

Multi-vehicle Stochastic Fundamental Diagram Consistent with Transportations Systems Theory

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Abstract

This paper describes a general approach to the specification the stable regime speed-flow function, for motorways, as a part of the stable regime Stochastic Fundamental Diagram consistent with main assumptions of Transportation Systems Theory. Main original elements are:

- Specification of speed-flow functions consistent with travel time function, such as BPR-like functions;
- Calibration from disaggregate data, say data from single vehicle trajectories;
- Specification of the speed r. v. distribution consistent with those used in RUT for route choice behavior modelling, such as Gamma, Inv-Gamma.

Keywords: fundamental diagram, stochastic FD, traffic flow models (macroscopic level)

1 Introduction

Fundamental Diagram can be dated back to 1935, when the first publication on speed-flow curves “A Study of Traffic Capacity”, presented at the 14th Annual Meeting of the Highway Research Board in 1935 [Gre35], was published. From this seminal paper the modern Traffic Flow Theory (TFT) evolved for studying: driving choice behavior, concerning interactions between users travelling on the same facility and their effects on speed, travel time and relationships with flow and density.

TFT (recent reviews in [Tre13; HCM16; Pac19]; see also [Dag97]) includes both descriptive-predictive models, Traffic Analysis, as well as decisional-prescriptive ones, Traffic Control. Under steady-state conditions the most commonly used (macroscopic) model to describe vehicles flowing along a street is the so-called Fundamental Diagram (FD) describing the relations among density, flow and (space average) speed. When steady state conditions do not hold, within-day dynamic macroscopic models are used, including FD as one of the main equations. It is well known that according to the speed-flow relationship in FD a value of flow less than capacity may occur under two different conditions: high speed and low density, stable regime, and low speed and high density, unstable regime.

Traffic and Transportation Theory, including TFT as well as Transportation Systems Theory (TST) begun with [War52]; TST studies the interactions between:

- driving behavior, concerning interactions between users travelling on the same facility and their effects on travel time, ...; arc travel time is generally modelled as a function of arc flows;
- routing behavior, usually modelled through Random Utility Theory.

2 Methodology

Two basic variables can be observed for each link of a highway: flow depending on spatial abscissa and density depending on time instant; speed can be averaged on space or time, the former depends on time instant, the latter on abscissa.

2.1 Main Definitions, Variables, Equations

Hypothesis 1 *Under steady-state conditions the flow during a time interval does not depend on the abscissa, the density and the (space average) speed over a stretch of space do not depend on the instant of time:*

- $f \geq 0$ be the (vehicular) flow;
- $k \geq 0$ be the (vehicular) density;
- $v \geq 0$ be the (space average) speed;
- f_{MAX} be the capacity, $f \leq f_{\text{MAX}}$ commonly assumed a function of geometrical characteristics of the infrastructure [HCM16] or calibrated;
- k_{MAX} be the maximum density, k_{jam} , $k \leq k_{\text{MAX}}$, depending on (average) vehicle length and minimum safety distance;
- v_o be the zero-flow or free-flow or maximum speed, v_{MAX} , $v \leq v_o$, commonly assumed a function of geometrical characteristics of the infrastructure, as well as weather and light conditions.

$$f = kv. \quad (1)$$

Hypothesis 2 Under steady-state conditions a relationship holds between speed and flow:

$$v = v(f) \in [0, v_o], \quad 0 \leq f \leq f_{\text{MAX}}. \quad (2)$$

Two regimes, with two different values of speed, correspond to each value of flow:

- stable regime, high speed and low density, $v(f)$ is monotone decreasing;
- unstable regime, low speed and high density, $v(f)$ is monotone increasing.

Hypothesis 3 The speed given by the stable regime speed-flow function is to be considered the mean of a random variable V , whose dispersion models several sources of uncertainty, such as motorway lay-out, non-steady-state conditions, heterogeneity of driving behavior and of vehicles' characteristics, variability of weather and light conditions, . . . , leading to so-called Stochastic Fundamental Diagram (S-FD) [Qu17; Wan13]:

$$v \leftarrow V \in [0, v_o] : E[V/f] = v(f), \quad 0 \leq f \leq f_{\text{MAX}}. \quad (3)$$

2.2 A Modelling Approach for Stable Regime Speed-flow Functions

In this paper some specifications of the stable regime speed-flow function (2), consistent with the following requirements, are described and compared with existing ones.

Requirement 1 The specification of the stable regime speed-flow functions has to allow to define proper travel time functions for transportation network analysis. Let:

- t be the travel time needed to traverse the link;
- $t_o = L/v_f$ be the zero-flow travel time, say the minimum travel time.

The well-known BPR-like travel time function, often used for transportation supply analysis and demand assignment is generated by the following stable regime speed-flow function:

$$v(f) = v_o / (1 + a(f/f_{\text{MAX}})^b), \quad 0 \leq f \leq f_{\text{MAX}}, \quad (4)$$

where a is the congestion factor, such that $1 + a = t(f_{\text{MAX}})/t_o$, and b a shape coefficient.

Requirement 2 The specification of the r. v. for speed is to be consistent with the (arc-based) specifications of route choice behavior models from RUT, based on disutility distributed as Normal, Gamma, Inv-Gamma, The most promising r. v. distribution for speed seems the Inv-Gamma, defined over a positive support, that leads to arc travel time distributed as Gamma.

Requirement 3 The calibration of the stable regime speed-flow function is carried out from disaggregate data, say observed data from single vehicle trajectories.

3 Results and Discussion

This section discussed results obtained from a data set containing the trajectories of all vehicles in both directions of a 3 lane section, 310 meters long, of the Italian A51 motorway; the time interval ranges from 9:10 a.m. to 1:10 p.m. The acquisition technique, based on video processing, allowed the detection of all vehicles in the stretch with a frequency of 1 Hz. For each trajectory, the time instant of crossing a virtual section located at the center of the section (Entry Gate) was detected.

The data have been processed and aggregated within a time interval Δt in order to obtain the following variables:

- Equivalent Flow: number of vehicles passing through the entry gate over Δt ,
- Spatial Mean Speed: average of the instantaneous measured speeds of all vehicles in the section over Δt ,
- Equivalent Density: average of the instantaneous densities, one for each second, over Δt ,
- Modeled Flow: flow computed as: (Spatial Mean Speed * Equivalent Density).

Flows and densities have been calculated using the following equivalence factors: Car = 1, Medium Vehicle = 1.5, Heavy Vehicle = 3, Motorcycle = 0.3, Bus = 3.

In Figure 1 observed data regarding the fundamental diagram is shown considering an aggregation of Δt equal to 15 seconds, while Figure 2 shows the same but considering an aggregation of Δt equal to 30 seconds.

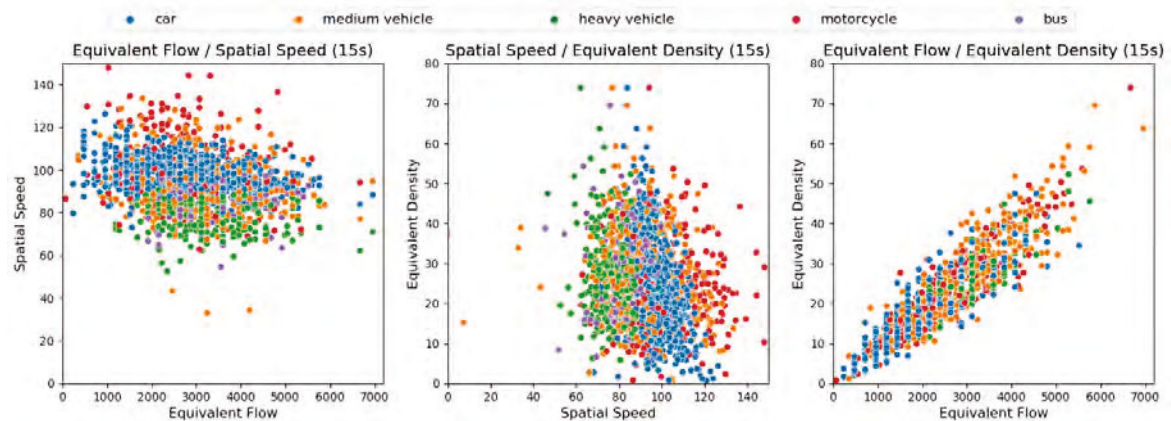


Figure 1: Fundamental diagram with 15-second time aggregation. a) Equivalent Flow - Spatial Mean Speed. b) Equivalent Flow - Spatial Speed. c) Equivalent Flow - Equivalent Density.

For both aggregation time intervals, the diagrams show different behaviors for vehicles of different classes; it can be observed that the speeds are lower for heavy vehicles than for cars and motorcycles while medium vehicle class seems to vaguely defined resulting in too dispersed data. All data may be considered referring to a hypocritical state and a steady-state

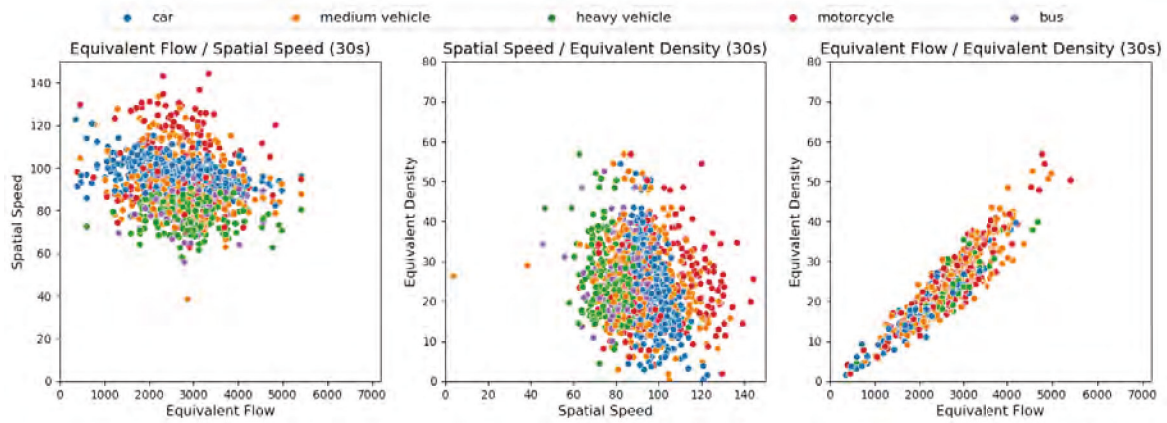


Figure 2: Fundamental diagram with 30-second time aggregation. a) Equivalent Flow - Spatial Mean Speed. b) Equivalent Flow - Spatial Speed. c) Equivalent Flow - Equivalent Density.

condition since observed flow is below the capacity and densities are very far from maximum values, as well as the very good collinearity between calculated flows as in (1), and observed flows (Figure 3).

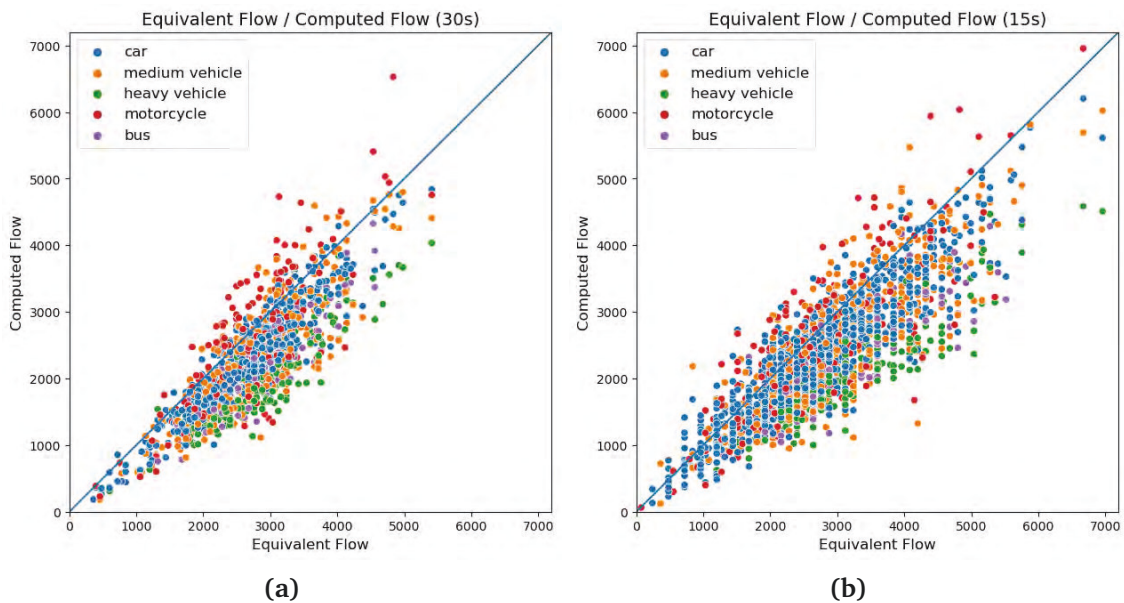


Figure 3: Comparison of equivalent and computed flows with aggregation at 15 and 30 seconds.

The values of the measured flows aggregated over a 15-second time interval have been successively aggregated into flow classes with a step size of 120 veh/h, considering only the speed of the cars and all variables refer to one lane, see Figure 4 where the vertical axis reports the average observed speed, the horizontal axis the flow classes. The whiskers indicate the value of the estimated standard deviation, while the value shown under the points on the graph indicates the numerosity of the class. A low value of speed is observed

for the first class and, the sample of this class has very few observation, therefore data of this class have not been considered in the analysis below..

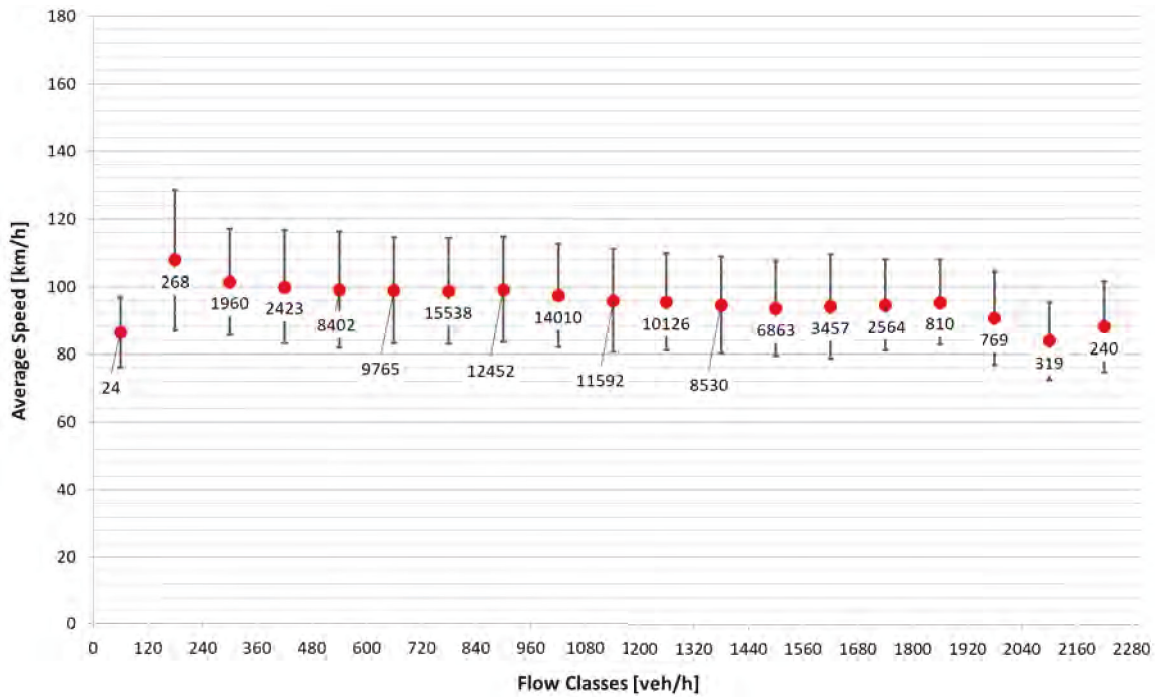


Figure 4: Average observed speed and its standard deviation, numerosity of the sample data for the different flow classes.

3.1 Calibration of the Speed-flow Function

This sub-section reports results obtained from the calibration of the fundamental diagram against the aggregate data reported in Figure 5, say the speed-flow function proposed by Greenshields, as a bench mark, or the one consistent with BPR function, Equation (4), reported below for reader's convenience:

$$\text{Greenshields' function: } v(f) = v_o(1 + (1 - f/f_{\text{MAX}})^{0.5})/2. \quad (5)$$

For Greenshields' function both the free-flow speed, the maximum flow, v_o and f_{MAX} , are calibrated; for BPR function different subsets of 4 parameters have been calibrated:

- BPR free flow speed and maximum flow for given values of parameters $a = 0.15$ and $b = 2.0$ as in the original BPR travel time function;
- BPR 2 parameters a and b for given values of free flow speed and maximum flow;
- BPR 3¹ parameter a , free flow speed and maximum flow, for $b = 1$;
- BPR 3² parameter a , free flow speed and maximum flow, for $b = 2$;
- BPR 4 parameters a and b , free flow speed and maximum flow.

The cases BPR 3^1 and BPR 3^2 have been considered since integer $b = 1, 2$ allows to derive the speed-density function solving a quadratic or a cubic equation, as described in details in a future paper.

The results of calibration are reported in Table 1, green boxes denote the parameters considered in calibration. All the BPR-like functions show performance similar to the Greenshields benchmark model (but the BPR case that performs quite worse). The free flow speed values for the BPR function are similar to Greenshields' one, but the capacity value is much closer to the commonly used values, presumably due to scaling effect of the congestion factor, a . Figure 5 shows the different functions calibrated; all the functions fall within the range of estimated standard deviation.

Table 1: Comparison of parameters and performance indicators of the different calibrations.

	Greenshields	BPR	BPR 2	BPR 3^1	BPR 3^2	BPR 4
v_{free}	104	100	110	106	102	106
f_{max}	4191	2164	2500	2589	2566	2198
a		0.15	0.25	0.22	0.22	0.19
b		4	0.65	1	2	0.90
$\sum(v_{\text{obs}} - v_{\text{mod}})^2$	83	137	86	85	98	85
RMSE	0.027	0.035	0.028	0.028	0.030	0.028

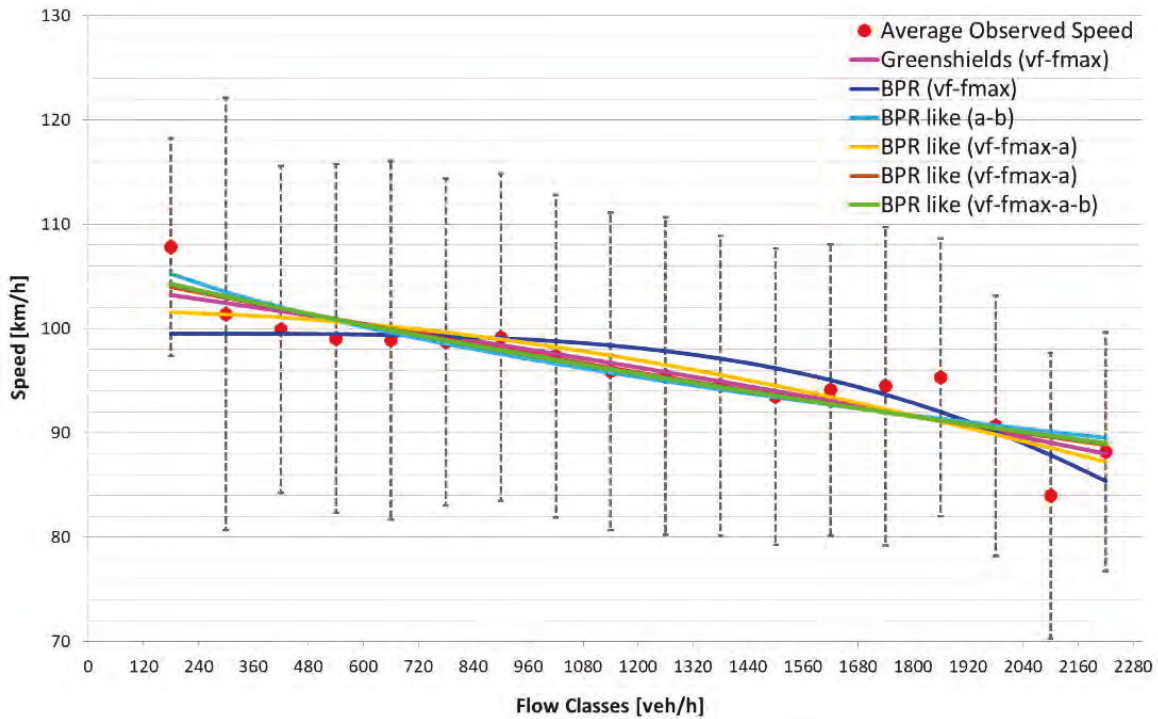


Figure 5: Calibrated function comparison with respect to observed speed.

For Greenshields' function, the maximum density (or jam density), k_{MAX} , can be computed starting from the free flow speed and the maximum flow, through equation: $f_{MAX} = v_o k_{MAX}/4$. The value 161 veh/km is obtained, consistent with the average vehicle lengths in Italy.

3.2 Stochasticity of the Speed-flow Function

After the calibration, dispersion of speed values for each flow class has been analysed. Figure 6 shows the frequencies of observed speed values only for 3 flow classes for brevity. Best fitting Normal, Gamma and Inv-Gamma pdf's (through Scipy routines) are also reported. For some classes (e.g. class 1200) very low skewness is observed, all the 3 pdf's perform in a similar way; but if skewness is significant (e.g. classes 600, 1920), Gamma and InvGamma outperform Normal as expected; Gamma and InvGamma are very close in all cases.

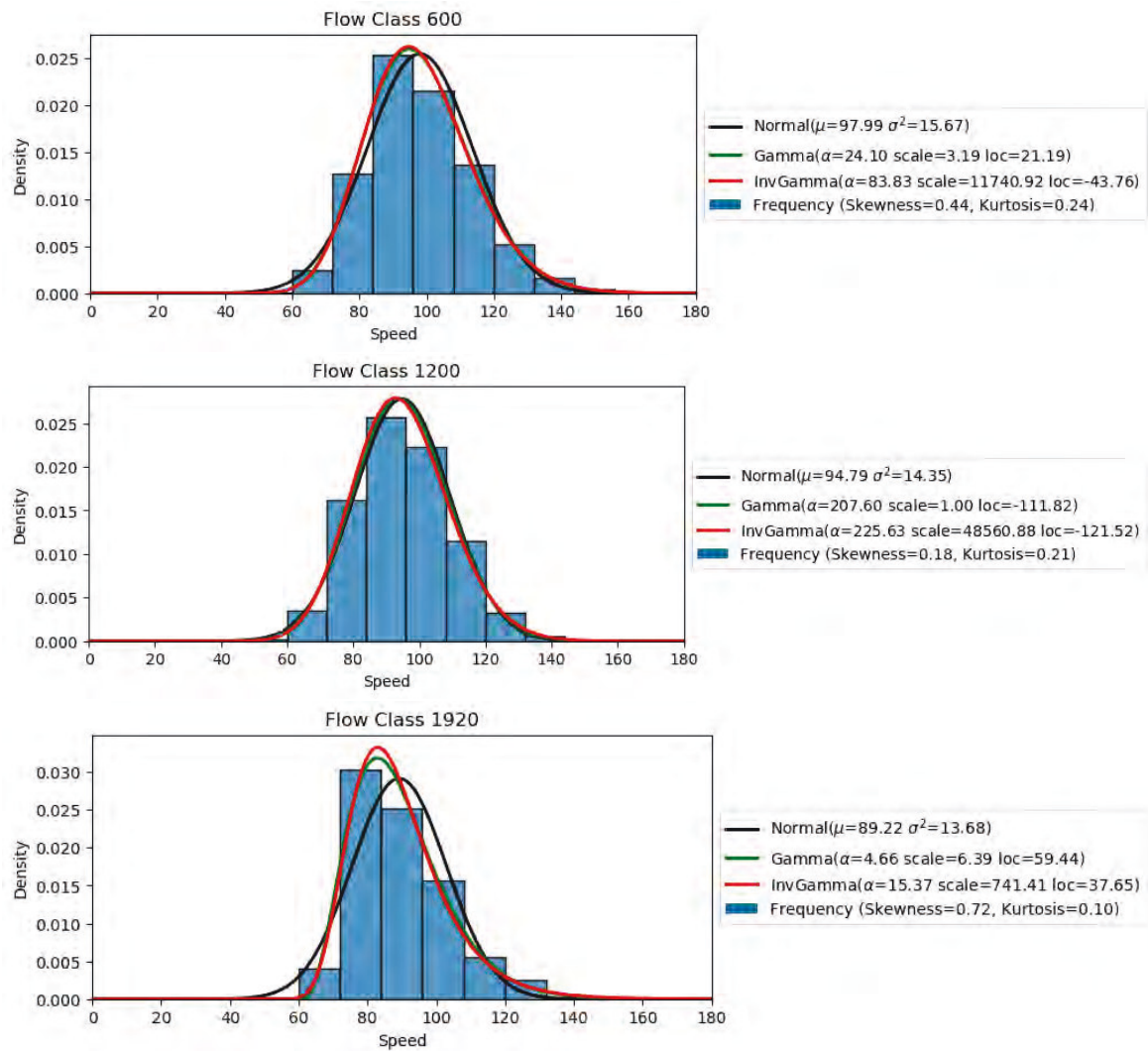


Figure 6: Frequencies of observed speed values and Normal, Gamma and InvGamma pdf's.

4 Conclusions

This paper discusses stable regime Stochastic Fundamental Diagram (sS-FD) consistent with Traffic Engineering as well as Transportation System Theory tools; major original contributions are:

- Specification of speed-flow functions consistent with BPR-like travel time function, largely used in Transportation Network Analysis;
- Calibration from disaggregate data, say data from single vehicle trajectories, supporting that BPR-like speed-flow functions perform as the benchmark Greenshields function;
- Specification of the speed r. v. distributions consistent with those used in RUT for route choice behavior modelling, such as Normal, Gamma, Inv-Gamma.

Worth of further research are:

- Derivation and calibration of speed-density functions;
- Comparison with other benchmark speed-flow functions;
- Application of formal tests of goodness of fit for stochastic analysis of speed dispersion;
- Estimation of emissions;
- Calibration of vehicle equivalence coefficients;
- Validation over hold-out samples for testing reproducibility over time or space;
- Analysis of data where flow values are close to capacity;
- Analysis of unstable regime.

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