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Sequential convergence with exceptional sets

Chaka Karieem Edwards

To the Graduate Council:

I am submitting herewith a thesis written by Chaka Karieem Edwards entitled "Sequential convergence with exceptional sets." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

G. S. Jordan, Major Professor

We have read this thesis and recommend its acceptance:

Suzanne Lenhart

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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and Dean of the Graduate School

Sequential Convergence with Exceptional Sets

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Chaka Karieem Edwards

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Abstract

The classical definition of convergence of a sequence $\{s_n\}$ of real numbers may be extended by permitting the defining inequality to fail on an infinite, but relatively small, exceptional set of integers n. In this thesis the cases of exceptional sets of linear density zero and logarithmic density zero are considered. Basic properties of classical convergence are shown to hold for these cases, an example is constructed to show that a set of logarithmic density zero need not have linear density zero, and for each case a Tauberian condition sufficient to deduce classical convergence is provided.

Contents

1 Introduction .

Sequences are a valuable tool in many areas of mathematics. For example, they are used in constructing the real numbers from the rational numbers, working with continuity of functions, and defining convergence of infinite series. In this thesis we will consider real-valued sequences defined on the positive integers $\mathbb{N}$ and two types of sequential convergence which differ from the classical concept of sequential convergence.

We begin with a definition of the classical concept of convergence.

Definition 1. *A sequence $\{s_n\}$ of real numbers is convergent (or converges) to the real number s, its limit, if for each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n \geq N, |s_n - s| < \varepsilon$.*

In 1951, H. Fast [2] introduced the concept of statistical convergence by allowing in the definition of classical convergence an exceptional set $E = E(\varepsilon)$ of linear density zero, that is, a set E for which $|E_n|/n$ goes to zero as n goes to infinity, where $E_n = \{k \leq n : k \in E\}$ and $|E_n|$ is the cardinality of E_n.

Definition 2. *A sequence $\{s_n\}$ of real numbers is statistically convergent (or statistically converges) to the real number s, its statistical* *limit, if for each* $\varepsilon > 0$, *the set* $E(\varepsilon) = \{k \in \mathbb{N} : |s_k - s| \geq \varepsilon\}$ *has linear density zero; i.e.,*

$$\lim_{n\to\infty} \left|\{k \leq n : |s_k - s| \geq \varepsilon\}\right| / n = 0. \tag{1}$$

For convenience, we will refer to statistical convergence as LinD-convergence and say that a statistically convergent sequence with statistical limit s is LinD-convergent with LinD-limit s.

Requiring $E(\varepsilon)$ to have linear density zero is simply a way of requiring that it is small relative to $\mathbb{N}$. Another way to require $E(\varepsilon)$ to be small is to ask that it have logarithmic density zero, that is,

$$\lim_{n\to\infty} \Big(\sum_{k\in E(\varepsilon)_n} 1/k \Big) / \Big(\sum_{k=1}^{n} 1/k \Big) = 0. \tag{2}$$

Recall [1, p.192] that

$$\sum_{k=1}^{n} 1/k = \log n + \gamma + \eta_n$$

where $\{\eta_n\}$ is a nonincreasing sequence with limit zero and γ is Euler's constant. Therefore, (2) may be rephrased as

$$\lim_{n\to\infty} \Big(\sum_{k\in E(\varepsilon)_n} 1/k \Big) / \log n = 0.$$

This leads to the following definition.

Definition 3. *A sequence $\{s_n\}$ of real numbers is LogD-convergent to the real number s, its LogD-limit, if for each $\varepsilon > 0$*

$$\lim_{n\to\infty} \Big(\sum_{k\in E(\varepsilon)_n} 1/k \Big) / \log n = 0,$$

where the set $E(\varepsilon)$ is defined in Definition 2.

In Section 2 we first describe some examples and then show that LinD-convergence of a sequence is equivalent to two other conditions. In particular, condition (iii) of Theorem 1 allows one to associate with a LinD-convergent sequence a single set E which has linear density zero and is independent of ε. The situation for LogD-convergence is similar. Using the existence of such sets is simpler than using the sets $E(\varepsilon)$ of Definitions 2 and 3. In Section 3 we prove some results for LinD-convergence of sequences that are analogous to basic results for the classical convergence of sequences. It will be clear that the results of Sections 2 and 3 also hold for LogD-convergence. In Section 4 the ideas of linear and logarithmic density are discussed and a set of logarithmic density zero, but not of linear density zero, is constructed to demonstrate that the two concepts of linear and logarithmic smallness are distinct. Finally in Section 5, a sequence which converges off a set of linear or logarithmic density zero and satisfies an additional condition is shown to converge in the classical sense.

2 Examples and Equivalent Conditions

It is obvious that any sequence $\{s_n\}$ which converges to a limit s in the classical sense also converges to s in the LinD sense because for each $\varepsilon > 0, |\{k \in \mathbb{N} : |s_k - s| \geq \varepsilon\}|$ is finite and every finite set clearly has linear density zero. The corresponding statement for LogD-convergence also is true.

The sequence $\{s_n\}$ defined by

$$s_n = \begin{cases} j & n = j^2, j = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

is clearly not classically convergent, but is LinD-convergent and LogD-convergent to zero. To establish the latter claim simply note that for $1 > \varepsilon > 0$ we have $|\{k \leq n : |s_k - 0| \geq \varepsilon\}| \leq \sqrt{n}$ and that $\left(\sum_{j=1}^{n} 1/j^2\right) / \log n^2$ goes to zero as n goes to infinity.

This sequence is obviously unbounded, but "wilder" behavior is possible. If the rationals are enumerated as $r_1, r_2, \ldots$ and s_{j^2} is redefined to be r_j, then the new sequence is still LinD- and LogD-convergent to zero, but its range is dense in the real numbers. As will be discussed in Section 4, a set of linear density zero has logarithmic density zero. Thus in these examples it is only necessary to establish LinD-convergence.

In the next section it will be shown that sequences which are LinD-convergent have several of the properties of classically convergent sequences. To prove these results we will use another characterization of LinD-convergence. To obtain it we will need an extension of the concept of a Cauchy sequence, which is important because it does not require knowledge of the limit of the sequence.

Definition 4. *A sequence $\{s_n\}$ is LinD-Cauchy if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \left|\{k \le n : |s_k - s_N| \ge \varepsilon\}\right| / n = 0.$$

The following theorem has been presented by J. A. Fridy ([3]).

Theorem 1. *Let $\{s_n\}$ be a sequence of real numbers. The following statements are equivalent:*

(i) $\{s_n\}$ is LinD-convergent;

(ii) $\{s_n\}$ is LinD-Cauchy;

(iii) there exist a sequence $\{t_n\}$ of real numbers and a real number s such that

$t_n \to s$ as $n \to \infty$ and $\lim_{n\to\infty} \left|\{k \le n : t_k \ne s_k\}\right| / n = 0$.

Proof. Let dens$E = 0$ mean that the linear density of E is zero. We first show that (i) implies (ii). Suppose $\{s_n\}$ is LinD-convergent to s. Then by Definition 2, for each $\delta > 0$

$$\lim_{n\to\infty} |\{k \le n : |s_k - s| \ge \delta\}| /n = 0.$$

Thus there exists $E(\delta) \subseteq \mathbb{N}$ such that dens$E(\delta) = 0$ and $|s_n - s| < \delta$ for $n \in \mathbb{N}\backslash E(\delta)$. Fix $\varepsilon > 0$; there exists $E(\varepsilon/2) \subseteq \mathbb{N}$, dens$E(\varepsilon/2) = 0$, such that $|s_k - s| < \varepsilon/2$ for $k \in \mathbb{N}\backslash E(\varepsilon/2)$. Fix $N \in \mathbb{N}$ such that $|s_N - s| < \varepsilon/2$. For $k \in \mathbb{N}\backslash E(\varepsilon/2), |s_k - s_N| \le |s_k - s| + |s - s_N| < \varepsilon$. Therefore for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} |\{k \le n : |s_k - s_N| \ge \varepsilon\}| /n = 0$$

and thus (i) implies (ii).

Next we show that (ii) implies (iii). Let $\varepsilon_0 = 1$. By Definition 4 there exists $N \in \mathbb{N}$ such that dens$\{k : s_k \notin [s_N - 1, s_N + 1]\} = 0$. Let $I = [s_N - 1, s_N + 1]$. Let $\varepsilon_1 = 1/2$. There exists $N(1)$ such that dens$\{k : s_k \notin [s_{N(1)} - 1/2, s_{N(1)} + 1/2]\} = 0$. Let $J_1 = [s_{N(1)} - 1/2, s_{N(1)} + 1/2]$. Put $I_1 = I \cap J_1$; then

$$|\{k \le n : s_k \notin I_1\}| /n \le |\{k \le n : s_k \notin I\}| /n + |\{k \le n : s_k \notin J_1\}| /n$$

and therefore $\lim_{n\to\infty} |\{k \le n : s_k \notin I_1\}| /n = 0$. Let $\varepsilon_2 = 1/2^2$. There exists $N(2)$ such that dens$\{k : s_k \notin [s_{N(2)} - 1/2^2, s_{N(1)} + 1/2^2]\} = 0$. Let $J_2 = [s_{N(2)} - 1/2^2, s_{N(1)} + 1/2^2]$ and put $I_2 = I_1 \cap J_2$; then dens$\{k : s_k \notin I_2\} = 0$. Continue in this fashion to get intervals I_j such that $I_1 \supseteq I_2 \supseteq \cdots$, dens$\{k : s_k \notin I_j\} = 0$ and $l(I_j) \le 1/2^{j-1}$.

Now by the Nested Interval Property [5, p. 32], $\cap_{j=1}^{\infty} I_j = \{s\}$ for some s. We next construct the desired sequence $\{t_n\}$. There exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|\{k \leq n : s_k \notin I_1\}|/n < 1$. Similarly, there exists $N_2 \in \mathbb{N}$, $N_2 > N_1$, such that $n \geq N_2$ implies $|\{k \leq n : s_k \notin I_2\}|/n < 1/2$. In general, there exists a sequence of integers N_j such that $0 < N_1 < N_2 < \cdots$ and $n \geq N_j$ implies $|\{k \leq n : s_k \notin I_j\}|/n < 1/j$. Put

$$t_k = \begin{cases} s_k & \text{if } k < N_1 \\ s_k & \text{if } N_j \leq k < N_{j+1} \text{ and } s_k \in I_j \\ s & \text{if } N_j \leq k < N_{j+1} \text{ and } s_k \notin I_j \end{cases}$$

Now fix $\varepsilon > 0$ and fix j_0 such that $1/2^{j_0-1} < \varepsilon$. For $j \geq j_0$ and $N_j \leq k < N_{j+1}$, if $s_k \in I_j$, then

$$|t_k - s| = |s_k - s| < 1/2^{j-1} \leq 1/2^{j_0-1} < \varepsilon,$$

and if $s_k \notin I_j$, then $|t_k - s| = |s - s| = 0 < \varepsilon$. Therefore for $k \geq N_{j_0}$, $|t_k - s| < \varepsilon$ and this implies $t_k \to s$ as $k \to \infty$.

It remains to show that $s_k = t_k$ except on a set of linear density zero. For $N_j \leq n < N_{j+1}$,

$$|\{k \leq n : s_k \neq t_k\}|/n \leq |\{k \leq n : s_k \notin I_j\}|/n.$$

Since the last term tends to zero as n tends to infinity,

$$\lim_{n \to \infty} |\{k : t_k \neq s_k\}|/n = 0,$$

so that (ii) implies (iii).

Finally we show (iii) implies (i). Fix $\varepsilon > 0$. Then

$$\{k \le n : |s_k - s| \ge \varepsilon\} \subseteq \{k \le n : s_k \ne t_k\} \cup \{k \le n : |t_k - s| \ge \varepsilon\}$$

and thus

$$\begin{aligned} 0 &\le \lim_{n\to\infty} |\{k \le n : |s_k - s| \ge \varepsilon\}|/n \\ &\le \lim_{n\to\infty} |\{k \le n : t_k \ne s_k\}|/n \\ &\quad + \lim_{n\to\infty} |\{k \le n : |t_k - s| \ge \varepsilon\}|/n \\ &= 0 \end{aligned}$$

by (iii). Therefore, $\{s_n\}$ is LinD-convergent to s and (iii) implies (i). The proof of Theorem 1 is now complete.

The definition of a LogD-cauchy sequence analogous to a LinD-Cauchy sequence may be stated easily. Then an analog of Theorem 1 also holds for LogD-convergence with obvious changes in the proof. Also, it is clear that (iii) is equivalent to the statement that there is a set $E \subseteq \mathbb{N}$ of linear density zero (or logarithmic density zero) such that $s_n \to s$ as $n \to \infty, n \notin E$. We shall use this rephrasing in the remainder of this thesis, and thereby avoid any need for the sets $E(\varepsilon)$ in Definitions 2 and 3.

3 Basic Results For Sequences Convergent Off Small Sets.

In this section basic results about the convergence of sequences are established in the context of linearly or logarithmically small exceptional sets. The first result concerns the uniqueness of LinD-limits.

Theorem 2. *Suppose $E, F \subseteq \mathbb{N}$ have linear density zero. If $s_n \to s$ as $n \to \infty, n \notin E$, and $s_n \to s^*$ as $n \to \infty, n \notin F$, then $s = s^*$.*

Proof. Let $\varepsilon > 0$. There exists $N = N(\varepsilon)$ such that $n \geq N, n \notin E$, imply $|s_n - s| < \varepsilon/2$ and $n \geq N, n \notin F$, imply $|s_n - s^*| < \varepsilon/2$. Since $|(E \cup F)_n|/n \leq |E_n|/n + |F_n|/n$ which goes to zero as n tends to infinity, $E \cup F$ has linear density zero and there are arbitrarily large $n \geq N$ such that $n \notin E \cup F$. For such n we have $0 \leq |s - s^*| \leq |s_n - s| + |s_n - s^*| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $s - s^* = 0$ or $s = s^*$ as claimed.

The next four results concern the behavior of LinD-convergent sequences under the operations of addition, scalar multiplication, multiplication and division.

Theorem 3. *Suppose $E, F \subseteq \mathbb{N}$ have linear density zero. If $s_n \to s$ as $n \to \infty, n \notin E$, and $t_n \to t$ as $n \to \infty, n \notin F$, then $s_n + t_n \to s + t$ as $n \to \infty, n \notin E \cup F$, where $E \cup F$ has linear density zero.*

Proof. Let $\varepsilon > 0$ and $N = N(\varepsilon)$ be such that $n \geq N, n \notin E$, imply $|s_n - s| < \varepsilon/2$ and $n \geq N, n \notin F$, imply $|t_n - t| < \varepsilon/2$. Put

$G = E \cup F$. Then G has linear density zero and for $n \geq N, n \notin G$, $|s_n + t_n - (s - t)| \leq |s_n - s| + |t_n - t| < 2\varepsilon/2 = \varepsilon$. Thus the result holds.

For the operation of scalar multiplication we have the following result.

Theorem 4. *Suppose $E \subseteq \mathbb{N}$ has linear density zero. If α is a complex number and $s_n \to s$, as $n \to \infty, n \notin E$, then $\alpha s_n \to \alpha s$ as $n \to \infty, n \notin E$.*

Proof. Let $\varepsilon > 0$ and $N = N(\varepsilon)$ be such that $n \geq N, n \notin E$, imply $|s_n - s| < \varepsilon/(1+|\alpha|)$. Then for $n \geq N, n \notin E$, $|\alpha s_n - \alpha s| = |\alpha||s_n - s| < \varepsilon$. Thus $\alpha s_n \to \alpha s$ as $n \to \infty, n \notin E$.

The next result shows that the product of two LinD-convergent sequences converges to the product of their LinD-limits.

Theorem 5. *Suppose $E, F \subseteq \mathbb{N}$ have linear density zero. If $s_n \to s$ as $n \to \infty, n \notin E$, and $t_n \to t$ as $n \to \infty, n \notin F$, then $s_n t_n \to st$ as $n \to \infty, n \notin E \cup F$, where $E \cup F$ has linear density zero.*

Proof. Since $t_n \to t$ as $n \to \infty, n \notin F$, there exists $m < \infty$ such that $|t_n| \leq m$ for all $n \geq 1, n \notin F$. Let $\varepsilon > 0$ and $N = N(\varepsilon)$ be such that $n \geq N, n \notin E$, imply $|s_n - s| < \varepsilon/[2(m+1)]$ and $n \geq N, n \in F$, imply $|t_n - t| < \varepsilon/[2(|s|+1)]$. Put $G = E \cup F$; then G has linear density zero and for $n \geq N, n \notin G$,

$$\begin{aligned}
|s_n t_n - st| &\leq |(s_n - s)t_n| + |(s(t_n - t)| \\
&\leq |s_n - s||t_n| + |s||t_n - t| \\
&\leq m|s_n - s| + |s||t_n - t| \\
&\leq \frac{m\varepsilon}{2(m+1)} + \frac{|s|\varepsilon}{2(|s|+1)} \\
&< \varepsilon.
\end{aligned}$$

Therefore $s_n t_n \to st$ as $n \to \infty, n \notin G$, where G has linear density zero.

The final theorem in this section concerns quotients of LinD-convergent sequences.

Theorem 6. *Suppose $E, F \subseteq \mathbb{N}$ are sets of linear density zero. If $s_n \to s$ as $n \to \infty, n \notin E$, and $t_n \to t$ as $n \to \infty, n \notin F$, where $t \neq 0$, then $s_n/t_n \to s/t$ as $n \to \infty, n \notin E \cup F$, where $E \cup F$ has linear density zero.*

Proof. Let $\varepsilon > 0$ and $N = N(\varepsilon)$ be such that $n \geq N, n \notin E$, imply $|s_n - s| < \varepsilon|t|/4$ and $n \geq N, n \notin F$, imply $|t_n| \geq |t|/2 > 0$ and $|t_n - t| < \varepsilon|t|^2/4(1+|s|)$. Put $G = E \cup F$; then G has linear density zero and for $n \geq N, n \notin G$,

$$\begin{aligned} |s_n/t_n - s/t| &= |(s_n t - t_n s)/t_n t| \\ &\leq [|t||s_n - s| + |s||t_n - t|] / |t_n||t| \\ &< 2\left[\varepsilon |t|^2/4 + \varepsilon |t|^2 |s|/4(1+s)\right] / |t|^2 \\ &< \varepsilon \end{aligned}$$

Therefore $s_n/t_n \to s/t$ as $n \to \infty, n \notin G$, where G has linear density zero.

The statements and proofs of the analogous results with linear density zero replaced by logarithmic density zero are essentially the same as those presented here and will not be given.

4 Distinction Between Linear and Logarithmic Density .

In this section we briefly discuss some relations between linear density and logarithmic density and present an example of a set E of logarithmic density zero which is not of linear density zero.

We have defined so far only what it means for a set E to be of linear or logarithmic density zero. If one requires only that the relevant limit exist with value α, not necessarily zero, then one says that the set E is of linear or logarithmic density α. Clearly, in each case $0 \leq \alpha \leq 1$. One also may consider lower and upper linear densities of E defined, respectively, by

$$\underline{\mathrm{dens}}E = \liminf_{n\to\infty} |E_n|/n, \qquad \overline{\mathrm{dens}}E = \limsup_{n\to\infty} |E_n|/n$$

and lower and upper logarithmic densities of E defined, respectively, by

$$\underline{\mathrm{logdens}}E = \liminf_{n\to\infty}\Big(\sum_{k\in E_n} 1/k\Big)/\log n,$$

$$\overline{\mathrm{logdens}}E = \limsup_{n\to\infty}\Big(\sum_{k\in E_n} 1/k\Big)/\log n.$$

One can show [4, p. 121] $0 \leq \underline{\mathrm{dens}}E \leq \underline{\mathrm{logdens}}E \leq \overline{\mathrm{logdens}}E \leq \overline{\mathrm{dens}}E$. It follows, of course, that a set of linear density zero also is a set of logarithmic density zero. The example we present next has logarithmic density zero and, hence, lower linear density zero, but its upper linear density is strictly positive.

The example set E involves two sequences $\{a_n\}$ and $\{b_n\}$ which we will construct so that $a_1 = 9 < b_1 = 10 < a_2 < b_2 < \cdots$. We take E to have the form

$$E = \cup_{n=1}^{\infty} \{j : a_n < j \leq b_n\}.$$

Put $x_n = |E_n|/n$, $y_n = \left(\sum_{j \in E_n} 1/j\right)/\log n$. We shall determine a_n and b_n so that $\limsup_{n\to\infty} x_n = 0.1$ and $\lim_{n\to\infty} y_n = 0$.

There are some useful observations that do not depend on any further restrictions on $\{a_n\}$ and $\{b_n\}$. First, note that x_n and y_n are decreasing for $b_{m-1} \leq n \leq a_m, m \geq 2$, since $|E_n|$ and $\sum_{j \in E_n} 1/j$ are constant for these n while n and $\log n$ increase. Also, for each $m = 1, 2, \ldots$ we have the inequalities

$$x_{a_m} < x_{a_m+1} < \cdots < x_{b_m}. \tag{3}$$

This result follows from the obvious inequality

$$q/n < (q+1)/(n+1)\ (0 \leq q < n,\ n = 1, 2, 3, \ldots),$$

since $a_1 = 9$ implies $|E_n| < n$.

A less obvious observation is that for each $m = 1, 2, \cdots$ we have the inequalities

(4) $$y_{a_m} < y_{a_m+1} < \cdots < y_{b_m}.$$

This result follows from

(5) $$q_n/\log n < \big(q_n + 1/(n+1)\big)/\log(n+1),$$

where $q_n = \sum_{j \in E_n} 1/j, a_m \leq n < b_m$.

To see the validity of this observation, note first that it is clearly true for $m = 1$ and then fix $m \geq 2$ and n so that $a_m \leq n < b_m$. Now note that (5) holds if and only if

(6) $$q_n\,[\log(n+1) - \log n] < \big(1/(n+1)\big)\log n.$$

By the Mean Value Theorem, there exists $s_n \in (n, n+1)$ such that

$$\log(n+1) - \log n = 1/s_n.$$

Thus our inequality (6) is equivalent to

(7) $$q_n/s_n < \big(1/(n+1)\big)\log n.$$

Since $b_1 = 10$, we have

$$q_n \leq \sum_{k=10}^{n} 1/k \leq \log n - \log 9 < \log n - 2.$$

Therefore (7) follows from

$$(\log n - 2)/s_n < \big(1/(n+1)\big)\log n$$

or, since $s_n > n$, from

$$\log n - 2 < \bigl(n/n+1\bigr) \log n \qquad (n \geq 10)$$

or, equivalently, from $\bigl(1/(n+1)\bigr) \log n < 2\,(n \geq 10)$, which follows from the easily established inequality $(1/n) \log n < 2\,(n \geq b_1 = 10)$. Thus the inequalities (3) are established.

From these observations it is clear that

$$x_{b_m} = \max\{x_n : a_m < n \leq a_{m+1}\}$$

and

$$y_{b_m} = \max\{y_n : a_m < n \leq a_{m+1}\}.$$

Thus, E will have the desired properties if $\lim_{m\to\infty} y_{b_m} = 0$ and $\limsup_{m\to\infty} x_{b_m} > 0$.

Note that

$$x_{b_m} = (1/b_m) \sum_{j=1}^{m} (b_j - a_j), \quad y_{b_m} = \bigl(1/\log b_m\bigr) \sum_{j=1}^{m} \sum_{k=a_j+1}^{b_j} 1/k$$

For each $m = 1, 2, 3, \ldots$ we wish to have $x_{b_m} = 0.1$. Then for $m \geq 2$, we will have

$$\sum_{j=1}^{m-1} (b_j - a_j) + (b_m - a_m) = 0.1 b_m$$

or, equivalently, $0.1b_{m-1} + b_m - a_m = 0.1b_m$. It follows that we will have

$$a_m = 0.9b_m + 0.1b_{m-1}.$$

Of course, we must determine the b_m in such a way that $a_m \in \mathbb{N}$. We will choose a strictly increasing sequence $\{p_j\}$ of positive integers with $p_1 = 1$ and take

$$b_m = \sum_{j=1}^{p_m} 10^j. \tag{8}$$

Then $a_m, b_m \in \mathbb{N}$ and

$$b_m > a_m > b_{m-1} \qquad (m \geq 2),$$

since $b_m > b_{m-1}$ and a_m is a strictly convex combination of b_m and b_{m-1}.

Recall that $p_1 = 1$. We need to choose $p_m, m \geq 2$, in (8) so that

$$\begin{aligned} y_{b_m} &= \Big(\sum_{j=1}^{m} \sum_{k=a_j+1}^{b_j} 1/k\Big)/\log b_m \\ &= \Big(\sum_{j=1}^{m-1} \sum_{k=a_j+1}^{b_j} 1/k\Big)/\log b_m + \Big(\sum_{k=a_m+1}^{b_m} 1/k\Big)/\log b_m \end{aligned} \tag{9}$$

goes to zero as $m \to \infty$. Suppose $p_1, \ldots, p_{m-1}$ have been chosen and thus $b_1, \ldots, b_{m-1}$ and $a_1, \ldots, a_{m-1}$ are determined. Replace a_m and b_m in (9) by a and b, respectively, and observe that the first term in the last line of (9) clearly goes to zero as b goes to infinity. The remaining term in the last line of (9) also goes to zero as b goes to infinity. To obtain this result, note that

$$0 \leq \sum_{k=a+1}^{b} 1/k \leq \log b - \log a.$$

Recall that $b > b_{m-1}$ and $a = 0.9b + 0.1b_{m-1}$; therefore

$$\begin{aligned} \log b &> \log a \\ &= \log\bigl(0.9b + 0.1b_{m-1}\bigr) \\ &> \log\bigl(0.9b\bigr) \\ &= \log 0.9 + \log b. \end{aligned}$$

It follows that as $b \to \infty$ we have $\log a/\log b \to 1$ and, consequently, $\bigl(\sum_{k=a+1}^{b} 1/k\bigr)/\log b \to 0$ as $b \to \infty$. Thus if we take p_m and, hence, $b_m = b$ sufficiently large, we will have $0 \leq y_{b_m} \leq 1/m$. It follows that E has logarithmic density zero and our construction is complete.

5 Tauberian Conditions for LinD- and LogD-Convergence.

In this section we combine LinD-convergence and LogD-convergence of a sequence with Tauberian conditions to deduce classical convergence. We begin with the linear density case.

Theorem 7. *Suppose $s_n \to s$ as $n \to \infty, n \notin E$, $densE = 0$ and $\{s_n\}$ satisfies the Tauberian condition*

$$s_m - s_n \to 0 \quad (m \geq n \to \infty, m/n \to 1).$$

Then $s_n \to s$ as $n \to \infty$.

Proof. Let $E = \cup_{n=1}^{\infty}\{j : a_n \leq j \leq b_n\}$ where $a_n \leq b_n$ and $b_n + 1 < a_{n+1}$ for $n = 1, 2, 3, \ldots$. Then $0 \leq 1 - a_n/b_n = (b_n - a_n)/b_n \leq |E_n|/b_n \to 0$ as $n \to \infty$; it follows that $a_n/b_n \to 1$ and, thus, $b_n/a_n \to 1$. Clearly for $m \in [a_n, b_n]$, as $n \to \infty$,

$$1 \leq m/a_n \leq b_n/a_n \to 1.$$

Fix $\varepsilon > 0$. There exist N_1 and $\eta > 0$ such that $m \geq n \geq N_1$ and $1 \leq m/n < 1 + \eta$ imply $|s_m - s_n| < \varepsilon/2$. There also exists $N_2 \geq N_1 - 1$ such that $n \geq N_2$ and $n \notin E$ imply

$$|s_n - s| < \varepsilon/2. \tag{10}$$

Finally, there exists $N_3 \geq N_2$ such that $n \geq N_3$ implies $a_n > N_2$ and $b_n/(a_n - 1) < 1 + \eta$. Thus for $n \geq N_3$ and $m \in [a_n, b_n]$ we have $a_n - 1 \geq N_2$, $m/(a_n - 1) < 1 + \eta$ and therefore

$$\begin{aligned} |s_m - s| &\leq |s_m - s_{a_n - 1}| + |s_{a_n - 1} - s| \\ &< \varepsilon. \end{aligned}$$

Combining this estimate with the one in (10) shows that $s_n \to s$ as $n \to \infty$ and the proof is complete.

For the logarithmic density case we use an analogous Tauberian condition.

Theorem 8. *Suppose $s_n \to s$ as $n \to \infty, n \notin E$, E of logarithmic density zero, and $\{s_n\}$ satisfies the Tauberian condition*

$$s_m - s_n \to 0 \quad (m \geq n \to \infty, \log m/\log n \to 1).$$

Then $s_n \to s$ as $n \to \infty$.

Proof. The proof parallels that of Theorem 7. Let $E = \cup_{n=1}^{\infty}\{j : a_n \leq j \leq b_n\}$ where $a_n \leq b_n$ and $b_n + 1 < a_{n+1}$ for $n = 1, 2, 3, \ldots$. We again use the Tauberian condition to estimate $|s_m - s_{a_n - 1}|$. For that purpose we need $(\log a_n)/\log b_n \to 1$ as $n \to \infty$, a result which follows from

$$
\begin{aligned}
(11) \qquad 0 &\le 1 - \log a_n/\log b_n = (\log b_n - \log a_n)/\log b_n \\
&< (1/\log b_n) \sum_{k=a_n}^{b_n} 1/k \\
&\le (1/\log b_n) \sum_{k \in E_{b_n}} 1/k \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

The second inequality in (11) is an application of

$$\log q - \log p = \int_p^q 1/t\,dt = \sum_{k=p+1}^{q} 1/k + \sum_{k=p}^{q-1} A_{k+1},$$

where A_{k+1} is the area of the planar region $\{(t, y) : k \le t \le k+1,\ 1/(k+1) \le y \le 1/t\}$, and the estimates

$$\sum_{k=p}^{q-1} A_{k+1} < 1/2 \sum_{k=p}^{q-1} \big(1/k - 1/(k+1)\big) = 1/2(1/p - 1/q) < 1/p.$$

It follows from (11) that for $m \in [a_n, b_n]$, as $n \to \infty$,

$$1 \le (\log m)/\log a_n \le (\log b_n)/\log a_n \to 1.$$

Obvious changes in the proof of Theorem 7 now yield Theorem 8.

Bibliography

[1] T.M. Apostol, *Mathematical Analysis* , Addison-Wesley Publishing Company, Reading, Mass. 2nd ed., 1974.

[2] H. Fast, *Sur la Convergence Statistique*, Colloq. Math. 2, 241-244 (1951).

[3] J. A. Fridy, *On Statistical Convergence*, Analysis 5, 301-313 (1985).

[4] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea Publishing Company, New York, N.Y., 2^{nd} Ed., 1975.

[5] W. R. Wade , *An Introduction to Analysis*, Prentice Hall, Suzlwood Cliffs, New Jersy, 1995.

Vita

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