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Research article

On subpolygroup commutativity degree of finite polygroups

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Abstract: Probabilistic group theory is concerned with the probability of group elements or group subgroups satisfying certain conditions. On the other hand, a polygroup is a generalization of a group and a special case of a hypergroup. This paper generalizes probabilistic group theory to probabilistic polygroup theory. In this regard, we extend the concept of the subgroup commutativity degree of a finite group to the subpolygroup commutativity degree of a finite polygroup P. The latter measures the probability of two random subpolygroups H, K of P commuting (i.e., $HK = KH$). First, using the subgroup commutativity table and the subpolygroup commutativity table, we present some results related to the new defined concept for groups and for polygroups. We then consider the special case of a polygroup associated to a group. We study the subpolygroup lattice and relate this to the subgroup lattice of the base group; this includes deriving an explicit formula for the subpolygroup commutativity degree in terms of the subgroup commutativity degree. Finally, we illustrate our results via non-trivial examples by applying the formulas that we prove to the associated polygroups of some well-known groups such as the dihedral group and the symmetric group.

Keywords: polygroup; subgroup commutativity degree; subpolygroup lattice; subpolygroup commutativity degree
Mathematics Subject Classification: 20N20

1. Introduction

Probability in finite groups has grabbed the interest of algebraists in the last few years. One of the concepts that have been studied is the probability that two group elements of a finite group G commute [17], denoted by $d(G)$. Another concept [10] is the relative commutativity degree of a subgroup H of a finite group G, denoted by $d(H, G)$. The latter measures the probability that an element of the subgroup H commutes with an element of the group G. Furthermore, subgroup commutativity degree of a finite group G [21] measures the probability of two random subgroups of G commuting. Other related work can be found in [7, 12, 13], and a survey on statistical group theory can be found in [8]. On the other hand, hypergroup theory, a generalization of group theory, is a field that was introduced by Marty [14] in 1934. Special classes of hypergroups are canonical hypergroups, introduced in 1970 [15,16], and quasi-canonical hypergroups, introduced in 1981 [3,4]. The latter was studied by Comer [5] in 1984 under the name polygroup. For details about hyperstructure theory and its applications, we refer to [6, 9]. Researchers involved in this field try to check the validity of the known results in group theory for hypergroups. Indeed, some generalizations have been accomplished, but the fact that the class of hypergroups is much larger than that of groups makes it more difficult to generalize many conceptsRecently, there has been a growing interest in the use of probability in finite polygroup theory. Some related concepts were introduced such as the commutativity degree of finite polygroups, and some related work can be found in [18–20].

Inspired by the subgroup commutativity degree of a finite group G, our paper generalizes this concept to finite polygroups, and it is organized as follows: After an introduction, Section 2 presents some results related to the subgroup commutativity degree of finite groups. Then, Section 3 presents some results on the subpolygroups lattice of a particular class of polygroups. Finally, Section 4 defines the subpolygroup commutativity degree of a finite polygroup and presents some related results and examples by using the subpolygroup commutativity table. Moreover, it considers a special class of polygroups and finds an explicit formula for the subpolygroup commutativity degree of these.

2. Subgroup commutativity degrees of finite groups

In [21], Tarnauceanu defined the subgroup commutativity degree of finite groups and found explicit formulas for the subgroup commutativity degrees of some special finite groups. In this section, we present some of their results and discuss some other related results.

Definition 2.1. *Let $(G, \cdot)$ be a finite group and $L(G)$ be the set of all subgroups of G. Then, the subgroup commutativity degree of G is defined as follows:*

$$sd(G) = \frac{|\{(H, K) \in L(G)^2 : HK = KH\}|}{|L(G)|^2}.$$

Remark 1. *If all subgroups of G are normal, for instance, if G is abelian or if G is the quaternion group Q_8, then $sd(G) = 1$.*

Remark 2. *Let $(G, \cdot)$ be a finite group. Then, $0 < sd(G) \leq 1$.*

Definition 2.2. *Let k be a positive integer and $(G, \cdot)$ be a finite group with distinct subgroups $H_1, \ldots, H_k$. Then, the subgroup commutativity table of G is defined in Table 1.*

Table 1. Subgroup commutativity table of G.

$\cdot$	H_1	H_2	$\dots$	H_k
H_1	H_{11}	H_{12}	$\dots$	H_{1k}
H_2	H_{21}	H_{22}	$\dots$	H_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
H_k	H_{k1}	H_{k2}	$\dots$	H_{kk}

Here, for all $1 \leq i, j \leq k$, $H_{ij} = \begin{cases} 1 & \text{if } H_i \cdot H_j = H_j \cdot H_i, \\ 0 & \text{otherwise.} \end{cases}$

Remark 3. *Let* $(G, \cdot)$ *be a finite group with subgroup commutativity table* (H_{ij})*. Then,*

$$sd(G) = \frac{\sum_{j=1}^{k} \sum_{i=1}^{k} H_{ij}}{|L(G)|^2}.$$

Example 1. *Let* S_3 *be the symmetric group on three letters. Then, the subgroup commutativity table of* S_3 *is given in Table 2.*

Table 2. Subgroup commutativity table of S_3.

	$\{(1)\}$	$\{(1),(12)\}$	$\{(1),(13)\}$	$\{(1),(23)\}$	$\{(1),(123),(132)\}$	S_3
$\{(1)\}$	1	1	1	1	1	1
$\{(1),(12)\}$	1	1	0	0	1	1
$\{(1),(13)\}$	1	0	1	0	1	1
$\{(1),(23)\}$	1	0	0	1	1	1
$\{(1),(123),(132)\}$	1	1	1	1	1	1
S_3	1	1	1	1	1	1

It is clear that $sd(S_3) = \frac{5}{6}$.

Proposition 2.1. *Let* $(G, \cdot)$ *be a finite group. Then,* $sd(G) = 1$, *or* $sd(G) \geq \frac{9}{|L(G)|^2}$.

Proof. Let G be a finite group with identity e and $M(G) = \{(H, K) \in L(G)^2 : HK = KH\}$. We consider the following cases.

If G is the trivial group, or G has no proper non-trivial subgroups, then $sd(G) = 1$. If G has a proper non-trivial subgroup H, then $(\{e\}, \{e\})$, $(\{e\}, H)$, $(\{e\}, G)$, $(H, \{e\})$, (H, H), (H, G), $(G, \{e\})$, (G, H), and (G, G) are all in $M(G)$. Thus, $|M(G)| \geq 9$, and hence, $sd(G) = \frac{|M(G)|}{|L(G)|^2} \geq \frac{9}{|L(G)|^2}$. $\square$

Example 2. *Let* S_3 *be the symmetric group on three letters. Then,* $sd(S_3) = \frac{5}{6} \geq \frac{9}{36} = \frac{9}{|L(S_3)|^2}$.

Proposition 2.2. *Let $(G, \cdot)$ be a finite group and R be the relation on G defined as follows:*

$$HRK \text{ if and only if } HK = KH.$$

Then, R is a reflexive and symmetric relation.

Proof. The proof is straightforward by using the subgroup commutativity table of $(G, \cdot)$. □

Theorem 2.1. *[21] Let G, G' be finite groups with coprime orders. Then,*

$$sd(G \times G') = sd(G)sd(G').$$

The dihedral group D_n is the symmetry group of a regular polygon with n sides. and it has the order $2n$. The generalized quaternion group Q_{2^m} can be expressed via the following presentation:

$$Q_{2^m} = \langle x, y : x^{2^{m-1}} = y^4 = 1, yxy^{-1} = x^{2^{m-1}-1} \rangle.$$

The quasi-dihedral group S_{2^m} with $m \geq 4$ can be expressed via the following presentation:

$$S_{2^m} = \langle x, y : x^{2^{m-1}} = y^2 = 1, y^{-1}xy = x^{2^{m-2}-1} \rangle.$$

In [21], Tarnauceanu found explicit formulas for the subgroup commutativity degrees of some finite groups such as the dihedral group, quasi-dihedral group, and generalized quaternion group. Let $n \geq 2$ be a positive integer and $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the decomposition of n as a product of prime factors. Then, $\tau(n)$, $\sigma(n)$ are the number and the sum of all divisors of n, respectivelym and

$$g(n) = \prod_{i=1}^{k} \frac{(2\alpha_i + 1)p_i^{\alpha_i+2} - (2\alpha_i + 3)p_i^{\alpha_i+1} + p_i + 1}{(p_i - 1)^2}.$$

Theorem 2.2. *[21] Let D_n be as above and $n = 2^\alpha n'$ with n' odd. Thenm*

$$sd(D_n) = \frac{\tau(n)^2 + 2\tau(n)\sigma(n) + [(\alpha - 1)2^{\alpha+3} + 9]g(n')}{(\tau(n) + \sigma(n))^2}.$$

Example 3. *Let D_4, D_6 be the symmetry groups of regular polygons with $4, 6$ sides, respectively. Then, $sd(D_4) = \frac{22}{25}$ and $sd(D_6) = \frac{101}{128}$.*

Theorem 2.3. *[21] Let $D_{2^{m-1}}$, Q_{2^m}, and S_{2^m} be as above. Then,*

$$sd(D_{2^{m-1}}) = \frac{(m-2)2^{m+2} + m2^{m+1} + (m-1)^2 + 8}{(m - 1 + 2^m)^2}, \qquad m \geq 2,$$

$$sd(Q_{2^m}) = \frac{(m-3)2^{m+1} + m2^m + (m-1)^2 + 8}{(m - 1 + 2^{m-1})^2}, \qquad m \geq 2,$$

$$sd(S_{2^m}) = \frac{(m-3)2^{m+1} + m2^m + (3m-2)2^{m-1} + (m-1)^2 + 8}{(m - 1 + 3 \cdot 2^{m-2})^2}, \qquad m \geq 4.$$

In particular, $\lim_{m\to\infty} sd(D_{2^{m-1}}) = \lim_{m\to\infty} sd(Q_{2^m}) = \lim_{m\to\infty} sd(S_{2^m}) = 0.$

3. Subpolygroup lattice of polygroups

In this section, we present some basic results and examples related to polygroup theory that are used throughout the paper. Moreover, we study the subpolygroup lattice for a particular class of polygroups.

3.1. Polygroups

Let P be a non-empty set and $\mathcal{P}^*(P)$ be the family of all non-empty subsets of P. A binary hyperoperation on P is a mapping $\circ : P \times P \to \mathcal{P}^*(P)$. The couple $(P, \circ)$ is called a hypergroupoid.

In the above definition, if H and K are two non-empty subsets of P and $p \in P$, then we define:

$$H \circ K = \bigcup_{\substack{h \in H \\ k \in K}} h \circ k, \ \ p \circ H = \{p\} \circ H \text{ and } H \circ p = H \circ \{p\}.$$

Definition 3.1. *[5] A polygroup is a system $\langle P, \cdot, e, ^{-1} \rangle$, where $(P, \cdot)$ is a hypergroupoid, $e \in P$, $^{-1} : P \to P$ is a unary operation on P, and the following axioms hold for all $x, y, z \in P$:*

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(2) $e \cdot x = x \cdot e = \{x\}$,
(3) $x \in y \cdot z$ *implies* $y \in x \cdot z^{-1}$ *and* $z \in y^{-1} \cdot x$.

For simplicity, we write x instead of $\{x\}$ for all x in the polygroup $\langle P, \cdot, e, ^{-1} \rangle$.

A canonical hypergroup $\langle P, \cdot, e, ^{-1} \rangle$ is a commutative polygroup, i.e., $x \cdot y = y \cdot x$ for all $x, y \in P$. For more details, we refer to [15, 16].

Example 4. *Let $P = \{m_0, m_1, m_2\}$ and $(P, \circ)$ be defined by Table 3.*

Table 3. The canonical hypergroup $(P, \circ)$.

$\circ$	m_0	m_1	m_2
m_0	m_0	m_1	m_2
m_1	m_1	$\{m_0, m_2\}$	$\{m_1, m_2\}$
m_2	m_2	$\{m_1, m_2\}$	$\{m_0, m_1\}$

The identity under $\circ$ is n_0, and $x^{-1} = x$ for all $x \in P$. Then, $\langle P, \circ, m_0, ^{-1} \rangle$ is a canonical hypergroup.

Example 5. *[11] Let $P' = \{n_0, n_1, n_2, n_3\}$ and $(P', \cdot)$ be defined by Table 4.*

Table 4. The polygroup $(P', \cdot)$.

$\cdot$	n_0	n_1	n_2	n_3
n_0	n_0	n_1	n_2	n_3
n_1	n_1	n_1	P'	n_3
n_2	n_2	$\{n_0, n_1, n_2\}$	n_2	$\{n_2, n_3\}$
n_3	n_3	$\{n_1, n_3\}$	n_3	P'

Then, $\langle P', \cdot, n_0, ^{-1} \rangle$ is a non-canonical hypergroup.

Remark 4. *Every group is a polygroup.*

Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup and $H \subseteq P$. Then, H is a subpolygroup of P if for all $x, y \in H$, we have $x \circ y \subseteq H$ and $x^{-1} \in H$.

Example 6. *Let $\langle P', \cdot, n_0, ^{-1} \rangle$ be the polygroup in Example 5. Then, $\{n_0\}$ and P' are the only subpolygroups of P', i.e., P' has no non-trivial proper subpolygroups.*

In [11], Jafarpour et al. described a method to get a polygroup from a group. Let $(G, \cdot)$ be a group, $a \notin G$, and $P_G = G \cup \{a\}$. Define "$\circ$" on P_G as follows:

(1) $a \circ a = e$;
(2) $e \circ x = x \circ e = x$ for all $x \in P_G$;
(3) $a \circ x = x \circ a = x$ for all $x \in P_G - \{e, a\}$;
(4) $x \circ y = x \cdot y$ for all $x, y \in G$ with $y \neq x^{-1}$;
(5) $x \circ x^{-1} = \{e, a\}$ for all $x \in P_G - \{e, a\}$.

Proposition 3.1. *[11] If $(G, \cdot)$ is a group, then $\langle P_G, \circ, e, ^{-1} \rangle$ is a polygroup where e and $^{-1}$ are the identity and inversion operations of G, respectively.*

Example 7. *Let $(\mathbb{Z}_3, +)$ be the group of integers modulo 3 under standard addition modulo 3. Then. $\langle P_{\mathbb{Z}_3}, \circ, 0, ^{-1} \rangle$ is a polygroup, and it is given by Table 5.*

Table 5. The associated polygroup $\langle P_{\mathbb{Z}_3}, \circ, 0, ^{-1} \rangle$.

$\circ$	0	1	2	a
0	0	1	2	a
1	1	2	$\{0, a\}$	1
2	2	$\{0, a\}$	1	2
a	a	1	2	0

Definition 3.2. *[9] Let $\langle P_1, \cdot_1, e_1, ^{-1} \rangle$, $\langle P_2, \cdot_2, e_2, ^{-1} \rangle$ be polygroups and $\psi : P_1 \to P_2$ be a function. Then,*

(1) ψ is a homomorphism if $\psi(x \cdot_1 y) \subseteq \psi(x) \cdot_2 \psi(y)$ for all $x, y \in P_1$, and ψ is a strong homomorphism if all these containments are equalities.
(2) ψ is an isomorphism if it is a bijective strong homomorphism. In this case, we say that P_1 and P_2 are isomorphic polygroups.

3.2. Subpolygroup lattice of P_G

We classify the subpolygroup lattice of P_G in relation to the subgroup lattice of G.

Theorem 3.1. *Let $(G, \cdot)$ be a group, P_G be its associated polygroup, and $\emptyset \neq N \subseteq P_G$. Then, N is a subpolygroup of P_G if and only if $N = \{e\}$ or $N = P_S$ for some subgroup S of G.*

Proof. Let S be a subgroup of G. Having $e \in S$ implies that $e \in P_S$, and hence, $P_S \neq \emptyset$. Let $x \in P_S$. Then,

$$x^{-1} = \begin{cases} a, & \text{if } x = a, \\ x^{-1}, & \text{if } x \in S, \end{cases} \in P_S.$$

For $x, y \in P_S$, we have

$$x \circ y = \begin{cases} xy, & \text{if } x, y \in S \text{ and } y \neq x^{-1}, \\ x, & \text{if } y = a \text{ and } x \in S, \\ y, & \text{if } x = a \text{ and } y \in S, \\ e, & \text{if } x = y = a, \\ \{e, a\}, & \text{if } x, y \in S \text{ and } y = x^{-1}, \end{cases} \subseteq P_S.$$

Thus, P_S is a subpolygroup of P_G.

Conversely, let $N \neq \{e\}$ be a subpolygroup of P_G. Then, there exists $x \neq e \in N$. Since N is a subpolygroup of P_G, it follows that $x^{-1} \in N$, and hence, $x \circ x^{-1} = \begin{cases} \{e, a\}, & \text{if } x \neq a, \\ \{a\}, & \text{otherwise}, \end{cases} \subseteq N$. Having $a \in N$ implies that we can write $N = S \cup \{a\}$ (with $a \notin S$). We need to show that S is a subgroup of G. Let $x \in S$. Then, $x \neq a \in N$, and hence, $x^{-1} \neq a \in N$. Thus, $x^{-1} \in S$. Let $x, y \in S$. Then, $x \circ y = \begin{cases} xy \neq a, & \text{if } y \neq x^{-1}, \\ \{e, a\}, & \text{otherwise}, \end{cases} \subseteq N$. Thus, $xy \in S$. □

Corollary 3.1. *Let n be a positive integer, $(\mathbb{Z}_n, +)$ be the group of integers modulo n under standard addition of integers modulo n, and S be a subpolygroup of the polygroup $P_{\mathbb{Z}_n}$. Then, $S = \{0\}$ or $S = P_{\langle k \rangle}$ for integers k that are divisors of n.*

Theorem 3.1 is important to construct the subpolygroup lattice of the associated polygroup P_G.

Example 8. *Let $(\mathbb{Z}_6, +)$ be the group of integers modulo 6 under standard addition modulo 6 and $P_{\mathbb{Z}_6}$ be its associated polygroup. Then, the subgroup lattice of $\mathbb{Z}_6$ is presented in Figure 1, and the subpolygroup lattice of $P_{\mathbb{Z}_6}$ is presented in Figure 2.*

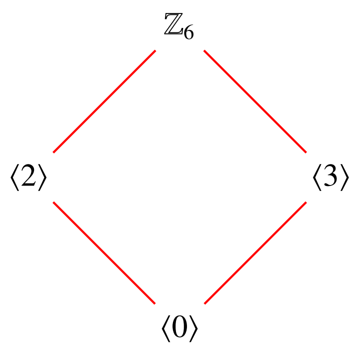

Figure 1. The subgroup lattice of the group $(\mathbb{Z}_6, +)$.

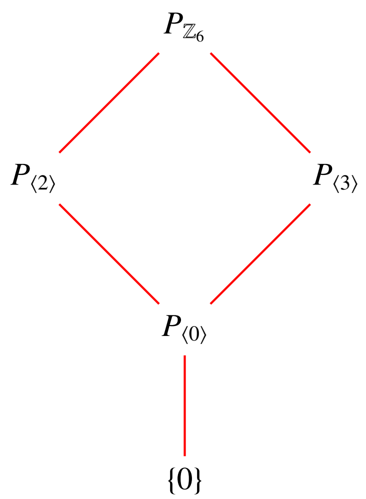

Figure 2. The subpolygroup lattice of the polygroup $P_{\mathbb{Z}_6}$.

Notation 1. *For a group G and a polygroup P, $L(G), L(P)$ denote the sets of subgroups of G and subpolygroups of P, respectively, and $|L(G)|, |L(P)|$ are their cardinalities.*

Corollary 3.2. *Let $(G, \cdot)$ be a finite group and $\langle P_G, \circ, e,^{-1}\rangle$ its associated polygroup. Then, $|L(P_G)| = |L(G)| + 1$.*

Lemma 3.1. *Let $(G, \cdot)$ be a group with subgroups H and K and $\langle P_G, \circ, e,^{-1}\rangle$ be the associated polygroup. Then, $HK = KH$ if and only if $P_H \circ P_K = P_K \circ P_H$.*

Proof. Let $HK = KH$. If $P_H \circ P_K = (H \cup \{a\}) \circ (K \cup \{a\}) = HK \cup \{a\}$ and $HK = KH$ then $P_H \circ P_K = KH \cup \{a\} = P_K \circ P_H$. Similarly, if $P_H \circ P_K = P_K \circ P_H$, then $HK = KH$. □

Corollary 3.3. *Let $(G, \cdot)$ be a group with subgroups H and K and $\langle P_G, \circ, e,^{-1}\rangle$ be the associated polygroup. Then, $HK \in L(G)$ if and only if $P_H \circ P_K \in L(P_G)$.*

Proof. This follows from Lemma 3.1. □

A lattice L is called modular if, for any $x, y, z \in L$ with $x \leq y$, $x \vee (y \wedge z) = y \wedge (x \vee z)$. For more details about lattice theory, we refer to [2]. We prove that under a certain condition, the lattice subpolygroup of the associated polygroup is modular.

Lemma 3.2. *Let $(G, \cdot)$ be a group and $\langle P_G, \circ, e,^{-1}\rangle$ be its associated polygroup. If $HK = KH$ for all subgroups H, K of G, then $(L(P_G), \wedge, \vee)$ is a lattice associated to P_G. Here,*

$$\begin{aligned} P_{H_i} \wedge P_{H_j} &= P_{H_i} \cap P_{H_j}, \\ P_{H_i} \vee P_{H_j} &= P_{H_i} \circ P_{H_j}, \quad i, j \in \{1, ..., |L(G)|\}. \end{aligned}$$

Proof. It is clear that $P_{H_i} \cap P_{H_j} \in L(P_G)$. Corollary 3.3 implies that $H_iH_j \in L(G)$ if and only if $P_{H_i} \circ P_{H_j} \in L(P_G)$. We need to prove that $P_{H_i} \cup P_{H_j} \subseteq P_{H_i} \circ P_{H_j}$. Let $x \in P_{H_i} \cup P_{H_j}$. Without loss of generality, we suppose that $x \in P_{H_i}$. Having $e \in P_{H_j}$, for any $j \in \{1, ..., |L(G)|\}$ implies that $x = x \circ e \subseteq P_{H_i} \circ P_{H_j}$. Now, we show that $P_{H_i} \circ P_{H_j}$ is the smallest polygroup which contains the subpolygroups P_{H_i} and P_{H_j}. Let $P_{H_k} \in L(P_G)$ such that $P_{H_i} \subseteq P_{H_k}$ and $P_{H_j} \subseteq P_{H_k}$. So, $P_{H_i} \circ P_{H_j} \subseteq P_{H_k} \circ P_{H_k} = P_{H_k}$. Therefore, $(L(P_G), \wedge, \vee)$ is a lattice associated to P_G. □

Theorem 3.2. *Let $(G, \cdot)$ be a group and $\langle P_G, \circ, e,^{-1} \rangle$ its associated polygroup. If $HK = KH$ for all subgroups H, K of G then $(L(P_G), \wedge, \vee)$ is a modular lattice associated to P_G.*

Proof. Let $P_{H_i}, P_{H_j}, P_{H_k} \in L(P_G)$, where $i, j, k \in \{1, ..., |L(G)|\}$ such that $P_{H_j} \subseteq P_{H_i}$. Any lattice satisfies the modularity inequality:

$$P_{H_j} \vee \left(P_{H_i} \wedge P_{H_j}\right) \subseteq P_{H_i} \wedge (P_{H_j} \vee P_{H_k}).$$

We show that $P_{H_i} \wedge (P_{H_j} \vee P_{H_k}) \subseteq P_{H_j} \vee \left(P_{H_i} \wedge P_{H_j}\right)$. Let $x \in P_{H_i} \wedge \left(P_{H_j} \vee P_{H_k}\right)$. Having $x \in P_{H_i}$ and $x \in P_{H_j} \circ P_{H_k}$ implies that there exist $y \in P_{H_j}, z \in P_{H_k}$ such that $x \in y \circ z$. The latter implies that $z \in y^{-1} \circ x$. Having $y^{-1} \in P_{H_j}$ implies that

$$z \in y^{-1} \circ x \subseteq P_{H_j} \circ P_{H_i} \subseteq P_{H_i} \circ P_{H_i} = P_{H_i} \text{ (because } P_{H_j} \subseteq P_{H_i}).$$

Having $z \in P_{H_k}$ and $z \in P_{H_i}$ implies that $x \in y \circ z \subseteq P_{H_j} \circ \left(P_{H_i} \cap P_{H_j}\right)$, and hence, $P_{H_i} \wedge (P_{H_j} \vee P_{H_k}) \subseteq P_{H_j} \vee \left(P_{H_i} \wedge P_{H_j}\right)$ for any $P_{H_i}, P_{H_j}, P_{H_k} \in L(P_G)$. □

Lemma 3.3. *Let G, G' be any finite non-trivial groups with identities e, e', respectively. Introduce elements a, b such that $\{a, b\} \cap (G \cup G') = \emptyset$ and $P_G = G \cup \{a\}$, $P_{G'} = G' \cup \{b\}$. Then, $A_1 = \{(e, e')\}$, $A_2 = \{(e, e'), (a, b)\}$, $A_3 = \{e\} \times P_{S'}$ (where S' is a subgroup of G'), $A_4 = P_S \times \{e'\}$ (where S is a subgroup of G), $A_5 = P_S \times P_{S'}$ (where S, S' are subgroups of G, G' respectively) are subpolygroups of $P_G \times P_{G'}$.*

Proof. The proof is straightforward. □

In group theory, it is well known that if G and G' are finite groups with coprime orders and A is a subgroup of $G \times G'$ then there exist subgroups S, S' of G, G' respectively such that $A = S \times S'$. This fact from group theory may not hold for polygroups. We illustrate this remark via Example 9.

Example 9. *Let G, G' be any non-trivial groups with identities e, e' respectively. Introduce elements a, b such that $\{a, b\} \cap (G \cup G') = \emptyset$ and $P_G = G \cup \{a\}$, $P_{G'} = G' \cup \{b\}$. Then, $M = \{(e, e'), (a, b)\}$, represented by Table 6, is a subpolygroup of $P_G \times P_{G'}$.*

Table 6. $M = \{(e, e'), (a, b)\}$.

$\star$	(e, e')	(a, b)
(e, e')	(e, e')	(a, b)
(a, b)	(a, b)	(e, e')

It is clear that M can not be written as a Cartesian product of two subpolygroups.

4. Subpolygroup commutativity degree of finite polygroups

In this section, inspired by the definition of the subgroup commutativity degree of finite groups [21], we define the subpolygroup commutativity degree of finite polygroups. First, we present some general results. Then, we make a complete study on special polygroups that are associated to finite groups. We find an explicit formula for the subpolygroup commutativity degree of these polygroups, and we present some interesting results.

Definition 4.1. *[1] Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup. Then,*

$$sd(P) = \frac{|\{(H, K) \in L(P)^2 : H \cdot K = K \cdot H\}|}{|L(P)|^2}.$$

It is clear that $0 < sd(P) \leq 1$.

Remark 5. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup. Then, $sd(P) = 1$ if P is commutative, or every two subpolygroups commute.*

Example 10. *Let $\langle P', \cdot, n_0, ^{-1} \rangle$ be the non-canonical hypergroup in Example 5. Since $\{n_0\}, P'$ are the only subpolygroups of P', it follows that $sd(P') = 1$.*

Proposition 4.1. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and H, K be subpolygroups of P. If $H \cdot K = K \cdot H$, then $H \cdot K \in L(P)$.*

Proof. The proof is straightforward. □

Proposition 4.2. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup. Then, $sd(P) = 1$ or $sd(P) \geq \frac{9}{|L(P)|^2}$.*

Proof. The proof is similar to that of Proposition 2.1. □

Corollary 4.1. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup. Then, the following statements hold.*

(1) If $|L(P)| \leq 3$, then $sd(P) = 1$.
(2) If $|L(P)| \leq 9$, then $sd(P) \geq \frac{1}{2}$.

Proof. This follows from Proposition 4.2. □

Proposition 4.3. *Let $\langle P_1, \cdot_1, e_1, ^{-1} \rangle$ and $\langle P_2, \cdot_2, e_2, ^{-1} \rangle$ be isomorphic finite polygroups. Then, $sd(P_1) = sd(P_2)$.*

Proof. The proof is straightforward. □

Remark 6. *The converse of Proposition 4.3 may not hold. The polygroups in Examples 4 and 5 have the same subpolygroup commutativity degree (which is equal to 1), but they are non-isomorphic polygroups.*

Definition 4.2. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup with distinct subpolygroups $H_1, \ldots, H_k$ where k is a positive integer. Then, the subpolygroup commutativity table of P is defined via Table 7.*

Table 7. Subpolygroup commutativity table of P.

$\cdot$	H_1	H_2	$\ldots$	H_k
H_1	H_{11}	H_{12}	$\ldots$	H_{1k}
H_2	H_{21}	H_{22}	$\ldots$	H_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
H_k	H_{k1}	H_{k2}	$\ldots$	H_{kk}

Here for all $1 \leq i, j \leq k$, $H_{ij} = \begin{cases} 1, & \text{if } H_i \cdot H_j = H_j \cdot H_i, \\ 0, & \text{otherwise.} \end{cases}$

Remark 7. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup with subpolygroup commutativity table (H_{ij}). Then,*

$$sd(P) = \frac{\sum_{j=1}^{k} \sum_{i=1}^{k} H_{ij}}{|L(P)|^2}.$$

Example 11. *Let S_3 be the symmetric group on three letters and P_{S_3} be its associated polygroup. Then, the subpolygroup commutativity table of P_{S_3} is given in Table 8.*

Table 8. Subpolygroup commutativity table of P_{S_3}.

$\circ$	$\{(1)\}$	$P_{\{(1)\}}$	$P_{\{(1),(12)\}}$	$P_{\{(1),(13)\}}$	$P_{\{(1),(23)\}}$	$P_{\{(1),(123),(132)\}}$	P_{S_3}
$\{(1)\}$	1	1	1	1	1	1	1
$P_{\{(1)\}}$	1	1	1	1	1	1	1
$P_{\{(1),(12)\}}$	1	1	1	0	0	1	1
$P_{\{(1),(13)\}}$	1	1	0	1	0	1	1
$P_{\{(1),(23)\}}$	1	1	0	0	1	1	1
$P_{\{(1),(123),(132)\}}$	1	1	1	1	1	1	1
P_{S_3}	1	1	1	1	1	1	1

It is clear that $sd(P_{S_3}) = \frac{43}{49}$.

Proposition 4.4. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup. Then, $sd(P) \geq \frac{5|L(P)|-6}{|L(P)|^2}$.*

Proof. Using the subpolygroup commutativity table, the first row and column, the last row and column and the diagonal are entirely 1's. So, we have at least $5|L(P)|-6$ of 1's after removing the repetitive 1's. Therefore, $sd(P) \geq \frac{5|L(P)|-6}{|L(P)|^2}$. □

Next, we find a formula for the subpolygroup commutativity degree of the polygroup P_G associated to a finite group G.

Theorem 4.1. *Let $(G, \cdot)$ be a finite group and P_G be its associated polygroup. Then,*

$$sd(P_G) = \frac{|L(G)|^2 sd(G) + 2|L(G)| + 1}{(|L(G)| + 1)^2}.$$

Proof. We have $sd(P_G) = \frac{|M(P_G)|}{|L(P_G)|^2}$ where $M(P_G) = \{(A, B) \in L^2(P_G) : A \circ B = B \circ A\}$ and $|L(P_G)| = |L(G)| + 1$. Lemma 3.1 implies that $M(P_G) = M_0 \cup M_1 \cup M_2$ where $M_0 = \{(A, \{e\}) : A \in L(P_G)\}$, $M_1 = \{(\{e\}, A) : A \in L(P_G)\}$ and $M_2 = \{(P_H, P_K) : HK = KH\}$. Having $|M_0| = |M_1| = |L(G)| + 1$, $M_0 \cap M_1 = \{(e, e)\}$, $M_0 \cap M_2 = M_1 \cap M_2 = \emptyset$, and $|M_2| = |L(G)|^2 sd(G)$ imply that

$$sd(P_G) = \frac{|L(G)|^2 sd(G) + 2|L(G)| + 1}{(|L(G)| + 1)^2}.$$

□

Example 12. *Let S_3 be the symmetric group on three letters and P_{S_3} be its associated polygroup. Having $sd(S_3) = \frac{5}{6}$ and $|L(S_3)| = 6$ implies that $sd(P_{S_3}) = \frac{30+2(6)+1}{7^2} = \frac{43}{49}$.*

Corollary 4.2. *Let $n = 2^\alpha n'$ with n' odd, D_n be the dihedral group, and P_{D_n} be its associated polygroup. Then,*

$$sd(P_{D_n}) = \frac{\tau(n)^2 + 2\tau(n)\sigma(n) + [(\alpha-1)2^{\alpha+3} + 9]g(n') + 2\tau(n) + 2\sigma(n) + 1}{(\tau(n) + \sigma(n) + 1)^2}.$$

Proof. The proof follows from Theorems 2.2 and 4.1. □

Corollary 4.3. *Let m be a positive integer, and $P_{D_{2^{m-1}}}$, $P_{Q_{2^m}}$, $P_{S_{2^m}}$ be the associated polygroups of $D_{2^{m-1}}$, Q_{2^m}, and S_{2^m}, respectively. Then,*

$$sd(P_{D_{2^{m-1}}}) = \frac{(m-2)2^{m+2} + (m+1)2^{m+1} + m^2 + 8}{(m+2^m)^2}, \qquad m \geq 2,$$

$$sd(P_{Q_{2^m}}) = \frac{(m-3)2^{m+1} + (m+1)2^m + m^2 + 8}{(m+2^{m-1})^2}, \qquad m \geq 2,$$

$$sd(P_{S_{2^m}}) = \frac{(m-3)2^{m+1} + m2^m + (3m+1)2^{m-1} + m^2 + 8}{(m+3\cdot 2^{m-2})^2}, \qquad m \geq 4.$$

In particular, $\lim_{m\to\infty} sd(P_{D_{2^{m-1}}}) = \lim_{m\to\infty} sd(P_{Q_{2^m}}) = \lim_{m\to\infty} sd(P_{S_{2^m}}) = 0$.

Proof. The proof follows from Theorems 2.3 and 4.1. □

Proposition 4.5. *Let $(G,\cdot)$ be a finite group and P_G be its associated polygroup. Then, $sd(P_G) \geq sd(G)$. Moreover, the equality holds if and only if $sd(G) = 1$.*

Proof. Having

$$sd(P_G) - sd(G) = \frac{(1 - sd(G))(2L(G) + 1)}{(|L(G)| + 1)^2} \geq 0$$

implies that $sd(P_G) \geq sd(G)$. □

Proposition 4.6. *Let $(G,\cdot)$ be a finite group and P_G be its associated polygroup. Then, $sd(P_G) \leq \frac{sd(G)+1}{2}$.*

Proof. If $|L(G)| \leq 3$, then $sd(G) = sd(P_G) = 1$, and the inequality holds. If $|L(G)|\rangle 3$, then $|L(G)|^2 - 2|L(G)| - 1\rangle 0$. Having $|L(G)|^2 - 2|L(G)| - 1 \geq 0$ implies that

$$sd(P_G) - \frac{sd(G) + 1}{2} = \frac{(sd(G) - 1)(|L(G)|^2 - 2|L(G)| - 1)}{2(|L(G)| + 1)^2} \leq 0,$$

and hence, $sd(P_G) \leq \frac{sd(G)+1}{2}$. □

5. Conclusions

This paper dealt with polygroup probability by introducing the subpolygroup commutativity degree of finite polygroups. Some basic results were elaborated, and some examples were presented. Moreover, an explicit formula for the subpolygroup commutativity degree of a particular class of finite polygroups was derived and applied to the polygroups associated to the dihedral group, to the quasi-dihedral group and to the generalized quaternion groups.

For future research, we raise some open problems.

(1) Find an explicit formula for the subpolygroup commutativity degree of other classes of finite polygroups.
(2) Given finite polygroups $P_1, \ldots, P_n$, find a necessary and sufficient condition so that $sd(P_1 \times \ldots \times P_n) = sd(P_1) \ldots sd(P_n)$.
(3) For a finite polygroup $\langle P, \cdot, e,^{-1} \rangle$, find a relationship between its commutativity degree and its subpolygroup commutativity degree.

Use of AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. Al-Tahan, A. Sonea, B. Davvaz, Subpolygroup commutativity degree of finite polygroups, *Proceedings of the 3rd International Conference on Symmetry*, 2021, 8–13.
2. G. Birhoff, *Lattice theory*, Providence: American Mathematical Soc., 1967.
3. P. Bonansinga, Sugli ipergruppi quasicanonici, *Atti Soc. Pelor. Sc. Fis. Mat. Nat.*, **27** (1981), 9–17.
4. P. Bonansinga, Ipergruppi debolmente quasicanonici, *Atti Sem. Mat. Fis. Univ. Modena*, **30** (1981), 286–298.
5. S. Comer, Polygroups derived from cogroups, *J. Algebra*, **89** (1984), 397–405. http://dx.doi.org/10.1016/0021-8693(84)90225-4
6. P. Corsini, V. Leoreanu, *Applications of hyperstructures theory*, New York: Springer, 2003. http://dx.doi.org/10.1007/978-1-4757-3714-1
7. A. Das, R. Nath, On generalized relative commutativity degree of a finite group, *Int. Electron. J. Algeb.*, **7** (2010), 140–151.
8. A. Das, R. Nath, M. Pournaki, A survey on the estimation of commutativity in finite groups, *SE Asian Bull. Math.*, **37** (2013), 161–180.
9. B. Davvaz, *Polygroup theory and related systems*, Singapore: World Scientific, 2013.

10. A. Erfanian, R. Rezaei, P. Lescot, On the relative commutativity degree of a subgroup of a finite group, *Commun. Algebra*, **35** (2007), 4183–4197. http://dx.doi.org/10.1080/00927870701545044
11. M. Jafarpour, H. Aghabozorgt, B. Davvaz, On nilpotent and solvable polygroups, *Bull. Iran. Math. Soc.*, **39** (2013), 487–499.
12. P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra*, **177** (1995), 847–869. http://dx.doi.org/10.1006/jabr.1995.1331
13. P. Lescot, Central extensions and commutativity degree, *Commun. Algebra*, **29** (2001), 4451–4460. http://dx.doi.org/10.1081/AGB-100106768
14. F. Marty, Sur une generalization de la notion de group, *Proceedings of 8th congress Math*, 1934, 45–49.
15. J. Mittas, Hypergroupes canoniques hypervalues, *CR Acad. Sci. Paris*, **271** (1970), 4–7.
16. J. Mittas, Hypergroupes canoniques, *Math. Balkanica*, **2** (1972), 165–179.
17. D. Rusin, What is the probability that two elements of a finite group commute? *Pac. J. Math.*, **82** (1979), 237–247. http://dx.doi.org/10.2140/pjm.1979.82.237
18. A. Sonea, HX-groups associated with the dihedral group D_n, *J. Mult.-Valued Log. S.*, **33** (2019),11–26.
19. A. Sonea, New aspects in polygroup theory, *An. St. Univ. Ovidius Constanta*, **28** (2020), 241–254. http://dx.doi.org/10.2478/auom-2020-0044
20. A. Sonea, I. Cristea, The class equation and the commutativity degree for complete hypergroups, *Mathematics*, **8** (2020), 2253. http://dx.doi.org/10.3390/math8122253
21. M. Tarnauceanu, Subgroup commutativity degrees of finite groups, *J. Algebra*, **321** (2009), 2508–2520. http://dx.doi.org/10.1016/j.jalgebra.2009.02.010