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On the Total Irregularity Strength of the Corona Product of a Path with Path

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Abstract: This paper deals with the totally irregular total labeling of the corona product of a path with path. The results gave the exact values of the total irregularity strength of $P_m \odot P_n$ for integer $2 \leq m \leq 3$ and $n \geq 3$.

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1. Introduction

In research of graph theory, graph labeling is recently becoming highly interesting area. It comes from the availability of open problems in graph labeling. The research of finding the general labeling for any graph, the exact value of any labeling parameter, or even verification of labeling and its parameter for small class of graph, are challenging and widely connect to many other research areas and problems.

Irregular k-labeling of a connected graph of order more than two, one of graph labeling that highly researched, is a map that assign every edge of a graph into k positive integer such that the weights of each vertex are distinct. The largest value of k for which a graph is irregular is called the irregularity strength, denoted as $s(G)$.

In this paper, we consider a finite, undirected, and simple graph. In [1], Tilukay, et al., the exact values of the total irregularity strength of fan, wheel, triangular book, and friendship graphs, are provided. It is clear that fan graph f_n is isomorphic to $P_1 \odot P_n$, cycle graph C_n is isomorphic to $P_1 \odot C_n$, and star graph S_n is isomorphic to $P_1 \odot S_n$. As an advanced study of the research in [1], the extension for P_m, where $2 \leq m \leq 3$, is analyzed to provide the more general exact values.

As mentioned in the first result of the totally irregular total labeling [2], the lower bound of the total irregularity strength of a graph (denoted by $ts(G)$) is the maximum of its total edge (or vertex) - irregularity strength; denoted by $tes(G)$ or $tvs(G)$, respectively, by Marzuki, et al. as follows.

$$ts(G) \geq \max\{tes(G), tvs(G)\}. \tag{1}$$

The exact values of the total edge irregular strength of the corona product of a path with a path, a cycle, and a star are given by Nurdin, et al. [3], as follow.

$$\begin{aligned} &tes(P_m \odot P_n) = \left\lceil \tfrac{2mn+1}{3} \right\rceil, \text{ for integer } m, n \geq 2; \\ &tes(P_m \odot C_n) = \left\lceil \tfrac{2mn+1}{3} \right\rceil, \text{ for integer } m, n \geq 2; \\ &tes(P_m \odot S_n) = \left\lceil \tfrac{2mn+1}{3} \right\rceil, \text{ for integer } m, n \geq 2. \end{aligned} \tag{2}$$

In other hand, the boundary of the total vertex irregularity strength of any graph is given by Baca, et al. in [4], as follow.

$$\left\lceil \frac{p+\delta(G)}{\Delta(G)+1} \right\rceil \leq tvs(G) \leq p + \Delta(G) - 2\delta(G) + 1, \tag{3}$$

Where p is the order of G, q is the size of G, $\delta(G)$ is the minimum degree of G, and $\Delta(G)$ is the maximum degree of G.

Marzuki, et al. [2] found that for path graph (with no order 5) and cycle graph, lower bound in equation (1) is equal to its tes. Moreover, for several Cartesian product graphs, Ramdani and Salman in [5], also found the same result. It is followed by many other results, such as Ramdani et al. [6] for the gear graph, the fungus graph with even order and the disjoint union of stars; Tilukay et al. [1] for that fan, wheel, triangular book, and friendship graphs; Jeyanti and Sudha [7] for double fans $DF_n, (n \geq 3)$, double triangular snakes $DT_p, (p \geq 3)$, joint-wheel graphs $WH_n, (n \geq 3)$, and $P_m + K_m, (m \geq 3)$; Tilukay, et al. for complete graph [8] and complete bipartite graph [9]. Further results of $tvs(G)$, $tes(G)$, and $ts(G)$ can be found in [10-17]. Since many previous results lead to $ts(G) = tes(G)$ for non-tree graphs, our results are suspected to have similar conclusion.

2. The Total Irregularity Strength of the Corona Product of a Path with Path

Using the axiomatic deductive and pattern detection method, we derive the following theorem.

Theorem 1. For every integer $2 \leq m \leq 3$, $n \geq 2$,

$$ts(P_m \odot P_n) = \left\lceil \frac{2mn+1}{3} \right\rceil.$$

Proof. The corona product of a path P_m and a path P_n resulting a graph $P_m \odot P_n$ of order $m(n+1)$ and size $2mn-1$. Follow from equation (1) and (2), we obtain $ts(P_m \odot P_n) \geq \left\lceil \frac{2mn+1}{3} \right\rceil$, for $2 \leq m \leq 3$, $n \geq 2$. Next, to conclude that it is the exact value of $ts(P_m \odot P_n)$, we need to prove that there is a totally irregular total $\left\lceil \frac{2mn+1}{3} \right\rceil -$ labeling of $P_m \odot P_n$. Let $V(P_m \odot P_n) = \{v_i, v_i^j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(P_m \odot P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq m-1\} \cup \{v_i v_i^j \mid 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{v_i^j v_i^{j+1} \mid 1 \leq i \leq m, 1 \leq j \leq n-1\}$. Let $\lambda: V \cup E \to \left\{1, 2, 3, \cdots, \left\lceil \frac{2mn+1}{3} \right\rceil \right\}$ and $r_i = \left\lceil \frac{2ni+1}{3} \right\rceil$ for $2 \leq i \leq m$. Define λ as follows.

$$\lambda(v_i) = \begin{cases} 1, & \text{for } i = 1; \\ r_i, & \text{for } 2 \leq i \leq m; \end{cases}$$

$$\lambda(v_i^j) = \begin{cases} j, & \text{for } i = 1, 1 \leq j \leq n; \\ r_i + j - n, & \text{for } 2 \leq i \leq m, 1 \leq j \leq n; \end{cases}$$

$$\lambda(v_i v_i^j) = \begin{cases} j, & \text{for } i = 1, 1 \le j \le n; \\ r_i + j - n, & \text{for } 2 \le i \le m, 1 \le j \le n; \end{cases}$$
$$\lambda(v_i^j v_i^{j+1}) = \begin{cases} 1, & \text{for } i = 1, 1 \le j \le n-1; \\ r_i, & \text{for } 2 \le i \le m, 1 \le j \le n-1; \end{cases}$$
$$\lambda(v_i v_{i+1}) = \begin{cases} 2n + 1 - r_2, & \text{for } i = 1; \\ 4n + 4 - r_2 - r_3, & \text{for } i = 2, n \ne 5; \\ n, & \text{for } i = 2\ n = 4, 5. \end{cases}$$

From the definition above, we have $\lambda(v_m^n) = \lambda(v_m v_m^n) = \lambda(v_m^n v_m^{n+1}) = r_m = \left\lceil \frac{2mn+1}{3} \right\rceil$ as the maximum label.

Next, we checked the vertex-weights and edge-weights as follows.

For the vertex-weight, we evaluate the functions above and obtain

$$w(v_1^j) = \begin{cases} 3, & \text{for } i = 1; \\ 2j + 2, & \text{for } 2 \le j \le n-1; \\ 2n + 1, & \text{for } i = n; \end{cases}$$
$$w(v_2^j) = \begin{cases} 2n + 3, & \text{for } i = 1; \\ 2n + r_2 + 2j + 1, & \text{for } 2 \le j \le n-1; \\ 4n + 1, & \text{for } i = n; \end{cases}$$
$$w(v_3^j) = \begin{cases} 4n + 3, & \text{for } i = 1; \\ 4n + r_3 + 2j + 1, & \text{for } 2 \le j \le n-1; \\ 6n + 1, & \text{for } i = n; \end{cases}$$
$$w(v_1) = \frac{n(n+5)}{2} - r_2 + 2;$$
$$w(v_2) = \begin{cases} \frac{n}{2}(2r_2 - n + 5) + 1, & \text{for } m = 2; \\ \frac{n}{2}(2r_2 - n + 13) - r_2 - r_3 + 3, & \text{for } m = 3, n \ne 5; \\ \frac{n}{2}(2r_2 - n + 5) + 6, & \text{for } m = 3, n = 5; \end{cases}$$
$$w(v_3) = \begin{cases} \frac{n}{2}(2r_3 - n + 9) + 5, & \text{for } m = 3, n \ne 5; \\ \frac{n}{2}(2r_3 - n + 3) + r_3, & \text{for } m = 3, n = 4, 5. \end{cases}$$

Next, we evaluate the edge-weights as follows.

$$w(v_1 v_1^j) = 2j + 1, 1 \le j \le n;$$
$$w(v_1^j v_1^{j+1}) = 2j + 2, 1 \le j \le n-1;$$
$$w(v_2 v_2^j) = \begin{cases} 2n + 2j + 1, 1 \le j \le n, n \equiv 2 \bmod 3; \\ 2n + 2j + 2, 1 \le j \le n, n \equiv 1 \bmod 3; \\ 2n + 2j + 3, 1 \le j \le n, n \equiv 0 \bmod 3; \end{cases}$$
$$w(v_2^j v_2^{j+1}) = \begin{cases} 2n + 2j + 2, 1 \le j \le n, n \equiv 2 \bmod 3; \\ 2n + 2j + 3, 1 \le j \le n, n \equiv 1 \bmod 3; \\ 2n + 2j + 4, 1 \le j \le n, n \equiv 0 \bmod 3; \end{cases}$$
$$w(v_3 v_3^j) = \begin{cases} 4n + 2j + 3, 1 \le j \le n, n \equiv 2 \bmod 3; \\ 4n + 2j + 2, 1 \le j \le n, n \equiv 1 \bmod 3; \\ 4n + 2j + 1, 1 \le j \le n, n \equiv 0 \bmod 3; \end{cases}$$
$$w(v_3^j v_3^{j+1}) = \begin{cases} 4n + 2j + 4, 1 \le j \le n, n \equiv 2 \bmod 3; \\ 4n + 2j + 3, 1 \le j \le n, n \equiv 1 \bmod 3; \\ 4n + 2j + 2, 1 \le j \le n, n \equiv 0 \bmod 3; \end{cases}$$
$$w(v_i v_{i+1}) = 2ni + 2, 1 \le i \le 2;$$

It can be checked that the vertex-weights and edge-weights form increasingly sub-sequences for which there is no two vertices of the same weight.

From the evaluation on vertex-weights and edge-weights above, we obtain that the corona product of a path P_m and a path P_n, $P_m \odot P_n$ with every integer $2 \le m \le 3$, $n \ge 2$, is a totally irregular total graph with the $ts(P_m \odot P_n) = \left\lceil \frac{2mn+1}{3} \right\rceil$. ■

3. Conclusion

By equation (3) and the result above, we can conclude that the corona product of a path P_m and a path P_n, $P_m \odot P_n$ with every integer $2 \le m \le 3$, $n \ge 2$, is a totally irregular total graph with the $ts(P_m \odot P_n) = \left\lceil \frac{2mn+1}{3} \right\rceil$. In other word, the total irregularity strength of $P_m \odot P_n$ with every integer $2 \le m \le 3$, $n \ge 2$ is equal to its total edge irregularity strength.

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