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Research article

Global behavior of a max-type system of difference equations of the second order with four variables and period-two parameters

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Abstract: In this paper, we study global behavior of the following max-type system of difference equations of the second order with four variables and period-two parameters

$$\begin{cases} x_n = \max\left\{A_n, \frac{z_{n-1}}{y_{n-2}}\right\}, \\ y_n = \max\left\{B_n, \frac{w_{n-1}}{x_{n-2}}\right\}, \\ z_n = \max\left\{C_n, \frac{x_{n-1}}{w_{n-2}}\right\}, \\ w_n = \max\left\{D_n, \frac{y_{n-1}}{z_{n-2}}\right\}, \end{cases} \quad n \in \{0, 1, 2, \cdots\},$$

where $A_n, B_n, C_n, D_n \in (0, +\infty)$ are periodic sequences with period 2 and the initial values $x_{-i}, y_{-i}, z_{-i}, w_{-i} \in (0, +\infty)$ $(1 \le i \le 2)$. We show that if $\min\{A_0C_1, B_0D_1, A_1C_0, B_1D_0\} < 1$, then this system has unbounded solutions. Also, if $\min\{A_0C_1, B_0D_1, A_1C_0, B_1D_0\} \ge 1$, then every solution of this system is eventually periodic with period 4.

Keywords: max-type system of difference equations; solution; eventual periodicity
Mathematics Subject Classification: 39A10, 39A11

1. Introduction

The purpose of this paper is to study the global behavior of the following max-type system of difference equations of the second order with four variables and period-two parameters

$$\begin{cases} x_n = \max\left\{A_n, \frac{z_{n-1}}{y_{n-2}}\right\}, \\ y_n = \max\left\{B_n, \frac{w_{n-1}}{x_{n-2}}\right\}, \\ z_n = \max\left\{C_n, \frac{x_{n-1}}{w_{n-2}}\right\}, \\ w_n = \max\left\{D_n, \frac{y_{n-1}}{z_{n-2}}\right\}, \end{cases} \quad n \in \mathbf{N}_0 \equiv \{0, 1, 2, \cdots\}, \tag{1.1}$$

where $A_n, B_n, C_n, D_n \in \mathbf{R}^+ \equiv (0, +\infty)$ are periodic sequences with period 2 and the initial values $x_{-i}, y_{-i}, z_{-i}, w_{-i} \in \mathbf{R}^+$ $(1 \leq i \leq 2)$. To do this we will use some methods and ideas which stems from [1,2]. For a more complex variant of the method, see [3]. A solution $\{(x_n, y_n, z_n, w_n)\}_{n=-2}^{+\infty}$ of (1.1) is called an eventually periodic solution with period T if there exists $m \in \mathbf{N}$ such that $(x_n, y_n, z_n, w_n) = (x_{n+T}, y_{n+T}, z_{n+T}, w_{n+T})$ holds for all $n \geq m$.

When $x_n = y_n$ and $z_n = w_n$ and $A_0 = A_1 = B_0 = B_1 = \alpha$ and $C_0 = C_1 = D_0 = D_1 = \beta$, (1.1) reduces to following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{\alpha, \frac{z_{n-1}}{x_{n-2}}\right\}, \\ z_n = \max\left\{\beta, \frac{x_{n-1}}{z_{n-2}}\right\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{1.2}$$

Fotiades and Papaschinopoulos in [4] investigated the global behavior of (1.2) and showed that every positive solution of (1.2) is eventually periodic.

When $x_n = z_n$ and $y_n = w_n$ and $A_n = C_n$ and $B_n = D_n$, (1.1) reduces to following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{A_n, \frac{y_{n-1}}{x_{n-2}}\right\}, \\ y_n = \max\left\{B_n, \frac{x_{n-1}}{y_{n-2}}\right\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{1.3}$$

Su et al. in [5] investigated the periodicity of (1.3) and showed that every solution of (1.3) is eventually periodic.

In 2020, Su et al. [6] studied the global behavior of positive solutions of the following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{A, \frac{y_{n-t}}{x_{n-s}}\right\}, \\ y_n = \max\left\{B, \frac{x_{n-t}}{y_{n-s}}\right\}, \end{cases} \quad n \in \mathbf{N}_0,$$

where $A, B \in \mathbf{R}^+$.

In 2015, Yazlik et al. [7] studied the periodicity of positive solutions of the max-type system of difference equations

$$\begin{cases} x_n = \max\left\{\frac{1}{x_{n-1}}, \min\left\{1, \frac{p}{y_{n-1}}\right\}\right\}, \\ y_n = \max\left\{\frac{1}{y_{n-1}}, \min\left\{1, \frac{p}{x_{n-1}}\right\}\right\}, \end{cases} \quad n \in \mathbf{N_0}, \tag{1.4}$$

where $p \in \mathbf{R}^+$ and obtained in an elegant way the general solution of (1.4).

In 2016, Sun and Xi [8], inspired by the research in [5], studied the following more general system

$$\begin{cases} x_n = \max\left\{\frac{1}{x_{n-m}}, \min\left\{1, \frac{p}{y_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\frac{1}{y_{n-m}}, \min\left\{1, \frac{q}{x_{n-t}}\right\}\right\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.5}$$

where $p, q \in \mathbf{R}^+$, $m, r, t \in \mathbf{N} \equiv \{1, 2, \cdots\}$ and the initial conditions $x_{-i}, y_{-i} \in \mathbf{R}^+$ $(1 \leq i \leq s)$ with $s = \max\{m, r, t\}$ and showed that every positive solution of (1.5) is eventually periodic with period $2m$.

In [9], Stević studied the boundedness character and global attractivity of the following symmetric max-type system of difference equations

$$\begin{cases} x_n = \max\left\{B, \frac{y_{n-1}^p}{x_{n-2}^p}\right\}, \\ y_n = \max\left\{B, \frac{x_{n-1}^p}{y_{n-2}^p}\right\}, \end{cases} \quad n \in \mathbf{N}_0,$$

where $B, p \in \mathbf{R}^+$ and the initial conditions $x_{-i}, y_{-i} \in \mathbf{R}^+$ $(1 \leq i \leq 2)$.

In 2014, motivated by results in [9], Stević [10] further study the behavior of the following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{B, \frac{y_{n-1}^p}{z_{n-2}^p}\right\}, \\ y_n = \max\left\{B, \frac{z_{n-1}^p}{x_{n-2}^p}\right\}, \\ z_n = \max\left\{B, \frac{x_{n-1}^p}{y_{n-2}^p}\right\}. \end{cases} \quad n \in \mathbf{N}_0, \tag{1.6}$$

where $B, p \in \mathbf{R}^+$ and the initial conditions $x_{-i}, y_{-i}, z_{-i} \in \mathbf{R}^+$ $(1 \leq i \leq 2)$, and showed that system (1.6) is permanent when $p \in (0, 4)$.

For more many results for global behavior, eventual periodicity and the boundedness character of positive solutions of max-type difference equations and systems, please readers refer to [11–30] and the related references therein.

2. Main results and proofs

In this section, we study the global behavior of system (1.1). For any $n \geq -1$, write

$$\begin{cases} x_{2n} = A_{2n} X_n, \\ y_{2n} = B_{2n} Y_n, \\ z_{2n} = C_{2n} Z_n, \\ w_{2n} = D_{2n} W_n, \\ x_{2n+1} = A_{2n+1} X'_n, \\ y_{2n+1} = B_{2n+1} Y'_n, \\ z_{2n+1} = C_{2n+1} Z'_n, \\ w_{2n+1} = D_{2n+1} W'_n. \end{cases}$$

Then, (1.1) reduces to the following system

$$\begin{cases} X_n = \max\left\{1, \frac{C_{2n-1}Z'_{n-1}}{A_{2n}B_{2n}Y_{n-1}}\right\}, \\ Y_n = \max\left\{1, \frac{D_{2n-1}W'_{n-1}}{B_{2n}A_{2n}X_{n-1}}\right\}, \\ Z'_n = \max\left\{1, \frac{A_{2n}X_n}{C_{2n+1}D_{2n+1}W'_{n-1}}\right\}, \\ W'_n = \max\left\{1, \frac{B_{2n}Y_n}{D_{2n+1}C_{2n+1}Z'_{n-1}}\right\}, \\ Z_n = \max\left\{1, \frac{A_{2n-1}X'_{n-1}}{C_{2n}D_{2n}W_{n-1}}\right\}, \\ W_n = \max\left\{1, \frac{B_{2n-1}Y'_{n-1}}{D_{2n}C_{2n}Z_{n-1}}\right\}, \\ X'_n = \max\left\{1, \frac{C_{2n}Z_n}{A_{2n+1}B_{2n+1}Y'_{n-1}}\right\}, \\ Y'_n = \max\left\{1, \frac{D_{2n}W_n}{B_{2n+1}A_{2n+1}X'_{n-1}}\right\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{2.1}$$

From (2.1) we see that it suffices to consider the global behavior of positive solutions of the following system

$$\begin{cases} u_n = \max\left\{1, \frac{bv_{n-1}}{aAU_{n-1}}\right\}, \\ U_n = \max\left\{1, \frac{BV_{n-1}}{aAu_{n-1}}\right\}, \\ v_n = \max\left\{1, \frac{au_n}{bBV_{n-1}}\right\}, \\ V_n = \max\left\{1, \frac{AU_n}{bBv_{n-1}}\right\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{2.2}$$

where $a, b, A, B \in \mathbf{R}^+$, the initial conditions $u_{-1}, U_{-1}, v_{-1}, V_{-1} \in \mathbf{R}^+$. If $(u_n, U_n, v_n, V_n, a, A, b, B) = (X_n, Y_n, Z'_n, W'_n, A_{2n}, B_{2n}, C_{2n-1}, D_{2n-1})$, then (2.2) is the first four equations of (2.1). If $(u_n, U_n, v_n, V_n, a, A, b, B) = (Z_n, W_n, X'_n, Y'_n, C_{2n}, D_{2n}, A_{2n-1}, B_{2n-1})$, then (2.2) is the next four equations of (2.1). In the following without loss of generality we assume $\mathbf{a} \leq \mathbf{A}$ and $\mathbf{b} \leq \mathbf{B}$. Let $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{\infty}$ be a positive solution of (2.2).

Proposition 2.1. If $ab < 1$, then there exists a solution $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{\infty}$ of (2.2) such that $u_n = v_n = 1$ for any $n \geq -1$ and $\lim_{n\longrightarrow\infty} U_n = \lim_{n\longrightarrow\infty} V_n = \infty$.

Proof. Let $u_{-1} = v_{-1} = 1$ and $U_{-1} = V_{-1} = \max\{\frac{b}{aA}, \frac{aA}{B}, \frac{a}{bB}\} + 1$. Then, from (2.2) we have

$$\begin{cases} u_0 = \max\left\{1, \frac{bv_{-1}}{aAU_{-1}}\right\} = 1, \\ U_0 = \max\left\{1, \frac{BV_{-1}}{aAu_{-1}}\right\} = \frac{BV_{-1}}{aA}, \\ v_0 = \max\left\{1, \frac{au_0}{bBV_{-1}}\right\} = 1, \\ V_0 = \max\left\{1, \frac{AU_0}{bBv_{-1}}\right\} = \frac{V_{-1}}{ab}, \end{cases}$$

and

$$\begin{cases} u_1 = \max\left\{1, \frac{bv_0}{aAU_0}\right\} = \max\left\{1, \frac{b}{BV_{-1}}\right\} = 1, \\ U_1 = \max\left\{1, \frac{BV_0}{aAu_0}\right\} = \max\left\{1, \frac{BV_{-1}}{aAab}\right\} = \frac{BV_{-1}}{aAab}, \\ v_1 = \max\left\{1, \frac{au_1}{bBV_0}\right\} = \max\left\{1, \frac{aab}{bBV_{-1}}\right\} = 1, \\ V_1 = \max\left\{1, \frac{AU_1}{bBv_0}\right\} = \max\left\{1, \frac{V_{-1}}{(ab)^2}\right\} = \frac{V_{-1}}{(ab)^2}. \end{cases}$$

Suppose that for some $k \in \mathbf{N}$, we have

$$\begin{cases} u_k = 1, \\ U_k = \frac{BV_{-1}}{aA(ab)^k}, \\ v_k = 1, \\ V_k = \frac{V_{-1}}{(ab)^{k+1}}. \end{cases}$$

Then,

$$\begin{cases} u_{k+1} = \max\left\{1, \frac{bv_k}{aAU_k}\right\} = \max\left\{1, \frac{b(ab)^k}{BV_{-1}}\right\} = 1, \\ U_{k+1} = \max\left\{1, \frac{BV_k}{aAu_k}\right\} = \max\left\{1, \frac{BV_{-1}}{aA(ab)^{k+1}}\right\} = \frac{BV_{-1}}{aA(ab)^{k+1}}, \\ v_{k+1} = \max\left\{1, \frac{au_{k+1}}{bBV_k}\right\} = \max\left\{1, \frac{a(ab)^{k+1}}{bBV_{-1}}\right\} = 1, \\ V_{k+1} = \max\left\{1, \frac{AU_{k+1}}{bBv_k}\right\} = \max\left\{1, \frac{V_{-1}}{(ab)^{k+2}}\right\} = \frac{V_{-1}}{(ab)^{k+2}}. \end{cases}$$

By mathematical induction, we can obtain the conclusion of Proposition 2.1. The proof is complete.

Now, we assume that $ab \geq 1$. Then, from (2.2) it follows that

$$\begin{cases} u_n = \max\left\{1, \frac{bv_{n-1}}{aAU_{n-1}}\right\}, \\ U_n = \max\left\{1, \frac{BV_{n-1}}{aAu_{n-1}}\right\}, \\ v_n = \max\left\{1, \frac{a}{bBV_{n-1}}, \frac{v_{n-1}}{ABU_{n-1}V_{n-1}}\right\}, \\ V_n = \max\left\{1, \frac{A}{bBv_{n-1}}, \frac{V_{n-1}}{abu_{n-1}v_{n-1}}\right\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{2.3}$$

Lemma 2.1. The following statements hold:

(1) For any $n \in \mathbf{N}_0$,

$$u_n,\ U_n,\ v_n,\ V_n \in [1, +\infty). \tag{2.4}$$

(2) If $ab \geq 1$, then for any $k \in \mathbf{N}$ and $n \geq k+2$,

$$\begin{cases} u_n = \max\left\{1, \frac{b}{aAU_{n-1}}, \frac{bv_k}{aA(AB)^{n-k-1}U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\}, \\ U_n = \max\left\{1, \frac{B}{aAu_{n-1}}, \frac{BV_k}{aA(ab)^{n-k-1}u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\}, \\ v_n = \max\left\{1, \frac{a}{bBV_{n-1}}, \frac{v_k}{(AB)^{n-k}U_{n-1}V_{n-1}\cdots U_kV_k}\right\}, \\ V_n = \max\left\{1, \frac{A}{bBv_{n-1}}, \frac{V_k}{(ab)^{n-k}u_{n-1}v_{n-1}\cdots u_kv_k}\right\}. \end{cases} \tag{2.5}$$

(3) If $ab \geq 1$, then for any $k \in \mathbf{N}$ and $n \geq k+4$,

$$\begin{cases} 1 \leq v_n \leq v_{n-2}, \\ 1 \leq V_n \leq \frac{A}{a}V_{n-2}, \\ 1 \leq u_n \leq \max\{1, \frac{b}{B}u_{n-2}, \frac{bv_k}{aA(AB)^{n-k-1}}\}, \\ 1 \leq U_n \leq \max\{1, \frac{B}{b}U_{n-2}, \frac{BV_k}{aA(ab)^{n-k-1}}\}. \end{cases} \tag{2.6}$$

Proof. (1) It follows from (2.2).

(2) Since $AB \geq ab \geq 1$, it follows from (2.2) and (2.3) that for any $k \in \mathbf{N}$ and $n \geq k+2$,

$$\begin{aligned} u_n &= \max\left\{1, \frac{bv_{n-1}}{aAU_{n-1}}\right\} \\ &= \max\left\{1, \frac{b}{aAU_{n-1}} \max\left\{1, \frac{a}{bBV_{n-2}}, \frac{v_{n-2}}{ABU_{n-2}V_{n-2}}\right\}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{1, \frac{b}{aAU_{n-1}}, \frac{bv_{n-2}}{ABaAU_{n-1}U_{n-2}V_{n-2}}\right\} \\
&= \max\left\{1, \frac{b}{aAU_{n-1}}, \frac{b}{ABaAU_{n-1}U_{n-2}V_{n-2}} \max\left\{1, \frac{a}{bBV_{n-1}}, \frac{v_{n-3}}{ABU_{n-3}V_{n-3}}\right\}\right\} \\
&= \max\left\{1, \frac{b}{aAU_{n-1}}, \frac{bv_{n-3}}{(AB)^2aAU_{n-1}U_{n-2}V_{n-2}U_{n-3}V_{n-3}}\right\} \\
&\cdots \\
&= \max\left\{1, \frac{b}{aAU_{n-1}}, \frac{bv_k}{aA(AB)^{n-k-1}U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\}.
\end{aligned}
$$

In a similar way, also we can obtain the other three formulas.

(3) By (2.5) one has that for any $k \in \mathbf{N}$ and $n \geq k+2$,

$$
\begin{cases}
u_n \geq \frac{b}{aAU_{n-1}}, \\
U_n \geq \frac{B}{aAu_{n-1}}, \\
v_n \geq \frac{a}{bBV_{n-1}}, \\
V_n \geq \frac{A}{bBv_{n-1}},
\end{cases}
$$

from which and (2.4) it follows that for any $n \geq k+4$,

$$
\begin{cases}
1 \leq u_n \leq \max\{1, \frac{b}{B}u_{n-2}, \frac{bv_k}{aA(AB)^{n-k-1}}\}, \\
1 \leq U_n \leq \max\{1, \frac{B}{b}U_{n-2}, \frac{BV_k}{aA(ab)^{n-k-1}}\}, \\
1 \leq v_n \leq \max\left\{1, \frac{av_{n-2}}{A}, v_{n-2}\right\} = v_{n-2}, \\
1 \leq V_n \leq \max\left\{1, \frac{AV_{n-2}}{a}, V_{n-2}\right\} = \frac{AV_{n-2}}{a}.
\end{cases}
$$

The proof is complete.

Proposition 2.2. If $ab = AB = 1$, then $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2.

Proof. By the assumption we see $a = A$ and $b = B$. By (2.5) we see that for any $k \in \mathbf{N}$ and $n \geq k+2$,

$$
\begin{cases}
u_n = \max\left\{1, \frac{b^3}{U_{n-1}}, \frac{b^3v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\}, \\
U_n = \max\left\{1, \frac{b^3}{u_{n-1}}, \frac{b^3V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\}, \\
v_n = \max\left\{1, \frac{a^3}{V_{n-1}}, \frac{v_k}{U_{n-1}V_{n-1}\cdots U_kV_k}\right\}, \\
V_n = \max\left\{1, \frac{a^3}{v_{n-1}}, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_kv_k}\right\}.
\end{cases}
\tag{2.7}
$$

(1) If $a = b = 1$, then it follows from (2.7) and (2.4) that for any $n \geq k+4$,

$$
\begin{cases}
u_n = \max\left\{1, \frac{v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\} \leq \max\left\{1, \frac{v_k}{U_{n-2}U_{n-3}V_{n-3}\cdots U_kV_k}\right\} = u_{n-1}, \\
U_n = \max\left\{1, \frac{V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\} \leq U_{n-1}, \\
v_n = \max\left\{1, \frac{v_k}{U_{n-1}V_{n-1}\cdots U_kV_k}\right\} \leq v_{n-1}, \\
V_n = \max\left\{1, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_kv_k}\right\} \leq V_{n-1}.
\end{cases}
\tag{2.8}
$$

We claim that $v_n = 1$ for any $n \geq 6$ or $V_n = 1$ for any $n \geq 6$. Indeed, if $v_n > 1$ for some $n \geq 6$ and $V_m > 1$ for some $m \geq 6$, then

$$
v_n = \frac{v_1}{U_{n-1}V_{n-1}\cdots U_1V_1} > 1, \quad V_m = \frac{V_1}{u_{m-1}v_{m-1}\cdots u_1v_1} > 1,
$$

which implies

$$1 \geq \frac{v_1}{U_{n-1}V_{n-1}\cdots U_1V_1}\frac{V_1}{u_{m-1}v_{m-1}\cdots u_1v_1} = V_m v_n > 1.$$

A contradiction.

If $v_n = 1$ for any $n \geq 6$, then by (2.8) we see $u_n = 1$ for any $n \geq 10$, which implies $U_n = V_n = V_{10}$. If $V_n = 1$ for any $n \geq 6$, then by (2.8) we see $U_n = 1$ for any $n \geq 10$, which implies $v_n = u_n = v_{10}$. Then, $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2.

(2) If $a < 1 < b$, then it follows from (2.7) that for any $n \geq k+4$,

$$\begin{cases} u_n = \max\left\{1, \frac{b^3}{U_{n-1}}, \frac{b^3 v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\}, \\ U_n = \max\left\{1, \frac{b^3}{u_{n-1}}, \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\}, \\ v_n = \max\left\{1, \frac{v_k}{U_{n-1}V_{n-1}\cdots U_kV_k}\right\} \leq v_{n-1}, \\ V_n = \max\left\{1, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_kv_k}\right\} \leq V_{n-1}. \end{cases} \tag{2.9}$$

It is easy to verify $v_n = 1$ for any $n \geq 6$ or $V_n = 1$ for any $n \geq 6$.

If $V_n = v_n = 1$ eventually, then by (2.9) we have

$$\begin{cases} 1 \geq \frac{v_k}{U_{n-1}V_{n-1}\cdots U_kV_k} \text{ eventually}, \\ 1 \geq \frac{V_k}{u_{n-1}v_{n-1}\cdots u_kv_k} \text{ eventually}. \end{cases}$$

Since $U_n \geq \frac{b^3}{u_{n-1}}$ and $u_n \geq \frac{b^3}{U_{n-1}}$, we see

$$\begin{cases} u_n = \max\left\{1, \frac{b^3}{U_{n-1}}, \frac{b^3 v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\} = \max\left\{1, \frac{b^3}{U_{n-1}}\right\} \leq u_{n-2} \text{ eventually}, \\ U_n = \max\left\{1, \frac{b^3}{u_{n-1}}, \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\} = \max\left\{1, \frac{b^3}{u_{n-1}}\right\} \leq U_{n-2} \text{ eventually}, \end{cases}$$

which implies

$$\begin{cases} u_{n-2} \geq u_n = \max\left\{1, \frac{b^3}{U_{n-1}}\right\} \geq \max\left\{1, \frac{b^3}{U_{n-3}}\right\} = u_{n-2} \text{ eventually}, \\ U_{n-2} \geq U_n = \max\left\{1, \frac{b^3}{u_{n-1}}\right\} \geq \max\left\{1, \frac{b^3}{u_{n-3}}\right\} = U_{n-2} \text{ eventually}. \end{cases}$$

If $V_n > 1 = v_n$ eventually, then by (2.9) we have

$$\begin{cases} 1 \geq \frac{v_k}{U_{n-1}V_{n-1}\cdots U_kV_k} \text{ eventually}, \\ V_n = \frac{V_k}{u_{n-1}v_{n-1}\cdots u_kv_k} > 1 \text{ eventually}. \end{cases}$$

Thus,

$$\begin{cases} u_n = \max\left\{1, \frac{b^3}{U_{n-1}}, \frac{b^3 v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_kV_k}\right\} = \max\left\{1, \frac{b^3}{U_{n-1}}\right\} \leq u_{n-2} \text{ eventually}, \\ U_n = \max\left\{1, \frac{b^3}{u_{n-1}}, \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\} = \max\left\{1, \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_kv_k}\right\} \leq U_{n-2} \text{ eventually}, \end{cases}$$

which implies

$$\begin{cases} u_{n-2} \geq u_n = \max\left\{1, \frac{b^3}{U_{n-1}}\right\} \geq \max\left\{1, \frac{b^3}{U_{n-3}}\right\} = u_{n-2} \text{ eventually}, \\ U_n = 1 \text{ eventually or } b^3 V_k \text{ eventually}. \end{cases}$$

If $V_n = 1 < v_n$ eventually, then by (2.9) we have $U_{n-2} = U_n$ eventually and $u_n = u_{n-1}$ eventually.

By the above we see that $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2.

(3) If $b < 1 < a$, then for any $k \in \mathbf{N}$ and $n \geq k+2$,

$$\begin{cases} u_n = \max\left\{1, \frac{b^3 v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_k V_k}\right\} \leq u_{n-1}, \\ U_n = \max\left\{1, \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_k v_k}\right\} \leq U_{n-1}, \\ v_n = \max\left\{1, \frac{a^3}{V_{n-1}}, \frac{v_k}{U_{n-1}V_{n-1}\cdots U_k V_k}\right\}, \\ V_n = \max\left\{1, \frac{a^3}{v_{n-1}}, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k}\right\}. \end{cases} \tag{2.10}$$

It is easy to verify $u_n = 1$ for any $n \geq 3$ or $U_n = 1$ for any $n \geq 3$.

If $u_n = U_n = 1$ eventually, then

$$\begin{cases} 1 \geq \frac{b^3 v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_k V_k} \text{ eventually}, \\ 1 \geq \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_k v_k} \text{ eventually}. \end{cases}$$

Thus, by (2.6) we have

$$\begin{cases} v_{n-2} \geq v_n = \max\left\{1, \frac{a^3}{V_{n-1}}, \frac{v_k}{U_{n-1}V_{n-1}\cdots U_k V_k}\right\} = \max\left\{1, \frac{a^3}{V_{n-1}}\right\} \geq v_{n-2} \text{ eventually}, \\ V_{n-2} \geq V_n = \max\left\{1, \frac{a^3}{v_{n-1}}, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k}\right\} = \max\left\{1, \frac{a^3}{v_{n-1}}\right\} \geq V_{n-2} \text{ eventually}. \end{cases}$$

If $u_n = 1 < U_n$ eventually, then

$$\begin{cases} 1 \geq \frac{b^3 v_k}{U_{n-1}U_{n-2}V_{n-2}\cdots U_k V_k} \text{ eventually}, \\ 1 < \frac{b^3 V_k}{u_{n-1}u_{n-2}v_{n-2}\cdots u_k v_k} = U_n \text{ eventually}. \end{cases}$$

Thus,

$$\begin{cases} v_{n-2} \geq v_n = \max\left\{1, \frac{a^3}{V_{n-1}}, \frac{v_k}{U_{n-1}V_{n-1}\cdots U_k V_k}\right\} = \max\left\{1, \frac{a^3}{V_{n-1}}\right\} \geq v_{n-2} \text{ eventually}, \\ V_n = \max\left\{1, \frac{a^3}{v_{n-1}}, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k}\right\} = \max\left\{1, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k}\right\} = 1 \text{ eventually or } V_k \text{ eventually}. \end{cases}$$

If $u_n > 1 = U_n$ eventually, then we have $V_n = V_{n-2}$ eventually and $v_n = 1$ eventually or $v_n = v_k$ eventually.

By the above we see that $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2.

Proposition 2.3. If $ab = 1 < AB$, then $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2.

Proof. Note that $U_n \geq \frac{B}{aAu_{n-1}}$ and $V_n \geq \frac{A}{bBv_{n-1}}$. By (2.5) we see that there exists $N \in \mathbf{N}$ such that for any $n \geq N$,

$$\begin{cases} u_n = \max\left\{1, \frac{b^2}{AU_{n-1}}\right\} \leq u_{n-2}, \\ U_n = \max\left\{1, \frac{B}{aAu_{n-1}}, \frac{BV_k}{aAu_{n-1}u_{n-2}v_{n-2}\cdots u_k v_k}\right\}, \\ v_n = \max\left\{1, \frac{a^2}{BV_{n-1}}\right\} \leq v_{n-2}, \\ V_n = \max\left\{1, \frac{A}{bBv_{n-1}}, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k}\right\}. \end{cases} \tag{2.11}$$

It is easy to verify that $u_n = 1$ for any $n \geq N+1$ or $v_n = 1$ for any $n \geq N+1$.

If $u_n = v_n = 1$ eventually, then by (2.11) we see that $U_n = U_{n-1}$ eventually and $V_n = V_{n-1}$ eventually.

If $u_{M+2n} > 1 = v_n$ eventually for some $M \in \mathbf{N}$, then by (2.11) and (2.4) we see that

$$\begin{cases} u_{M+2n} = \frac{b^2}{AU_{M+2n-1}} > 1 \text{ eventually}, \\ U_{M+2n+1} = \max\left\{1, \frac{B}{b}U_{M+2n-1}, \frac{BV_k}{aAu_{M+2n}u_{M+2n-1}v_{M+2n-1}\cdots u_k v_k}\right\} \geq \frac{B}{b}U_{M+2n-1} \text{ eventually}, \\ v_n = \max\left\{1, \frac{a^2}{BV_{n-1}}\right\} = 1 \text{ eventually}, \\ V_n = \max\left\{1, \frac{A}{bBv_{n-1}}, \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k}\right\} \leq V_{n-1} \text{ eventually}. \end{cases}$$

By (2.11) we see that U_n is bounded, which implies $B = b$.

If $U_{M+2n-1} \leq \frac{BV_k}{aAu_{M+2n}u_{M+2n-1}v_{M+2n-1}\cdots u_k v_k}$ eventually, then

$$U_{M+2n+1} = \frac{BV_k}{aAu_{M+2n}u_{M+2n-1}v_{M+2n-1}\cdots u_k v_k} \leq U_{M+2n-1} \text{ eventually}.$$

Thus, $U_{M+2n+1} = U_{M+2n-1}$ eventually and $u_{M+2n} = u_{M+2n-2}$ eventually. Otherwise, we have $U_{M+2n+1} = U_{M+2n-1}$ eventually and $u_{M+2n} = u_{M+2n-2}$ eventually. Thus, $V_n = V_{n-1} = \max\left\{1, \frac{A}{bB}\right\}$ eventually since $\lim_{n\longrightarrow\infty} \frac{V_k}{u_{n-1}v_{n-1}\cdots u_k v_k} = 0$. By (2.2) it follows $U_{M+2n} = U_{M+2n-2}$ eventually and $u_{M+2n+1} = u_{M+2n-1}$ eventually.

If $v_{M+2n} > 1 = u_n$ eventually for some $M \in \mathbf{N}$, then we may show that $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2. The proof is complete.

Proposition 2.4. If $ab > 1$, then $\{(u_n, U_n, v_n, V_n)\}_{n=-1}^{+\infty}$ is eventually periodic with period 2.

Proof. By (2.5) we see that there exists $N \in \mathbf{N}$ such that for any $n \geq N$,

$$\begin{cases} u_n = \max\left\{1, \frac{b}{aAU_{n-1}}\right\}, \\ U_n = \max\left\{1, \frac{B}{aAu_{n-1}}\right\}, \\ v_n = \max\left\{1, \frac{a}{bBV_{n-1}}\right\}, \\ V_n = \max\left\{1, \frac{A}{bBv_{n-1}}\right\}. \end{cases} \tag{2.12}$$

If $a < A$, then for $n \geq 2k + N$ with $k \in \mathbf{N}$,

$$v_n = \max\left\{1, \frac{a}{bBV_{n-1}}\right\} \leq \max\left\{1, \frac{a}{A}v_{n-2}\right\} \leq \cdots \leq \max\left\{1, (\frac{a}{A})^k v_{n-2k}\right\},$$

which implies $v_n = 1$ eventually and $V_n = \max\left\{1, \frac{A}{bB}\right\}$ eventually.

If $a = A$, then

$$\begin{cases} v_n = \max\left\{1, \frac{a}{bBV_{n-1}}\right\} \leq v_{n-2} \text{ eventually}, \\ V_n = \max\left\{1, \frac{A}{bBv_{n-1}}\right\} \leq V_{n-2} \text{ eventually}. \end{cases}$$

Which implies

$$\begin{cases} v_{n-2} \geq v_n = \max\left\{1, \frac{a}{bBV_{n-1}}\right\} \geq \max\left\{1, \frac{a}{bBV_{n-3}}\right\} = v_{n-2} \text{ eventually}, \\ V_{n-2} \geq V_n = \max\left\{1, \frac{A}{bBv_{n-1}}\right\} \geq \max\left\{1, \frac{A}{bBv_{n-3}}\right\} = V_{n-2} \text{ eventually}. \end{cases}$$

Thus, V_n, v_n are eventually periodic with period 2. In a similar way, we also may show that U_n, u_n are eventually periodic with period 2. The proof is complete.

From (2.1), (2.2), Proposition 2.1, Proposition 2.2, Proposition 2.3 and Proposition 2.4 one has the following theorem.

Theorem 2.1. (1) If $\min\{A_0C_1, B_0D_1, A_1C_0, B_1D_0\} < 1$, then system (1.1) has unbounded solutions.

(2) If $\min\{A_0C_1, B_0D_1, A_1C_0, B_1D_0\} \geq 1$, then every solution of system (1.1) is eventually periodic with period 4.

3. Conclusions

In this paper, we study the eventual periodicity of max-type system of difference equations of the second order with four variables and period-two parameters (1.1) and obtain characteristic conditions of the coefficients under which every positive solution of (1.1) is eventually periodic or not. For further research, we plan to study the eventual periodicity of more general max-type system of difference equations by the proof methods used in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflict of interest in this article.

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