3 matroids listed above. Therefore, Pappus, non-Pappus, and (Non-Pappus\9) $+e$ matroids are minor minimal with respect to the half-plane property. In Section 4.2, we confirm that they are the only minimal forbidden minors for the half-plane property of rank 3 on 9 elements.

Remark 4.1.8. A matroid M with ground set E is called *Rayleigh*, if

$$\Delta_{ij}(h_M)(x) \geq 0$$

for all $i, j \in E$ and all x in the *nonnegative orthant*. It is clear from Theorem 3.0.1 that the half-plane property implies Rayleigh. The converse is not true. For instance every matroid of rank 3 is Rayleigh [62]. When searching for critical points of Rayleigh differences using the Julia package "Homotopy-Continuation.jl", we found that the matroids $\mathcal{M}_k$ for k from the following list are not Rayleigh:

$$\{768, 816, 821, 825, 878, 879, 882, 883, 891, 894, 895, 896, 910, 911, 912\}.$$

Note that the matroid $\mathcal{M}_{912}$ is the matroid S_8 [50, p. 648].

Furthermore, for fixed indices i, j, we can express h_M as $h_M = ax_ix_j + bx_i + cx_j + d$ where $a, b, c, d \in \mathbb{R}[x_1, \ldots, x_n]$ are polynomials that do not depend on x_i and x_j. Then we have $\Delta_{i,j}h_M = ad - bc$, and the supremum of the ratio $\frac{bc}{ad}(x)$ over all x in the nonnegative orthant and all i, j is called the *correlation constant* of the matroid M. If M is not Rayleigh, then the correlation constant is larger than 1. It was conjectured in [32] that the correlation constant cannot exceed $\frac{8}{7}$. Our computations support their conjecture, see also Table 4.3

SOS-Rayleigh and Weakly Determinantal Matroids

Concerning the SOS-Rayleigh property, the tests applied using Algorithm 2 shows that among the 287 matroids with the half-plane property, 256 matroids are SOS-Rayleigh, i.e., for those matroids *all* Rayleigh differences are a sum of squares.

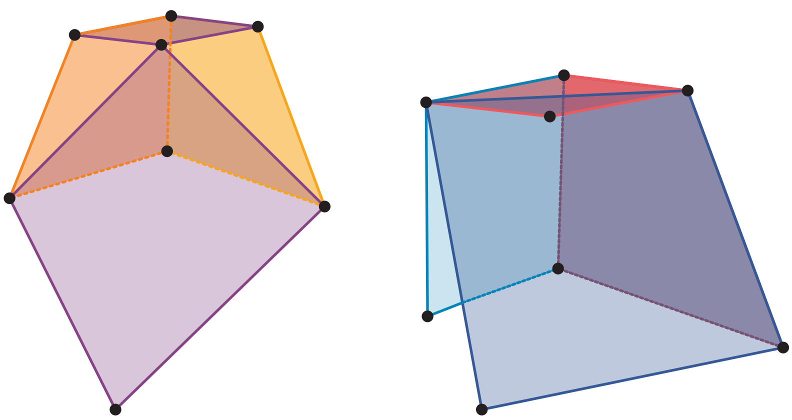
Remark 4.1.9. For the matroids $\mathcal{M}_k$ where k is in

$$\{393, 395, 397, 399, 400, 401, 403, 404, 405, 410, 411, 412, 421, 423, 433, 664, 717\}$$

there are some indices i, j such that the SDP solver cannot find positive semidefinite rational Gram matrices during the SOS test on $\Delta_{i,j}$ but only positive semidefinite Gram matrices having floating points as entries (i.e, they are approximate). These numerical results indicate that these matroids might be SOS-Rayleigh but we do not have a symbolic certificate for that. However, there is always another pair of indices for which we can find a rational positive semidefinite Gram matrix, so this issue does not affect the proof of the half-plane property.

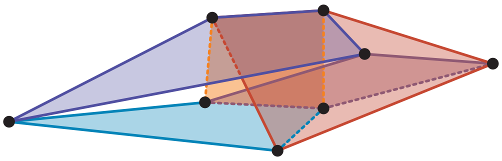
	(i,j)	Points p	$\Delta_{i,j}(p)$	$\frac{ad}{bc}(p)$
$\mathcal{M}_{768}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{1151}{50000}, \frac{1151}{50000}, \frac{11}{12500}, \frac{121}{6250}, \frac{121}{6250})$	$-\frac{1370445758816552 77}{244140625.10^{18}}$	$\frac{7087627531101625600}{7086494931300950763}$
$\mathcal{M}_{816}$	$(5,2)$	$(x_0,x_1,x_3,x_4,x_6,x_7)$ $(1, \frac{256}{625}, \frac{2117}{5000}, \frac{493}{10000}, \frac{1071}{10000}, \frac{373}{10000})$	$-\frac{44144993633423974179}{10^{24}}$	$\frac{16778353097134052416 30}{16757331671594326714 31}$
$\mathcal{M}_{821}$	$(5,2)$	$(x_0,x_1,x_3,x_4,x_6,x_7)$ $(1, \frac{2337}{2500}, \frac{81}{200}, \frac{81}{200}, \frac{799}{10000}, \frac{799}{10000})$	$-\frac{24447003560277364844 9}{25.10^{22}}$	$\frac{2106739084957900875}{2101723183203601984}$
$\mathcal{M}_{825}$	$(5,2)$	$(x_0,x_1,x_3,x_4,x_6,x_7)$ $(1, \frac{23}{250}, \frac{1}{10}, \frac{7}{1000}, \frac{13}{250}, \frac{1}{250})$	$-\frac{428814189}{390625.10^{10}}$	$\frac{17057286016}{17041404009}$
$\mathcal{M}_{878}$	$(0,5)$	$(x_1,x_2,x_3,x_4,x_6,x_7)$ $(1, \frac{2673}{10000}, \frac{1411}{5000}, \frac{147}{2000}, \frac{141}{1250}, \frac{63}{10000})$	$-\frac{7274235177449194527}{10^{24}}$	$\frac{10731828475989312509 30}{10723745992458813404 27}$
$\mathcal{M}_{879}$	$(0,5)$	$(x_1,x_2,x_3,x_4,x_6,x_7)$ $(1, \frac{17}{25}, \frac{61}{20}, \frac{32}{5}, \frac{12}{25}, \frac{7}{100})$	$-\frac{167603609619}{625.10^{8}}$	$\frac{6212735812250}{6206528271153}$
$\mathcal{M}_{882}$	$(0,5)$	$(x_1,x_2,x_3,x_4,x_6,x_7)$ $(1, \frac{47}{500}, \frac{51}{500}, \frac{11}{1000}, \frac{7}{125}, \frac{1}{500})$	$-\frac{1444139463}{125.10^{15}}$	$\frac{574575099022}{574093719201}$
$\mathcal{M}_{883}$	$(0,5)$	$(x_1,x_2,x_3,x_4,x_6,x_7)$ $(1, \frac{13}{100}, \frac{13}{100}, \frac{1}{50}, \frac{1}{25}, \frac{1}{250})$	$-\frac{423857}{15625.10^{8}}$	$\frac{208804463}{208562259}$
$\mathcal{M}_{891}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{19}{100}, \frac{141}{100}, \frac{141}{100}, \frac{19}{100}, \frac{19}{100})$	$-\frac{3136366917}{2.10^{11}}$	$\frac{2377183363}{2371329748}$
$\mathcal{M}_{894}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{37}{500}, \frac{73}{1000}, \frac{7}{1000}, \frac{73}{1000}, \frac{3}{1000})$	$-\frac{55652830199}{10^{18}}$	$\frac{88756978376279}{88701325546080}$
$\mathcal{M}_{895}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{37}{1000}, \frac{9}{250}, \frac{1}{500}, \frac{9}{250}, \frac{1}{1000})$	$-\frac{673510083}{25.10^{16}}$	$\frac{1113113239348}{1112439729265}$
$\mathcal{M}_{896}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{3}{1000}, \frac{3}{1000}, \frac{3}{1000}, \frac{3}{1000}, \frac{3}{1000})$	$-\frac{29157813}{10^{25}}$	$\frac{3202760405}{3202400432}$
$\mathcal{M}_{910}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{3}{100}, \frac{3}{100}, \frac{1}{500}, \frac{3}{100}, \frac{1}{1000})$	$-\frac{1784831}{25.10^{14}}$	$\frac{5380022943}{5378238112}$
$\mathcal{M}_{911}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{1}{50}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{50}, \frac{1}{1000})$	$-\frac{480249}{25.10^{14}}$	$\frac{263808}{263687}$
$\mathcal{M}_{912}$	$(0,7)$	$(x_1,\ldots,x_6)$ $(1, \frac{1}{4000}, \frac{1}{4000}, \frac{1}{4000}, \frac{1}{4000}, \frac{1}{4000})$	$-\frac{1}{64.10^{15}}$	$\frac{16010001}{16008001}$

Table 4.3: List of non-Rayleigh matroids on 8 elements and their correlation constants.

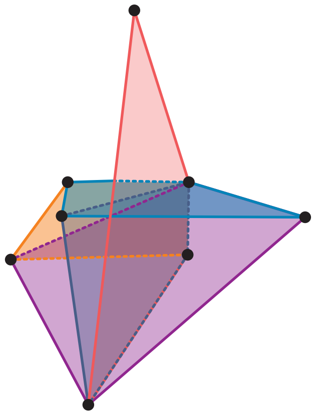


(a) $\mathcal{M}_{411}$ (b) $\mathcal{M}_{423}$

Figure 4.4: Examples of matroids with the HPP whose SOS-Rayleigh property could not be certified.



(a) $\mathcal{M}_{409}$



(b) $\mathcal{M}_{438}$

Figure 4.5: Examples of matroids that are not weakly determinantal.

Example 4.1.10. The matroid $\mathcal{M}_{411}$, shown in Figure 4.4a, has only four circuit hyperplanes: $\{1,2,3,4\}$, $\{1,2,5,6\}$,$\{1,3,5,7\}$ and $\{4,6,7,8\}$. It is one of the 17 matroids whose SOS-Rayleigh property could not be certified, but is strongly suspected.

Mayhew and Royle in [45] show that among the matroids on eight elements, there are 44 non-representable matroids, and 39 of them are Vámos-like. The remaining 5 matroids are four relaxations of P_8 and a relaxation of L_8. Our tests concluded that out of these 44 matroids, which are not representable over any field, only the Vámos matroid has the half-plane property. Furthermore, the minor-minimal matroids from Theorem 4.1.1 all satisfy Ingleton's inequalities. Therefore Criterion 11 was not helpful for this case.

For finding further instances of matroids with the half-plane property that are not weakly determinantal, we use the Criterion 9 by applying Algorithm 2. At the end of our tests, we found 14 matroids, including the Vámos matroid $\mathcal{M}_{502}$, for which some Rayleigh differences are a sum of squares but not all. In particular, they have the half-plane property but are not weakly determinantal.

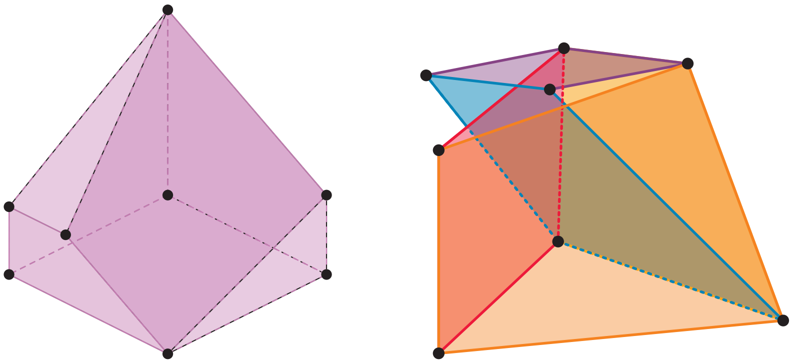
Theorem 4.1.11 (Theorem 5.13 in [39])**.** *The matroids $\mathcal{M}_k$ for k in*

$$\{409, 413, 414, 415, 417, 418, 419, 438, 440, 445, 498, 500, 501, 502\}$$

have the half-plane property but are not SOS-Rayleigh. In particular, these matroids are not weakly determinantal.

For each of the matroids M in the list, we produce symbolic certificates for the existence of a pair i,j such that $\Delta_{i,j}h_M$ is not an sum of squares. The certificates were produced by applying Lemma 3.0.6 and showing that the Gram spectrahedron of the corresponding Rayleigh difference is empty.

Example 4.1.12. The matroid $\mathcal{M}_{409}$ shown in Figure 4.5a with circuit hyperplanes $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{3,4,5,7\}$, $\{3,4,6,8\}$ and $\{1,2,7,8\}$ is not SOS-Rayleigh, therefore it is not weakly determinantal. A non-SOS certificate was produced for $\Delta_{27}h_{\mathcal{M}_{409}}$.



(a) $\mathcal{M}_{417}$ (b) $\mathcal{M}_{498}$

Figure 4.6: More examples of matroids that are not weakly determinantal.

Properties	**Count**
Half-plane property	287
Certified SOS-Rayleigh	256
Half-plane property but not SOS-Rayleigh	14
Not having the HPP	22

Table 4.4: Summary of the classification of simple connected matroids on 8 elements of rank 3 or 4 all of whose proper minors have the half-plane property.

Example 4.1.13. Another non-weakly determinantal matroid is $\mathcal{M}_{438}$ illustrated in Figure 4.5b. It has non-bases $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{3,4,5,7\}$, $\{3,4,6,8\}$, $\{1,2,7,8\}$. A non-SOS certificate was produced for $\Delta_{63}h_{\mathcal{M}_{438}}$. The matrix that certifies the emptiness of the Gram spectrahedron of $\Delta_{63}h_{\mathcal{M}_{438}}$ has eigenvalues

$$990.535,\ 725.807,\ 560.358,\ 346.281,\ 240.059,\ 546.759,\ 107.165,\ 81.6063,$$
$$217.501,\ 120.738,\ 3.63865,\ 5.95991,\ 1.22469,\ 0.0918248,\ 0.200458,\ 0.302676,$$
$$0.971657,\ 0.754552,\ 0.709615,\ 0.719758.$$

Example 4.1.14. Matroid $\mathcal{M}_{498}$ depicted in Figure 4.6b is not weakly determinantal. Its non-bases are $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{3,4,5,6\}$, $\{1,3,7,8\}$ and $\{2,3,7,8\}$. Like other 13 non-SOS-Rayleigh matroids listed in Theorem 4.1.11, it is a paving matroid, i.e., its circuits have size at least $\mathrm{rk}(\mathcal{M}_{498}) = 4$.

Remark 4.1.15. Let M be a matroid that is not weakly determinantal listed in Theorem 4.1.11 and $i, j \in [n]$ be the pair of indices for which $\Delta_{ij}h_M$ is not a sum of squares. Since M has the half-plane property, by Theorem 3.0.1 $\Delta_{ij}h_M$ is non negative on $\mathbb{R}^n$. In particular it is non-negative, but not a sum of squares. Therefore, matroids listed in Theorem 4.1.11 give rise to new non-negative, non-SOS polynomials.

Example 4.1.16. For the matroid M_{417} the Rayleigh difference

$$\begin{aligned}\Delta_{27}h_{M_{417}} =& x_1^2x_4^2x_5^2+x_1x_3x_4^2x_5^2+x_3^2x_4^2x_5^2-x_1^2x_3x_4x_5x_6+x_1^2x_4^2x_5x_6+x_1x_3x_4^2x_5x_6+x_3^2x_4^2x_5x_6+\\
& x_3^2x_4x_5^2x_6+x_1x_4^2x_5^2x_6+x_3x_4^2x_5^2x_6+x_1^2x_3^2x_6^2+x_1^2x_3x_4x_6^2+x_1x_3^2x_4x_6^2+x_1^2x_4^2x_6^2+\\
& x_1x_3x_4^2x_6^2+x_3^2x_4^2x_6^2+x_1x_3^2x_5x_6^2+x_1x_3x_4x_5x_6^2+x_3^2x_4x_5x_6^2+x_1x_4^2x_5x_6^2+x_3x_4^2x_5x_6^2+\\
& x_3^2x_5^2x_6^2+x_3x_4x_5^2x_6^2+x_4^2x_5^2x_6^2+x_1^2x_4^2x_5x_8+x_1x_3x_4^2x_5x_8+x_3^2x_4^2x_5x_8+x_1^2x_4x_5^2x_8+\\
& x_3^2x_4x_5^2x_8+x_1^2x_3^2x_6x_8+x_1^2x_3x_4x_6x_8+x_1x_3^2x_4x_6x_8+x_1^2x_4^2x_6x_8+x_3^2x_4^2x_6x_8+\\
& x_1x_3^2x_5x_6x_8+x_1^2x_4x_5x_6x_8+x_1x_3x_4x_5x_6x_8+2\,x_3^2x_4x_5x_6x_8+x_3^2x_5^2x_6x_8+\\
& x_1x_4x_5^2x_6x_8+x_1x_3x_4x_6^2x_8+x_3^2x_4x_6^2x_8+x_1x_3x_5x_6^2x_8+x_3^2x_5x_6^2x_8+x_3x_4x_5x_6^2x_8+\\
& x_3x_5^2x_6^2x_8+x_4x_5^2x_6^2x_8+x_1^2x_3^2x_8^2+x_1^2x_3x_4x_8^2+x_1x_3^2x_4x_8^2+x_1^2x_4^2x_8^2+x_1x_3x_4^2x_8^2+\\
& x_3^2x_4^2x_8^2+x_1^2x_3x_5x_8^2+x_1x_3^2x_5x_8^2+x_1^2x_4x_5x_8^2+x_1x_3x_4x_5x_8^2+x_3^2x_4x_5x_8^2+x_1^2x_5^2x_8^2+\\
& x_1x_3x_5^2x_8^2+x_3^2x_5^2x_8^2+x_1^2x_3x_6x_8^2+x_1x_3^2x_6x_8^2+x_1^2x_4x_6x_8^2+x_1x_3x_4x_6x_8^2+\\
& x_3^2x_4x_6x_8^2+x_1^2x_5x_6x_8^2+2\,x_1x_3x_5x_6x_8^2+x_3^2x_5x_6x_8^2+x_1x_5^2x_6x_8^2+x_3x_5^2x_6x_8^2+\\
& x_1^2x_6^2x_8^2+x_1x_3x_6^2x_8^2+x_3^2x_6^2x_8^2+x_1x_5x_6^2x_8^2+x_3x_5x_6^2x_8^2+x_5^2x_6^2x_8^2\end{aligned}$$

is non-negative on $\mathbb{R}^8$, however it is not a sum of squares. The circuit hyperplanes of $\mathcal{M}_{417}$ are given by the quadrilateral faces of the heptahedron depicted in Figure 4.6a.

We provide the list of simple matroids on 8 elements of rank 3 and 4 that are HPP, that are not HPP, some matroids that are not SOS-Rayleigh, and matroids for which we could not detect whether they are SOS-Rayleigh at `https://zenodo.org/record/6108027` .

4.2 Matroids on 9 Elements

In this section, we apply Algorithm 1 on simple and connected matroids on 9 elements of rank 3 and 4. We use the list of all simple matroids on nine elements provided by Matsumoto, Moriyama, Imai, and Bremner in [44]. They encode the set of bases of matroids after ordering the subsets of the ground set with respect to the reverse lexicographic order.

Definition 4.2.1. Two distinct r-element sets S_1, S_2 are said to have the relation $S_1 \prec S_2$ with respect to the *reverse lexicographic order* if $\max(S_1) < \max(S_2)$ or $\max(S_1) = \max(S_2) = a$ and $S_1 \setminus \{a\} \prec S_2 \setminus \{a\}$.

Let n and r be fixed, and all size r subsets of $[n]$ be increasingly ordered in the reverse lexicographic order. We then encode each subset among the ordered subsets with the character "$*$" if it appears in the collection of bases of a given matroid M (on $E = [n]$ of rank r), and we encode the subsets that are not bases of M with the character "0". That way, every matroid can be identified by the fingerprint given by the sequence of symbols constructed above.

Example 4.2.2. The fingerprint

$$**0*************************0$$

represents the matroid on 8 elements of rank 2 that has all size two subsets of $E = [8]$ as bases except the subsets $\{2,3\}$ and $\{7,8\}$ represented on the 3rd and 28th positions respectively.

(a) A matroid with the HPP for which it is unknown whether it is SOS-Rayleigh.

(b) A matroid with the HPP that is SOS-Rayleigh.

Figure 4.7: Examples of two matroids on 9 elements.

```
0******0******0*************************************0**********************0********
0******0***********************0****************************************************
0******0***********************0************************************************0***
0******0***********************0****************0***********************************
0******0******0*************************************0*******************************
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```

Figure 4.8: Fingerprints of ten matroids on 9 elements of rank 3 for which SOS-Raileigh property could not be detected

After excluding forbidden minors, we conduct an SOS test on the Rayleigh differences of simple connected matroids on 9 elements of rank 3. Our tests show that the only minimal forbidden minors for the half-plane property on 9 elements of rank 3 are the Pappus, non-Pappus, and (non-Pappus$\backslash$9)+e matroids. The remaining 116 matroids have the half-plane property. For ten of these matroids, there are some indices (i, j) such that for $\Delta_{i,j}$ the SDP solver can only find positive semidefinite Gram matrices with floating point entries (i.e., they are approximate). On the other hand, we could not certify that they are not SOS-Rayleigh. The numerical data from the SDP outputs suggests that such $\Delta_{i,j}$ are sum of squares with non-rational coefficients only.

Example 4.2.3. The matroid M shown in Figure 4.7a on ground set $E = [9]$ with non-bases $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{2, 4, 6\}$, $\{3, 7, 8\}$ and $\{5, 7, 9\}$ is one of the ten matroids of rank 3 with the HPP for which our algorithm cannot prove or disprove that they are SOS-Rayleigh.

There are 185982 simple matroids on 9 elements of rank 4. We first extend the list of excluded minors (by adding those of rank 3 on 9 elements and of rank 4 on 8 elements) and eliminate matroids having one of the 35 forbidden minors as a minor. By applying a sum of squares test described in Algorithm 1 on the Rayleigh differences of the remaining connected matroids, we found that 4125 matroids have the half-plane property. For each, there is a pair (i, j) such that the SDP solver could find positive semidefinite rational Gram matrices during the SOS test on $\Delta_{i,j}$. In addition, there are 819 matroids for which the SDP solvers could only find positive semi-definite Gram matrices with floating point entries (i.e., they are approximate). The numerical results suggest they might

Properties	Count
Half-plane property	116
Undetected SOS-Rayleigh	10
Not having the HPP	3

Table 4.5: Summary of the classification of simple connected matroids on 9 elements of rank 3 all of whose proper minors have the half-plane property.

Properties	Count
Half-plane property	4125
Candidates for having the HPP	819
Not having the HPP	1218
Undetected	556

Table 4.6: Summary of the test results on simple connected matroids on 9 elements of rank 4 all of whose proper minors have the half-plane property.

have the half-plane property, but our computations do not provide a proof. We list those matroids under the category "Candidates for having the half-plane property".

We found 1218 matroids that do not have the half-plane property, all of whose proper minors have the half-plane property. By using the Julia package "Homotopy Continuation" on the second part of Algorithm 1, for each such M, we were able to find indices (i, j) and $x \in \mathbb{R}^7$ such that $\Delta_{i,j}(h_M)(x) < 0$.

Remark 4.2.4. Our experiments do not provide a complete classification with respect to the half-plane property for matroids on 9 elements with rank 4. There are 556 matroids that neither pass the SOS test nor the test for finding negative points. We suspect that the Rayleigh differences of those matroids are non-negative, but not sum of squares. However, our computations do not provide a proof. These are listed under the name "Undetected". Due to computational time constraints, we did not perform tests using Algorithm 2 on matroids on 9 elements of rank 4: Once we found a sum of squares representation for one pair of indices, we did not continue to check the remaining ones.

Below we list some examples of matroids on 9 elements of rank 4.

Example 4.2.5. Matroid M with non bases $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 5, 7\}$, $\{3, 4, 6, 8\}$, $\{2, 4, 7, 8\}$, $\{2, 6, 7, 9\}$, and $\{5, 6, 8, 9\}$ is one of the 556 matroids for which Algorithm 1 could not detect whether it has the half-plane property. The SDP solver could not find any SOS decomposition, and we were not able to find any points for which one of its Rayleigh differences takes a negative value. Its combinatorial structure is illustrated in Figure 4.11.

Example 4.2.6. Matroid M' with non bases $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, $\{1, 3, 5, 7\}$, $\{2, 4, 5, 8\}$, $\{3, 5, 6, 9\}$ and $\{4, 7, 8, 9\}$ is a candidate for having the half-plane property. It has at least one Rayleigh difference for which the SDP solver could find a Gram matrix with floating point entries, and there are no Rayleigh

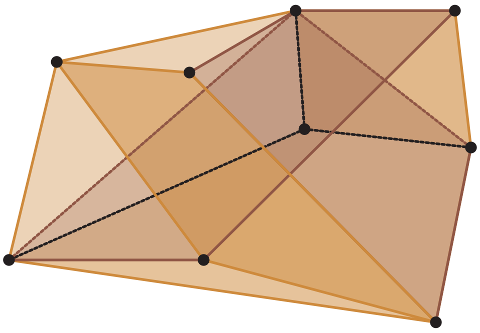

Figure 4.9: Matroid M'' whose non bases elements are given by the depicted quadrilaterals is a candidate for having the HPP.

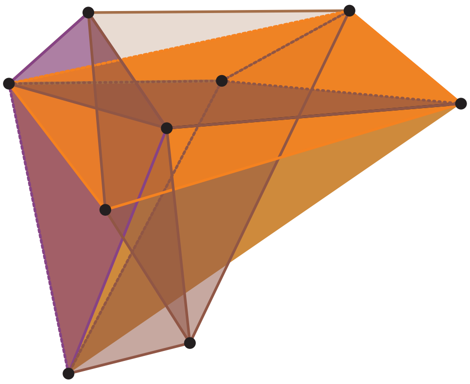

Figure 4.10: A matroid on 9 elements that does not have the HPP.

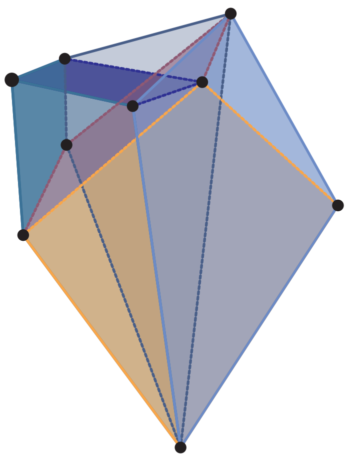

Figure 4.11: A matroid on 9 elements whose HPP could not be detected.

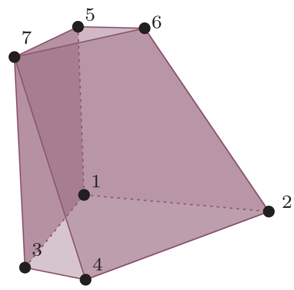

Figure 4.12: The quadrilateral faces of the shown polytope represent the non bases of a matroid on 9 elements with the HPP.

differences for which the solver could find a Gram matrix with rational entries. We suspect that it has the half-plane property, however we don't have a proof. The combinatorial structure of M' is depicted in Figure 4.9.

Example 4.2.7. Matroid M'' with non bases $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,3,5,7\}$, $\{1,4,6,7\}$, $\{4,5,7,8\}$, $\{1,6,7,9\}$ and $\{2,3,8,9\}$ does not have the half-plane property. The Rayleigh difference $\Delta_{21}h_{M''} \in \mathbb{R}[x_3,\ldots,x_9]$ takes the value $-7641.00833538555809163215183/2$ at

$$(1, 6.4556, 10.268, 78.1388, -5.1065, -6.688, 4.6387).$$

It is depicted in Figure 4.10.

Example 4.2.8. Matroid M''' with non bases $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,3,5,6\}$ and $\{2,4,6,7\}$ illustrated in Figure 4.12 has the half-plane property. In particular, every 4-element subset $S \subset [9]$ that contains 8, 9 or both is a basis.

We provide data-base description of matroids that appear in our test results at `https://zenodo.org/record/6108027`.

Chapter 5

Conclusion and Future Perspectives

Throughout the manuscript, we revisited the properties of homogeneous polynomials, some operations one may apply to them, and the operations that preserve their properties. The minor closedness of the half-plane property, being weakly determinantal, being SOS-Rayleigh, and being spectrahedral had a special highlight ([18, 38, 39],§ 2.2). In particular, these results imply that for each of the properties, one can start searching for small matroids that do not have it. Using this fact and also the criteria for the half-plane property and being weakly determinantal ([10, 63]), we suggest an algorithm to test the half-plane property and weak determinantal representability of a matroid.

In summary, given a matroid M, we follow the following steps in order to test the HPP:

- Check whether M has any of the forbidd en minors as a minor.
  - If it has a forbidden minor as a minor then it does not have the HPP.
  - If it does not have such a minor, apply Algorithm 1.
    - Test the non-negativity of the Rayleigh differences. If all proper minors of M have the HPP, it is enough to prove the non-negativity of one of the Rayleigh differences.
    - If the test does not give a positive result, try to find points on which a Rayleigh difference takes a negative value to disprove the HPP.
- If any of above methods does not work, try to find some other methods.

For being weakly determinantal, we only have methods to disprove it, one of which yields to Algorithm 2, and the other one is about showing that a matroid does not satisfy the Ingleton inequalities.

Using these algorithms, we classified the matroids on 8 elements with respect to the half-plane property. Our tests on matroids on 8 and 9 elements yield a long list of forbidden minors for the half-plane property, making testing the bigger matroids' half-plane property more feasible. In addition, we found 14 matroids on 8 elements of rank 4, including the Vámos matroid, that are not weakly determinantal. All these contributions suggest the following further research directions.

5.1 Spectrahedral Matroids

In the context of the generalized Lax conjecture, we are interested in the spectrahedral representability of the hyperbolicity cones. In particular, the result on the minor closedness of the spectrahedral representability suggests a search for a counterexample starting from matroids on smaller elements. From the results of Bránden [12] and Amini [2] that support the conjecture, we know that uniform matroids and graphical matroids are spectrahedral. In addition, the hyperbolicity cone of polynomials with a determinantal representation are spectrahedral and matroids that are representable over every field (thus over $\mathbb{R}$) have a determinantal representation (see § 2.1). Therefore, by construction, $\sqrt[6]{1}$-matroids are spectrahedral (note that there are $\sqrt[6]{1}$-matroids that are only weakly determinantal). In order to implement these results, we use the following characterizations:

Theorem 5.1.1 (Theorem 14.7.9 in [50]). *A matroid M is a $\sqrt[6]{1}$-matroid if and only if it has no minor isomorphic to any of $U_{25}, U_{35}, F_7, F_7^*, F_7^-, \left(F_7^-\right)^*$ or P_8.*

Theorem 5.1.2 (Theorem 10.1.1 in [50]). *A matroid is regular (thus representable over $\mathbb{R}$) if and only if it has no minor isomorphic to U_{24}, F_7 and F_7^**

Theorem 5.1.3 (Theorem 10.3.1 in [50]). *A matroid M is graphical if and only if it has no minor isomorphic to any of U_{24}, F_7, F_7^*, $M(K_5)^*$ or $M(K_{3,3})^*$.*

Matroids $M(K_5)$ and $M(K_{3,3})$ are constructed from the complete graph K_5 and the 3-connected graph $K_{3,3}$ respectively. Since graphical matroids have the half-plane property and it is preserved under taking duals (Proposition 1.2.28), $M(K_5)^*$ and $M(K_{3,3})^*$ have the half-plane property. Among the forbidden minors mentioned above, F_7, F_7^-, P_8 and their duals are forbidden minors for the half-plane property. Therefore, matroids with a minor isomorphic to any of them are not hyperbolic. Thus, matroids with the half-plane property with any minor isomorphic to any of U_{24}, U_{25}, U_{35}, $M(K_5)^*$ or $M(K_{3,3})^*$ are candidates for our search for spectrahedrality. Since proper minors of a matroid M have at least one less element than M, we start searching on matroids on 5 elements.

We use the command `allMatroids(5)` in Macaulay2 with the package "Matroids" ([16, 17]) to obtain the list of all non-isomorphic matroids on 5 elements. The basis generating polynomial of the direct sum of two spectrahedral matroids is the product of their basis generating polynomials. The hyperbolicity cone of the product is the intersection of the hyperbolicity cones of the components. Since the intersection of spectrahedral cones is again spectrahedral, it is enough to consider connected matroids. Moreover, a non-simple matroid either has a linearity space in its hyperbolicity cone, or its hyperbolicity cone is intersection of a hyperbolicity cone with some coordinate hyperplanes. Therefore, we can also restrict our attention to simple matroids. By conducting a test on the minors of simple non-isomorphic matroids from the list, we conclude the following:

Theorem 5.1.4. *Every matroid on at most 5 elements is spectrahedral.*

Proof. Note that all matroids on at most 6 elements have the half-plane property ([18, Proposition 10.10]). There are 3 non-isomorphic simple connected matroids on 5 elements that have U_{24} as a minor; namely U_{35}, U_{25} and one more matroid which does not have neither U_{25} nor U_{35} as a minor. By Theorem 5.1.1, the latter is a $\sqrt[6]{1}$-matroid. The claim follows from the results on the spectrahedrality of uniform and $\sqrt[6]{1}$-matroids ([12, 2]). □

Then, the next step is checking the minors of simple matroids on 6 elements. This time we use the command `allMatroids(6)` and obtain the list $\mathcal{L}$ of all non-isomorphic matroids on 6 elements. There are 5 simple connected matroids on 6 elements that have the matroid U_{25} as a minor (thus they are not $\sqrt[6]{1}$-matroids), two of which are U_{36} and U_{26} and the other 3 are

- the matroid P_6 ($\mathcal{L}_{31}$ (counting starts from 0)) with non-basis $\{1, 2, 3\}$,
- the matroid Q_6 ($\mathcal{L}_{33}$) with non-bases $\{1, 2, 3\}$ and $\{1, 4, 5\}$,
- the matroid $\mathcal{L}_{39}$ with non-bases $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

All of them have the matroid U_{24} as a minor, therefore by Theorem 5.1.2, they are not regular. Moreover, P_6 and Q_6 have U_{35} as a minor, while $\mathcal{L}_{39}$ does not. We can further restrict the list by applying some more results. The basis generating polynomial of a nice transversal matroid corresponds to the matching polynomial of the bipartite graph it is constructed from (see Example 1.2.19). By [18, Corollary 10.3], those matroids have the half-plane property, and since they have graphical interpretation, by [2], they are spectrahedral. In [18, Example 10.7], Choe et al. show that the matroids P_6 and Q_6 are nice transversal and, thus by [2], spectrahedral.

Moreover the matroid $\mathcal{L}_{69}$ with non-bases $\{2, 3, 4, 5\}$ does not have U_{25} as a minor, but has the matroids U_{35} and U_{24} as its minors. Therefore, it is not regular, it is not $\sqrt[6]{1}$ and it is not graphical.

Hence, we are left with matroids $\mathcal{L}_{39}$ and $\mathcal{L}_{69}$ whose spectrahedrality needs to be shown.

Question 5. *Are the matroids $\mathcal{L}_{39}$ and $\mathcal{L}_{69}$ spectrahedral? In other words, are all matroids on at most 6 elements spectrahedral?*

By the criteria given in [30], in order to show the spectrahedrality of $\mathcal{L}_{39}$, One needs to find a hyperbolic polynomial g with $C_{h_{\mathcal{L}_{39}}} \subseteq C_g$ such that $h_{\mathcal{L}_{39}} \cdot g$ has a determinantal representation. Moreover, the following result shows that for smooth hyperbolic polynomials, one can always find another hyperbolic polynomial such that their product has a determinantal representation (without ensuring the inclusion condition on their hyperbolicity cones).

Theorem 5.1.5 (Theorem 6 in [36])**.** *Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be hyperbolic with respect to some $e \in \mathbb{R}^n_{\geq 0}$. Assume that h has no real singularities (i.e. $\nabla h(v) \neq 0$ for all $0 \neq v \in \mathbb{R}^n$). Then there is a hyperbolic polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$, such that $h \cdot g$ has a determinantal representation.*

Kummer, for instance, applied this result in [35] to show that the hyperbolicity cone of the specialized Vámos matroid is spectrahedral. The construction of the method used in the mentioned paper can be transferred to our case as follows:

Let p be a vector of size m with polynomial entries in $\mathbb{R}[x_1, \dots, x_6]$ of degree $d' \geq \deg(h_{\mathcal{L}_{39}}) - 1 = 2$ such that p has at least one component that is not divisible by $h_{\mathcal{L}_{39}}$. Further, let $A_1, \dots, A_n \in \mathrm{Sym}_{\mathbb{R}}(m)$ and q be a vector of size m with polynomial entries of size $d' - \deg(h_{\mathcal{L}_{39}}) + 1$. Fix p and consider the following SDP program

$$(A_1x_1 + \cdots + A_nx_n) \cdot p = q \cdot h_{\mathcal{L}_{39}}$$

in the entries of A_i and the coefficients of the entries of q. If this program has a solution, then there exists a hyperbolic polynomial g with $g \cdot h_{\mathcal{L}_{39}} = \lambda \det(A_1x_1 + \cdots + A_nx_n)$. Finding a suitable p and the desired polynomial g and exploring the inclusion of the hyperbolicity cones is ongoing work.

One can further consider the following questions:

Question 6. *Are all matroids on at most* 7 *elements with the HPP spectrahedral?*

Question 7. *Are all matroids on at most* 8 *elements with the HPP spectrahedral? Especially, are all* 14 *non-weakly determinantal matroids listed in Theorem 4.1.11 spectrahedral?*

Question 8. *Is there a computationally feasible algorithm to check spectrahedral representability of matroids with the HPP?*

Another approach could be finding a particular (feasible) characterization of spectrahedral cones and checking whether hyperbolicity cones also satisfy them. For example in [1], Allamigeon et al. introduce tropical spectrahedra, taking a tropical geometric approach for a characterization.

5.2 Non-negative Non-SOS Polynomials

Non-negative polynomials and non-negativity certificates have an important role in convex optimization ([40], [6], [41]). In particular, the certification of non-negativity of polynomials that are not a sum of squares is of interest. For instance, there are studies (see, e.g., [33, 34, 22]), on non-negative polynomials supported on circuits that establish new methods (sums of non negative circuit polynomials (SONC) method) for producing non-negativity certificates.

Each of the 14 matroids listed in Theorem 4.1.11 has at least one Rayleigh difference that is non-negative but is not a sum of squares of polynomials. Further, the non-negativity of each such polynomial was certified via a criterion ([63]) on Rayleigh differences, namely by finding another Rayleigh difference that is a sum of squares. In other words, matroids that are not weakly determinantal give rise to a new way of producing non-negative, non-SOS polynomials. This suggests further research on the common properties of these Rayleigh differences, such as their support structure.

Definition 5.2.1. Let $h \in \mathbb{R}[x_1, \dots, x_n] := \mathbb{R}[\mathbf{x}]$ be a polynomial and $\mathcal{S}$ be its support such that the vertices of its Newton polytope $\mathrm{Newt}(h)$ are in $(2\mathbb{N})^n$. Then h is called a *circuit polynomial* if it is of the form

$$h(\mathbf{x}) = \sum_{i=1}^{m} c_i \mathbf{x}^{\alpha_i} - d\mathbf{x}^{\beta}$$

where $m \leq n+1$, $c_i \in \mathbb{R}_{>0}$ for $i \in [m]$, $d \in \mathbb{R}^*$, $\{\alpha_1, \ldots, \alpha_m\}$ is the set of vertices of $\mathrm{Newt}(h)$, and β is in the interior of $\mathrm{Newt}(h)$.

Example 5.2.2. Consider the matroid $\mathcal{M}_{417}$ and $\Delta_{27} h_{\mathcal{M}_{417}}$ as in Example 4.1.16. The Newton polytope of $\Delta_{27} h_{\mathcal{M}_{417}}$ has 15 vertices given in the columns of

$$\begin{pmatrix} 2 & 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Moreover, $\Delta_{27} h_{\mathcal{M}_{417}}$ has 81 exponents and only the monomial with the exponent $\{2,0,1,1,1,1,0,0\}$ has a negative coefficient -1. Clearly it is not of the form of a circuit polynomial, but it might be a sum of non-negative circuit polynomials.

Question 9. *Are non-negative non-SOS polynomials produced by non-SOS-Rayleigh matroids sums of non-negative circuit polynomials?*

5.3 Completing the Classification of Matroids on 9 Elements and More

While Algorithm 1 was sufficient for the classification of the half-plane property of matroids on at most 8 elements, one needs further methods to classify matroids on 9 elements with respect to the half-plane property. Investigating the matroids whose half-plane property could not be detected, and coming up with new methods to test the HPP are among the future perspectives.

Question 10. *Are there other computationally feasible methods for testing the half-plane property of matroids?*

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