2023

Finite Matroidal Spaces and Matrological Spaces

Ziyad M. Hamad
West Virginia University, zmbusaif@mix.wvu.edu

Finite Matroidal Spaces and Matrological Spaces

Ziyad M. Hamad

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in
Mathematics

Jerzy Wojciechowski, Ph.D., Chair
Hong-Jian Lai, Ph.D.
John Goldwasser, Ph.D.
Rong Luo, Ph.D.
Elaine Eschen, Ph.D.

Department of Mathematics

Morgantown, West Virginia
2023

Keywords: Finite matroidal spaces, Matrological spaces, Finitary matroidal closure operators, Topological closure operators with the exchange property, Common closure operators

ABSTRACT

Finite Matroidal Spaces and Matrological Spaces

Ziyad Hamad

The purpose of this thesis is to present new different spaces as attempts to generalize the concept of topological vector spaces. A topological vector space, a well-known concept in mathematics, is a vector space over a field $\mathbb{F}$ with a topology that makes the addition and scalar multiplication operations of the vector space continuous functions. The field $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$ with their standard topologies. Since every vector space is a finitary matroid, we define two spaces called finite matroidal spaces and matrological spaces by replacing the linear structure of the topological vector space with a finitary matroidal structure. The idea is to combine a finitary matroidal closure operator like the linear closure operator with a topological closure operator into a single closure operator called a common closure operator. Therefore, one may take a set with a finitary matroidal closure operator and a topological closure operator like the topological vector space. The study starts with basic definitions, some fundamental properties and a collection of examples. The finite matroidal spaces and matrological spaces are then presented. Furthermore, the idea of a common closure operator is introduced and then a discussion is given of when to obtain from a set and a common closure operator a finite matroidal space or a matrological space. Finally, relationships of topological vector spaces with both finite matroidal spaces and topological vector spaces are presented.

DEDICATION

To my parents, siblings, wife and beloved children.
I am forever grateful and honored for having you in my life.

And

to Abdullah, Osama and my best friend, Hamza.
I will remember you all, as long as I live.

Acknowledgments

First and foremost, I thank Allah for giving me the ability to perform and complete this dissertation.

I would like to express my deepest and most sincere gratitude to my supervisor, Professor Jerzy Wojciechowski who thought of the original idea for this dissertation, inspired my interest in the subject, and patiently encouraged me along the way. I thank him for all the efforts he put into supervising me throughout the dissertation.

I would also like to thank the other committee members, Professor Hong-Jian Lai, Professor John Goldwasser, Professor Rong Luo and Professor Elaine Eschen for their expertise and insightful feedback in preparing this dissertation.

Furthermore, I would like to acknowledge the Department of Mathematics at West Virginia University for providing an excellent study environment and I am grateful for the helpful and productive advice of faculty and colleagues in the department.

Words can hardly describe my thanks and appreciation to my parents who have been my source of inspiration, support, and guidance.

My appreciation goes to my brothers, sisters, relatives and friends who shared their words of advice and encouragement to finish this study.

Last, but not least, I would like to express my appreciation and gratitude to my wife who has been a constant source of support and encouragement during the challenges of graduate school and life.

Contents

Chapter 1

Introduction

The theory of topological vector spaces is one of the important areas of Functional Analysis concerning the study of a vector space endowed with a topology. Functional analysis is a large branch of mathematics which is essentially an attempt to combine ideas from Topology and Linear Algebra. The theory of topological vector spaces has important applications connected with Differential Calculus and Measure Theory in infinite-dimensional spaces [4]. The importance of the theory of topological vector spaces arises not only in mathematics but also in engineering and physics [5].

The purpose of this thesis is to attempt to generalize the concept of topological vector spaces. As known, a matroid is a structure that generalizes the notion of linear independence in vector spaces and graphs. Matroids are one of the richest and most useful ideas of our day [23]. In the same idea, we aim to find a space that generalizes the topological vector space and create a new branch of mathematics as an attempt to combine ideas from Topology and Matroid Theory. A topological vector space is a vector space over a field and a topological space at the same time provided that the vector space operations (addition and scalar multiplication) are topologically continuous functions. This means that topological vector spaces are endowed with two structures: a linear structure and a topological structure. Since every vector space is a finitary matroid, the usual strategy for generalizing the concept of topological vector spaces is to replace the linear structure of a topological vector space with a finitary matroidal structure.

In this research, we will proceed to introduce two new spaces called finite matroidal spaces and matrological spaces. Ideally, we would define a closure operator on a set that behaves as a matroidal closure operator on finite subsets and behaves as a topological closure operator on the infinite subsets. The main research point is to assess whether topological vector spaces are finite matroidal spaces and matrological spaces. One very important concept in the generalization, which will be used extensively throughout this thesis, is the common closure operator. The common closure operator is actually an operator defined on a set with a finitary matroidal structure and a topological structure by combining the finitary matroidal closure operator and the topological closure operator. This research will concentrate on investigating when combining a finitary matroidal closure operator and a topological closure operator on a given set produces a finite matroidal space or a matrological space. In particular, we will discuss what happens in the case of a topological vector space.

This thesis is organized as follows: Chapter 2 is concerned with important basic definitions and concepts about spaces in general. Pre-closure operators and equivalent axioms for defining spaces will be introduced as derived set operators, closure systems, neighborhood bases, neighborhood systems and simplicial complexes. Afterward, we provide important definitions and theorems from Matroid Theory, General Topology and Functional Analysis which are essential materials for our research. Chapter 3 will be an introduction to finite matroidal spaces and matrological spaces where basic definitions and main theorems will be given. Comparisons will be made between our new spaces and the old spaces such as pre-independence spaces [34] and exchange systems [8] that show the difference of our new spaces. Two important key theorems in our research that tell when a matroid or topological space is a matrological space will be presented. Chapter 4 is devoted to the study of common closure operators and their importants in combining finitary matroidal structures and topological structures. In Chapter 5, we will discuss the relationships of topological vector spaces with both finite matroidal spaces and matrological spaces to find out which space is a generalization of the concept of topological vector spaces. Finally, we state our conclusions in Chapter 6.

Chapter 2

Foundations

This thesis relies on a deep background from Matroid Theory, General Topology and Functional Analysis. This chapter is dedicated to introducing the concepts and results that are used in this research.

2.1 Spaces

In general, spaces are defined using operators as in [18], [15] and [34]. In this section, after we define spaces using pre-closure operators, equivalent axioms for defining spaces will be introduced as derived set operators, closure systems, neighborhood bases, neighborhood systems and simplicial complexes.

2.1.1 Pre-closure Operators

Definition 1. An operator on a set X is a function $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$.

Definition 2. Let $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ be an operator on a set X and $Y \subseteq X$.

1. The restriction of cl to Y, denoted by $\mathrm{cl}_{\restriction Y}$, is an operator $\mathrm{cl}_{\restriction Y} : \mathscr{P}(Y) \to \mathscr{P}(Y)$ defined by
$$\mathrm{cl}_{\restriction Y}(A) = \{y \in Y : y \in \mathrm{cl}(A)\} \text{ for each } A \subseteq Y.$$

2. The contraction of cl to Y, denoted by $\mathrm{cl}._Y$, is an operator $\mathrm{cl}._Y : \mathscr{P}(Y) \to \mathscr{P}(Y)$ defined by
$$\mathrm{cl}._Y(A) = \{y \in Y : y \in \mathrm{cl}(A \cup (X \setminus Y))\} \text{ for each } A \subseteq Y.$$

In other words, if $A \subseteq Y$ and $y \in Y$, then

$$y \in \mathrm{cl}_{\restriction Y}(A) \text{ if and only if } y \in \mathrm{cl}(A).$$
$$y \in \mathrm{cl}_{\cdot Y}(A) \text{ if and only if } y \in \mathrm{cl}(A \cup (X \setminus Y)).$$

Definition 3. Let X be any set. An operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ is called a pre-closure on X if and only if

(CL1) $A \subseteq \mathrm{cl}(A)$ for all $A \subseteq X$.

(CL2) $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ for all $A \subseteq B \subseteq X$.

cl is called a closure operator on X if it also satisfies the following condition:

(CL3) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all $A \subseteq X$.

Definition 4. A space (X, cl) is a set X together with a pre-closure operator cl on X.

Theorem 5. *Let cl be a pre-closure (closure) operator on a set X and $Y \subseteq X$. Then $\mathrm{cl}_{\restriction Y}$ and $\mathrm{cl}_{\cdot Y}$ are pre-closure (closure) operators on Y.*

Proof. First, we show that $\mathrm{cl}_{\restriction Y}$ is a closure operator on Y. So, we want to prove that $\mathrm{cl}_{\restriction Y}$ satisfies (CL1), (CL2) and (CL3). Let $A \subseteq Y$ and $y \in A$. Then $y \in Y$ and $y \in \mathrm{cl}(A)$. Thus, $y \in \mathrm{cl}_{\restriction Y}(A)$. Therefore, $A \subseteq \mathrm{cl}_{\restriction Y}(A)$, and hence (CL1) holds. Let $A \subseteq B \subseteq Y$ and $y \in \mathrm{cl}_{\restriction Y}(A)$. Then $y \in Y$ and $y \in \mathrm{cl}(A)$. Since $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, we have $y \in \mathrm{cl}(B)$. Thus, $y \in \mathrm{cl}_{\restriction Y}(B)$. Therefore $\mathrm{cl}_{\restriction Y}(A) \subseteq \mathrm{cl}_{\restriction Y}(B)$, and hence (CL2) holds. Let $A \subseteq Y$ and $y \in Y$. Then

$$y \in \mathrm{cl}_{\restriction Y}\left(\mathrm{cl}_{\restriction Y}(A)\right)$$

if and only if

$$y \in \mathrm{cl}\left(\mathrm{cl}_{\restriction Y}(A)\right)$$

if and only if

$$y \in \mathrm{cl}(\mathrm{cl}(A))$$

if and only if

$$y \in \mathrm{cl}(A)$$

if and only if

$$y \in \mathrm{cl}_{\restriction Y}(A).$$

Therefore, $\mathrm{cl}_{\restriction Y}\left(\mathrm{cl}_{\restriction Y}(A)\right) = \mathrm{cl}_{\restriction Y}(A)$, and hence (CL3) holds. Thus, $\mathrm{cl}_{\restriction Y}$ is a closure operator on Y.

Now, we show that $\mathrm{cl}._Y$ is a closure operator on Y. So, we want to prove that $\mathrm{cl}._Y$ satisfies (CL1), (CL2) and (CL3). Let $A \subseteq Y$ and $y \in A$. Then $y \in Y$ and $y \in \mathrm{cl}(A) \subseteq \mathrm{cl}(A \cup (X \setminus Y))$. Thus, $y \in \mathrm{cl}._Y(A)$. Therefore, $A \subseteq \mathrm{cl}._Y(A)$, and hence (CL1) holds. Let $A \subseteq B \subseteq Y$ and $y \in \mathrm{cl}._Y(A)$. Then $y \in Y$ and $y \in \mathrm{cl}(A \cup (X \setminus Y))$. Since $\mathrm{cl}(A \cup (X \setminus Y)) \subseteq \mathrm{cl}(B \cup (X \setminus Y))$, we have $y \in \mathrm{cl}(B \cup (X \setminus Y))$. Thus, $y \in \mathrm{cl}._Y(B)$. Therefore $\mathrm{cl}._Y(A) \subseteq \mathrm{cl}._Y(B)$, and hence (CL2) holds. Let $A \subseteq Y$ and $y \in Y$. Then

$$y \in \mathrm{cl}._Y(\mathrm{cl}._Y(A))$$

if and only if

$$y \in \mathrm{cl}(\mathrm{cl}._Y(A) \cup (X \setminus Y)) = \mathrm{cl}(\mathrm{cl}(A \cup (X \setminus Y)))$$

if and only if

$$y \in \mathrm{cl}(A \cup (X \setminus Y))$$

if and only if

$$y \in \mathrm{cl}._Y(A).$$

Therefore, $\mathrm{cl}._Y(\mathrm{cl}._Y(A)) = \mathrm{cl}._Y(A)$, and hence (CL3) holds. Thus, $\mathrm{cl}._Y$ is a closure operator on Y. □

Theorem 6. *Let cl be a pre-closure operator on a set X and $cl^* : \mathscr{P}(X) \to \mathscr{P}(X)$ be an operator on X defined by*

$$cl^*(A) = A \cup \{x \in X \setminus A : x \notin cl(X \setminus (A \cup \{x\}))\} \textit{ for all } A \subseteq X.$$

Then cl^* is also a pre-closure operator on X.

Proof. We want to show that cl satisfies (CL1) and (CL2). Let $A \subseteq X$. It is clear, from the definition of cl^*, that $A \subseteq \mathrm{cl}^*(A)$, and hence (CL1) holds. Now, let $A \subseteq B \subseteq X$ and let $x \in \mathrm{cl}^*(A)$. If $x \in A$, then $x \in B$. By (CL1), $x \in \mathrm{cl}^*(B)$. If $x \in X \setminus A$, then

$$x \notin \mathrm{cl}(X \setminus (A \cup \{x\})).$$

Since $A \subseteq B$, then

$$A \cup \{x\} \subseteq B \cup \{x\}$$

which leads to

$$X \setminus (B \cup \{x\}) \subseteq X \setminus (A \cup \{x\}).$$

By (CL2),

$$\mathrm{cl}(X \setminus (B \cup \{x\})) \subseteq \mathrm{cl}(X \setminus (A \cup \{x\})).$$

Since we have $x \notin \mathrm{cl}(X \setminus (A \cup \{x\}))$, then

$$x \notin \mathrm{cl}(X \setminus (B \cup \{x\})).$$

So, $x \in \mathrm{cl}^*(B)$, and hence (CL2) holds. Thus, cl^* is also a pre-closure operator on X. □

Definition 7. Let cl be a pre-closure operator on a set X. The pre-closure operator cl^* on X, as defined in Theorem 6, is called a dual of the pre-closure operator cl on X.

Theorem 8. *Let cl and cl' be pre-closure operators on a set X. Then cl and cl' are dual of each other if and only if for each disjoint sets $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ with $X = A \cup B \cup \{x\}$ we have*

$$x \in cl'(A) \text{ if and only if } x \notin cl(B).$$

Proof. Suppose that for each disjoint sets $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ with $X = A \cup B \cup \{x\}$ we have

$$x \in \mathrm{cl}'(A) \text{ if and only if } x \notin \mathrm{cl}(B).$$

We want to show that $\mathrm{cl}' = \mathrm{cl}^*$. Let $A \subseteq X$ and $x \in X$. Take $B = X \setminus (A \cup \{x\})$. So, we have $A, B \subseteq X$ are disjoint and $x \in X \setminus (A \cup B)$ with $X = A \cup B \cup \{x\}$. Thus,

$$x \in \mathrm{cl}'(A) \text{ if and only if } x \notin \mathrm{cl}(B) = \mathrm{cl}(X \setminus (A \cup \{x\})).$$

Therefore, $\mathrm{cl}' = \mathrm{cl}^*$.

Now, let $A, B \subseteq X$ be disjoint and $x \in X \setminus (A \cup B)$ with $X = A \cup B \cup \{x\}$. By Theorem 6, we have $x \in \mathrm{cl}^*(A)$ if and only if $x \notin \mathrm{cl}(X \setminus (A \cup \{x\})) = \mathrm{cl}(B)$. □

Corollary 9. *Let cl be a pre-closure operator on a set X. Then cl^* is the unique pre-closure on X such that for each disjoint sets $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ with $X = A \cup B \cup \{x\}$ we have*

$$x \in cl^*(A) \text{ or } x \in cl(B) \text{ and not in both.}$$

Proof. This is an immediate consequence of Theorem 8. □

Remark 10. The dual of a closure operator on a set X is not necessarily a closure operator on X.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topology on X. Then the topological closure operator on X is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases}$$

for all $A \subseteq X$. Using Theorem 6, the dual of the pre-closure cl on X is

$$\mathrm{cl}^*(A) = \begin{cases} X \setminus \{0\} & \text{if } 0 \notin A \text{ and } X \setminus A \text{ is infinite} \\ X & \text{if } 0 \in A \text{ or } X \setminus A \text{ is finite} \end{cases}$$

for all $A \subseteq X$. Now, take $A = \varnothing$. Then

$$X \setminus \{0\} = \mathrm{cl}^*(\varnothing) \neq \mathrm{cl}^*(\mathrm{cl}^*(\varnothing)) = \mathrm{cl}^*(X \setminus \{0\}) = X.$$

Thus, although we have cl is a closure operator on X, cl^* is a pre-closure operator not a closure operator on X. □

Theorem 11. *Let cl be a pre-closure operator on a set X. Then $cl^{**} := (cl^*)^* = cl$.*

Proof. Let $A \subseteq X$ and $x \in X$. Assume, without loss of generality, that $x \in X \setminus A$. Then

$$x \in (\mathrm{cl}^*)^*(A)$$

if and only if

$$x \notin \mathrm{cl}^*(X \setminus (A \cup \{x\}))$$

if and only if

$$x \in \mathrm{cl}(X \setminus ([X \setminus (A \cup \{x\})] \cup \{x\}))$$

if and only if

$$x \in \mathrm{cl}(A).$$

Then $(\mathrm{cl}^*)^* = \mathrm{cl}$. □

Theorem 12. *Let cl be a pre-closure operator on a set X and $Y \subseteq X$. Then*

1. $cl^*_{\restriction Y} = (cl._Y)^*$.

2. $cl^*._Y = \left(cl_{\restriction Y}\right)^*$.

Proof. First, we prove part (1). Let $A \subseteq Y$ and $y \in Y$. Assume, without loss of generality, that $y \in Y \setminus A$. Then

$$y \in \mathrm{cl}^*_{\restriction Y}(A)$$

if and only if

$$y \in \mathrm{cl}^*(A)$$

if and only if

$$y \notin \mathrm{cl}(X \setminus (A \cup \{y\}))$$

if and only if

$$y \notin \mathrm{cl}([Y \setminus (A \cup \{y\})] \cup (X \setminus Y))$$

if and only if

$$y \notin \mathrm{cl}._Y(Y \setminus (A \cup \{y\}))$$

if and only if

$$y \in (\mathrm{cl}._Y)^*(A).$$

Then $\mathrm{cl}^*_{\restriction Y} = (\mathrm{cl}._Y)^*$. Now, we prove part (2). By Theorem 11 and then part (1), we have

$$\mathrm{cl}^*._Y = (\mathrm{cl}^*._Y)^{**} = ((\mathrm{cl}^*._Y)^*)^* = \left(\mathrm{cl}^{**}_{\restriction Y}\right)^* = \left(\mathrm{cl}_{\restriction Y}\right)^*.$$

Then $\mathrm{cl}^*._Y = \left(\mathrm{cl}_{\restriction Y}\right)^*$. □

Theorem 13. *Let cl be a pre-closure operator on a set X and $Z \subseteq Y \subseteq X$. Then*

1. $\left(cl_{\restriction Y}\right)_{\restriction Z} = cl_{\restriction Z}$.

2. $(cl._Y)._Z = cl._Z$.

Proof. First, we prove part (1). Let $A \subseteq Z$ and $z \in Z$. Assume, without loss of generality, that $z \in Z \setminus A$. Then

$$z \in \left(\mathrm{cl}_{\restriction Y}\right)_{\restriction Z}(A)$$

if and only if

$$z \in \mathrm{cl}_{\restriction Y}(A)$$

if and only if

$$z \in \mathrm{cl}(A)$$

if and only if

$$z \in \mathrm{cl}_{\restriction Z}(A).$$

Then $\left(\mathrm{cl}_{\restriction Y}\right)_{\restriction Z} = \mathrm{cl}_{\restriction Z}$. Now, we prove part (2). Let $A \subseteq Z$ and $z \in Z$. Assume, without loss of generality, that $z \in Z \setminus A$. Then

$$z \in (\mathrm{cl}_{\cdot Y})_{\cdot Z}(A)$$

if and only if

$$z \in \mathrm{cl}_{\cdot Y}(A \cup (Y \setminus Z))$$

if and only if

$$z \in \mathrm{cl}([A \cup (Y \setminus Z)] \cup (X \setminus Y))$$

if and only if

$$z \in \mathrm{cl}(A \cup (X \setminus Z))$$

if and only if

$$z \in \mathrm{cl}_{\cdot Z}(A).$$

Then $(\mathrm{cl}_{\cdot Y})_{\cdot Z} = \mathrm{cl}_{\cdot Z}$. □

Definition 14. Let cl be a pre-closure operator on a set X and $Y \subseteq X$. Define

1. $\mathrm{cl}_{\setminus Y} := \mathrm{cl}_{\restriction (X \setminus Y)}$ and we say that Y is removed out.

2. $\mathrm{cl}_{/Y} := \mathrm{cl}_{\cdot (X \setminus Y)}$ and we say that Y is contracted out.

In other words, if $A \subseteq X \setminus Y$ and $x \in X \setminus Y$, then

$$x \in \mathrm{cl}_{\setminus Y}(A) \text{ if and only if } x \in \mathrm{cl}(A).$$
$$x \in \mathrm{cl}_{/Y}(A) \text{ if and only if } x \in \mathrm{cl}(A \cup Y).$$

If $Y, Z \subseteq X$ with $Y \cap Z = \varnothing$, then define

$$\mathrm{cl}_{\setminus Y/Z} := \left(\mathrm{cl}_{\setminus Y}\right)_{/Z}$$
$$\mathrm{cl}_{/Y\setminus Z} := \left(\mathrm{cl}_{/Y}\right)_{\setminus Z}$$

The next theorem says that the order of taking restrictions and contractions does not matter.

Theorem 15. *Let cl be a pre-closure operator on a set X and $Y, Z \subseteq X$ with $Y \cap Z = \varnothing$. Then* $cl_{\setminus Y/Z} = cl_{/Z\setminus Y}$.

Proof. Let $A \subseteq X \setminus (Y \cup Z)$ and $x \in X \setminus (Y \cup Z)$. Assume, without loss of generality, that $x \notin A$. Then

$$x \in \mathrm{cl}_{\setminus Y/Z}(A)$$

if and only if

$$x \in \left(\mathrm{cl}_{\setminus Y}\right)_{/Z}(A)$$

if and only if

$$x \in \mathrm{cl}_{\setminus Y}(A \cup Z)$$

if and only if

$$x \in \mathrm{cl}(A \cup Z)$$

if and only if

$$x \in \mathrm{cl}_{/Z}(A)$$

if and only if

$$x \in \left(\mathrm{cl}_{/Z}\right)_{\setminus Y}(A)$$

if and only if

$$x \in \mathrm{cl}_{/Z\setminus Y}(A).$$

Then $\mathrm{cl}_{\setminus Y/Z} = \mathrm{cl}_{/Z\setminus Y}$. □

Definition 16. Let cl be a pre-closure operator on a set X.

1. $A \subseteq X$ is called closed (or cl-closed) in X if and only if $\mathrm{cl}(A) = A$.

2. $A \subseteq X$ is called open (or cl-open) in X if and only if $X \setminus A$ is closed in X.

3. $A \subseteq X$ is called a spanning (or dense) set in X if and only if $\mathrm{cl}(A) = X$.

4. $I \subseteq X$ is called an independent (or cl-independent or discrete) set in X if and only if $x \notin \mathrm{cl}(I \setminus \{x\})$ for each $x \in I$.

5. $B \subseteq X$ is called a base in X if and only if B is a maximal[1] independent set in X.

[1] B is a maximal independent set in X if and only if B is independent in X but $B \cup \{x\}$ is not independent in X for each $x \in X \setminus B$.

6. $D \subseteq X$ is called a dependent (or cl-dependent) set in X if and only if D is not independent in X.

7. $C \subseteq X$ is called a circuit in X if and only if C is a minimal[2] dependent set in X.

2.1.2 Derived Set Operators

We define derived set operators, and then we see how to induce pre-closure operators and derived set operators from each other.

Definition 17. Let X be any set. An operator $\partial : \mathscr{P}(X) \to \mathscr{P}(X)$ is called a derived set operator on X if and only if

(DR1) $\partial(A) \subseteq \partial(B)$ for all $A \subseteq B \subseteq X$.

(DR2) If $x \in \partial(A)$, then $x \in \partial(A \setminus \{x\})$ for all $A \subseteq X$.

An element $x \in \partial(A)$ is called a limit (accumulation) point of A and $\partial(A)$, the set of all limit points of A, is called the derived set of A.

Theorem 18. *Let ∂ be a derived set operator on a set X and define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = A \cup \partial(A) \text{ for all } A \subseteq X.$$

Then cl is a pre-closure operator on X.

Proof. We want to show that cl satisfies (CL1) and (CL2). Let $A \subseteq X$ and $x \in A$. Then $x \in A \cup \partial(A) = \mathrm{cl}(A)$. So, $A \in \mathrm{cl}(A)$, and hence cl satisfies (CL1). Let $A \subseteq B \subseteq X$ and $x \in \mathrm{cl}(A)$. Thus, by (DR1), we get

$$x \in A \cup \partial(A) \subseteq B \cup \partial(B) = \mathrm{cl}(B).$$

So, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. Hence cl satisfies (CL2). Therefore, cl is a pre-closure operator on X. □

Definition 19. A pre-closure operator on a set X as defined in Theorem 18 is called a pre-closure operator on X induced by the derived set operator ∂.

[2] C is a minimal dependent set in X if and only if C is not independent in X but $C \setminus \{x\}$ is independent in X for each $x \in C$.

Theorem 20. *Let cl be a pre-closure operator on a set X and define $\partial : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$\partial(A) = \{x \in X : x \in cl(A \setminus \{x\})\} \text{ for all } A \subseteq X.$$

Then ∂ is a derived set operator on X.

Proof. We want to show that ∂ satisfies (DR1) and (DR2). Let $A \subseteq B \subseteq X$ and $x \in \partial(A)$. So, $x \in \mathrm{cl}(A \setminus \{x\})$. By (CL2), $x \in \mathrm{cl}(B \setminus \{x\})$. Therefore, $x \in \partial(B)$. Thus, $\partial(A) \subseteq \partial(B)$, and hence ∂ satisfies (DR1). Let $A \subseteq X$ and $x \in \partial(A)$. Thus,

$$x \in \mathrm{cl}(A \setminus \{x\}) = \mathrm{cl}((A \setminus \{x\}) \setminus \{x\}).$$

This means that $x \in \partial(A \setminus \{x\})$. Therefore, ∂ satisfies (DR2). Hence cl is a derived set operator on X. □

Definition 21. A derived set operator on a set X as defined in Theorem 20 is called a derived set operator on X induced by the pre-closure operator cl.

The following theorem shows that pre-closure operators and derived set operators work nicely with each other.

Theorem 22. *Let X be any set.*

1. *Let ∂ be a derived set operator on X and cl be the pre-closure operator on X induced by ∂. Then ∂ is the derived set operator on X induced by cl.*

2. *Let cl be a pre-closure operator on a set X and ∂ be the derived set operator on X induced by cl. Then cl is the pre-closure operator on X induced by ∂.*

Proof. First, we prove part (1). Let ∂' be the derived set operator on X induced by cl. We want to show that $\partial' = \partial$. Let $A \subseteq X$ and $x \in \partial'(A)$. By Theorem 20, we have $x \in \mathrm{cl}(A \setminus \{x\})$. By Theorem 18, we get $x \in (A \setminus \{x\}) \cup \partial(A \setminus \{x\})$. Since $x \notin A \setminus \{x\}$, then $x \in \partial(A \setminus \{x\})$. Using (DR1), we obtain $x \in \partial(A)$. Thus, $\partial'(A) \subseteq \partial(A)$. Now, let $A \subseteq X$ and $x \in \partial(A)$. Using (DR2), we obtain $x \in \partial(A \setminus \{x\})$. So,

$$x \in (A \setminus \{x\}) \cup \partial(A \setminus \{x\}).$$

By Theorem 18, we get $x \in \mathrm{cl}(A \setminus \{x\})$. By Theorem 20, we have $x \in \partial'(A)$. Thus, $\partial(A) \subseteq \partial'(A)$. Therefore, $\partial'(A) = \partial(A)$ for each $A \subseteq X$.

Now, we prove part (2). Let cl' be the pre-closure operator on X induced by ∂. We want to show that $\mathrm{cl}' = \mathrm{cl}$. Let $A \subseteq X$ and $x \in \mathrm{cl}'(A)$. Assume, without loss of generality, that $x \in \mathrm{cl}'(A) \setminus A$. By Theorem 18, we get $x \in A \cup \partial(A)$. Since $x \notin A$, then

$$x \in \partial(A) = \partial(A \setminus \{x\}).$$

By Theorem 20, we have $x \in \mathrm{cl}((A \setminus \{x\}) \setminus \{x\}) = \mathrm{cl}(A)$. Thus, $\mathrm{cl}'(A) \subseteq \mathrm{cl}(A)$. Now, let $A \subseteq X$ and $x \in \mathrm{cl}(A)$. Assume, without loss of generality, that $x \in \mathrm{cl}(A) \setminus A$. Since $x \notin A$, then $x \in \mathrm{cl}(A \setminus \{x\})$. By Theorem 20, we have

$$x \in \partial(A) \subseteq A \cup \partial(A).$$

By Theorem 18, we get $x \in \mathrm{cl}'(A)$. Thus, $\mathrm{cl}(A) \subseteq \mathrm{cl}'(A)$. Therefore, $\mathrm{cl}'(A) = \mathrm{cl}(A)$ for each $A \subseteq X$. □

Theorem 23. *Let ∂ be a derived set operator on a set X and cl be the pre-closure operator on X induced by ∂. Then cl is a closure operator on X if and only if*

$$\partial(A \cup \partial(A)) \subseteq A \cup \partial(A) \text{ for each } A \subseteq X.$$

Proof. Suppose that cl is a closure operator on X. We want to show that for each $A \subseteq X$, $\partial(A \cup \partial(A)) \subseteq A \cup \partial(A)$. Suppose, by way of contradiction, that there is $A \subseteq X$ and $x \in X$ such that $x \in \partial(A \cup \partial(A)) \setminus A \cup \partial(A)$. So, we have

$$x \in (A \cup \partial(A)) \cup \partial(A \cup \partial(A)) \text{ and } x \notin A \cup \partial(A).$$

By Theorem 18, we get

$$x \in \mathrm{cl}(\mathrm{cl}(A)) \text{ and } x \notin \mathrm{cl}(A).$$

Thus, $\mathrm{cl}(\mathrm{cl}(A)) \neq \mathrm{cl}(A)$, which leads cl is not a closure operator on X, contradiction. Therefore, $\partial(A \cup \partial(A)) \subseteq A \cup \partial(A)$ for each $A \subseteq X$.

Now, suppose that

$$\partial(A \cup \partial(A)) \subseteq A \cup \partial(A) \text{ for each } A \subseteq X.$$

By (CL2), we have

$$\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A)) \text{ for each } A \subseteq X. \tag{2.1.1}$$

We just need to show that $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$ for each $A \subseteq X$. Let $A \subseteq X$ and $x \in \mathrm{cl}(\mathrm{cl}(A))$. By Theorem 18, we get

$$x \in \mathrm{cl}(A) \cup \partial(\mathrm{cl}(A)).$$

If $x \in \mathrm{cl}(A)$, then it is done. If $x \in \partial(\mathrm{cl}(A))$, then $x \in \partial(A \cup \partial(A))$. Using the assumption, we get

$$x \in \partial(A \cup \partial(A)) \subseteq A \cup \partial(A) = \mathrm{cl}(A).$$

Then $x \in \mathrm{cl}(A)$. Thus,

$$\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A) \text{ for each } A \subseteq X. \tag{2.1.2}$$

By (2.1.1) and (2.1.2), we obtain

$$\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A) \text{ for each } A \subseteq X.$$

Hence cl is a closure operator on X. □

2.1.3 Closure Systems

Now, we define closure systems, and then we see how to induce pre-closure operators and closure systems from each other.

Definition 24. Let X be any set and $\mathscr{F} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. $\mathscr{F}$ is called a closure system on X if and only if

(F1) $X \in \mathscr{F}$.

(F2) $\bigcap \mathscr{A} \in \mathscr{F}$ for each $\mathscr{A} \subseteq \mathscr{F}$ with $\mathscr{A} \neq \varnothing$.

Theorem 25. *Let $\mathscr{F}$ be a closure system on a set X and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ be defined by*

$$cl(A) = \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X. \tag{2.1.3}$$

Then cl is a closure operator on X.

Proof. We want to show that cl satisfies (CL1), (CL2) and (CL3). Let $A \subseteq X$ and $x \in A$. For each $C \in \mathscr{F}$ with $A \subseteq C$, we have $x \in C$. Than

$$x \in \bigcap \{C \in \mathscr{F} : A \subseteq C\} = \mathrm{cl}(A).$$

Therefore, $A \subseteq \mathrm{cl}(A)$, and hence (CL1) holds. Let $A \subseteq B \subseteq X$. Thus,

$$\{C \in \mathscr{F} : B \subseteq C\} \subseteq \{C \in \mathscr{F} : A \subseteq C\}.$$

Therefore,

$$\bigcap \{C \in \mathscr{F} : A \subseteq C\} \subseteq \bigcap \{C \in \mathscr{F} : B \subseteq C\}.$$

So,

$$\mathrm{cl}(A) \subseteq \mathrm{cl}(B).$$

Hence (CL2) holds. Let $A \subseteq X$. By (CL1) and (CL2), we get

$$\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A)).$$

So, we only need to show that $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$. Note, By (F2), that

$$\mathrm{cl}(A) = \bigcap \{C \in \mathscr{F} : A \subseteq C\} \in \mathscr{F}.$$

Thus,

$$\mathrm{cl}(A) \in \{C \in \mathscr{F} : \mathrm{cl}(A) \subseteq C\}. \tag{2.1.4}$$

From (2.1.3), we have

$$\mathrm{cl}(\mathrm{cl}(A)) = \bigcap \{C \in \mathscr{F} : \mathrm{cl}(A) \subseteq C\}.$$

Using (2.1.4), we get

$$\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A),$$

and hence (CL3) holds. Thus, cl is a closure operator on X. □

Definition 26. In Theorem 25, the closure operator cl is called the closure operator on X induced by the closure system $\mathscr{F}$.

Example 27. Consider $X = \mathbb{N}$ and $\mathscr{F} = \{C \subseteq X : C \text{ is finite}\} \cup \{X\}$. Then the closure operator cl on X induced by $\mathscr{F}$ is

$$\begin{aligned} \mathrm{cl}(A) &= \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X \\ &= \begin{cases} A & A \text{ is finite} \\ X & A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X. \end{aligned}$$

Theorem 28. *Let cl be a pre-closure (not necessarily closure) operator on a set X and*

$$\begin{aligned}\mathscr{F} &= \{C \subseteq X : C \text{ is cl-closed in } X\} \\ &= \{C \subseteq X : cl(C) = C\}.\end{aligned}$$

Then $\mathscr{F}$ is a closure system on X.

Proof. We want to prove that $\mathscr{F}$ satisfies (F1) and (F2). Since X is the whole set, $\mathrm{cl}(X) = X$. So, $X \in \mathscr{F}$, and hence (F1) holds. Let $\mathscr{A} \subseteq \mathscr{F}$ with $\mathscr{A} \neq \varnothing$. We want to show that $\mathrm{cl}\left(\bigcap \mathscr{A}\right) = \bigcap \mathscr{A}$. By (CL2), we have $\bigcap \mathscr{A} \subseteq \mathrm{cl}\left(\bigcap \mathscr{A}\right)$. So, we only need to show that $\mathrm{cl}\left(\bigcap \mathscr{A}\right) \subseteq \bigcap \mathscr{A}$. We claim that $\mathrm{cl}\left(\bigcap \mathscr{A}\right) \subseteq A$ for each $A \in \mathscr{A}$. To show this claim, note that

$$\bigcap \mathscr{A} \subseteq A \text{ for each } A \in \mathscr{A}.$$

By (CL2), we get

$$\mathrm{cl}\left(\bigcap \mathscr{A}\right) \subseteq \mathrm{cl}(A) \text{ for each } A \in \mathscr{A}.$$

For each $A \in \mathscr{A}$, we have $A \in \mathscr{F}$. So, $\mathrm{cl}(A) = A$ for each $A \in \mathscr{A}$. Then we have

$$\mathrm{cl}\left(\bigcap \mathscr{A}\right) \subseteq A \text{ for each } A \in \mathscr{A}.$$

Thus,

$$\mathrm{cl}\left(\bigcap \mathscr{A}\right) \subseteq \bigcap \mathscr{A}.$$

Thus, $\mathrm{cl}\left(\bigcap \mathscr{A}\right) = \bigcap \mathscr{A}$. Therefore, $\bigcap \mathscr{A} \in \mathscr{F}$, and hence $\mathscr{F}$ satisfies (F2). So, $\mathscr{F}$ is a closure system on X. □

Definition 29. In Theorem 28, the closure system $\mathscr{F}$ is called the closure system on X induced by the pre-closure operator cl.

Theorems 25 and 28 show that pre-closure operators do not work nicely with closure systems. It turns out if we have a pre-closure operator cl but not a closure operator on a set and we induce a closure system $\mathscr{F}$, then $\mathscr{F}$ will not induce the pre-closure operator cl.

Example 30. Take $X = \{1, 2\}$ and define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} \{1\} & \text{if } A = \varnothing \\ X & \text{if } A \neq \varnothing \end{cases} \quad \text{for all } A \subseteq X.$$

Clearly, cl is a pre-closure operator on X. But cl is not a closure operator on X because

$$\{1\} = \mathrm{cl}(\varnothing) \neq \mathrm{cl}(\mathrm{cl}(\varnothing)) = \mathrm{cl}(\{1\}) = X.$$

The closure system $\mathscr{F}$ on X induced by cl is

$$\mathscr{F} = \{C \subseteq X : \mathrm{cl}(C) = C\} = \{X\}.$$

The closure operator cl$'$ on X induced by $\mathscr{F}$ is

$$\begin{aligned} \mathrm{cl}'(A) &= \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X \\ &= X \text{ for all } A \subseteq X. \end{aligned}$$

Now,

$$\{1\} = \mathrm{cl}(\varnothing) \neq \mathrm{cl}'(\varnothing) = X.$$

Hence $\mathrm{cl}' \neq \mathrm{cl}$.

Now, if we add the axiom that cl is a closure operator and not just a pre-closure operator, we will have the following theorem that shows that closure operators and closure systems work nicely with each other.

Theorem 31. *Let X be any set.*

1. *Let $\mathscr{F}$ be a closure system on X and cl be the closure operator on X induced by $\mathscr{F}$. Then $\mathscr{F}$ is the closure system on X induced by cl.*

2. *Let cl be a closure operator on X and $\mathscr{F}$ be the closure system on X induced by cl. Then cl is the closure operator on X induced by $\mathscr{F}$.*

Proof. First, we prove part (1). By Theorem 25, $\mathscr{F}$ induces the closure operator

$$\mathrm{cl}(A) = \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X \tag{2.1.5}$$

Let $\mathscr{F}'$ be the closure system on X induced by cl. By Theorem 28, we have

$$\mathscr{F}' = \{A \subseteq X : \mathrm{cl}(A) = A\}.$$

We want to show that $\mathscr{F}' = \mathscr{F}$. Let $A \in \mathscr{F}'$. So, $A \subseteq X$ and $\mathrm{cl}(A) = A$. By (2.1.5), we have

$$\bigcap \{C \in \mathscr{F} : A \subseteq C\} = A.$$

By (F2), we get

$$A = \bigcap \{C \in \mathscr{F} : A \subseteq C\} \in \mathscr{F}.$$

So,

$$\mathscr{F}' \subseteq \mathscr{F}. \tag{2.1.6}$$

Now, let $A \in \mathscr{F}$. By (2.1.5), we have

$$\mathrm{cl}(A) = \bigcap \{C \in \mathscr{F} : A \subseteq C\}.$$

Since $A \in \mathscr{F}$ and $A \subseteq A$, then

$$\mathrm{cl}(A) = A.$$

and hence $A \in \mathscr{F}'$. So

$$\mathscr{F} \subseteq \mathscr{F}'. \tag{2.1.7}$$

By (2.1.6) and (2.1.7), we get $\mathscr{F}' = \mathscr{F}$.

Now, we prove part (2). By Theorem 28, cl induces the closure system

$$\mathscr{F} = \{C \subseteq X : \mathrm{cl}(C) = C\}.$$

Let cl' be the closure operator on X induced by $\mathscr{F}$. By Theorem 25, we have

$$\mathrm{cl}'(A) = \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X.$$

We want to show that $\mathrm{cl}' = \mathrm{cl}$. Let $A \subseteq X$. By (CL2), for each $C \in \mathscr{F}$ with $A \subseteq C$ we have

$$\mathrm{cl}(A) \subseteq \mathrm{cl}(C) = C.$$

Thus,

$$\mathrm{cl}(A) \subseteq \bigcap \{C \in \mathscr{F} : A \subseteq C\} = \mathrm{cl}'(A).$$

So,

$$\mathrm{cl}(A) \subseteq \mathrm{cl}'(A). \tag{2.1.8}$$

By (CL1) and (CL3), $A \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. So, we have $\mathrm{cl}(A) \in \mathscr{F}$ with $A \subseteq \mathrm{cl}(A)$. Thus,

$$\bigcap \{C \in \mathscr{F} : A \subseteq C\} \subseteq \mathrm{cl}(A).$$

So,

$$\mathrm{cl}'(A) \subseteq \mathrm{cl}(A). \tag{2.1.9}$$

By (2.1.8) and (2.1.9), we get

$$\mathrm{cl}'(A) = \mathrm{cl}(A).$$

Then $\mathrm{cl}' = \mathrm{cl}$. □

Theorem 32. *Let $\mathscr{F}$ be a closure system on a set X. Then any element $C \in \mathscr{F}$ is closed in X.*

Proof. Let $C \in \mathscr{F}$ and cl be the closure operator on X induced by $\mathscr{F}$. Then

$$\mathrm{cl}(C) = \cap \{D \in \mathscr{F} : C \subseteq D\}.$$

Since $C \in \mathscr{F}$ and $C \subseteq C$, then $\mathrm{cl}(C) = C$. Thus C is closed in X. □

2.1.4 Neighborhood Bases

This section introduces an equivalent approach to get pre-closure operators and closure operators on a given set. For each element in the set, we assign a family of subsets called a neighborhood base.

Definition 33. Let X be any set. The function $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ is called a neighborhood base on X if and only if for each $x \in X$ and for each $A \in \mathscr{N}(x)$, we have $x \in A$. An element $A \in \mathscr{N}(x)$ is called a basic neighborhood of $x \in X$.

Therefore, for each $x \in X$, $\mathscr{N}(x)$ is the collection of all basic neighborhoods of x.

Example 34. Let $X = \{x, y\}$ and define $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}(x) = \{\{x\}\}$$
$$\mathscr{N}(y) = \{\{y\}\}.$$

Then $\mathscr{N}$ is a neighborhood base on X.

Example 35. Let $X = \{x, y\}$ and define $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}(x) = \{\{x\}, \{y\}\}$$
$$\mathscr{N}(y) = \{\{x, y\}\}.$$

Then $\mathscr{N}$ is not a neighborhood base on X because $\{y\} \in \mathscr{N}(x)$ but $x \notin \{y\}$.

Example 36. Let $X = \mathbb{N}$ and define $\mathscr{N} : \mathbb{N} \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}(n) = \{\{1, 2, 3, \ldots, n\}, \{2, 3, \ldots, n\}, \ldots, \{n-1, n\}, \{n\}\} \text{ for all } n \in \mathbb{N}.$$

Then $\mathscr{N}$ is a neighborhood base on X.

The next two theorems tell us how to get a neighborhood base from a pre-closure operator on a given set and vice versa.

Theorem 37. *Let cl be a pre-closure operator on a set X. Define $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by*

$$\mathscr{N}(x) = \{A \subseteq X : x \notin cl(X \setminus A)\} \text{ for all } x \in X. \tag{2.1.10}$$

Then $\mathscr{N}$ is a neighborhood base on X.

Proof. Let $x \in X$ and $A \in \mathscr{N}(x)$. By (2.1.10), we have $x \notin \mathrm{cl}(X \setminus A)$. By (CL1), we get $x \notin X \setminus A$. Thus, $x \in A$, and hence $\mathscr{N}$ is a neighborhood base on X. □

Definition 38. The neighborhood base $\mathscr{N}$ on X, defined in Theorem 37, is called the neighborhood base on X induced by cl.

Theorem 39. *Let $\mathscr{N}$ be a neighborhood base on a set X and define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left\{x \in X : A \cap A' \neq \varnothing \text{ for each } A' \in \mathscr{N}(x)\right\} \text{ for all } A \subseteq X. \tag{2.1.11}$$

Then cl is a pre-closure operator on X.

Proof. We want to show that cl satisfies (CL1) and (CL2). Let $A \subseteq X$ and $x \in A$. For each $A' \in \mathscr{N}(x)$, we have $\{x\} \subseteq A \cap A'$ which means $A \cap A' \neq \varnothing$. By (2.1.11), we get $x \in \mathrm{cl}(A)$. So, $A \subseteq \mathrm{cl}(A)$, and hence (CL1) holds. Let $A \subseteq B \subseteq X$ and $x \in \mathrm{cl}(A)$. We need to show that $x \in \mathrm{cl}(B)$. Let $A' \in \mathscr{N}(x)$. Using (2.1.11), we have $A \cap A' \neq \varnothing$. Since $A \subseteq B$, we get $B \cap A' \neq \varnothing$. So, for each $A' \in \mathscr{N}(x)$, we have $B \cap A' \neq \varnothing$. By (2.1.11), we get $x \in \mathrm{cl}(B)$. Therefore, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and hence cl satisfies (CL2). Then cl is a pre-closure operator on X. □

Definition 40. The pre-closure operator cl on X, defined in Theorem 39, is called the pre-closure operator on X induced by $\mathscr{N}$.

Example 41. Consider $X = \{1,2\}$ and define a neighborhood base $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}(1) = \{X\}$$
$$\mathscr{N}(2) = \{\{2\}, X\}.$$

The pre-closure operator on X induced by $\mathscr{N}$ is

$$\operatorname{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{1\} & \text{if } A = \{1\} \\ X & \text{otherwise} \end{cases}$$

for all $A \subseteq X$.

The following theorem shows that when a pre-closure operator on a set induces a neighborhood base, the induced neighborhood base induces the same pre-closure operator. However, when a neighborhood base on a set induces a pre-closure operator, the induced pre-closure operator may not induce the same neighborhood base.

Theorem 42. *Let X be any set.*

1. *Let cl be a pre-closure operator on X and $\mathscr{N}$ be the neighborhood base on X induced by cl. Then cl is the pre-closure operator on X induced by $\mathscr{N}$.*

2. *Let $\mathscr{N}$ be a neighborhood base on X and $\mathscr{N}'$ be the neighborhood base on X induced by the pre-closure operator cl on X induced by $\mathscr{N}$. Then*

$$\mathscr{N}(x) \subseteq \mathscr{N}'(x) \text{ for each } x \in X,$$

and equality may not hold.

Proof. First, we prove part (1). By Theorem 37, the neighborhood base $\mathscr{N}$ on X induced by cl is

$$\mathscr{N}(x) = \{A \subseteq X : x \notin \operatorname{cl}(X \setminus A)\} \text{ for all } x \in X. \tag{2.1.12}$$

Let cl' be the pre-closure operator on X induced by $\mathscr{N}$. By Theorem 39, we have

$$\mathrm{cl}'(A) = \left\{x \in X : A \cap A' \neq \varnothing \text{ for each } A' \in \mathscr{N}(x)\right\} \text{ for all } A \subseteq X. \tag{2.1.13}$$

We want to show that $\mathrm{cl}' = \mathrm{cl}$ which means we want to show that $\mathrm{cl}'(A) = \mathrm{cl}(A)$ for all $A \subseteq X$. Let $A \subseteq X$ and $x \in \mathrm{cl}'(A)$. By (2.1.13), we have

$$A \cap A' \neq \varnothing \text{ for each } A' \in \mathscr{N}(x).$$

Suppose, by way of contradiction, that $x \notin \mathrm{cl}(A)$. So, $x \notin \mathrm{cl}(X \setminus (X \setminus A))$. By (2.1.12), we get $X \setminus A \in \mathscr{N}(x)$. But

$$A \cap (X \setminus A) = \varnothing.$$

By (2.1.13), we get $x \notin \mathrm{cl}'(A)$, a contradiction. Thus, $x \in \mathrm{cl}(A)$. Thus,

$$\mathrm{cl}'(A) \subseteq \mathrm{cl}(A). \tag{2.1.14}$$

Now, let $x \in \mathrm{cl}(A)$. Suppose, by way of contradiction, that $x \notin \mathrm{cl}'(A)$. By (2.1.13), there is $A' \in \mathscr{N}(x)$ such that $A \cap A' = \varnothing$. By (2.1.12), we get $x \notin \mathrm{cl}(X \setminus A')$. Since $A \cap A' = \varnothing$, we have

$$A \subseteq \left(X \setminus A'\right).$$

By (CL2), we get

$$\mathrm{cl}(A) \subseteq \mathrm{cl}\left(X \setminus A'\right).$$

Since we have $x \notin \mathrm{cl}(X \setminus A')$, then $x \notin \mathrm{cl}(A)$, a contradiction. Thus, $x \in \mathrm{cl}'(A)$. Thus,

$$\mathrm{cl}(A) \subseteq \mathrm{cl}'(A). \tag{2.1.15}$$

By (2.1.14) and (2.1.15), we get

$$\mathrm{cl}'(A) = \mathrm{cl}(A) \text{ for each } A \subseteq X.$$

Hence $\mathrm{cl}' = \mathrm{cl}$.

Now, we show part (2). Let $x \in X$ and $A \in \mathscr{N}(x)$. Since

$$(X \setminus A) \cap A = \varnothing,$$

then, by Theorem 39, we have $x \notin \mathrm{cl}(X \setminus A)$. By Theorem 37, we get $A \in \mathcal{N}'(x)$. So, $\mathcal{N}(x) \subseteq \mathcal{N}'(x)$ for each $x \in X$. Now, let X be any set and define a neighborhood base $\mathcal{N} : X \to \mathcal{P}(\mathcal{P}(X))$ by

$$\mathcal{N}(x) = \{\{x\}\} \text{ for each } x \in X.$$

$\mathcal{N}$ induces the pre-closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned}\mathrm{cl}(A) &= \left\{x \in X : A \cap A' \neq \varnothing \text{ for each } A' \in \mathcal{N}(x)\right\} \text{ for all } A \subseteq X.\\ &= A \text{ for all } A \subseteq X.\end{aligned}$$

But cl induces the neighborhood base $\mathcal{N}' : X \to \mathcal{P}(\mathcal{P}(X))$ defined by

$$\begin{aligned}\mathcal{N}'(x) &= \{A \subseteq X : x \notin \mathrm{cl}(X \setminus A)\} \text{ for each } x \in X\\ &= \{A \subseteq X : x \in A\} \text{ for each } x \in X.\end{aligned}$$

Therefore,

$$\mathcal{N}(x) \subsetneq \mathcal{N}'(x) \text{ for each } x \in X.$$

□

In the proof of Theorem 42 part (2), the different neighborhood bases $\mathcal{N}$ and $\mathcal{N}'$ induce the same pre-closure operator. The next theorem provides a sufficient condition of when we induce the same pre-closure operator from different neighborhood bases.

Definition 43. Any neighborhood bases $\mathcal{N}$ and $\mathcal{N}'$ are called equivalent if and only if for each $x \in X$, the following hold:

1. for each $A \in \mathcal{N}(x)$, there is $A' \in \mathcal{N}'(x)$ with $A' \subseteq A$.

2. for each $A' \in \mathcal{N}'(x)$, there is $A \in \mathcal{N}(x)$ with $A \subseteq A'$.

Theorem 44. *Let $\mathcal{N}$, $\mathcal{N}'$ be neighborhood bases on a set X and cl, cl$'$ be the pre-closure operators on X induced by $\mathcal{N}, \mathcal{N}'$, respectively. Then $\mathrm{cl} = \mathrm{cl}'$ if and only if $\mathcal{N}$ and $\mathcal{N}'$ are equivalent.*

Proof. Recall that the pre-closure operators cl, cl$'$ on X induced by $\mathcal{N}$, $\mathcal{N}'$ are

$$\begin{aligned}\mathrm{cl}(Y) &= \{x \in X : Y \cap A \neq \varnothing \text{ for each } A \in \mathcal{N}(x)\} \text{ for all } Y \subseteq X.\\ \mathrm{cl}'(Y) &= \{x \in X : Y \cap A' \neq \varnothing \text{ for each } A' \in \mathcal{N}'(x)\} \text{ for all } Y \subseteq X.\end{aligned} \tag{2.1.16}$$

Assume that $\mathrm{cl} = \mathrm{cl}'$. We want to show that the conditions (1) and (2) in Definition 43 hold. Let $x \in X$ and $A \in \mathscr{N}(x)$. We know that

$$(X \setminus A) \cap A = \varnothing.$$

By (2.1.16), $x \notin \mathrm{cl}(X \setminus A)$. Since $\mathrm{cl} = \mathrm{cl}'$, we get $x \notin \mathrm{cl}'(X \setminus A)$. By (2.1.16), there is $A' \in \mathscr{N}'(x)$ such that

$$(X \setminus A) \cap A' = \varnothing.$$

So, $A' \subseteq A$. So, the condition (1) holds. Similarly, the condition (2) also holds.

Now, assume that $\mathscr{N}$ and $\mathscr{N}'$ are equivalent. So, for each $x \in X$ the conditions (1) and (2) in Definition 43 hold. We want to show that $\mathrm{cl} = \mathrm{cl}'$. So, we want to show that $\mathrm{cl}(Y) = \mathrm{cl}'(Y)$ for each $Y \subseteq X$. Let $Y \subseteq X$ and $x \notin \mathrm{cl}'(Y)$. By (2.1.16), there is $A' \in \mathscr{N}'(x)$ such that

$$Y \cap A' = \varnothing.$$

By the condition (2), there is $A \in \mathscr{N}(x)$ with $A \subseteq A'$. Thus,

$$Y \cap A = \varnothing.$$

Thus, $x \notin \mathrm{cl}(Y)$. Therefore, $\mathrm{cl}(Y) \subseteq \mathrm{cl}'(Y)$. By symmetry, $\mathrm{cl}'(Y) \subseteq \mathrm{cl}(Y)$. Hence $\mathrm{cl}(Y) = \mathrm{cl}'(Y)$ for each $Y \subseteq X$. □

In Example 41, the induced pre-closure operator cl from the neighborhood base $\mathscr{N}$ is a closure operator on X. This is not true in general.

Remark 45. Let $\mathscr{N}$ be a neighborhood base on a set X. Then the pre-closure operator on X induced by $\mathscr{N}$ might not be a closure operator on X.

Proof. Consider $X = \{1, 2\}$ and define a neighborhood base $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}(1) = \varnothing$$
$$\mathscr{N}(2) = \{X\}.$$

The pre-closure operator on X induced by $\mathscr{N}$ is

$$\mathrm{cl}(A) = \begin{cases} \{1\} & \text{if } A = \varnothing \\ X & \text{if } A \neq \varnothing \end{cases} \quad \text{for all } A \subseteq X.$$

In Example 30 we proved that cl is not a closure operator on X. □

In Theorem 39 and Remark 45, we saw that the pre-closure operator induced from a neighborhood base on a set may not be a closure operator. Now, we discuss what axioms we can add to a neighborhood base to induce a closure operator, not just a pre-closure operator.

Definition 46. Let $\mathscr{N}$ be a neighborhood base on a set X, cl be the pre-closure operator induced by $\mathscr{N}$ and $\mathscr{F}$ be the closure system induced by cl. We say $\mathscr{F}$ is the closure system induced by $\mathscr{N}$.

Theorem 47. *Let $\mathscr{N}$ be a neighborhood base on a set X and $\mathscr{F}$ be the closure system induced by $\mathscr{N}$. Then $U \subseteq X$ is open with respect to $\mathscr{F}$ if and only if for each $x \in U$, there is $A \in \mathscr{N}(x)$ with $A \subseteq U$.*

Proof. Recall that the pre-closure operator on X induced by $\mathscr{N}$ is

$$\mathrm{cl}(Y) = \{x \in X : Y \cap A \neq \varnothing \text{ for each } A \in \mathscr{N}(x)\} \text{ for all } Y \subseteq X. \tag{2.1.17}$$

Suppose that $U \subseteq X$ is open with respect to $\mathscr{F}$. So, $X \setminus U$ is closed in X. Therefore, $\mathrm{cl}(X \setminus U) = X \setminus U$ which means $x \notin \mathrm{cl}(X \setminus U)$ for each $x \in U$. By (2.1.17), for each $x \in U$ there is $A \in \mathscr{N}(x)$ such that

$$(X \setminus U) \cap A \neq \varnothing,$$

which leads to $A \subseteq U$.

Now, suppose that for each $x \in U$ there is $A \in \mathscr{N}(x)$ with $A \subseteq U$. We want to show that $U \subseteq X$ is open in X with respect to $\mathscr{F}$. So, we want to show that $X \setminus U$ is closed in X. This means that we want to show that $\mathrm{cl}(X \setminus U) = X \setminus U$. By (CL1), we know that

$$X \setminus U \subseteq \mathrm{cl}(X \setminus U). \tag{2.1.18}$$

Suppose, by way of contradiction, that $\mathrm{cl}(X \setminus U) \nsubseteq X \setminus U$. So, there is $x \in \mathrm{cl}(X \setminus U) \setminus (X \setminus U)$. Thus, $x \in U$. By the assumption, there is $A \in \mathscr{N}(x)$ such that $A \subseteq U$. Thus,

$$(X \setminus U) \cap A \neq \varnothing,$$

which leads to $x \notin \mathrm{cl}(X \setminus U)$, a contradiction. Therefore,

$$\mathrm{cl}(X \setminus U) \subseteq X \setminus U. \tag{2.1.19}$$

By (2.1.18) and (2.1.19), we get

$$\mathrm{cl}(X \setminus U) = X \setminus U.$$

So, $U \subseteq X$ is open in X with respect to $\mathscr{F}$. □

Definition 48. A neighborhood base $\mathscr{N}$ on a set X is called an open (closed) neighborhood base on X if and only if for each $x \in X$, each $A \in \mathscr{N}(x)$ is open (closed) with respect to the closure system $\mathscr{F}$ induced by $\mathscr{N}$.

Theorem 49. *Let $\mathscr{N}$ be a neighborhood base on a set X and cl be the pre-closure operator on X induced by $\mathscr{N}$. Then cl is a closure operator on X if and only if $\mathscr{N}$ is equivalent to an open neighborhood base on X.*

Proof. Recall that the pre-closure operator cl on X induced by $\mathscr{N}$ is

$$\mathrm{cl}(Y) = \{x \in X : Y \cap A \neq \varnothing \text{ for each } A \in \mathscr{N}(x)\} \text{ for all } Y \subseteq X. \tag{2.1.20}$$

Suppose that cl is a closure operator on X.

Claim 50. For each $x \in X$ and $A \in \mathscr{N}(x)$, there is $A' \subseteq A$ such that $x \in A'$ and A' is open in X with respect to the closure system $\mathscr{F}$ induced by $\mathscr{N}$.

Proof. Let $x \in X$ and $A \in \mathscr{N}(x)$. We know that

$$(X \setminus A) \cap A = \varnothing.$$

By (2.1.20), $x \notin \mathrm{cl}(X \setminus A)$. Since cl is a closure operator on X, we have

$$\mathrm{cl}(\mathrm{cl}(X \setminus A)) = \mathrm{cl}(X \setminus A).$$

So, $\mathrm{cl}(X \setminus A)$ is closed in X. Take

$$A' = X \setminus \mathrm{cl}(X \setminus A).$$

By (CL1), we get

$$A' \subseteq X \setminus (X \setminus A) = A.$$

Thus, $A' \subseteq A$ such that $x \in A'$ and A' is open in X. □

Using Claim 50, define $\mathscr{N}' : X \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}'(x) = \left\{A' \subseteq X : A' \text{ is the open subset of } A \in \mathscr{N}(x)\right\} \text{ for each } x \in X.$$

For each $x \in X$ and for each $A' \in \mathscr{N}'(x)$, we have $x \in A'$ and A' is open in X. So, $\mathscr{N}'$ is an open neighborhood base on X. Now, we want to show that $\mathscr{N}$ and $\mathscr{N}'$ are equivalent. Let $x \in X$. We want to show that the conditions (1) and (2) in Definition 43 hold. By Claim 50, for each $A \in \mathscr{N}(x)$ there is $A' \in \mathscr{N}'(x)$ with $A' \subseteq A$. So, the condition (1) holds. Let $A' \in \mathscr{N}'(x)$. Thus, A' is open in X and $x \in A'$. By Theorem 47, there is $A \in \mathscr{N}(x)$ such that $A \subseteq A'$. So, the condition (2) holds as well. Therefore, $\mathscr{N}$ is equivalent to $\mathscr{N}$'.

Now, suppose that there is an open neighborhood base $\mathscr{N}'$ on X such that $\mathscr{N}$ is equivalent to $\mathscr{N}'$. By Theorem 44, $\mathscr{N}$ and $\mathscr{N}'$ induce the same pre-closure cl on X. Thus, let

$$\begin{aligned}\mathrm{cl}(Y) &= \{x \in X : Y \cap A \neq \varnothing \text{ for each } A \in \mathscr{N}(x)\} \text{ for all } Y \subseteq X.\\ &= \left\{x \in X : Y \cap A' \neq \varnothing \text{ for each } A' \in \mathscr{N}'(x)\right\} \text{ for all } Y \subseteq X.\end{aligned}$$

We want to show that cl is a closure operator on X. By (CL1), we know that

$$\mathrm{cl}(Y) \subseteq \mathrm{cl}(\mathrm{cl}(Y)) \text{ for each } Y \subseteq X. \tag{2.1.21}$$

So, we only need to show that $\mathrm{cl}(\mathrm{cl}(Y)) \subseteq \mathrm{cl}(Y)$ for each $Y \subseteq X$. Suppose, by way of contradiction, that there is $Y \subseteq X$ such that

$$\mathrm{cl}(\mathrm{cl}(Y)) \nsubseteq \mathrm{cl}(Y).$$

So, there is $x \in \mathrm{cl}(\mathrm{cl}(Y)) \setminus \mathrm{cl}(Y)$. Since $x \notin \mathrm{cl}(Y)$, by (2.1.20), there is $A \in \mathscr{N}(x)$ such that

$$Y \cap A = \varnothing.$$

Since $\mathscr{N}$ and $\mathscr{N}'$ are equivalent, there is $A' \in \mathscr{N}'(x)$ such that $A' \subseteq A$. Since $\mathscr{N}'$ is an open neighborhood base on X, then A' is open in X. So, $X \setminus A'$ is closed in X and $Y \cap A' = \varnothing$. Thus, we have $X \setminus A'$ is closed in X and $Y \subseteq X \setminus A'$. Therefore,

$$X \setminus A' \in \{C \in \mathscr{F} : Y \subseteq C\},$$

and then

$$\mathrm{cl}(Y) = \bigcap\{C \in \mathscr{F} : Y \subseteq C\} \subseteq X \setminus A'.$$

Thus,

$$A' \cap \mathrm{cl}(Y) = \varnothing.$$

By (2.1.20), $x \notin \mathrm{cl}(\mathrm{cl}(Y))$, a contradiction. Therefore,

$$\mathrm{cl}(\mathrm{cl}(Y)) \subseteq \mathrm{cl}(Y) \text{ for each } Y \subseteq X. \tag{2.1.22}$$

By (2.1.21) and (2.1.22), we obtain

$$\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y) \text{ for each } Y \subseteq X.$$

Hence cl is a closure operator on X. □

Corollary 51. *Let cl be a pre-closure operator on a set X. Then cl is a closure operator on X if and only if cl is induced by an open neighborhood base on X.*

Proof. Let $\mathscr{N}$ be the neighborhood base on X induced by cl, see Theorem 37. By Theorem 42, cl is pre-closure operator on X induced by $\mathscr{N}$. Now, suppose that cl is a closure operator on X. By Theorem 49, $\mathscr{N}$ is equivalent to an open neighborhood base $\mathscr{N}'$ on X. By Theorem 44, $\mathscr{N}$ and $\mathscr{N}'$ induce cl. Thus, cl is induced by the open neighborhood base $\mathscr{N}'$ on X.

Now, suppose that there is an open neighborhood base $\mathscr{N}$ on X such that $\mathscr{N}$ induces cl. Then $\mathscr{N}$ is equivalent to itself. By Theorem 49, cl is a closure operator on X. □

2.1.5 Neighborhood Systems

In Theorem 42, we saw that when a neighborhood base on a set induces a pre-closure operator, the induced pre-closure operator does not have to induce the same neighborhood base. In this section, we add one axiom to a neighborhood base to solve the problem.

Definition 52. Let X be any set. A neighborhood base $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ is called a neighborhood system on X if and only if for each $x \in X$ and if $A \in \mathscr{N}(x)$ and $B \subseteq X$ with $A \subseteq B$, then $B \in \mathscr{N}(x)$. An element $A \in \mathscr{N}(x)$ is called a neighborhood of $x \in X$.

Therefore, for each $x \in X$, $\mathscr{N}(x)$ is the collection of all neighborhoods of x.

Example 53. Let X be any infinite set and define a neighborhood base $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by

$$\mathscr{N}(x) = \{A \subseteq X : x \in A \text{ and } X \setminus A \text{ is finite}\} \text{ for all } x \in X.$$

Let $x \in X$ and let $A \in \mathscr{N}(x)$ and $B \subseteq X$ with $A \subseteq B$. So, $x \in A$ and $X \setminus A$ is finite. Thus, $x \in B$ and $X \setminus B$ is finite. Therefore, $B \in \mathscr{N}(x)$. Hence $\mathscr{N}$ is a neighborhood system on X.

Theorem 54. *Let cl be a pre-closure operator on a set X. Then the neighborhood base $\mathscr{N}$ on X induced by cl is a neighborhood system on X.*

Proof. By Theorem 37, we have

$$\mathscr{N}(x) = \{A \subseteq X : x \notin \mathrm{cl}(X \setminus A)\} \text{ for each } x \in X.$$

We want to show that the neighborhood base $\mathscr{N}$ is a neighborhood system on X. Let $x \in X$ and let $A \in \mathscr{N}(x)$ and $B \subseteq X$ with $A \subseteq B$. So, $x \notin \mathrm{cl}(X \setminus A)$. Since $A \subseteq B$, then $X \setminus B \subseteq X \setminus A$. By (CL2), we get $\mathrm{cl}(X \setminus B) \subseteq \mathrm{cl}(X \setminus A)$. Thus, $x \notin \mathrm{cl}(X \setminus B)$. Therefore, $B \in \mathscr{N}(x)$. Hence $\mathscr{N}$ is a neighborhood system on X. □

Theorem 55. *Let X be any set.*

1. *Let cl be a pre-closure operator on X and $\mathscr{N}$ be the neighborhood system on X induced by cl. Then cl is the pre-closure operator on X induced by $\mathscr{N}$.*

2. *Let $\mathscr{N}$ be a neighborhood system on X and cl be the pre-closure operator on X induced by $\mathscr{N}$. Then $\mathscr{N}$ is the neighborhood system on X induced by cl.*

Proof. Part (1) is an immediate consequence of Theorems 42 and 54. Now, we prove part (2). Let $\mathscr{N}'$ be the neighborhood system on X induced by cl. By Theorem 42, $\mathscr{N}'$ also induces the pre-closure operator cl. We want to show that $\mathscr{N} = \mathscr{N}'$. From Theorem 42 part (2), we have

$$\mathscr{N}(x) \subseteq \mathscr{N}'(x) \text{ for each } x \in X. \tag{2.1.23}$$

Now, let $x \in X$ and let $A' \in \mathscr{N}'(x)$. By Theorem 44 part (2), there is $A \in \mathscr{N}(x)$ such that $A \subseteq A'$. Since $A \in \mathscr{N}(x)$, then $A' \in \mathscr{N}(x)$. So,

$$\mathscr{N}'(x) \subseteq \mathscr{N}(x) \text{ for each } x \in X. \tag{2.1.24}$$

By (2.1.23) and (2.1.24), we obtain $\mathscr{N} = \mathscr{N}'$. □

Example 56. Let X be any infinite set. The following pre-closure operator cl and neighborhood system $\mathscr{N}$ on X induce each other.

$$\mathrm{cl}(A) = \begin{cases} A & A \text{ is finite} \\ X & A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

$$\mathscr{N}(x) = \{A \subseteq X : x \in A \text{ and } X \setminus A \text{ is finite}\} \text{ for all } x \in X.$$

2.1.6 Simplicial Complexes

In this section we discuss how to obtain a pre-closure operator on a set from a simplicial complex and how to get a simplicial complex on a set from a pre-closure operator.

Definition 57. Let X be any set and $\mathscr{I} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. $\mathscr{I}$ is called a simplicial complex on X if and only if

(I1) $\varnothing \in \mathscr{I}$.

(I2) If $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.

Example 58. Consider $X = \{a, b, c\}$ and

$$\mathscr{I}_1 = \{\varnothing, \{a\}, \{c\}, \{a, c\}\}$$
$$\mathscr{I}_2 = \{\varnothing, \{a\}, \{b\}, \{a, c\}\}.$$

Then $\mathscr{I}_1$ is a simplicial complex on X but $\mathscr{I}_2$ is not a simplicial complex on X.

Theorem 59. *Let cl be a pre-closure operator on a set X and let*

$$\begin{aligned}\mathscr{I} &= \{I \subseteq X : I \text{ is an independent set in } X\} \\ &= \{I \subseteq X : x \notin cl(I \setminus \{x\}) \text{ for each } x \in I\}.\end{aligned}$$

Then $\mathscr{I}$ is a simplicial complex on X.

Proof. We want to show that $\mathscr{I}$ satisfies (I1) and (I2). Suppose, by way of contradiction, that $\varnothing \notin \mathscr{I}$. So, there is $x \in \varnothing$ such that $x \in \mathrm{cl}(\varnothing \setminus \{x\})$. But $x \in \varnothing$ is a contradiction. Thus, $\varnothing \in \mathscr{I}$, and hence $\mathscr{I}$ satisfies (I1). Let $I \in \mathscr{I}$ and $J \subseteq I$. Thus, $x \notin \mathrm{cl}(I \setminus \{x\})$ for each $x \in I$. By (CL2), since $J \subseteq I$, we have

$$\mathrm{cl}(J \setminus \{x\}) \subseteq \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I.$$

Therefore, $x \notin \text{cl}(J \setminus \{x\})$ for each $x \in J$. So, $J \in \mathscr{I}$, and hence $\mathscr{I}$ satisfies (I2). Then $\mathscr{I}$ is a simplicial complex on X. □

Definition 60. The simplicial complex $\mathscr{I}$ defined in Theorem 59 is called a simplicial complex on X induced by the pre-closure operator cl.

Theorem 61. *Let $\mathscr{I}$ be a simplicial complex on a set X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = A \cup \{x \in X \setminus A : \text{ there is } I \subseteq A \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\} \tag{2.1.25}$$

for all $A \subseteq X$. Then cl is a pre-closure operator on X.

Proof. We want to show that cl satisfies (CL1) and (CL2). It is clear from (2.1.25) that $A \subseteq \text{cl}(A)$ for all $A \subseteq X$. Hence cl satisfies (CL1). Let $A \subseteq B \subseteq X$ and let $x \in \text{cl}(A)$. If $x \in B$. Using (CL1), $x \in \text{cl}(B)$. If $x \in X \setminus B$. Since $A \subseteq B$, $x \in X \setminus A$. By (2.1.25), there is $I \subseteq A$ such that $I \in \mathscr{I}$ and $I \cup \{x\} \notin \mathscr{I}$. Since $A \subseteq B$, then $I \subseteq A$ such that $I \in \mathscr{I}$ and $I \cup \{x\} \notin \mathscr{I}$. By (2.1.25), $x \in \text{cl}(B)$. Therefore, $\text{cl}(A) \subseteq \text{cl}(B)$, and hence cl satisfies (CL2). Then cl is a pre-closure operator on X. □

Definition 62. The pre-closure operator cl defined in Theorem 61 is called a pre-closure operator on X induced by the simplicial complex $\mathscr{I}$.

We have seen that pre-closure operators work nicely, for example, with neighborhood systems. It turns out that when we have a pre-closure operator cl on a set and we induce a neighborhood base $\mathscr{N}$ from cl and then we induce a pre-closure operator cl$'$ from $\mathscr{N}$ we get cl$'$ = cl. In addition, if we have a neighborhood base $\mathscr{N}$ on a set and we induce a pre-closure operator cl from $\mathscr{N}$ and then we induce a neighborhood base $\mathscr{N}'$ from cl, we get $\mathscr{N}' = \mathscr{N}$. The next theorem shows in general that pre-closure operators and simplicial complexes do not work nicely with each other, and we will see in Section 2.2.3 that they work nicely with each other when we add more axioms which will be the axioms of matroids.

Remark 63. Let X be a set.

1. Let cl be a pre-closure operator on X and cl$'$ be the pre-closure operator on X induced by the simplicial complex $\mathscr{I}$ on X induced by cl. Then cl$' \neq$ cl.

2. Let $\mathscr{I}$ be a simplicial complex on X and $\mathscr{I}'$ be the simplicial complex on X induced by the pre-closure operator cl on X induced by $\mathscr{I}$. Then $\mathscr{I} \subseteq \mathscr{I}'$ but equality may not hold.

Proof. To show part (1), consider $X = \{1, 2\}$ and a pre-closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{1\} & \text{if } A = \{1\} \\ X & \text{otherwise} \end{cases} \text{ for all } A \subseteq X.$$

By Theorem 59, the simplicial complex $\mathscr{I}$ on X induced by cl is

$$\begin{aligned} \mathscr{I} &= \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} \\ &= \{\varnothing, \{1\}, \{2\}\}. \end{aligned}$$

By Theorem 61, the pre-closure operator cl' on X induced by $\mathscr{I}$ is

$$\begin{aligned} \mathrm{cl}'(Y) &= Y \cup \{x \in X \setminus Y : \text{ there is } I \subseteq Y \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\} \\ &= \begin{cases} \varnothing & \text{if } Y = \varnothing \\ X & \text{if } Y \neq \varnothing \end{cases} \text{ for all } Y \subseteq X. \end{aligned}$$

Take $Y = \{1\} \subseteq X$. We get

$$X = \mathrm{cl}'(\{1\}) \neq \mathrm{cl}(\{1\}) = \{1\}.$$

Thus, $\mathrm{cl}' \neq \mathrm{cl}$.

Now, we show part (2). By Theorem 61, the pre-closure operator cl on X induced by $\mathscr{I}$ is

$$\mathrm{cl}(Y) = Y \cup \{x \in X \setminus Y : \text{ there is } I \subseteq Y \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\}$$

for each $Y \subseteq X$. By Theorem 59, the simplicial complex $\mathscr{I}'$ on X induced by cl is

$$\mathscr{I}' = \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\}$$

Let $I \in \mathscr{I}$. We want to show that $I \in \mathscr{I}'$. So, we want to show that $x \notin \mathrm{cl}(I \setminus \{x\})$ for each $x \in I$. Suppose, by way of contradiction, that there is $x \in I$ such that $x \in \mathrm{cl}(I \setminus \{x\})$. Therefore, there is $J \subseteq I \setminus \{x\}$ such that $J \in \mathscr{I}$ and $J \cup \{x\} \notin \mathscr{I}$. This contradicts that $\mathscr{I}$ is a simplicial complex on X because $J \subseteq I \setminus \{x\}$ leads to $J \cup \{x\} \subseteq I$, and then $J \cup \{x\} \in \mathscr{I}$. Thus, $x \notin \mathrm{cl}(I \setminus \{x\})$ for each $x \in I$. Therefore, $I \in \mathscr{I}'$. Hence $\mathscr{I} \subseteq \mathscr{I}'$.

Now, consider $X = \{1, 2, 3, \ldots\}$ and the simplicial complex

$$\mathscr{I} = \{I \subseteq X : I \text{ is finite}\}.$$

By Theorem 61, the pre-closure operator cl on X induced by $\mathscr{I}$ is

$$\begin{aligned} \mathrm{cl}(Y) &= Y \cup \{x \in X \setminus Y : \text{ there is } I \subseteq Y \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\} \\ \mathrm{cl}(Y) &= Y \text{ for all } Y \subseteq X. \end{aligned}$$

By Theorem 59, the simplicial complex $\mathscr{I}'$ on X induced by cl is

$$\mathscr{I}' = \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} = \mathscr{P}(X).$$

Now, note that $X \in \mathscr{I}'$ but $X \notin \mathscr{I}$. Hence $\mathscr{I} \subsetneq \mathscr{I}'$. □

Theorem 64. *Let $\mathscr{I}$ be a simplicial complex on a set X. Define*

$$\mathscr{I}^* = \{\varnothing\} \cup \{A \subseteq X : \textit{ there is maximal } B \in \mathscr{I} \textit{ such that } A \cap B = \varnothing\}.$$

Then $\mathscr{I}^*$ is a simplicial complex on X.

Proof. $\varnothing \in \mathscr{I}^*$. So, $\mathscr{I}^*$ satisfies (I1). We want to show that $\mathscr{I}^*$ satisfies (I2). Let $A \in \mathscr{I}^*$ and $A' \subseteq A$. If $A' = \varnothing$, then $A' \in \mathscr{I}^*$. If $A' \neq \varnothing$, then also $A \neq \varnothing$. So, there is maximal $B \in \mathscr{I}$ such that $A \cap B = \varnothing$. Since $A' \subseteq A$, then $A' \cap B = \varnothing$. Thus, $A' \in \mathscr{I}^*$. So, $\mathscr{I}^*$ satisfies (I1). Hence $\mathscr{I}^*$ is a simplicial complex on X. □

Definition 65. The simplicial complex $\mathscr{I}^*$ as defined in Theorem 64 is called the dual simplicial complex of $\mathscr{I}$.

Example 66. Let X be any set and take $\mathscr{I} = \{A \subseteq X : A \text{ is finite}\}$. Then the dual simplicial complex of $\mathscr{I}$ is $\mathscr{I}^* = \{\varnothing\}$.

2.2 Matroid Theory

2.2.1 Vector Spaces

This section introduces definitions and theorems from linear algebra that are used in this thesis. The proofs and more details can be found in [3] and [12]. Throughout this thesis, the field $\mathbb{F}$ is usually $\mathbb{R}$ or $\mathbb{C}$, and the elements of $\mathbb{F}$ are called scalars.

Definition 67. Let X be a set.

1. An addition operation on X is a function $+ : X \times X \longrightarrow X$ defined by

$$+(u, v) = u + v \text{ for each } u, v \in X.$$

2. A scalar multiplication operation on X is a function $\cdot : \mathbb{F} \times X \longrightarrow X$ defined by

$$\cdot(\lambda, v) = \lambda v \text{ for each } v \in X \text{ and each scalar } \lambda.$$

Definition 68. A vector space over a field $\mathbb{F}$ is a set X along with an addition operation and a scalar multiplication operation such that the following conditions hold:

(VS1) $u + v = v + u$ for all $u, v \in X$.

(VS2) For all $u, v, w \in X$ and for all scalars λ, μ,

$$(u + v) + w = u + (v + w)$$
$$(\lambda\mu)v = \lambda(\mu v).$$

(VS3) There is an element $0 \in X$ such that $v + 0 = v$ for all $v \in X$.

(VS4) For each $v \in X$, there is $w \in X$ such that $v + w = 0$.

(VS5) $1v = v$ for all $v \in X$.

(VS6) For all $u, v \in X$ and for all scalars λ, μ,

$$\lambda(u + v) = \lambda u + \lambda v$$
$$(\lambda + \mu)v = \lambda v + \mu v.$$

The elements of the vector space X are called vectors.

Example 69. The set $\mathbb{F}^{\mathbb{N}} = \left\{(x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{F}\right\}$ of all sequences of elements of $\mathbb{F}$ with the addition and scalar multiplication operations

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n + y_n)_{n\in\mathbb{N}}$$
$$\lambda(x_n)_{n\in\mathbb{N}} = (\lambda x_n)_{n\in\mathbb{N}}$$

is a vector space over $\mathbb{F}$ called the space of all sequences.

Definition 70. Let X be a vector space over a field $\mathbb{F}$. A subset $Y \subseteq X$ is called a vector subspace of X if and only if Y is also a vector space over $\mathbb{F}$ with the operations of addition and scalar multiplication defined on X.

Theorem 71. *Let X be a vector space over a field $\mathbb{F}$ and $Y \subseteq X$. Then Y is a vector subspace of X if and only if Y satisfies the following three conditions for the operations of addition and scalar multiplication defined on X.*

1. $0 \in Y$.

2. $u + v \in Y$ *for each* $u, v \in Y$.

3. $\lambda v \in Y$ *for each* $v \in Y$ *and for each scalar* λ.

Definition 72. A sequence space is any vector subspace of $\mathbb{F}^{\mathbb{N}}$.

Example 73. The following subsets of $\mathbb{F}^{\mathbb{N}}$ are sequence spaces.

1. For $0 < p < \infty$, $\ell_p = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$

2. For $p = \infty$, $\ell_\infty = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is a bounded sequence}\right\}$.

3. $c = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : \lim\limits_{n\to\infty} x_n \text{ exists}\right\}$.

4. $c_0 = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : \lim\limits_{n\to\infty} x_n = 0\right\}$.

5. $c_{00} = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} : x_n = 0 \text{ for all but a finite number of } n\right\}$.

Note that $c_{00} \subseteq \ell_p \subseteq \ell_q \subseteq c_0 \subseteq c \subseteq \ell_\infty$ for all $p \leq q$.

Theorem 74. *Let X be a vector space over a field $\mathbb{F}$ and $Y_1, \ldots, Y_n$ be vector subspaces of X. Then the sum*

$$Y_1 + \cdots + Y_n = \{y_1 + \cdots + y_n : y_1 \in Y_1, \ldots, y_n \in Y_n\}$$

is the smallest vector subspace of X containing $Y_1, \ldots, Y_n$.

Definition 75. Let $Y_1, \ldots, Y_n$ be vector subspaces of a vector space X. The sum $Y_1 + \cdots + Y_n$ is called a direct sum, denoted by $Y_1 \oplus \cdots \oplus Y_n$, if and only if for each $x \in Y_1 + \cdots + Y_n$ there are unique $y_1 \in Y_1, \ldots, y_n \in Y_n$ such that $x = y_1 + \cdots + y_n$.

Theorem 76. *Let $Y_1, \dots, Y_n$ be vector subspaces of a vector space X. Then the sum $Y_1 + \cdots + Y_n$ is a direct sum if and only if for all $y_1 \in Y_1, \dots, y_n \in Y_n$ if $y_1 + \cdots + y_n = 0$ then $y_1 = \cdots = y_n = 0$.*

Theorem 77. *Let Y_1, Y_2 be vector subspaces of a vector space X. Then $Y_1 + Y_2$ is a direct sum if and only if $Y_1 \cap Y_2 = \{0\}$.*

Definition 78. Let X be a vector space over a field $\mathbb{F}$ and Y be a nonempty subset of X. A vector $v \in X$ is called a linear combination of Y if there is a finite number of vectors $v_1, \dots, v_n \in Y$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$.

Definition 79. Let X be a vector space over a field $\mathbb{F}$ and Y be a nonempty subset of X. The linear span of Y, denoted $\operatorname{span}(Y)$, is the set of all linear combinations of the vectors in Y. The linear span of $\varnothing$ is defined to be $\operatorname{span}(\varnothing) = \{0\}$.

Theorem 80. *The linear span of any subset Y of a vector space X is the smallest vector subspace of X containing Y.*

Theorem 81. *Let X be a vector space over the field $\mathbb{F}$ and define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(Y) = span(Y) \text{ for all } Y \subseteq X.$$

Then cl is a closure operator on X.

Definition 82. A closure operator cl on a vector space X as defined in Theorem 81 is called the linear closure operator on X.

Theorem 83. *Let cl be the linear closure of a vector space X. Then any vector subspace is cl-closed in X.*

Definition 84. A vector space X over a field $\mathbb{F}$ is called finite-dimensional if and only if there is a finite subset $F \subseteq X$ such that $\operatorname{span}(F) = X$. X is called infinite-dimensional if it is not finite-dimensional.

Definition 85. Let X be a vector space over $\mathbb{F}$ and $A \subseteq X$.

1. A is balanced if and only if $\lambda x \in A$ for each $x \in A$ and for each scalar $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$.

2. A is absorbent if and only if for each $x \in X$ there is $a > 0$ such that $\lambda x \in A$ for each scalar $\lambda \in \mathbb{F}$ with $|\lambda| \leq a$.

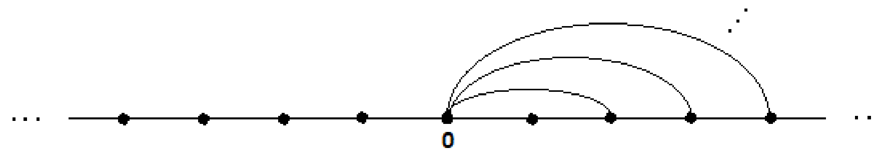

Figure 2.2.1: The Bean Graph.

2.2.2 Graphs

This section presents definitions and theorems from graph theory that are used in this thesis. The proofs and more details can be found in [11] and [35].

Definition 86. A graph is a triple $G = \langle V, E, \tau \rangle$ consisting of disjoint, possibly infinite, sets V and E together with a function $\tau : E \to \mathcal{P}(V)$ satisfying

$$|\tau(e)| \in \{1, 2\} \text{ for each } e \in E.$$

The elements of V are called the vertices of G and the elements of E are called the edges of G.

The vertex set of a graph G is referred to as $V(G)$ and its edge set as $E(G)$. If $V(G) = \varnothing$ and $E(G) = \varnothing$, we say that G is an empty graph.

Example 87. The graph $G = \langle V, E, \tau \rangle$ in Figure 2.2.1 is called the Bean Graph [14] where

$$\begin{aligned} V &= \mathbb{Z} \\ E &= \{\{n, n+1\} : n \in \mathbb{Z}\} \cup \{\{0, n\} : n \geq 2\} \\ \tau(e) &= e \text{ for all } e \in E \end{aligned}$$

Definition 88. A vertex x is called incident with an edge e if $x \in e$. The two vertices incident with an edge are called its endvertices or ends, and we say that an edge joins its ends. An edge $e = \{x, y\}$ is usually written as xy or yx.

Definition 89. Let G be a graph and $X, Y \subseteq V(G)$. If $x \in X$ and $y \in Y$, then xy is called an X–Y edge. The set of all X–Y edges in a set $E(G)$ is denoted by $E(X, Y)$.

Definition 90. Two vertices x, y of a graph G are called adjacent if xy is an edge of G. Two different edges are called adjacent if they have an endvertix in common.

Definition 91. If all the vertices of a graph G are pairwise adjacent, then G is called complete. A complete graph on n vertices is denoted by K^n.

Definition 92. A subgraph G' of a graph G, written as $G' \subseteq G$, is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If $G' \subseteq G$ and $G' \neq G$, then G' is a proper subgraph of G.

Definition 93. A path is a non-empty graph P such that

$$V(P) = \{x_0, x_1, \ldots, x_k\}$$
$$E(P) = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$$

where the x_i are all distinct. The vertices x_0 and x_k are linked by P and are called its endvertices or ends. We say that P is a path from x_0 to x_k and write $P = x_0x_1 \cdots x_k$. If a graph G has a subgraph P which is a path from u to v, we say that G has a path from u to v.

Definition 94. The number of edges of a path is its length, and the path of length k is denoted by P^k. Note that when $k = 0$ we have $P^0 = K^1$.

Definition 95. If $P = x_0x_1 \cdots x_{k-1}$ is a path and $k \geqslant 3$, then the graph $C = P \cup \{x_{k-1}x_0\}$ is called a (finite) cycle and denoted by $x_0x_1 \cdots x_{k-1}x_0$. If a graph G has a subgraph which is a cycle, we say that G contains a cycle.

Definition 96. The length of a cycle is its number of edges, and the cycle of length k is called a k-cycle and denoted by C^k.

Definition 97. A (finite) circuit of a graph G is the edge set of a cycle of G.

Definition 98. A graph G is called connected if and only if G is non-empty and for any two vertices $u, v \in V(G)$ there is a path in G between u and v.

Definition 99. A forest is a graph that does not contain any cycles. A connected forest is called a tree.

Definition 100. A subdivision of a graph G is any graph obtained from G by subdividing some or all of its edges by drawing new vertices on those edges.

Definition 101. A graph G is called locally finite if and only if each vertex of G has finite degree.

Definition 102. Let G be a graph. A nonempty subset $F \subseteq E(G)$ is called a cut in G if and only if there is a partition $\{V_1, V_2\}$ of V such that $F = E(V_1, V_2)$. A minimal non-empty cut in G is called a bond.

Definition 103. A ray is an infinite graph R such that

$$V(R) = \{x_0, x_1, x_2, \ldots\}$$
$$E(R) = \{x_0x_1, x_1x_2, x_2x_3, \ldots\};$$

and a double ray is an infinite graph R such that

$$V(R) = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$$
$$E(R) = \{\ldots, x_{-1}x_0, x_0x_1, x_1x_2, \ldots\};$$

in both cases the x_i are all distinct.

Definition 104. Let G be a graph. A nonempty subset $N \subseteq E(G)$ is called a nibble in G if and only if there is a partition $\{V_1, V_2\}$ of V such that $N = E(V_1, V_2)$ and there are no rays R in G such that $V(R) \subseteq V_1$.

2.2.3 Matroids

Remark 63 shows that pre-closure operators and simplicial complexes do not work nicely with each other. In this section, we add more axioms on pre-closure operators and simplicial complexes to make them work nicely with each other. We then define matroids using these axioms.

Definition 105. Let cl be a pre-closure operator on a set X. We say that cl has the exchange property on X if and only if for every $A \subseteq X$ and every distinct $x, y \in X \setminus A$ such that $y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$ we have $x \in \mathrm{cl}(A \cup \{y\})$.

Theorem 106. *Let cl be a pre-closure operator on a set X and $Y \subseteq X$. If cl has the exchange property on X, then $cl_{\restriction Y}$ and $\mathrm{cl}._Y$ have the exchange property on Y.*

Proof. Let $A \subseteq Y$ and $x, y \in Y \setminus A$ be distinct such that

$$y \in \mathrm{cl}_{\restriction Y}(A \cup \{x\}) \setminus \mathrm{cl}_{\restriction Y}(A).$$

Then

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

Since cl has the exchange property on X, $x \in \mathrm{cl}(A \cup \{y\})$. It follows that $x \in \mathrm{cl}_{\restriction Y}(A \cup \{y\})$. Therefore, $\mathrm{cl}_{\restriction Y}$ has the exchange property of Y.

Now, let $A \subseteq Y$ and $x, y \in Y \setminus A$ be distinct such that

$$y \in \mathrm{cl}._Y(A \cup \{x\}) \setminus \mathrm{cl}._Y(A).$$

Then

$$y \in \mathrm{cl}(A \cup \{x\} \cup (X \setminus Y)) \setminus \mathrm{cl}(A \cup (X \setminus Y)).$$

Since cl has the exchange property on X, $x \in \mathrm{cl}(A \cup \{y\} \cup (X \setminus Y))$. It follows that $x \in \mathrm{cl}._Y(A \cup \{y\})$. Thus, $\mathrm{cl}._Y$ has the exchange property on Y. □

Remark 107. Let cl be a pre-closure operator on a set X. If cl has the exchange property on X, then cl^* may not have the exchange property on X.

Proof. Take $X = \{1, 2\}$ and the pre-closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}(A) = \begin{cases} \{1\} & \text{if } A = \varnothing \\ X & \text{if } A \neq \varnothing \end{cases} \text{ for all } A \subseteq X.$$

cl has the exchange property on X. By Theorem 6, $\mathrm{cl}^* : \mathscr{P}(X) \to \mathscr{P}(X)$ is defined by

$$\mathrm{cl}^*(A) = A \cup \{x \in X \setminus A : x \notin \mathrm{cl}(X \setminus (A \cup \{x\}))\} \text{ for all } A \subseteq X.$$
$$= \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{2\} & \text{if } A = \{2\} \\ X & \text{otherwise} \end{cases} \text{ for all } A \subseteq X.$$

Now, we have

$$2 \in \mathrm{cl}^*(\varnothing \cup \{1\}) \setminus \mathrm{cl}^*(\varnothing) \text{ but } 1 \notin \mathrm{cl}^*(\varnothing \cup \{2\}).$$

Therefore, cl^* does not have the exchange property on X. □

Remark 108. If cl is just a pre-closure operator on a set X, then cl^* may not have the exchange property on X.

Proof. Consider the same example in the proof of Remark 107. In Example 30, we showed that the pre-closure operator cl is not a closure operator on X, and in Remark 107, we showed that cl^* does not have the exchange property on X. □

Theorem 109. *Let cl be a pre-closure operator on a set X. If cl is a closure operator on X, then cl^* has the exchange property on X.*

Proof. Suppose that cl is a closure operator on X. We want to show that cl^* has the exchange property on X. Let $A \subseteq X$ and $x, y \in Y \setminus A$ be distinct such that $y \in \mathrm{cl}^*(A \cup \{x\}) \setminus \mathrm{cl}^*(A)$. By Theorem 6, we get

$$y \notin \mathrm{cl}(X \setminus ((A \cup (x)) \cup \{y\})) \text{ and } y \in \mathrm{cl}(X \setminus (A \cup \{y\})). \tag{2.2.1}$$

Suppose, by way of contradiction, that $x \notin \mathrm{cl}^*(A \cup \{y\})$. So,

$$x \in \mathrm{cl}(X \setminus ((A \cup (y)) \cup \{x\})). \tag{2.2.2}$$

Let $C = X \setminus (A \cup \{x, y\}) \subseteq X$. From (2.2.1) and (2.2.2), we obtain

$$y \notin \mathrm{cl}(C), y \in \mathrm{cl}(C \cup \{x\}) \text{ and } x \in \mathrm{cl}(C).$$

Since $x \in \mathrm{cl}(C)$ then, using (CL1), we have

$$C \cup \{x\} \subseteq \mathrm{cl}(C).$$

By (CL2), we get

$$\mathrm{cl}(C \cup \{x\}) \subseteq \mathrm{cl}(\mathrm{cl}(C)).$$

Since $y \in \mathrm{cl}(C \cup \{x\})$, then $y \in \mathrm{cl}(\mathrm{cl}(C))$. Thus, we have

$$y \in \mathrm{cl}(\mathrm{cl}(C)) \text{ and } y \notin \mathrm{cl}(C).$$

Therefore, $\mathrm{cl}(\mathrm{cl}(C)) \neq \mathrm{cl}(C)$. So, cl is not a closure operator on X, a contradiction. Thus, $x \in \mathrm{cl}^*(A \cup \{y\})$, and hence cl^* has the exchange property on X. □

In the following remark we show that the converse of Theorem 109 may not hold.

Remark 110. Let cl be a pre-closure operator on a set X. If cl^* has the exchange property on X, then cl is not necessarily a closure operator on X.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Define $\mathrm{cl} : P(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} X \setminus \{0\} & \text{if } 0 \notin A \text{ and } X \setminus A \text{ is infinite} \\ X & \text{if } 0 \in A \text{ or } X \setminus A \text{ is finite} \end{cases} \quad \text{for all } A \subseteq X.$$

cl is a pre-closure operator on X. Using Theorem 6, the dual of cl is the standard topological closure operator on X. So,

$$\mathrm{cl}^*(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

We want to show that cl^* has the exchange property on X. We claim that for each $A \subseteq X$ and each distinct $x, y \in Y \setminus A$, if $y \in \mathrm{cl}^*(A \cup \{x\})$ then $y \in \mathrm{cl}^*(A)$. To prove the claim, suppose by way of contradiction that there is $A \subseteq X$ and distinct $x, y \in Y \setminus A$ such that

$$y \in \mathrm{cl}^*(A \cup \{x\}) \setminus \mathrm{cl}^*(A).$$

Since cl^* is a topological closure operator on X, we have

$$y \in [\mathrm{cl}^*(A) \cup \mathrm{cl}^*(\{x\})] \setminus \mathrm{cl}^*(A).$$

Thus, $y \in \mathrm{cl}^*(\{x\}) = \{x\}$. Then $x = y$, a contradiction. Thus, cl^* has the exchange property on X. In Remark 10, we proved that cl is not a closure operator on X. □

Remark 110 shows that the exchange property of a pre-closure operator is too weak in the sense that when we have a pre-closure operator and its dual has the exchange property, this does not have to lead to the pre-closure operator being a closure operator. We are going to define a stronger property on pre-closure operators that provides the equivalence in Theorem 109.

Definition 111. Let cl be a pre-closure operator on a set X. We say that cl has the strong exchange property on X if and only if for each disjoint $A, B \subseteq X$ and each $y \in X \setminus (A \cup B)$ such that $y \in \mathrm{cl}(A \cup B) \setminus \mathrm{cl}(A)$, there is $x \in B$ with

$$x \in \mathrm{cl}(A \cup (B \setminus \{x\}) \cup \{y\}).$$

Theorem 112. *Let cl be a pre-closure operator on a set X and $Y \subseteq X$. If cl has the strong* *exchange property on X, then $\mathrm{cl}_{\restriction Y}$ and $\mathrm{cl}._{Y}$ have the strong exchange property on Y.*

Proof. Let $A, B \subseteq Y$ be disjoint and $y \in Y \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}_{\restriction Y}(A \cup B) \setminus \mathrm{cl}_{\restriction Y}(A).$$

Then $A, B \subseteq X$ are disjoint and $y \in X \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}(A \cup B) \setminus \mathrm{cl}(A).$$

Since cl has the strong exchange property on X, there is $x \in B$ with

$$x \in \mathrm{cl}(A \cup (B \setminus \{x\}) \cup \{y\}).$$

It follows that

$$x \in \mathrm{cl}_{\restriction Y}(A \cup (B \setminus \{x\}) \cup \{y\}).$$

Therefore, $\mathrm{cl}_{\restriction Y}$ has the strong exchange property of Y.

Now, let $A, B \subseteq Y$ be disjoint and $y \in Y \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}._{Y}(A \cup B) \setminus \mathrm{cl}._{Y}(A).$$

Therefore,

$$y \in \mathrm{cl}(A \cup B \cup (X \setminus Y)) \setminus \mathrm{cl}(A \cup (X \setminus Y)).$$

So, $A \cup (X \setminus Y), B \subseteq X$ are disjoint and $y \in X \setminus [(A \cup (X \setminus Y)) \cup B]$ such that

$$y \in \mathrm{cl}((A \cup (X \setminus Y)) \cup B) \setminus \mathrm{cl}(A \cup (X \setminus Y)).$$

Since cl has the strong exchange property on X, there is $x \in B$ with

$$x \in \mathrm{cl}((A \cup (X \setminus Y)) \cup (B \setminus \{x\}) \cup \{y\}).$$

So,

$$x \in \mathrm{cl}(A \cup (B \setminus \{x\}) \cup \{y\} \cup (X \setminus Y)).$$

It follows that

$$x \in \mathrm{cl}._{Y}(A \cup (B \setminus \{x\}) \cup \{y\}).$$

Therefore, $\mathrm{cl}._Y$ has the strong exchange property of Y. □

Theorem 113. *Let cl be a pre-closure operator on a set X. If cl has the strong exchange property on X, then cl has the exchange property on X.*

Proof. Suppose that cl has the strong exchange property on X. We want to show that cl has the exchange property on X. Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

Take $B = \{x\}$. So, we have $A, B \subseteq Y$ are disjoint and $y \in Y \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}(A \cup B) \setminus \mathrm{cl}(A).$$

Since cl has the strong exchange property on X, there is $x \in B$ such that

$$x \in \mathrm{cl}(A \cup (B \setminus \{x\}) \cup \{y\}),$$

and then $x \in \mathrm{cl}(A \cup \{y\})$. Therefore, cl has the exchange property on X. □

Remark 114. Let cl be a pre-closure operator on a set X. If cl has the exchange property on X, then cl might not have the strong exchange property on X.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topology. So, the topological closure operator on X is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

In Remark 110, we showed that cl has the exchange property on X. Now, take

$$A = \varnothing, B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \text{ and } y = 0.$$

So, we have $A, B \subseteq X$ are disjoint and $y \in X \setminus (A \cup B)$ such that

$$0 \in \mathrm{cl}(\varnothing \cup B) \setminus \mathrm{cl}(\varnothing).$$

But

$$\frac{1}{n} \notin \mathrm{cl}\left(\varnothing \cup \left(B \setminus \left\{\frac{1}{n}\right\}\right) \cup \{0\}\right) \text{ for each } n \in \mathbb{N}.$$

Thus, cl does not have the strong exchange property on X. □

Theorem 115. *Let X be a finite set and cl be a pre-closure operator on X. Then cl has the strong exchange property on X if and only if cl has the exchange property on X.*

Proof. In Theorem 113, we proved in general that if cl has the strong exchange property on X then cl has the exchange property on X. Now, suppose that cl has the exchange property on X. We want to show that cl has the strong exchange property on X. Let $A, B \subseteq X$ be disjoint and $y \in X \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}(A \cup B) \setminus \mathrm{cl}(A).$$

Since B is finite, $B = \{b_1, b_2, \dots, b_n\}$ for some $n \in \mathbb{N}$. Let i be the smallest element in $\{1, 2, \dots, n\}$ such that

$$y \in \mathrm{cl}(A \cup \{b_1, b_2, \dots, b_i\}).$$

Let $A' = A \cup \{b_1, b_2, \dots, b_{i-1}\}$. Thus,

$$y \in \mathrm{cl}\left(A' \cup \{b_i\}\right) \setminus \mathrm{cl}\left(A'\right).$$

Since cl has the exchange property on X,

$$b_i \in \mathrm{cl}\left(A' \cup \{y\}\right).$$

Therefore, we have $b_i \in B$ such that

$$b_i \in \mathrm{cl}(A \cup \{b_1, b_2, \dots, b_{i-1}\} \cup \{y\}) = \mathrm{cl}(A \cup (B \setminus \{b_i\}) \cup \{y\}).$$

Hence cl has the strong exchange property on X. □

Now, we are going to see that the strong exchange property on pre-closure operators provides the equivalence in Theorem 109.

Theorem 116. *Let cl be a pre-closure operator on a set X. Then cl is a closure operator on X if and only if cl^* has the strong exchange property on X.*

Proof. Suppose that cl is a closure operator on X. We want to show that cl^* has the strong exchange property on X. Let $A, B \subseteq X$ be disjoint and $y \in X \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}^*(A \cup B) \setminus \mathrm{cl}^*(A).$$

By Theorem 6, we get

$$y \notin \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\})) \text{ and } y \in \mathrm{cl}\,(X \setminus (A \cup \{y\}))\,. \tag{2.2.3}$$

We want to show that there is $x \in B$ such that

$$x \in \mathrm{cl}^*\,(A \cup (B \setminus \{x\}) \cup \{y\})\,.$$

By Theorem 6, it suffices to show that there is $x \in B$ such that

$$x \notin \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))\,.$$

And this suffices to show that $B \nsubseteq \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))$. Suppose, by way of contradiction, that

$$B \subseteq \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))\,.$$

Note that

$$X \setminus (A \cup B \cup \{y\}) \subseteq X \setminus (A \cup \{y\})\,. \tag{2.2.4}$$

By (CL2), we get

$$\mathrm{cl}\,(X \setminus (A \cup B \cup \{y\})) \subseteq \mathrm{cl}\,(X \setminus (A \cup \{y\}))\,. \tag{2.2.5}$$

This means, from (2.2.3), (2.2.4), and (2.2.5), that when we add B to $X \setminus (A \cup B \cup \{y\})$ we find $y \in \mathrm{cl}\,(X \setminus (A \cup \{y\}))$. Since we assumed

$$B \subseteq \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))\,,$$

then, by (CL2), we get

$$\mathrm{cl}\,(B) \subseteq \mathrm{cl}\,(\mathrm{cl}\,(X \setminus (A \cup B \cup \{y\})))\,,$$

and thus, $y \in \mathrm{cl}\,(\mathrm{cl}\,(X \setminus (A \cup B \cup \{y\})))$. But, from (2.2.3), we have $y \notin \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))$. Therefore,

$$\mathrm{cl}\,(\mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))) \neq \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))\,,$$

and this contradicts that cl is a closure operator on X. Therefore, $B \nsubseteq \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))$. So, there is $x \in B$ such that $x \notin \mathrm{cl}\,(X \setminus (A \cup B \cup \{y\}))$. Thus, we have showed that there is $x \in B$ such that

$$x \in \mathrm{cl}^*\,(A \cup (B \setminus \{x\}) \cup \{y\})\,.$$

Hence cl^* has the strong exchange property on X.

Now, suppose that cl^* has the strong exchange property on X. We want to show that cl is a closure operator on X. By (CL2), we have

$$\mathrm{cl}(Y) \subseteq \mathrm{cl}(\mathrm{cl}(Y)) \text{ for all } Y \subseteq X. \tag{2.2.6}$$

Suppose, by way of contradiction, that there $Y \subseteq X$ such that $\mathrm{cl}(\mathrm{cl}(Y)) \nsubseteq \mathrm{cl}(Y)$. So, there is $y \in \mathrm{cl}(\mathrm{cl}(Y)) \setminus \mathrm{cl}(Y)$. Let

$$B = \mathrm{cl}(Y) \setminus Y$$
$$A = X \setminus (B \cup Y \cup \{y\})$$

Then we have $A, B \subseteq X$ are disjoint and $y \in X \setminus (A \cup B)$. Note that $Y = X \setminus (A \cup B \cup \{y\})$ and $\mathrm{cl}(Y) = X \setminus (A \cup \{y\})$. Since $y \in \mathrm{cl}(\mathrm{cl}(Y)) \setminus \mathrm{cl}(Y)$, then

$$y \in \mathrm{cl}(X \setminus (A \cup \{y\})) \text{ and } y \notin \mathrm{cl}(X \setminus (A \cup B \cup \{y\})).$$

By Theorem 6, we get

$$y \in \mathrm{cl}^*(A \cup B) \setminus \mathrm{cl}^*(A).$$

Since cl^* has the strong exchange property on X, there is $x \in B$ such that

$$x \in \mathrm{cl}^*(A \cup (B \setminus \{x\}) \cup \{y\}).$$

Again, by Theorem 6, we get

$$x \notin \mathrm{cl}(X \setminus ((A \cup (B \setminus \{x\}) \cup \{y\})) \cup \{x\}).$$

So,

$$x \notin \mathrm{cl}(X \setminus (A \cup B \cup \{y\})) = \mathrm{cl}(Y),$$

which contradicts that $B = \mathrm{cl}(Y) \setminus Y$. So,

$$\mathrm{cl}(\mathrm{cl}(Y)) \subseteq \mathrm{cl}(Y) \text{ for all } Y \subseteq X. \tag{2.2.7}$$

By (2.2.6) and (2.2.7), we get

$$\mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y) \text{ for all } Y \subseteq X.$$

Hence cl is a closure operator on X. □

Remark 117. Let $\mathscr{I}$ be a simplicial complex on a set X and cl be the pre-closure operator on X induced by $\mathscr{I}$. Then cl does not have to be a closure operator on X.

Proof. Consider $X = \{1,2,3\}$ and the simplicial complex

$$\mathscr{I} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{2,3\}\}.$$

By Theorem 61, the pre-closure operator cl on X induced by $\mathscr{I}$ is

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{1,2\} & \text{if } A = \{2\} \\ \{1,3\} & \text{if } A = \{3\} \\ X & \text{otherwise.} \end{cases} \quad \text{for all } A \subseteq X$$

Now, note that

$$\mathrm{cl}(\{2\}) = \{1,2\}$$
$$\mathrm{cl}(\mathrm{cl}(\{2\})) = \mathrm{cl}(\{1,2\}) = X$$

Thus, $\mathrm{cl}(\mathrm{cl}(\{2\})) \neq \mathrm{cl}(\{2\})$. Therefore, cl is not a closure operator on X. □

Theorem 118. *Let $\mathscr{I}$ be a simplicial complex on a set X and cl be the pre-closure operator on X induced by $\mathscr{I}$. Then cl has the exchange property on X.*

Proof. Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

By Theorem 61, there is $I \subseteq A \cup \{x\}$ such that

$$I \in \mathscr{I} \text{ and } I \cup \{y\} \notin \mathscr{I}, \tag{2.2.8}$$

and we have

$$J \notin \mathscr{I} \text{ or } J \cup \{y\} \in \mathscr{I} \text{ for each } J \subseteq A. \tag{2.2.9}$$

So, we must have $x \in I$, otherwise $I \subseteq A$ such that $I \in \mathscr{I}$ and $I \cup \{y\} \notin \mathscr{I}$. This contradicts (2.2.9). We want to show that $x \in \mathrm{cl}(A \cup \{y\})$. Take $I' = (I \setminus \{x\}) \cup \{y\}$. Since $I \subseteq A \cup \{x\}$, then $(I \setminus \{x\}) \subseteq A$. Thus,

$$I' \subseteq A \cup \{y\}. \tag{2.2.10}$$

Also, since $I \in \mathscr{I}$ and $\mathscr{I}$ is a simplicial complex on X, then $(I \setminus \{x\}) \in \mathscr{I}$. By (2.2.9), we get

$$I' = (I \setminus \{x\}) \cup \{y\} \in \mathscr{I}. \tag{2.2.11}$$

Now,

$$I' \cup \{x\} = ((I \setminus \{x\}) \cup \{y\}) \cup \{x\} = I \cup \{y\}.$$

By (2.2.8), we have

$$I' \cup \{x\} \notin \mathscr{I}. \tag{2.2.12}$$

By (2.2.10), (2.2.11) and (2.2.12), we obtain $x \in \mathrm{cl}(A \cup \{y\})$. Therefor, cl has the exchange property on X. □

Remark 119. Let $\mathscr{I}$ be a simplicial complex on a set X and cl be the pre-closure operator on X induced by $\mathscr{I}$. Then cl may not have the strong exchange property on X.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and the simplicial complex

$$\mathscr{I} = \{I \subseteq X : I \text{ is finite or } 0 \notin I\}.$$

$\mathscr{I}$ induces the topological closure operator

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

In Remark 114, we showed that cl does not have the strong exchange property on X. □

In Remark 63, we showed that when a pre-closure operator induces a simplicial complex, the simplicial complex might not induce the pre-closure operator. Remark 117 and Theorem 118 urge us to add the axiom that cl has the exchange property.

Remark 120. Let cl be a pre-closure operator with the exchange property on a set X and cl' be the pre-closure operator on X induced by the simplicial complex $\mathscr{I}$ on X induced by cl. Then

$$\mathrm{cl}'(Y) \subseteq \mathrm{cl}(Y) \text{ for each } Y \subseteq X,$$

and equality does not have to hold.

Proof. Suppose that cl has the exchange property on X. Let $Y \subseteq X$ and $x \in \mathrm{cl}'(Y)$. Assume, without loss of generality, that $x \in \mathrm{cl}'(Y) \setminus Y$. By Theorem 61, there is $I \subseteq Y$ such that

$$I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}.$$

By Theorem 59, since $I \cup \{x\} \notin \mathscr{I}$, there is $y \in I \cup \{x\}$ such that

$$y \in \mathrm{cl}((I \cup \{x\}) \setminus \{y\}) = \mathrm{cl}((I \setminus \{y\}) \cup \{x\}).$$

If $y = x$. Therefore, $x \in \mathrm{cl}(I)$ and by (CL2) we get $x \in \mathrm{cl}(Y)$. If $y \neq x$. Since $y \in I \cup \{x\}$ and x, y are distinct, then $y \in I$. By Theorem 59, since $I \in \mathscr{I}$, we get

$$y \notin \mathrm{cl}(I \setminus \{y\}).$$

Thus, we have $I \setminus \{y\} \subseteq X$ and $x, y \in X \setminus (I \setminus \{y\})$ are distinct such that

$$y \in \mathrm{cl}((I \setminus \{y\}) \cup \{x\}) \setminus \mathrm{cl}(I \setminus \{y\}).$$

Since cl has the exchange property on X, then

$$x \in \mathrm{cl}((I \setminus \{y\}) \cup \{y\}) = \mathrm{cl}(I).$$

By (CL2), we get $x \in \mathrm{cl}(Y)$. Thus, $\mathrm{cl}'(Y) \subseteq \mathrm{cl}(Y)$ for each $Y \subseteq X$.

Now, consider $X = \mathbb{N}$ and the pre-closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}(Y) = \begin{cases} Y & \text{if } Y \text{ is finite} \\ X & \text{if } Y \text{ is infinite} \end{cases} \quad \text{for all } Y \subseteq X.$$

Note that cl has the exchange property on X. By Theorem 59, the simplicial complex $\mathscr{I}$ on X induced by cl is

$$\begin{aligned} \mathscr{I} &= \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} \\ &= \{I \subseteq X : I \text{ is finite}\}. \end{aligned}$$

By Theorem 61, the pre-closure operator cl' on X induced by $\mathscr{I}$ is

$$\begin{aligned}\mathrm{cl}'(Y) &= Y \cup \{x \in X \setminus Y : \text{ there is } I \subseteq Y \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\} \\ \mathrm{cl}'(Y) &= Y \text{ for all } Y \subseteq X.\end{aligned}$$

Now, take $Y = \{2, 4, 6, \ldots\} \subseteq X$. Then

$$Y = \mathrm{cl}'(Y) \subsetneq \mathrm{cl}(Y) = X.$$

Thus, the equality does not hold. □

So, we got one inclusion and still need more axioms to get the other inclusion. The next remark partially solved this problem for the independent subsets.

Remark 121. Let cl be a pre-closure operator with the exchange property on a set X and cl' be the pre-closure operator on X induced by the simplicial complex $\mathscr{I}$ on X induced by cl. Then for each $I \in \mathscr{I}$, we have

1. $\mathrm{cl}'(I) = \mathrm{cl}(I)$.

2. For each $x \in X \setminus I$, we have $x \in \mathrm{cl}(I)$ if and only if $I \cup \{x\} \notin \mathscr{I}$.

Proof. Let $I \in \mathscr{I}$. We first show part (1). By Remark 120, we know that $\mathrm{cl}'(I) \subseteq \mathrm{cl}(I)$. Let $x \in \mathrm{cl}(I)$. Assume, without loss of generality, that $x \in \mathrm{cl}(I) \setminus I$. Thus, $x \in \mathrm{cl}((I \cup \{x\}) \setminus \{x\})$. By Theorem 59, we have $I \cup \{x\} \notin \mathscr{I}$. By Theorem 61, we get $x \in \mathrm{cl}'(I)$. So, $\mathrm{cl}'(I) \subseteq \mathrm{cl}(I)$, and therefore $\mathrm{cl}'(I) = \mathrm{cl}(I)$. Now, we show part (2). Let $x \in X \setminus I$. Then $x \in \mathrm{cl}(I)$ if and only if $x \in \mathrm{cl}'(I)$ if and only if $I \cup \{x\} \notin \mathscr{I}$. □

Definition 122. Let cl be a pre-closure operator on a set X and $B \subseteq Y \subseteq X$.

1. Y is spanned by B if and only if $Y \subseteq \mathrm{cl}(B)$.

2. B is a maximal independent subset of Y if and only if B is independent and for every independent B' with $B \subseteq B' \subseteq Y$ we have $B' = B$.

Theorem 123. *Let cl be a pre-closure operator on a set X and $B \subseteq Y \subseteq X$ with B is independent. If Y is spanned by B, then B is a maximal independent subset of Y.*

Proof. Suppose that Y is spanned by B. We want to show that B is a maximal independent subset of Y. Suppose, by way of contradiction, that B is not a maximal independent subset of Y. Thus, there is independent B' with $B \subseteq B' \subseteq Y$ such that $B' \neq B$. Let $x \in B' \setminus B \subseteq Y$. This leads to $B \subseteq B' \setminus \{x\}$. Since B' is independent, $x \notin \mathrm{cl}(B' \setminus \{x\})$. By (CL2), $x \notin \mathrm{cl}(B)$. So, $Y \nsubseteq \mathrm{cl}(B)$, which contradicts that Y is spanned by B. Therefore, B is a maximal independent subset of Y. $\square$

The converse of Theorem 123 is not always true.

Remark 124. Let cl be a pre-closure operator on a set X and $B \subseteq Y \subseteq X$ with B is independent. If B is a maximal independent subset of Y, then Y may not be spanned by B.

Proof. Consider $X = \{1, 2\}$ and define a closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{1\} & \text{if } A = \{1\} \\ X & \text{otherwise} \end{cases} \quad \text{for all } A \subseteq X.$$

The set of independent sets of cl is the simplicial complex

$$\mathscr{I} = \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} = \{\varnothing, \{1\}, \{2\}\}$$

$B = \{1\}$ is a maximal independent subset of $Y = X$, but

$$X = Y \nsubseteq \mathrm{cl}(B) = \mathrm{cl}(\{1\}) = \{1\}.$$

Thus, Y is not spanned by B. $\square$

Note that the pre-closure operator in the proof of Remark 124 does not have the exchange property.

Theorem 125. *Let cl be a pre-closure operator that has the exchange property on a set X and $B \subseteq Y \subseteq X$ with B is independent. Then Y is spanned by B if and only if B is a maximal independent subset of Y.*

Proof. In Theorem 123 we proved that if Y is spanned by B then B is a maximal independent subset of Y. Now, suppose that B is a maximal independent subset of Y. We want to show that $Y \subseteq \mathrm{cl}(B)$. Let $x \in Y$. By (CL1), we can assume, without loss of generality, that $x \in Y \setminus B$. Since B is maximal independent, $B \cup \{x\}$ is not independent. So, there is

$y \in B \cup \{x\}$ such that

$$y \in \mathrm{cl}\left((B \cup \{x\}) \setminus \{y\}\right) = \mathrm{cl}\left((B \setminus \{y\}) \cup \{x\}\right).$$

If $y = x$, then $x \in \mathrm{cl}(B)$ and we are done. If $y \neq x$, then $y \in B$. Since B is independent, then

$$y \notin \mathrm{cl}(B \setminus \{y\}).$$

So, we have $B \setminus \{y\} \subseteq X$ and $x, y \in X \setminus (B \setminus \{y\})$ are distinct such that

$$y \in \mathrm{cl}\left((B \setminus \{y\}) \cup \{x\}\right) \setminus \mathrm{cl}(B \setminus \{y\}).$$

Since cl has the exchange property on X, we get

$$x \in \mathrm{cl}\left((B \setminus \{y\}) \cup \{y\}\right) = \mathrm{cl}(B).$$

Thus, $Y \subseteq \mathrm{cl}(B)$ and hence Y is spanned by B. □

Definition 126. Let cl be a pre-closure operator on a set X. We say that cl has the maximality property on X if and only if for every $A \subseteq Y \subseteq X$ such that A is independent, there is a maximal independent subset B of Y such that $A \subseteq B$.

Example 127. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topology. So, the topological closure operator cl on X is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

The set of independent sets of cl is the simplicial complex

$$\begin{aligned} \mathscr{I} &= \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} \\ &= \{I \subseteq X : I \text{ is finite or } 0 \notin I\}. \end{aligned}$$

Take $Y = X$ and $A = \{0\}$. So $A \subseteq Y \subseteq X$ such that A is independent but for each independent subset I of Y with $A \subseteq I$, we have I is not maximal independent. Therefore, cl does not have the maximality property on X.

Theorem 128. *Let X be a finite set and cl be a pre-closure operator on X. Then cl has the maximality property on X.*

Proof. Let $A \subseteq Y \subseteq X$ such that A is independent. Since X is finite, then $Y \setminus A$ is finite. So, $Y \setminus A = \{y_1, y_2, \ldots, y_n\}$ for some $n \in \mathbb{N}$. Let i be the largest element in $\{1, 2, \ldots, n\}$ such that $A \cup \{y_1, y_2, \ldots, y_i\}$ is independent. Thus, $A \cup \{y_1, y_2, \ldots, y_i\}$ is a maximal independent subset of Y such that $A \subseteq A \cup \{y_1, y_2, \ldots, y_i\}$. Therefore, cl has the maximality property on X. □

We proved in Remark 120 that when a pre-closure operator with the exchange property induces a simplicial complex, the simplicial complex might not induce the pre-closure operator, but we get one inclusion. Even if we add an axiom that the pre-closure operator also has the maximality property, the pre-closure operator might not be induced.

Remark 129. Let cl be a pre-closure operator on a set X with the exchange property and maximality property on X and let $\mathscr{I}$ be the simplicial complex on X induced by cl. Then the pre-closure operator cl might not be induced by $\mathscr{I}$.

Proof. Let cl$'$ be the pre-closure operator on X induced by $\mathscr{I}$. In Remark 120, we proved, in case cl has the exchange property on X, that

$$\mathrm{cl}'(Y) \subseteq \mathrm{cl}(Y) \text{ for each } Y \subseteq X,$$

but the equality might not hold. Now, consider $X = \{1, 2, 3\}$ and define a pre-closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(Y) = \begin{cases} \varnothing & \text{if } Y = \varnothing \\ \{1,2\} & \text{if } Y \in \{\{1\}, \{2\}\} \\ \{3\} & \text{if } Y = \{3\} \\ X & \text{otherwise} \end{cases} \quad \text{for all } Y \subseteq X.$$

cl has the exchange property and maximality property on X. By Theorem 59, the simplicial complex $\mathscr{I}$, the set of independent sets of cl, induced by cl is

$$\begin{aligned} \mathscr{I} &= \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} \\ &= \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}. \end{aligned}$$

By Theorem 61, the pre-closure operator cl' on X induced by $\mathscr{I}$ is

$$\mathrm{cl}'(Y) = Y \cup \{x \in X \setminus Y : \text{ there is } I \subseteq Y \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\}$$

$$\mathrm{cl}'(Y) = \begin{cases} \varnothing & \text{if } Y = \varnothing \\ \{1,2\} & \text{if } Y \in \{\{1\},\{2\},\{1,2\}\} \\ \{3\} & \text{if } Y = \{3\} \\ X & \text{otherwise} \end{cases} \quad \text{for all } Y \subseteq X.$$

But

$$\{1,2\} = \mathrm{cl}'(Y) \neq \mathrm{cl}(\{1,2\}) = X.$$

Thus, $\mathrm{cl}' \neq \mathrm{cl}$, and hence $\mathscr{I}$ does not induce cl. □

Note that the pre-closure operator cl we used in Remark 129 is not a closure operator on X because

$$\{1,2\} = \mathrm{cl}(\{1\}) \neq \mathrm{cl}(\mathrm{cl}(\{1\})) = \mathrm{cl}(\{1,2\}) = X.$$

So, let us see what happens if we start with a pre-closure operator that is a closure operator and has the exchange property and maximality property.

Definition 130. A matroidal closure operator cl on a set X is a closure operator with the exchange property and maximality property on X.

That is, an operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ is a matroidal closure operator on X if and only if

(CL1) $A \subseteq \mathrm{cl}(A)$ for all $A \subseteq X$.

(CL2) $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ for all $A \subseteq B \subseteq X$.

(CL3) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all $A \subseteq X$.

(CL4) cl has the exchange property on X.

(CLM) cl has the maximality property on X.

Theorem 131. *Let cl be a matroidal closure operator on a set X and $\mathscr{I}$ be the simplicial complex on X induced by cl. Then $\mathscr{I}$ induces cl.*

Proof. Let cl' be the pre-closure operator on X induced by $\mathscr{I}$. By Theorem (59) and Theorem (61), we have

$$\mathscr{I} = \{I \subseteq X : x \notin \mathrm{cl}(I \setminus \{x\}) \text{ for each } x \in I\} \tag{2.2.13}$$

$$\mathrm{cl}'(Y) = Y \cup \{x \in X \setminus Y : \text{ there is } I \subseteq Y \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\} \tag{2.2.14}$$

for each $Y \subseteq X$. We want to show that $\mathrm{cl}' = \mathrm{cl}$. By Remark 120, we get

$$\mathrm{cl}'(Y) \subseteq \mathrm{cl}(Y) \text{ for each } Y \subseteq X. \tag{2.2.15}$$

Let $x \in \mathrm{cl}(Y)$. Let $B \in \mathscr{I}$ be a maximal subset of Y. Since cl has the exchange property on X then, by Theorem 125, we have $Y \subseteq \mathrm{cl}(B)$. By (CL2) and (CL3), we get $\mathrm{cl}(Y) \subseteq \mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$. Thus,

$$x \in \mathrm{cl}(B) = \mathrm{cl}((B \cup \{x\}) \setminus \{x\}).$$

By (2.2.13), we obtain $B \cup \{x\} \notin \mathscr{I}$. By (2.2.14), we get $x \in \mathrm{cl}'(B)$. By (CL2), $x \in \mathrm{cl}'(Y)$. So,

$$\mathrm{cl}(Y) \subseteq \mathrm{cl}'(Y) \text{ for each } Y \subseteq X. \tag{2.2.16}$$

By (2.2.15) and (2.2.16), we have

$$\mathrm{cl}'(Y) = \mathrm{cl}(Y) \text{ for each } Y \subseteq X,$$

and hence $\mathrm{cl}' = \mathrm{cl}$. □

Definition 132. A matroid is a pair $(X, \mathscr{I})$ where X is a set and $\mathscr{I}$ is a simplicial complex on X induced by a matroidal closure operator on X.

Example 133. Consider any set X with the simplicial complex $\mathscr{I} = \mathscr{P}(X)$. The matroidal closure operator cl on X defined by

$$\mathrm{cl}(Y) = Y \text{ for all } Y \subseteq X$$

induces $\mathscr{I} = \mathscr{P}(X)$. Thus, $(X, \mathscr{P}(X))$ is a matroid.

When a set with a simplicial complex is given, it is not always easy to tell if there is a matroidal closure operator inducing the simplicial complex. So, it is not easy to tell that the set with the simplicial complex is a matroid. To deal with this problem, we will add axioms on simplicial complexes that tell whether a given set with a simplicial complex is a matroid or not without looking for a matroidal closure operator that induces the simplicial complex.

Remark 134. Let $\mathscr{I}$ be a simplicial complex on a set X such that the pre-closure operator induced by $\mathscr{I}$ is a matroidal closure operator on X. It does not follow that $(X, \mathscr{I})$ is a matroid.

Proof. Consider $X = \mathbb{N}$ and the simplicial complex

$$\mathscr{I} = \{I \subseteq X : I \text{ is finite}\}.$$

By Theorem 61, the pre-closure operator cl on X induced by $\mathscr{I}$ is

$$\operatorname{cl}(Y) = Y \text{ for all } Y \subseteq X.$$

cl is a matroidal closure operator on X and, by Theorem 59, the simplicial complex on X induced by cl is $\mathscr{P}(X)$. But suppose, by way of contradiction, that $(X, \mathscr{I})$ is a matroid. Then there is a matroidal closure operator cl' on X induces $\mathscr{I}$, which is the family of all cl'-independent sets. Take $Y = X$ and $A = \{1\}$. So. $A \subseteq Y$ is cl'-independent but for each cl'-independent subset I of Y with $A \subseteq I$, we have I is not maximal cl'-independent. So, cl' does not have the maximality property on X, contradiction. Therefore, $(X, \mathscr{I})$ is not a matroid. □

In Remark 63 we showed that when a simplicial complex induces a pre-closure operator, the pre-closure operator might not induce the simplicial complex, but we get one inclusion. The following definition provides an axiom on simplicial complexes that gives the other inclusion.

Definition 135. Let $\mathscr{I}$ be a simplicial complex on a set X. We say $\mathscr{I}$ has the maximality property on X if and only if for every $A \subseteq Y \subseteq X$ with $A \in \mathscr{I}$ there is $B \in \mathscr{I}$ such that B is maximal[3] and $A \subseteq B \subseteq Y$.

Theorem 136. *Let cl be a pre-closure operator on a set X. Then cl has the maximality property on X if and only if the simplicial complex induced by cl has the maximality property on X.*

[3]Let $\mathscr{I}$ be a simplicial complex on a set X and $B \subseteq Y \subseteq X$. We say $B \in \mathscr{I}$ is a maximal subset of Y if and only if for each $B' \in \mathscr{I}$ with $B \subseteq B' \subseteq Y$ we have $B = B'$.

Proof. It is clear because the family of all cl-independent subsets of X is the simplicial complex induced by cl. $\square$

Remark 137. Let $\mathscr{I}$ be a simplicial complex on a set X such that the pre-closure operator induced by $\mathscr{I}$ has the maximality property on X, then $\mathscr{I}$ may not have the maximality property on X.

Proof. Consider $X = \mathbb{N}$ and the simplicial complex

$$\mathscr{I} = \{I \subseteq X : I \text{ is finite}\}.$$

By Theorem 61, the pre-closure operator induced by $\mathscr{I}$ is

$$\mathrm{cl}(A) = A \text{ for all } A \subseteq X.$$

The family of all cl-independent subsets of X is the simplicial complex $\mathscr{I}' = \mathscr{P}(X)$. So, cl has the maximality property on X. Now, take $A = \varnothing$ and $Y = X$. For each $I \in \mathscr{I}$ such that $A \subseteq I \subseteq Y$, we have I is not maximal. So, $\mathscr{I}$ does not have the maximality property on X. $\square$

The example we used in Remarks 134 and 137 encourages us to add the maximality property as an axiom on simplicial complexes.

Theorem 138. *Let $\mathscr{I}$ be a simplicial complex with the maximality property on a set X and cl be the pre-closure operator on X induced by $\mathscr{I}$. Then $\mathscr{I}$ is the simplicial complex on X induced by cl.*

Proof. Let $\mathscr{I}'$ be the simplicial complex on X induced by cl on X induced by $\mathscr{I}$. We want to show that $\mathscr{I}' = \mathscr{I}$. By Remark 63, we know that $\mathscr{I} \subseteq \mathscr{I}'$. So, we just need to show that $\mathscr{I}' \subseteq \mathscr{I}$. Let $A \in \mathscr{I}'$. Suppose, by way of contradiction, that $A \notin \mathscr{I}$. By the maximality property of $\mathscr{I}$, since $\varnothing \subseteq A \subseteq X$ and $\varnothing \in \mathscr{I}$, there is $B \in \mathscr{I}$ such that B is maximal and $B \subseteq A$. Since $A \notin \mathscr{I}$, then $B \neq A$. Let $a \in A \setminus B$. Since $B \in \mathscr{I}$ is maximal in A, then $B \cup \{a\} \notin \mathscr{I}$. By Theorem 61, we get $a \in \mathrm{cl}(B)$. Since $B \subseteq A \setminus \{a\}$ then, by (CL2), we have $a \in \mathrm{cl}(A \setminus \{a\})$. By Theorem 59, $A \notin \mathscr{I}'$ which is a contradiction. So, $A \in \mathscr{I}$ and hence $\mathscr{I}' \subseteq \mathscr{I}$. Then $\mathscr{I}' = \mathscr{I}$. $\square$

Example 139. Let $X = X_1 \cup X_2$ with X_1 and X_2 both infinite and disjoint. Let

$$\mathscr{I} = \{I \subseteq X : I \cap X_1 = \varnothing \text{ or } I \cap X_2 = \varnothing\} = \mathscr{P}(X_1) \cup \mathscr{P}(X_2).$$

Let $I \subseteq Y \subseteq X$ with $I \in \mathscr{I}$. Assume, without lose of generality, that $I \subseteq X_1$. Take $B = Y \cap X_1$. Thus, $B \in \mathscr{I}$ such that B is maximal with $I \subseteq B \subseteq Y$. So, the simplicial complex $\mathscr{I}$ has the maximality property on X. By Theorem 61, the pre-closure operator induced by $\mathscr{I}$ is

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ A \cup X_2 & \text{if } A \subseteq X_1 \\ A \cup X_1 & \text{if } A \subseteq X_2 \\ X & \text{if } A \cap X_1 \neq \varnothing \text{ and } A \cap X_2 \neq \varnothing \end{cases}$$

for each $A \subseteq X$. By Theorem 59, cl induces $\mathscr{I}$.

Theorem 140. *Let $\mathscr{I}$ be a simplicial complex with the maximality property on a set X and cl be the pre-closure operator on X induced by $\mathscr{I}$. Then cl has the strong exchange property on X.*

Proof. Let $A, B \subseteq X$ be disjoint and $y \in X \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}(A \cup B) \setminus \mathrm{cl}(A).$$

By Theorem 61, there is $I \subseteq A \cup B$ such that $I \in \mathscr{I}$ but $I \cup \{y\} \notin \mathscr{I}$. Since $y \notin \mathrm{cl}(A)$, then $I \nsubseteq A$. Let $J = A \cap I$. So, $J \subseteq A$ and by (I2) we have $J \in \mathscr{I}$. Since $y \notin \mathrm{cl}(A)$, then $J \cup \{y\} \in \mathscr{I}$. Note that $J \cup \{y\} \subseteq I \cup \{y\}$. By the maximality property of $\mathscr{I}$, there is $B' \in \mathscr{I}$ such that B' is maximal and $J \cup \{y\} \subseteq B' \subsetneqq I \cup \{y\}$. Take $x \in I \cup \{y\} \setminus B'$. So, we have $x \in B$ and $B' \subseteq A \cup (B \setminus \{x\}) \cup \{y\}$ with $B' \in \mathscr{I}$. Since B' is maximal in $I \cup \{y\}$, then $B' \cup \{x\} \notin \mathscr{I}$. By Theorem 61, we get

$$x \in \mathrm{cl}(A \cup (B \setminus \{x\}) \cup \{y\}).$$

Therefore, cl has the strong exchange property on X. □

Theorem 138 shows that adding the axiom of maximality property on simplicial complexes makes simplicial complexes work nicely with pre-closure operators. However, the following remark shows that is not enough to tell whether a given set with a simplicial complex is a matroid or not without looking for a matroidal closure operator that induces the simplicial complex.

Remark 141. Let $\mathscr{I}$ be a simplicial complex with the maximality property on a set X. Then $(X, \mathscr{I})$ may not be a matroid.

Proof. Let $X = X_1 \cup X_2$ with X_1 and X_2 both infinite and disjoint. Let

$$\mathscr{I} = \{I \subseteq X : I \cap X_1 = \varnothing \text{ or } I \cap X_2 = \varnothing\} = \mathscr{P}(X_1) \cup \mathscr{P}(X_2).$$

In Example 139, we proved that the simplicial complex $\mathscr{I}$ has the maximality property on X. Suppose, by way of contradiction, that $(X, \mathscr{I})$ is a matroid. So, there is a matroidal closure operator cl on X induces $\mathscr{I}$, which is the family of all cl-independent sets. Let $x, y \in X_1$ with $x \neq y$ and $z \in X_2$. So, $\{x, y\} \in \mathscr{I}$ and $\{x, z\}, \{y, z\}, \{x, y, z\} \notin \mathscr{I}$. By Theorem 59, we have

$$\begin{gathered} y \notin \mathrm{cl}(\{x\}) \text{ and } x \notin \mathrm{cl}(\{y\}) \\ x \in \mathrm{cl}(\{z\}) \text{ or } z \in \mathrm{cl}(\{x\}) \\ y \in \mathrm{cl}(\{z\}) \text{ or } z \in \mathrm{cl}(\{y\}) \\ x \in \mathrm{cl}(\{y, z\}) \text{ or } y \in \mathrm{cl}(\{x, z\}) \text{ or } z \in \mathrm{cl}(\{x, y\}). \end{gathered}$$

So, cl is not a closure operator or does not have the exchange property on X, contradiction. Thus, $(X, \mathscr{I})$ is not a matroid. □

We now add a new axiom on simplicial complexes that tells whether a given set with a simplicial complex is a matroid or not without looking for a matroidal closure operator that induces the simplicial complex.

Definition 142. Let $\mathscr{I}$ be a simplicial complex on a set X. We say that $\mathscr{I}$ has the augmentation property on X if and only if for each $A, B \in \mathscr{I}$ such that B is maximal but A is not, there is $x \in B \setminus A$ such that $A \cup \{x\} \in \mathscr{I}$.

Example 143. Let $X = \{1, 2, 3\}$ and $\mathscr{I} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 3\}\}$. Take $A = \{1\}$ and $B = \{2\}$. So, $A, B \in \mathscr{I}$ with B is maximal but A is not. $B \setminus A = \{2\}$ but $A \cup \{2\} = \{1, 2\} \notin \mathscr{I}$. Therefore, the augmentation property fails.

Remark 144. Let cl be a pre-closure operator with the exchange property on a set X and $\mathscr{I}$ be the simplicial complex on X induced by cl. Then $\mathscr{I}$ might not have the augmentation property on X.

Proof. Consider $X = \{1,2,3\}$ and define a pre-closure operator $\text{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\text{cl}(Y) = \begin{cases} \varnothing & \text{if } Y = \varnothing \\ \{1,2\} & \text{if } Y = \{1\} \\ \{2,3\} & \text{if } Y = \{3\} \\ X & \text{otherwise} \end{cases} \quad \text{for all } Y \subseteq X.$$

The pre-closure operator cl has the exchange property on X. By Theorem 59, the simplicial complex $\mathscr{I}$ induced by cl is

$$\begin{aligned} \mathscr{I} &= \{I \subseteq X : x \notin \text{cl}(I \setminus \{x\}) \text{ for each } x \in I\} \\ &= \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,3\}\}. \end{aligned}$$

In Example 143, we proved $\mathscr{I}$ does not have the augmentation property on X. □

Note that the pre-closure operator cl in Remark 144 is not a closure operator on X.

Theorem 145. *Let cl be a closure operator with the exchange property on a set X and $\mathscr{I}$ be the simplicial complex on X induced by cl. Then $\mathscr{I}$ has the augmentation property on X.*

Proof. Let $A, B \in \mathscr{I}$ such that B is maximal but A is not. We want to show that there is $x \in B \setminus A$ such that $A \cup \{x\} \in \mathscr{I}$. Since $A \in \mathscr{I}$ is not maximal, there is $x \in X \setminus A$ such that $A \cup \{x\} \in \mathscr{I}$. If $x \in B$, then we are done. If $x \notin B$. By Theorem 125, since B is maximal in $\mathscr{I}$ which is the family of all cl-independent sets of X, we have $x \in \text{cl}(B)$. If $B \subseteq \text{cl}(A)$ then, using (CL2) and (CL3), we have

$$\text{cl}(B) \subseteq \text{cl}(\text{cl}(A)) = \text{cl}(A).$$

Thus, $x \in \text{cl}(A)$. By Remark 121, we have $A \cup \{x\} \notin \mathscr{I}$, contradiction. Therefore, $B \nsubseteq \text{cl}(A)$. So, there is $y \in B$ such that $y \notin \text{cl}(A)$. By Remark 121, $y \in B \setminus A$ such that $A \cup \{y\} \in \mathscr{I}$. Hence $\mathscr{I}$ has the augmentation property on X. □

Corollary 146. *Let $M = (X, \mathscr{I})$ be a matroid. Then $\mathscr{I}$ has the augmentation property on X.*

Proof. This is an immediate consequence of Theorem 145 because the matroidal closure operator that induces $\mathscr{I}$ is a closure operator with the exchange property on a set X. □

Remark 147. Let cl be a pre-closure operator on a set X and $\mathscr{I}$ be the set of all cl-independent subsets of X. Then $(X, \mathscr{I})$ does not have to have circuits.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topology. Then the topological closure operator on X is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

So, the set of all cl-independent subsets of X is the simplicial complex

$$\mathscr{I} = \{I \subseteq X : I \text{ is finite or } 0 \notin I\}.$$

The set of all dependent subsets of X is

$$\mathscr{P}(X) \setminus \mathscr{I} = \{D \subseteq X : D \text{ is infinite and } 0 \in D\}.$$

Let $D \in X \setminus \mathscr{I}$. So, D is infinite and $0 \in D$. Take $x \in D$ with $x \neq 0$. Now, $D \setminus \{x\}$ is also infinite and $0 \in D \setminus \{x\}$. Thus, $D \setminus \{x\} \in X \setminus \mathscr{I}$, and hence D is not a minimal dependent subset of X. So, there is no circuits in $(X, \mathscr{I})$. □

Remark 148. There is a matroid $(X, \mathscr{I})$ such that it does not have circuits.

Proof. Consider the matroid $(X, \mathscr{P}(X))$. All subsets of X are independent in $(X, \mathscr{P}(X))$. Thus, $(X, \mathscr{P}(X))$ does not have dependent subsets and hence it does not have circuits. □

Theorem 149. *Let $M = (X, \mathscr{I})$ be a matroid. If D is a dependent subset of M, then there is a circuit C of M such that $C \subseteq D$.*

Proof. The proof can be found in [9]. □

Theorem 150. *Let $M = (X, \mathscr{I})$ be a matroid and $\mathscr{C} \subseteq \mathscr{P}(X)$ be the set of all circuits of M. Then the matroidal closure operator cl on X induces $\mathscr{I}$ is*

$$cl(A) = A \cup \{x \in X \setminus A : \text{ there is } C \in \mathscr{C} \text{ such that } x \in C \subseteq A \cup \{x\}\}$$

for each $A \subseteq X$.

Proof. Let cl be the the matroidal closure operator cl on X that induces $\mathscr{I}$. Let $x \in \mathrm{cl}(A) \setminus A$. By Theorem 61 and Theorem 131, there is $I \subseteq A$ such that $I \in \mathscr{I}$ and $I \cup \{x\} \notin \mathscr{I}$. By Theorem 149, there is $C \in \mathscr{C}$ such that $C \subseteq I \cup \{x\}$. Thus, $x \in C$ otherwise, by (I2), $C \in \mathscr{I}$

which is a contradiction. So, we have

$$x \in C \subseteq I \cup \{x\} \subseteq A \cup \{x\}.$$

Now, Let $x \in X \setminus A$ such that there is $C \in \mathscr{C}$ with $x \in C \subseteq A \cup \{x\}$. Take $I = C \setminus \{x\}$. So, $I \subseteq A$ such that $I \in \mathscr{I}$ and $I \cup \{x\} = C \notin \mathscr{I}$. By Theorem 61 and Theorem 131, we get $x \in \mathrm{cl}(A)$. □

Theorem 151. *Let cl be a matroidal closure operator on a set X. Then cl* is also a matroidal closure operator on X.*

Proof. Let $M = (X, \mathscr{I})$ be the matroid in which $\mathscr{I}$ is induced by cl. By Theorem 6, cl^* is a pre-closure on X. We need to show that cl^* is a closure operator and has the exchange property and maximality property on X. First, we show that cl^* is a closure operator on X. By Theorem 116, it suffices to show that cl has the strong exchange property on X. Let $A, B \subseteq X$ be disjoint and $y \in X \setminus (A \cup B)$ such that

$$y \in \mathrm{cl}(A \cup B) \setminus \mathrm{cl}(A).$$

By Theorem 150, there is a circuit C of M such that

$$y \in C \subseteq (A \cup B) \cup \{y\}.$$

Since $y \notin \mathrm{cl}(A)$, we have $C \cap B \neq \varnothing$. Let $x \in C \cap B$. Note that

$$x \in C \subseteq (A \cup (B \setminus \{x\}) \cup \{y\}) \cup \{x\}.$$

So, $x \in B$ such that

$$x \in \mathrm{cl}(A \cup (B \setminus \{x\}) \cup \{y\}).$$

So, cl has the strong exchange property on X, and hence cl^* is a closure operator on X. By Theorem 109, since cl is a closure operator on X, cl^* has the exchange property on X. It remains to show that cl^* has the maximality property on X. Let $A \subseteq Y \subseteq X$ and A be cl^*-independent. Let B be a maximal cl-independent subset of $X \setminus Y$ and let D be a maximal cl-independent subset of $X \setminus A$ such that $B \subseteq D$. Let $J = Y \setminus D$.

Claim 152. J is a maximal cl^*-independent subset of Y.

Proof. First, we prove that J is a cl^*-independent subset of Y. Let $x \in J$. We want to show that $x \notin \mathrm{cl}^*(J \setminus \{x\})$. By (CL2), it suffices to show that $x \notin \mathrm{cl}^*(X \setminus (D \cup \{x\}))$. By Theorem 8, it suffices to show that $x \in \mathrm{cl}(D)$. If $x \in J \setminus A$. Since D is a maximal cl-independent subset of $X \setminus A$ then, by Theorem 125, $X \setminus A \subseteq \mathrm{cl}(D)$. Since $J \setminus A \subseteq X \setminus A$, we get $x \in \mathrm{cl}(D)$. If $x \in A$. Since A is cl*-independent, $x \notin \mathrm{cl}^*(A \setminus \{x\})$. By Theorem 8, we have $x \in \mathrm{cl}(X \setminus A)$. Since D is a maximal cl-independent subset of $X \setminus A$ then, by Theorem 125, $X \setminus A \subseteq \mathrm{cl}(D)$. By (CL2) and (CL3), we get

$$x \in \mathrm{cl}(X \setminus A) \subseteq \mathrm{cl}(\mathrm{cl}(D)) = \mathrm{cl}(D).$$

Thus, $x \notin \mathrm{cl}^*(J \setminus \{x\})$ and hence J is a cl*-independent subset of Y. Now, we show that the cl*-independent subset J of Y is maximal. Suppose, by way of contradiction, that J is a cl*-independent subset of Y but not maximal. So, there is $a \in Y \cap D$ such that $J \cup \{a\}$ is a cl*-independent subset of Y. Therefore,

$$a \notin \mathrm{cl}^*((J \cup \{a\}) \setminus \{a\}) = \mathrm{cl}^*(J).$$

By Theorem 8, we get

$$a \in \mathrm{cl}(((X \setminus Y) \cup (Y \cap D)) \setminus \{a\}).$$

Note that since B is a maximal cl-independent subset of $X \setminus Y$ then, by Theorem 125, we have $X \setminus Y \subseteq \mathrm{cl}(B)$. Since $B \subseteq D$, we obtain

$$((X \setminus Y) \cup (Y \cap D)) \setminus \{a\} \subseteq \mathrm{cl}(D \setminus \{a\}).$$

By (CL2) and (CL3), we get

$$\mathrm{cl}(((X \setminus Y) \cup (Y \cap D)) \setminus \{a\}) \subseteq \mathrm{cl}(\mathrm{cl}(D \setminus \{a\})) = \mathrm{cl}(D \setminus \{a\}).$$

Thus, $a \in \mathrm{cl}(D \setminus \{a\})$ which contradicts that D is a cl-independent. Hence J is a maximal cl*-independent subset of Y. □

From Claim 152, we have J is a maximal cl*-independent subset of Y such that $A \subseteq J$. Therefore, cl* has the maximality property on X. Then cl* is a matroidal closure operator on X. □

Definition 153. Let $M = (X, \mathscr{I})$ be a matroid and cl be the matroidal closure operator on X that induces the simplicial complex $\mathscr{I}$. The matroid $M^* = (X, \mathscr{I}^*)$ is called the dual matroid of M where the simplicial complex $\mathscr{I}^*$ is induced by the matroidal closure operator cl^* on X.

Theorem 154. *Let $M = (X, \mathscr{I})$ be a matroid. Then $M^* = (X, \mathscr{I}^*)$ be the dual matroid of M if and only if*

$$\mathscr{I}^* = \{\varnothing\} \cup \{A \subseteq X : \text{ there is maximal } B \in \mathscr{I} \text{ such that } A \cap B = \varnothing\}.$$

Proof. The proof can be found in [9]. □

Example 155. Let X be a set. The matroids $(X, \{\varnothing\})$ and $(X, \mathscr{P}(X))$ are duals of each other.

Theorem 156. *Let cl be a matroidal closure operator on a set X and $Y \subseteq X$. Then $\mathrm{cl}_{\restriction Y}$ and $\mathrm{cl}_{\cdot Y}$ are matroidal closure operators on Y.*

Proof. By Theorem 5, $\mathrm{cl}_{\restriction Y}$ is a closure operators on Y. By Theorem 106, $\mathrm{cl}_{\restriction Y}$ has the exchange property on Y. We just need to show that $\mathrm{cl}_{\restriction Y}$ has the maximality property on Y. Let $A \subseteq Z \subseteq Y$ and A be $\mathrm{cl}_{\restriction Y}$-independent. So, $x \notin \mathrm{cl}_{\restriction Y}(A \setminus \{x\})$ for each $x \in A$. Thus, $x \notin \mathrm{cl}(A \setminus \{x\})$ for each $x \in A$. So, we have $A \subseteq Z \subseteq Y$ and A is cl-independent. Since cl has the maximality property on X, there is a maximal cl-independent subset B of Z such that $A \subseteq B$. Therefore, B is a maximal $\mathrm{cl}_{\restriction Y}$-independent subset of Z such that $A \subseteq B$. Therefore, $\mathrm{cl}_{\restriction Y}$ has the maximality property on Y. Hence $\mathrm{cl}_{\restriction Y}$ is a matroidal closure operators on Y. By Theorem 151, $\mathrm{cl}_{\cdot Y} = \left(\mathrm{cl}^*_{\restriction Y}\right)^*$ is also a matroidal closure operators on Y. □

Theorem 157. *Let $\mathscr{I}$ be a simplicial complex with the maximality property and the augmentation property on a set X and cl be the pre-closure operator on X induced by $\mathscr{I}$. Then*

1. *$\mathscr{I}^*$ has the maximality property on X.*
2. *$\mathscr{I}^*$ induces cl^*.*
3. *cl^* has the strong exchange property on X.*
4. *cl^* induces $\mathscr{I}^*$.*

Proof. The proof can be found in [9]. □

Now, we will see that the axioms of maximality property and augmentation property on simplicial complexes are enough to tell whether a given set with a simplicial complex is a matroid or not without looking for a matroidal closure operator that induces the simplicial complex.

Corollary 158. *Let X be a set and $\mathscr{I} \subseteq \mathscr{P}(X)$. Then $(X, \mathscr{I})$ is a matroid if and only if*

(I1) $\varnothing \in \mathscr{I}$.

(I2) If $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.

(I3) $\mathscr{I}$ has the augmentation property on X.

(IM) $\mathscr{I}$ has the maximality property on X.

Proof. Suppose that $(X, \mathscr{I})$ is a matroid. So, there is a matroidal closure operator cl on X such that cl induces $\mathscr{I}$. By Theorems 59, 145 and 136, $\mathscr{I}$ satisfies (I1), (I2), (I3) and (IM). Now, suppose that $\mathscr{I}$ satisfies (I1), (I2), (I3) and (IM). By Theorem 61, $\mathscr{I}$ induces a pre-closure operator cl on X. We want to show that cl is a matroidal closure operator on X such that cl induces $\mathscr{I}$. By Theorem 157, cl^* has the strong exchange property on X. By Theorem 116, cl is a closure operator on X. By Theorem 138, cl induces $\mathscr{I}$. By Theorem 118, cl has the exchange property on X. By Theorem 136, cl has the maximality property on X. Thus, cl is a matroidal closure operator on X such that cl induces $\mathscr{I}$. Hence $(X, \mathscr{I})$ is a matroid. □

Example 159. Let X be any set and $n \in \mathbb{N}$.

1. If $\mathscr{I} = \{I \subseteq X : |I| \leq n\}$, then $(X, \mathscr{I})$ is a matroid called a uniform matroid of rank n.
2. If $\mathscr{I}^* = \{I \subseteq X : |X \setminus I| \geq n\}$, then $(X, \mathscr{I}^*)$ is also a matroid called a uniform matroid of co-rank n.

Clearly, $(X, \mathscr{I})$ and $(X, \mathscr{I}^*)$ are duals of each other.

Definition 160. If X is a finite set and $(X, \mathscr{I})$ is a matroid, we say $(X, \mathscr{I})$ is a finite matroid.

Theorem 161. *Let X be a finite set and $\mathscr{I} \subseteq \mathscr{P}(X)$. Then $(X, \mathscr{I})$ is a matroid if and only if*

(I1) $\varnothing \in \mathscr{I}$.

(I2) If $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.

(I3') If $I, J \in \mathscr{I}$ with $|I| \lneqq |J|$, there is $x \in J \setminus I$ such that $I \cup \{x\} \in \mathscr{I}$.

Proof. The proof can be found in [9]. □

Definition 162. Let $M = (X, \mathscr{I})$ be a matroid.

1. M is called a finitary matroid if and only if every circuit of M is finite.

2. M is called a cofinitary matroid if and only if M^* is finitary.

Theorem 163. *An operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ is a finitary matroidal closure operator on a set X if and only if*

(CL1) $A \subseteq cl(A)$ *for all* $A \subseteq X$.

(CL2) $cl(A) \subseteq cl(B)$ *for all* $A \subseteq B \subseteq X$.

(CL3) $cl(cl(A)) = cl(A)$ *for all* $A \subseteq X$.

(CL4) cl *has the exchange property on* X.

(CL5) If $A \subseteq X$ *and* $x \in cl(A)$, *then there is a finite subset* $F \subseteq A$ *such that* $x \in cl(F)$.

Proof. The proof can be found in [34] and [9]. □

Theorem 164. *Let X be a set, V be vector space over the field $\mathbb{F}$ and $f : X \to V$ be a function. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \{x \in X : f(x) \text{ is a linear combination of elements in } f[A]\} \text{ for all } A \subseteq X$$

where $f[A] = \{f(a) : a \in A\} \subseteq V$. Then cl is a finitary matroidal closure operator on X.

Proof. The proof can be found in [9]. □

Corollary 165. *Let X be vector space over the field $\mathbb{F}$. Then the linear closure operator cl on X is a finitary matroidal closure operator on X.*

Proof. Consider the identity function $id : X \to X$. By Theorem 164, the pre-closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\begin{aligned}\mathrm{cl}(A) &= \{x \in X : id(x) \text{ is a linear combination of elements in } id[A]\} \text{ for all } A \subseteq X \\ &= \{x \in X : x \text{ is a linear combination of elements in } A\} \text{ for all } A \subseteq X\end{aligned}$$

is a finitary matroidal closure operator on X. cl is the linear closure operator on X. □

Theorem 166. *Let X be any set and $\mathscr{I} \subseteq \mathscr{P}(X)$. Then $(X, \mathscr{I})$ is a finitary matroid if and only if*

(I1) $\varnothing \in \mathscr{I}$.

(I2) *If* $I \in \mathscr{I}$ *and* $J \subseteq I$, *then* $J \in \mathscr{I}$.

(I3') *If* $I, J \in \mathscr{I}$ *are finite with* $|I| \lneqq |J|$, *there is* $x \in J \setminus I$ *such that* $I \cup \{x\} \in \mathscr{I}$.

(I4) *If* $I \subseteq X$ *and for each finite* $J \subseteq I$ *we have* $J \in \mathscr{I}$, *then* $I \in \mathscr{I}$.

Proof. The proof can be found in [9]. □

Remark 167. The dual of a finitary matroid is not necessarily finitary.

Proof. Let X be an infinite set and $n \in \mathbb{N}$. The uniform matroid $(X, \mathscr{I})$ of rank n is a finitary matroid, where

$$\mathscr{I} = \{I \subseteq X : |I| \leq n\}.$$

But the dual $(X, \mathscr{I}^*)$, the uniform matroid of co-rank n, is not a finitary matroid, where

$$\mathscr{I}^* = \{I \subseteq X : |X \setminus I| \geq n\}$$

because for every finite subset $J \subseteq X$ we have $|X \setminus I| \geq n$. So, for every finite subset $J \subseteq X$ we have $J \in \mathscr{I}^*$ but $X \notin \mathscr{I}^*$. □

Theorem 168. *Let* G *be a graph and* $X = E(G)$. *Define*

$$\mathscr{I} = \{I \subseteq X : I \textit{ contains no circuits of } G\}.$$

Then $(X, \mathscr{I})$ *is a finitary matroid.*

Proof. The proof can be found in [9]. □

Definition 169. The finitary matroid $(X, \mathscr{I})$ defined in Theorem 168 is called the finite cycle matroid of G.

Theorem 170. *Let* $M = (X, \mathscr{I})$ *be a finite cycle matroid of a graph* G *where* $X = E(G)$. *If* $M^* = (X, \mathscr{I}^*)$ *is the dual of* M, *then*

$$\mathscr{I}^* = \{I \subseteq X : I \textit{ contains no cuts of } G\}.$$

Proof. The proof can be found in [9]. □

Definition 171. The matroid $M^* = (X, \mathscr{I}^*)$ defined in Theorem 170 is called the bond matroid of G.

Corollary 172. *The circuits of the bond matroid of a graph G are the bonds of G.*

Definition 173. Let G be a graph. $C \subseteq E(G)$ is called an algebraic circuit of G if and only if it is a circuit of G or an edge set of a double ray of G.

Theorem 174. ***(Higgs)*** *Let G be a graph, $X = E(G)$ and $\mathscr{C}$ be the collection of all algebraic circuits of G. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = A \cup \{x \in X \setminus A : \text{ there is } C \in \mathscr{C} \text{ such that } x \in C \subseteq A \cup \{x\}\}$$

for all $A \subseteq X$. Then cl is a matroidal closure operator on X if and only if G contains no subgraph isomorphic to a subdivision of the Bean graph, Figure 2.2.1.

Proof. The proof can be found in [14]. □

Note that the matroidal closure operator cl induces the simplicial complex

$$\mathscr{I} = \{I \subseteq X : I \text{ contains no algebraic circuits of } G\}.$$

Definition 175. The matroid $(X, \mathscr{I})$ defined in Theorem 174 is called the algebraic cycle matroid where $\mathscr{I}$ is induced by the matroidal closure operator cl on X.

Example 176. Consider the Bean graph G, see Figure 2.2.1. Let $X = E(G)$ and

$$\mathscr{I} = \{I \subseteq X : I \text{ contains no algebraic circuits of } G\}.$$

By Theorem 174, $(X, \mathscr{I})$ is not a matroid.

Corollary 177. *If G is a locally finite graph and $X = E(G)$, then $(X, \mathscr{I})$ is a matroid where*

$$\mathscr{I} = \{I \subseteq X : I \textit{ contains no algebraic circuits of } G\}.$$

Theorem 178. *Let G be a graph and $X = E(G)$. If $M = (X, \mathscr{I})$ is an algebraic cycle matroid of G, then $M^* = (X, \mathscr{I}^*)$ is the dual of M such that*

$$\mathscr{I}^* = \{I \subseteq X : I \textit{ contains no nibbles of } G\}.$$

Proof. The proof can be found in [9]. □

Definition 179. The matroid $M^* = (X, \mathscr{I}^*)$ defined in Theorem 178 is called the nibble matroid of G.

Remark 180. Let $\langle G, * \rangle$ be a group and $\mathrm{cl} : \mathscr{P}(G) \to \mathscr{P}(G)$ be the algebraic closure on G defined by

$$\begin{aligned} \mathrm{cl}(A) &= \text{ The subgroup generated by } A \\ &= \cap \{H \subseteq G : H \text{ is a subgroup of } G \text{ containing } A\} \end{aligned}$$

for all $A \subseteq G$. Then cl is not a matroidal closure operator on G.

Proof. Consider the symmetric group $\langle S_4, \circ \rangle$. Take $A = \{e\}$, $x = (1234)$ and $y = (13)(24)$. So, $A \subseteq S_4$ and $x, y \in S_4 \setminus A$ are distinct.

$$\begin{aligned} \mathrm{cl}(A) &= \mathrm{cl}(\{e\}) = \{e\} \\ \mathrm{cl}(A \cup \{x\}) &= \mathrm{cl}(\{e, (1234)\}) = \{e, (1234), (13)(24), (1432)\} \\ \mathrm{cl}(A \cup \{y\}) &= \mathrm{cl}(\{e, (13)(24)\}) = \{e, (13)(24)\} \end{aligned}$$

Therefore, $y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$ but $x \notin \mathrm{cl}(A \cup \{y\})$. Thus, cl does not have the exchange property on S_4. Then cl is not a matroidal closure on S_4. □

2.3 Topology

This section introduces definitions and theorems from topology that are used in this thesis. The proofs and examples can be found in [39], [21] and [19].

2.3.1 Metric Spaces

Definition 181. A metric space $\langle X, d \rangle$ is a set X together with a function $d : X \times X \to \mathbb{R}$ such that

(M1) $d(x, y) \geq 0$ for each $x, y \in X$;

(M2) $d(x, y) = 0$ if and only if $x = y$ for each $x, y \in X$;

(M3) $d(x, y) = d(x, y)$ for each $x, y \in X$;

(M4) $d(x,z) \leq d(x,y) + d(y,z)$ for each $x, y, z \in X$.

d is said to be a metric on X and the real number $d(x,y)$ is called the distance between x and y in X.

We say that $\langle X, d \rangle$ is a pseudometric space and d is a pseudometric on X if and only if d satisfies (M1), (M3) and (M4) with (M2) replaced by the weaker axiom

(M2') $d(x,x) = 0$ for each $x \in X$.

Example 182. Consider the set of all real scalars or complex scalars $\mathbb{F}$. Define $d : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{R}$ by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{F}^n$. Then $\langle \mathbb{F}^n, d \rangle$ is a metric space.

Definition 183. The metric d defined in Example 182 is called the usual metric on $\mathbb{F}^n$.

Example 184. In Example 73, we saw that $\ell_p, \ell_\infty, c, c_0, c_{00}$ are vector subspaces of $\mathbb{F}^{\mathbb{N}}$. Here we define a metric on each of these sequence spaces.

1. For $p = \infty$, define a metric $d : \ell_\infty \times \ell_\infty \to \mathbb{R}$ by

   $$d((x_n), (y_n)) = \sup\{|x_n - y_n| : n \in \mathbb{N}\} \text{ for all } (x_n), (y_n) \in \ell_\infty.$$

   d is also a metric on the vector subspaces c_0 and c_{00}.

2. For $1 \leq p < \infty$, define a metric $d : \ell_p \times \ell_p \to \mathbb{R}$ by

   $$d((x_n), (y_n)) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \text{ for all } (x_n), (y_n) \in \ell_p.$$

   d is also a metric on the vector subspace c_{00}.

3. For $0 < p < 1$, define a metric $d : \ell_p \times \ell_p \to \mathbb{R}$ by

   $$d((x_n), (y_n)) = \sum_{n=1}^{\infty} |x_n - y_n|^p \text{ for all } (x_n), (y_n) \in \ell_p.$$

4. Define a pseudometric $d : c \times c \to \mathbb{R}$ by

   $$d((x_n), (y_n)) = \lim_{n \to \infty} |x_n - y_n| \text{ for all } (x_n), (y_n) \in c.$$

Definition 185. Let $\langle X, d\rangle$ be a metric (pseudometric) space and $x \in X$. For each $\epsilon > 0$, the set

$$B(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$$

is called the ϵ-disc about x.

Theorem 186. *Let $\langle X, d\rangle$ be a metric (pseudometric) space and define $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ by*

$$\mathscr{N}(x) = \{B(x,\epsilon) : \epsilon > 0\} \text{ for each } x \in X.$$

Then $\mathscr{N}$ is a neighborhood base on X.

Theorem 187. *Let $\langle X, d\rangle$ be a metric (pseudometric) space. A subset U is open in X if and only if for each $x \in U$, there is $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.*

Proof. Let $\mathscr{N}$ be the neighborhood base on X. By Theorem 47, $U \subseteq X$ is open if and only if for each $x \in U$, there is $B(x,\epsilon) \in \mathscr{N}(x)$ such that $B(x,\epsilon) \subseteq U$ if and only if for each $x \in U$, there is $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$. □

Theorem 188. *Let $\langle X, d\rangle$ be a metric (pseudometric) space. For each $x \in X$ and each $\epsilon > 0$, the ϵ-disc $B(x,\epsilon)$ about x is open.*

Theorem 189. *Let $\langle X, d\rangle$ be a metric (pseudometric) space. Then*

1. *$\varnothing$ and X are both open.*

2. *The finite intersections of open sets are open.*

3. *The unions of open sets are open.*

Definition 190. Let $\langle X, d\rangle$ be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{d} x$, if and only if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Definition 191. Let $\langle X, d\rangle$ be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called Cauchy if and only if for each $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, y_m) < \epsilon \text{ for each } n, m \geq n_0.$$

Theorem 192. *Every convergent sequence in a metric space is Cauchy.*

Remark 193. Not every Cauchy sequence in a metric space converges.

Definition 194. A metric space is called complete if and only if every Cauchy sequence is convergent.

Theorem 195. *Every convergent sequence in a metric space is bounded.*

Definition 196. Let $\langle X, d\rangle$ be a metric space. A subset $D \subseteq X$ is called dense if and only if for each $x \in X$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in D that converges to x.

2.3.2 Topological Spaces

In Theorem 189, we saw that the open sets of a metric space have important properties. Here we define topological spaces based on those properties.

Definition 197. Let X be any set. A collection $\tau \subseteq \mathscr{P}(X)$ of subsets of X is called a topology for X if and only if

(O1) $\varnothing$ and X are both in τ.

(O2) The finite intersections of elements of τ are in τ.

(O3) The unions of elements of τ are in τ.

We say that (X, τ) is a topological space and the elements of τ are called open sets of X.

Theorem 198. *Any metric (pseudometric) space is a topological space.*

Proof. Let $\langle X, d\rangle$ be a metric (pseudometric) space and let τ_d be the collection of all open sets, see Theorem 187, of $\langle X, d\rangle$. That is,

$$\tau_d = \{U \subseteq X : \text{for each } x \in U \text{ there is } \epsilon > 0 \text{ such that } B(x, \epsilon) \subseteq U\}.$$

By Theorem 189, (X, τ_d) is a topological space. □

Definition 199. Let $\langle X, d\rangle$ be a metric (pseudometric) space and τ_d be the induced topology as defined in Theorem 198. We say that τ_d is a metric (pseudometric) topology on X.

Definition 200. The metric topology induced by the usual metric on any subset of $\mathbb{F}^n$ is called the usual topology.

Example 201. The usual metric $d : \mathbb{F} \times \mathbb{F} \to \mathbb{R}$ defined by

$$d(x, y) = |x - y| \text{ for all } x, y \in \mathbb{F}$$

induces the usual topology on $\mathbb{F}$.

Definition 202. A topological space (X, τ) is called metrizable (pseudometrizable) if and only if there is a metric (pseudometric) d on X such that τ is the metric (pseudometric) topology τ_d.

Example 203. Let X be any set and $\tau = \{\varnothing, X\}$. τ is a topology for X called the indiscrete (trivial) topology for X.

Example 204. Let X be any set. $(X, \mathscr{P}(X))$ is a topological space called the discrete space.

Example 205. Let $X = \{1, 2\}$ and $\tau = \{\varnothing, \{1\}, X\}$. (X, τ) is a topological space called the Sierpinski space.

Example 206. Let X be any set and $x \in X$. Define

$$\tau = \{U \subseteq X : X \setminus U \text{ is finite or } x \in X \setminus U\}.$$

(X, τ) is a topological space called the Uncountable Fort Space [32].

Definition 207. Let X be a topological space. $A \subseteq X$ is called closed in X if and only if $X \setminus A$ is open in X.

Example 208. In Example 206, the collection of all closed sets in the Uncountable Fort Space is

$$\mathscr{F} = \{A \subseteq X : A \text{ is finite or } x \in A\}.$$

Theorem 209. *(Closed Axioms)*

1. *Let X be a topological space and $\mathscr{F} \subseteq \mathscr{P}(X)$ be the collection of all closed sets in X. Then $\mathscr{F}$ satisfies the following:*

   (F1') $\varnothing$ and X are both in $\mathscr{F}$.

   (F2) The intersections of elements of $\mathscr{F}$ are in $\mathscr{F}$.

   (F3) The finite unions of elements of $\mathscr{F}$ are in $\mathscr{F}$.

2. *Let X be a set and $\mathscr{F} \subseteq \mathscr{P}(X)$ satisfying (F1'), (F2) and (F3). Then*

$$\tau = \{X \setminus C : C \in \mathscr{F}\}$$

is the unique topology for X in which the collection of all closed sets is $\mathscr{F}$.

Remark 210. Theorem 209 shows that the collection $\mathscr{F}$ of all closed sets in a topological space X is a closure system on X. By Theorem 25, $\mathscr{F}$ induces the closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}(A) = \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X.$$

Definition 211. The closure operator cl on a topological space X defined in Remark 210 is called the topological closure operator on X.

Example 212. Consider the Uncountable Fort Space, see Example 206 and Example 208. The topological closure operator cl on X is

$$\begin{aligned} \mathrm{cl}(A) &= \bigcap \{C \in \mathscr{F} : A \subseteq C\} \text{ for all } A \subseteq X \\ &= \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{x\} & \text{if } A \text{ is infinite} \end{cases} \text{ for all } A \subseteq X. \end{aligned}$$

Theorem 213. *(Closure Axioms)*

1. *Let X be a topological space and cl be the topological closure operator on X. Then cl satisfies the following:*

   (CL1) $A \subseteq cl(A)$ *for all* $A \subseteq X$.

   (CL2") $cl(A \cup B) = cl(A) \cup cl(B)$ *for all* $A, B \subseteq X$.

   (CL3) $cl(cl(A)) = cl(A)$ *for all* $A \subseteq X$.

   (CL4") $cl(\varnothing) = \varnothing$.

2. *Let X be a set and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ be an operator on X satisfying (CL1), (CL2"), (CL3) and (CL4"). Define*

$$\mathscr{F} = \{C \subseteq X : cl(C) = C\}.$$

Then $\tau = \{X \setminus C : C \in \mathscr{F}\}$ is the unique topology for X in which the collection of all closed sets is $\mathscr{F}$. Moreover the topological closure operator on (X,τ) is cl.

Remark 214. The axiom (CL2”) is stronger than the axiom (CL2).

Proof. Suppose that cl is an operator on X satisfying (CL2”). To show that cl satisfies (CL2), let $A \subseteq B \subseteq X$. So, we have $A, B \setminus A \subseteq X$. By (CL2”), we have

$$\mathrm{cl}(A \cup (B \setminus A)) = \mathrm{cl}(A) \cup \mathrm{cl}(B \setminus A).$$

Since $A \subseteq B$, we get $\mathrm{cl}(B) = \mathrm{cl}(A) \cup \mathrm{cl}(B \setminus A)$. Thus, $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. Hence cl satisfies (CL2). □

Theorem 215. *Let cl be the topological closure operator on a topological space X. Then there is an open neighborhood base $\mathscr{N}$ on X such that*

$$cl(A) = \left\{x \in X : A \cap A' \neq \varnothing \text{ for each } A' \in \mathscr{N}(x)\right\} \text{ for all } A \subseteq X.$$

Proof. This is an immediate consequence of Corollary 51 because cl is a closure operator on X. □

We know, from Theorem 44, that a topological closure operator on a topological space might be induced by equivalent neighborhood bases.

Theorem 216. *Let (X,τ) be a topological space. If a neighborhood base $\mathscr{N}$ on X induces the topological closure operator on X, then $\mathscr{N}$ induces the topology τ as follows:*

$$\tau = \{U \subseteq X : \text{for each } x \in U \text{ there is } A \in \mathscr{N}(x) \text{ such that } A \subseteq U\}.$$

Proof. This is an immediate consequence of Theorem 47. □

Definition 217. Let (X,τ) be a topological space. If a neighborhood base $\mathscr{N}$ on X induces the topology τ as in Theorem 216, we say that $\mathscr{N}$ is a neighborhood base for the topology τ on X.

Theorem 218. *(Neighborhood base Axioms)*

1. *Let (X,τ) be a topological space and $\mathscr{N}$ be a neighborhood base for the topology τ on X. Then for each $x \in X$, we have*

(V1) If $A \in \mathscr{N}(x)$, then $x \in A$.

(V2) If $A_1, A_2 \in \mathscr{N}(x)$, then there is $A \in \mathscr{N}(x)$ such that $A \subseteq A_1 \cap A_2$.

(V3) If $A \in \mathscr{N}(x)$, then there is $A_0 \in \mathscr{N}(x)$ such that for each $y \in A_0$ there is $B \in \mathscr{N}(y)$ with $B \subseteq A$.

2. *Let X be a set and $\mathscr{N} : X \to \mathscr{P}(\mathscr{P}(X))$ be a function on X such that for each $x \in X$, the axioms (V1), (V2) and (V3) hold. Define*

$$\tau = \{U \subseteq X : \text{for each } x \in U \text{ there is } A \in \mathscr{N}(x) \text{ such that } A \subseteq U\}.$$

Then τ is a topology on X in which the neighborhood base is $\mathscr{N}$.

In Theorem 44, we showed in general that if different neighborhood bases on a set induce pre-closure operators, then the induced pre-closure operators on the set are identical if and only if the neighborhood bases are equivalent. Here, in particular, if the induced pre-closure operators are topological closure operators on the set, then the topological closure operators are the same if and only if the neighborhood bases are equivalent.

Theorem 219. *(Hausdorff Criterion) Let $\mathscr{N}$, $\mathscr{N}'$ be neighborhood bases on a set X and τ, τ' be the topologies on X in which the neighborhood bases are $\mathscr{N}$, $\mathscr{N}'$ respectively. Then $\tau = \tau'$ if and only if $\mathscr{N}$, $\mathscr{N}'$ are equivalent, that is, for each $x \in X$, the following hold:*

1. *for each $A \in \mathscr{N}(x)$, there is $A' \in \mathscr{N}'(x)$ with $A' \subseteq A$.*

2. *for each $A' \in \mathscr{N}'(x)$, there is $A \in \mathscr{N}(x)$ with $A \subseteq A'$.*

Definition 220. Let (X, τ) be a topological space. A subcollection $\mathscr{B} \subseteq \tau$ is called a base for τ if and only if

$$\tau = \left\{\bigcup \mathscr{A} : \mathscr{A} \subseteq \mathscr{B}\right\}.$$

Theorem 221. *Let X be a set and $\mathscr{B} \subseteq \mathscr{P}(X)$. Then $\mathscr{B}$ is a base for a topology on X if and only if*

1. *$X = \bigcup \mathscr{B}$.*

2. *For each $A, B \in \mathscr{B}$ and each $x \in A \cap B$, there is $C \in \mathscr{B}$ with $x \in C \subseteq A \cap B$.*

Theorem 222. *Let (X, τ) be a topological space and $Y \subseteq X$. Define $\tau' \subseteq \mathscr{P}(Y)$ as*

$$\tau' = \{U \cap Y : U \in \tau\}.$$

Then τ' is a topology on Y.

Definition 223. A topology τ' on a subset Y of a topological space (X,τ), as defined in Theorem 222, is called the subspace (relative) topology on Y and the the topological space (Y,τ') is referred to as topological subspace of (X,τ).

Notations When talking about multiple topological spaces we denote to the neighborhood base of a topological space X by $\mathcal{N}_X$ and to the topological closure operator of X by cl_X.

Definition 224. Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous at a point $x \in X$ if and only if for each $V \in \mathcal{N}_Y(f(x))$ there is $U \in \mathcal{N}_X(x)$ such that $f(U) \subseteq V$.

Definition 225. Let X and Y be topological spaces. We say that a function $f : X \to Y$ is continuous on X if and only if f is continuous at each point $x \in X$.

Theorem 226. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are all equivalent:*

1. *f is continuous.*
2. *For each open set U in Y, $f^{-1}(U)$ is an open set in X.*
3. *For each closed set C in Y, $f^{-1}(C)$ is a closed set in X.*
4. *For each $A \subseteq X$, we have $f(cl_X(A)) \subseteq cl_Y(f(A))$.*

Definition 227. Let A be any set and X_α be a set for each $\alpha \in A$. The Cartesian product of the sets X_α, denoted $\prod_{\alpha\in A} X_\alpha$ or $\prod X_\alpha$, is the set of all functions $x : A \to \bigcup_{\alpha\in A} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for each $\alpha \in A$.

The value $x(\alpha)$ is usually denoted by x_α and is called the αth coordinate of x.

Example 228. Let $A = \{1,2\}$ and X_1, X_2 be any sets. Then the Cartesian product of X_1 and X_2 is the set

$$X = \prod_{\alpha\in A} X_\alpha = X_1 \times X_2 = \{(x_1,x_2) : x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$

Theorem 229. *Let A be any set and (X_α,τ_α) be a topological space for each $\alpha \in A$. Then the collection*

$$\mathscr{B} = \left\{ \prod_{\alpha\in A} U_\alpha : U_\alpha \in \tau_\alpha \textit{ and the set } \{\alpha \in A : U_\alpha \neq X_\alpha\} \textit{ is finite} \right\}$$

is a base for a topology on the Cartesian product $X = \prod X_\alpha$.

Definition 230. Let A be any set and (X_α, τ_α) be a topological space for each $\alpha \in A$. The topology defined on the Cartesian product $X = \prod X_\alpha$ in Theorem 229 is called the product (Tychonoff) topology on $X = \prod X_\alpha$.

Corollary 231. *Let A be a finite set and (X_α, τ_α) be a topological space for each $\alpha \in A$. Then the collection*

$$\mathscr{B} = \left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \in \tau_\alpha \right\}$$

is a base for the product topology on the Cartesian product $X = \prod X_\alpha$.

Example 232. Let $A = \{1,2\}$ and X_1, X_2 be any topological spaces. Then

$$\mathscr{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is a base for the product topology on $X_1 \times X_2$.

Definition 233. A topological space X is called a T_0-space if and only if for every distinct points $x, y \in X$, there is an open set U in X such that U contains exactly one of the points x, y.

Theorem 234. *Every topological subspace of a T_0-space is T_0.*

Definition 235. A topological space X is called a T_1-space if and only if for any distinct points $x, y \in X$, there are open sets U and V in X such that $x \in U \setminus V$ and $y \in V \setminus U$.

Theorem 236. *Every T_1-space is a T_0-space.*

Theorem 237. *Let X be a topological space. Then the following conditions are equivalent:*

1. *X is a T_1-space.*

2. *$\{x\}$ is closed in X for each $x \in X$.*

3. *If $A \subseteq X$ and $\mathscr{U}$ is the family of all open sets in X that contain A, then $\bigcap \mathscr{U} = A$.*

Theorem 238. *Every topological subspace of a T_1-space is T_1.*

Theorem 239. *Let X be a finite set. The T_1-topology on X is discrete.*

Definition 240. A topological space X is called a T_2-space (Hausdorff space) if and only if for any distinct points $x, y \in X$, there are disjoint open sets U and V in X such that $x \in U$ and $x \in V$.

Theorem 241. *Every T_2-space is a T_1-space.*

Theorem 242. *Every metrizable space is Hausdorff.*

Theorem 243. *Every topological subspace of a Hausdorff space is Hausdorff.*

Definition 244. Let X be a topological space. X is called regular if and only if for every closed set $C \subseteq X$ and every $x \notin C$, there are disjoint open sets U and V in X such that $C \subseteq U$ and $x \in V$.

Theorem 245. *Let X be a topological space. The following conditions are equivalent:*

1. *X is regular.*
2. *For each open set U in X and $x \in U$, there is an open set V in X containing x such that $cl(V) \subseteq U$.*
3. *There is a closed neighborhood base $\mathcal{N}$ of X.*

Remark 246. A regular space does not have to be T_1.

Definition 247. A topological space is T_3 if and only if it is T_1 and regular.

Theorem 248. *Any T_3-space is a T_2-space.*

Theorem 249. *Every topological subspace of a regular space is regular and every topological subspace of a T_3-space is T_3.*

Definition 250. A topological space X is called completely regular if and only if for every closed set $C \subseteq X$ and every $x \notin C$, there is a continuous function $f : X \to [0,1]$ with $f(x) = 0$ and $f[C] \subseteq \{1\}$.

Theorem 251. *Any completely regular space is regular.*

Definition 252. A topological space X is called Tychonoff if and only if X is T_1 and completely regular.

Theorem 253. *Every Tychonoff space is T_3.*

Theorem 254. *Every topological subspace of a completely regular space is completely regular and every topological subspace of a Tychonoff space is Tychonoff.*

Definition 255. Let X be a topological space. X is called normal if and only if for every disjoint closed sets $C, D \subseteq X$, there are disjoint open sets U and V in X such that $C \subseteq U$ and $D \subseteq V$.

Theorem 256. *Any regular space is normal.*

Definition 257. A topological space X is called T_4 if and only if X is T_1 and normal.

Theorem 258. *Any T_4-space is T_3.*

Definition 259. A topological space X is called completely normal if and only if every topological subspace of X is normal.

Definition 260. Let X be a topological space. A collection $\mathscr{A} \subseteq \mathscr{P}(X)$ is called a cover (covering) of X if and only if $\bigcup \mathscr{A} = X$. A subcollection $\mathscr{A}' \subseteq \mathscr{A}$ is called a subcover of $\mathscr{A}$ if and only if $\mathscr{A}'$ is cover of X. An open cover of X is a cover of X consisting of open sets.

Definition 261. A topological space X is called compact if and only if every open cover of X has a finite subcover.

Definition 262. Let X be a topological space. A subset A is called compact in X if and only if A is compact with respect to the subspace topology on A.

Definition 263. A topological space X is called locally compact if and only if there is a neighborhood base $\mathscr{N}$ of X such that for each $x \in X$, each $A \in \mathscr{N}(x)$ is a compact set in X.

2.4 Functional Analysis

This section introduces definitions and theorems from functional analysis that are used in this thesis. More details can be found in [30], [28] and [10].

2.4.1 Banach Spaces

Definition 264. A normed space $\langle X, \|\cdot\| \rangle$ is a vector space over $\mathbb{F}$ with a function $\|\cdot\| : X \to \mathbb{R}$, called a norm such that

(N1) $\|x\| \geq 0$ for each $x \in X$.

(N2) $\|x\| = 0$ if and only if $x = 0$ for each $x \in X$.

(N3) $\|\lambda x\| = |\lambda| \|x\|$ for each $x \in X$ and for each $\lambda \in \mathbb{F}$.

(N4) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$.

If the axiom (N2) is replaced by the weaker axiom

(N2') If $x = 0$, then $\|x\| = 0$

we say that $\langle X, \|\cdot\|\rangle$ is a seminormed space and the function $\|\cdot\| : X \to \mathbb{R}$ is a seminorm. The real number $\|x\|$ is called the norm of x.

Theorem 265. *Every normed (seminormed) space is a metric (pseudometric) space.*

Proof. Let $\langle X, \|\cdot\|\rangle$ be a normed (seminormed) space. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \|x - y\| \text{ for all } x, y \in X.$$

Then $\langle X, d\rangle$ is a metric (pseudometric) space. □

Definition 266. The metric (pseudometric) space defined in Theorem 265 is called a metric space induced by a norm (seminorm).

Example 267. In Example 73, we saw that $\ell_p, \ell_\infty, c, c_0, c_{00}$ are vector subspaces of $\mathbb{F}^{\mathbb{N}}$. Here we define a norm on each of these sequence spaces.

1. For $p = \infty$, define a norm $\|\cdot\|_\infty : \ell_\infty \to \mathbb{R}$ by

   $$\|(x_n)\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\} \text{ for all } (x_n) \in \ell_\infty.$$

   $\|\cdot\|_\infty$ is also a norm on the vector subspaces c, c_0 and c_{00}.

2. For $1 \leq p < \infty$, define a metric $\|\cdot\|_p : \ell_p \to \mathbb{R}$ by

   $$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \text{ for all } (x_n) \in \ell_p.$$

$\|\cdot\|_p$ is also a metric on the vector subspace c_{00}. Note that for $0 < p < 1$, the function $\|\cdot\|_p$ is not a norm on ℓ_p because we have $(1,0,0,\ldots),(0,1,0,\ldots) \in \ell_p$ but

$$\|(1,0,0,\ldots)+(0,1,0,\ldots)\|_p = 2^{\frac{1}{p}}$$
$$\|((1,0,0,\ldots))\|_p + \|((0,1,0,\ldots))\|_p = 2.$$

Thus,

$$\|(1,0,0,\ldots)+(0,1,0,\ldots)\|_p \nleq \|((1,0,0,\ldots))\|_p + \|((0,1,0,\ldots))\|_p$$

which means that (N4) does not hold.

3. For $0 < p < 1$, define a norm $\|\cdot\|_p : \ell_p \to \mathbb{R}$ by

$$\|(x_n)\|_p = \sum_{n=1}^{\infty} |x_n|^p \text{ for all } (x_n) \in \ell_p.$$

4. Define a function $\|\cdot\| : c \to \mathbb{R}$ by

$$\|(x_n)\| = \lim_{n\to\infty} |x_n| \text{ for all } (x_n) \in c.$$

$\|\cdot\|$ is not a norm on c because $\left\|\left(\frac{1}{n}\right)_{n\in\mathbb{N}}\right\| = \lim_{n\to\infty}\left|\frac{1}{n}\right| = 0$ but $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} \neq (0)_{n\in\mathbb{N}}$ which means that (N2) fails.

Definition 268. A Banach space is a normed space such that the metric space induced by the norm is complete.

Example 269. The normed spaces $\langle \ell_\infty, \|\cdot\|_\infty\rangle$, $\langle c, \|\cdot\|_\infty\rangle$ and $\langle c_0, \|\cdot\|_\infty\rangle$ are Banach spaces. The normed space $\left\langle \ell_p, \|\cdot\|_p\right\rangle$ is a Banach space for all $p < \infty$. The normed space $\langle c_{00}, \|\cdot\|_\infty\rangle$ is dense in c_0 and the normed space $\left\langle c_{00}, \|\cdot\|_p\right\rangle$ is dense in ℓ_p for all $p < \infty$.

2.4.2 Hilbert Spaces

Definition 270. An inner product space $\langle X, (\cdot,\cdot)\rangle$ is a vector space X over $\mathbb{F}$ together with a function $(\cdot,\cdot) : X \times X \to \mathbb{F}$, called the inner product such that

(IP1) $(x,x) \geq 0$ for each $x \in X$ with $(x,x) = 0$ if and only if $x = 0$.

(IP2) $(x,y) = \overline{(y,x)}$ for each $x,y \in X$.

(IP3) $(x, \lambda y) = \lambda(x, y)$ for each $x, y \in X$ and for each $\lambda \in \mathbb{F}$.

(IP4) $(x, y+z) = (x, y) + (x, z)$ for each $x, y, z \in X$.

The scalar (x, y) is called the inner product of x and y.

Theorem 271. *Every inner product space is a normed space.*

Proof. Let $\langle X, (\cdot,\cdot)\rangle$ be an inner product space. Define $\|\cdot\| : X \to \mathbb{R}$ by

$$\|x\| = \sqrt{(x,x)} \text{ for all } x \in X.$$

Then $\langle X, \|\cdot\|\rangle$ is a normed space. □

Definition 272. The normed space defined in Theorem 271 is called a normed space induced by an inner product.

Corollary 273. *Every inner product space is a metric space.*

Proof. This is an immediate consequence of Theorems 271 and 265. □

Definition 274. The metric space defined in Corollary 273 is called the metric space induced by an inner product.

Definition 275. A Hilbert space is an inner product space such that the metric space induced by the inner product is complete.

Example 276. The set ℓ_2 of all sequences $(x_n)_{n\in\mathbb{N}}$ of complex numbers with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and the inner product on ℓ_2 is defined by

$$\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \sum_{n=1}^{\infty} \overline{x_n} y_n.$$

2.4.3 Topological Vector Spaces

This section introduces definitions and theorems on topological vector spaces that are used in this thesis. More details can be found in [22], [29] and [4].

Definition 277. A topological vector space is a vector space X over $\mathbb{F}$ equipped with a topology τ that makes the addition operation and scalar multiplication operation continuous functions. The topology τ is said to be a vector topology on X.

Some authors such as Rudin [30] require that the topology of the topological vector space is T_1.

Definition 278. Let X be a topological vector space. If the vector topology of X is Hausdorff, we say that X is a Hausdorff topological vector space.

Example 279. Any vector space over $\mathbb{F}$ endowed with the trivial topology is a topological vector space.

Example 280. The discrete topology on any vector space $X \neq \{0\}$ is not a vector topology on X.

Theorem 281. *Any normed space equipped with the metric topology induced by the norm is a Hausdorff topological vector space.*

The vector topology of a topological vector space is completely determined by the neighborhood base of any of its vectors, particularly by the neighborhood base of the zero vector.

Theorem 282. *Let X be a topological vector space over $\mathbb{F}$ and $\mathcal{N}$ be a neighborhood base for the vector topology on X. Then*

$$\mathcal{N}(x) = \{x + A : A \in \mathcal{N}(0)\} \text{ for each } x \in X.$$

Theorem 283. *Let X be a topological vector space over $\mathbb{F}$ and $\mathcal{N}$ be a neighborhood base for the vector topology on X. For each $A \in \mathcal{N}(0)$ and each $\alpha \in \mathbb{F} \setminus \{0\}$, we have $\alpha A \in \mathcal{N}(0)$.*

Theorem 284. *Let X be a topological vector space over $\mathbb{F}$ and $\mathcal{N}$ be a neighborhood base for the vector topology on X. For each $A \in \mathcal{N}(0)$, we have*

1. *A is absorbent.*

2. *There is $A' \in \mathcal{N}(0)$ such that $A' + A' \subseteq A$.*

3. *There is a balanced $A' \in \mathcal{N}(0)$ such that $A' \subseteq A$.*

Corollary 285. *Let X be a topological vector space over $\mathbb{F}$. Then there is a neighborhood base $\mathcal{N}$ for the vector topology on X such that each $A \in \mathcal{N}(0)$ is balanced.*

Theorem 286. *Let X be a topological vector space over $\mathbb{F}$ and $\mathcal{N}$ be a neighborhood base for the vector topology on X. For each $A \in \mathcal{N}(0)$, there is $A' \in \mathcal{N}(0)$ such that $cl(A') \subseteq A$ where cl is the topological closure operator of X.*

Corollary 287. *Any topological vector space over $\mathbb{F}$ is regular.*

Theorem 288. *Let X be a topological vector space over $\mathbb{F}$ and $\mathscr{N}$ be a neighborhood base for the vector topology on X. If $A \in \mathscr{N}(0)$ is balanced, then $cl(A)$ is balanced where cl is the topological closure operator of X.*

Corollary 289. *Let X be a topological vector space over $\mathbb{F}$. Then there is a neighborhood base $\mathscr{N}$ for the vector topology on X such that each $A \in \mathscr{N}(0)$ is balanced closed.*

As we said, once we know the basic neighborhoods of 0, we know the basic neighborhoods of any vector in a topological vector space. Therefore, we need some criteria on a neighborhood base defined on a vector space which ensures that it is the neighborhood base for some vector topology on the vector space.

Theorem 290. *Let X be a vector space over $\mathbb{F}$ and $\mathscr{N}$ be a family of subsets of X such that*

1. *Every $A \in \mathscr{N}$ is balanced and absorbent.*
2. *For every $A_1, A_2 \in \mathscr{N}$, there is $A \in \mathscr{N}$ such that $A \subseteq A_1 \cap A_2$.*
3. *For every $A \in \mathscr{N}$, there is $A' \in \mathscr{N}$ such that $A' + A' \subseteq A$.*

Then there is a vector topology τ on X such that $\mathscr{N}$ is a neighborhood base for τ.

Theorem 291. *Let X be a topological vector space over $\mathbb{F}$ and $A \subseteq X$. If A is a vector subspace of X, then $cl(A)$ is also a vector subspace of X where cl is the topological closure operator of X.*

Theorem 292. *Let X be a Hausdorff topological vector space over $\mathbb{F}$ and $A \subseteq X$. If A is a finite-dimensional vector subspace of X, then A is closed.*

Theorem 293. *Let X be a topological vector space over $\mathbb{F}$. If M is a closed vector subspace of X and N is a finite-dimensional vector subspace of X, then*

$$M + N = \{m + n : m \in M \text{ and } n \in N\}$$

is a closed vector subspace of X.

Theorem 294. *Let X be a Hausdorff topological vector space over $\mathbb{F}$. Then X is locally compact if and only if X is finite-dimensional.*

Chapter 3

New Spaces

The purpose of this thesis is to find a space that generalizes the concept of topological vector spaces. This chapter introduces two new spaces called finite matroidal spaces and matrological spaces. Since any topological vector space is equipped with a linear structure and a topological structure and any vector space is a finitary matroid, we define a finite matroidal space and a matrological space as a set with a pre-closure operator that behaves as a matroidal closure operator on the finite subsets and behaves as a topological closure operator on the infinite subsets.

3.1 Finite Matroidal Spaces

The theory of finite matroids was introduced by Whitney [38], and general matroids were most often defined as finitary matroids. But, as we saw in Remark 167, the class of finitary matroids is not closed under duality. As a consequence, Rado [27] asked for a suitable extension of general matroids with duality. To answer Rado's problem, Higgs [15] proposed possible approaches to general matroids. He first defined what he called a matroid as a set with a pre-closure operator cl such that cl and cl^* are closure operators. In the same idea, we define a finite matroidal space as a space that possesses the same property as its dual. More precisely, a finite matroidal space is a space in which every finite subset is a matroid in the space and in its dual.

Definition 295. A space (X, cl) is called a finite matroidal space if and only if $\text{cl}_{\restriction F}$ and $\text{cl}._F$ are matroidal closure operators on every finite $F \subseteq X$.

Example 296. Let X be any set and define $\text{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\text{cl}(A) = A \text{ for each } A \subseteq X.$$

Then cl is a matroidal closure operator on X. By Theorem 156, $\text{cl}_{\restriction F}$ and $\text{cl}._F$ are matroidal closure operators on every finite $F \subseteq X$. In fact,

$$\text{cl}_{\restriction F}(A) = A \text{ for each } A \subseteq F$$
$$\text{cl}._F(A) = A \text{ for each } A \subseteq F.$$

Hence (X, cl) is a finite matroidal space.

Example 297. Let X be an infinite set. Define $\text{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\text{cl}(A) = \begin{cases} A & \text{if } X \setminus A \text{ is infinite} \\ X & \text{if } X \setminus A \text{ is finite} \end{cases} \text{ for each } A \subseteq X.$$

cl is a closure operator on X. So, (X, cl) is a space. For every finite $F \subseteq X$, we have

$$\text{cl}_{\restriction F}(A) = A \text{ for each } A \subseteq F$$
$$\text{cl}._F(A) = F \text{ for each } A \subseteq F.$$

Then $\text{cl}_{\restriction F}$ and $\text{cl}._F$ are matroidal closure operators on F. Therefore, (X, cl) is a finite matroidal space.

Example 298. Let X be any set with more than one element and $x_0 \in X$. Define $\text{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\text{cl}(A) = \begin{cases} A & \text{if } x_0 \notin A \\ X & \text{if } x_0 \in A \end{cases} \text{ for each } A \subseteq X.$$

Then cl is a closure operator operator on X. So, (X, cl) is a space. Consider $F = \{x_0, y\} \subseteq X$ where $y \neq x_0$. Let $A = \varnothing$ and $x_0, y \in F \setminus A$. We have

$$\text{cl}_{\restriction F}(\varnothing) = \varnothing$$
$$\text{cl}_{\restriction F}(\{x_0\}) = \{x_0, y\}$$
$$\text{cl}_{\restriction F}(\{y\}) = \{y\}.$$

Thus, $y \in \text{cl}_{\restriction F}(A \cup \{x_0\}) \setminus \text{cl}_{\restriction F}(A)$ but $x_0 \notin \text{cl}_{\restriction F}(A \cup \{y\})$. Therefore, $\text{cl}_{\restriction F}$ does not have the exchange property on F. So, $\mathrm{cl}_{\restriction F}$ is not a matroidal closure operators on F, and hence (X, cl) is not a finite matroidal space.

Example 299. Let X be an infinite set and $x_0 \in X$. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \neq X \setminus \{x_0\} \\ X & \text{if } A = X \setminus \{x_0\} \end{cases} \quad \text{for each } A \subseteq X.$$

cl is a closure operator on X. So, (X, cl) is a space. Note that for each finite $F \subseteq X$, we have

$$\mathrm{cl}_{\restriction F}(A) = A \text{ for all } A \subseteq F.$$

Thus, $\mathrm{cl}_{\restriction F}$ is a matroidal closure operator on F. Now, take $F = \{x_0, y\} \subseteq X$ with $y \neq x_0$. But

$$\begin{aligned} \mathrm{cl}_{\cdot F}(\varnothing) &= \varnothing \\ \mathrm{cl}_{\cdot F}(\varnothing \cup \{y\}) &= \{x_0, y\} \\ \mathrm{cl}_{\cdot F}(\varnothing \cup \{x_0\}) &= \{x_0\}. \end{aligned}$$

Then $x_0 \in \mathrm{cl}_{\cdot F}(\varnothing \cup \{y\}) \setminus \mathrm{cl}_{\cdot F}(\varnothing)$ but $y \notin \mathrm{cl}_{\cdot F}(\varnothing \cup \{x_0\})$. Therefore, $\mathrm{cl}_{\cdot F}$ does not have the exchange property on F. So, $\mathrm{cl}_{\cdot F}$ is not a matroidal closure operators on F, and hence (X, cl) is not a finite matroidal space.

Example 300. Consider $X = \{1, 2\}$ and define a pre-closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{1\} & \text{if } A = \{1\} \\ X & \text{otherwise} \end{cases} \quad \text{for each } A \subseteq X.$$

Clearly, cl is a closure operator on X but does not have the exchange property on X. Therefore, cl is not a matroidal closure operator on X. Hence (X, cl) is not a finite matroidal space. Using Theorem 6, the dual of cl is

$$\mathrm{cl}^*(A) = \begin{cases} \{2\} & \text{if } A = \varnothing \\ X & \text{otherwise} \end{cases} \quad \text{for each } A \subseteq X.$$

Clearly, cl^* has the exchange property on X but is not a closure operator on X. Therefore, cl^* is not a matroidal closure operator on X. Hence (X, cl^*) is not a finite matroidal space.

Example 301. Take $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topological closure operator on X. Then

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

(X, cl) is a space. Let $F \subseteq X$ be finite. Thus,

$$\mathrm{cl}_{\restriction F}(A) = A \text{ for all } A \subseteq F.$$

Therefore, $\mathrm{cl}_{\restriction F}(A)$ is a matroidal closure operator on F. We still need to show $\mathrm{cl}_{\cdot F}$ is a matroidal closure operator on F. If $0 \in F$, then

$$\mathrm{cl}_{\cdot F}(A) = A \text{ for all } A \subseteq F.$$

If $0 \notin F$, then

$$\mathrm{cl}_{\cdot F}(A) = \begin{cases} A & \text{if } 0 \in A \\ A \cup \{0\} & \text{if } 0 \notin A \end{cases} \quad \text{for all } A \subseteq F.$$

In both cases, $\mathrm{cl}_{\cdot F}$ is a matroidal closure operator on F. Therefore, (X, cl) is a finite matroidal space. Using Theorem 6, the dual of standard topological closure on X is $\mathrm{cl}^* : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}^*(A) = \begin{cases} X \setminus \{0\} & \text{if } 0 \notin A \text{ and } X \setminus A \text{ is infinite} \\ X & \text{if } 0 \in A \text{ or } X \setminus A \text{ is finite} \end{cases} \quad \text{for each } A \subseteq X.$$

(X, cl^*) is a space. Let $F \subseteq X$ be finite. If $0 \in F$, then

$$\mathrm{cl}^*_{\restriction F}(A) = \begin{cases} F \setminus \{0\} & \text{if } 0 \notin A \\ F & \text{if } 0 \in A \end{cases} \quad \text{for each } A \subseteq F.$$

If $0 \notin F$, then

$$\mathrm{cl}^*_{\restriction F}(A) = F \text{ for all } A \subseteq F.$$

In both cases, $\mathrm{cl}^*_{\restriction F}$ is a matroidal closure operator on F. Also, we get

$$\mathrm{cl}^*._F(A) = F \text{ for all } A \subseteq F.$$

$\mathrm{cl}^*._F$ is a matroidal closure operator on F. Therefore, (X, cl^*) is a finite matroidal space.

The next theorem shows that a finite matroidal space can be defined as a space in which each finite set is a matroid in the space and in its dual.

Theorem 302. *Let (X, cl) be a space. (X, cl) is a finite matroidal space if and only if $cl_{\restriction F}$ and $cl^*_{\restriction F}$ are matroidal closure operators on each finite $F \subseteq X$.*

Proof. Let (X, cl) be a finite matroidal space. Let $F \subseteq X$ be finite. Then $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on F. By Theorems 12 and Theorem 151, $\mathrm{cl}^*_{\restriction F} = (\mathrm{cl}._F)^*$ is a matroidal closure operator on F. Suppose that $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}^*_{\restriction F}$ are matroidal closure operators on every finite $F \subseteq X$. By Theorems 12 and 151, $\mathrm{cl}._F = \left(\mathrm{cl}^*_{\restriction F}\right)^*$ is a matroidal closure operator on F. Hence (X, cl) is a finite matroidal space. □

Example 303. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topology on X. Then the topological closure operator on X is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

(X, cl) is a space. Let $F \subseteq X$ be finite. As in Example 301, $\mathrm{cl}_{\restriction F}$ is a matroidal closure operator on F. We still need to show $\mathrm{cl}._F$ is a matroidal closure operator on F. By Theorem 302, it suffices to show that $\mathrm{cl}^*_{\restriction F}$ is a matroidal closure operators on F. By Theorem 109, cl^* has the exchange property on X since cl is a closure operator on X. By Theorem 106, $\mathrm{cl}^*_{\restriction F}$ has the exchange property on F. It remains to show that $\mathrm{cl}^*_{\restriction F}$ is a closure operator on F. Using Theorem 6, the dual pre-closure of cl on X is

$$\mathrm{cl}^*(A) = \begin{cases} X \setminus \{0\} & \text{if } 0 \notin A \text{ and } X \setminus A \text{ is infinite} \\ X & \text{if } 0 \in A \text{ or } X \setminus A \text{ is finite} \end{cases} \quad \text{for each } A \subseteq X.$$

As in Example 301, $\mathrm{cl}^*_{\restriction F}$ is a closure operator on F. Therefore, $\mathrm{cl}^*_{\restriction F}$ is a matroidal closure operator on F. Hence (X, cl) is a finite matroidal space.

Note that cl^* is a pre-closure on X but it is not a closure operator on X because

$$\mathrm{cl}^*(\mathrm{cl}^*(\varnothing)) = \mathrm{cl}^*(X \setminus \{0\}) = X$$
$$\mathrm{cl}^*(\varnothing) = X \setminus \{0\}.$$

Then

$$\mathrm{cl}^*(\mathrm{cl}^*(\varnothing)) \neq \mathrm{cl}^*(\varnothing).$$

So, cl^* is not a topological closure operator on X, although cl is a topological closure operator on X. Thus the dual of a topological space is not necessarily a topological space. The following theorem says that the dual of a finite matroidal space is also a finite matroidal space.

Theorem 304. *Let (X, cl) be a finite matroidal space and let cl^* be the dual pre-closure operator of cl on X defined by*

$$cl^*(A) = A \cup \{x \in X \setminus A : x \notin cl(X \setminus (A \cup \{x\}))\} \text{ for each } A \subseteq X.$$

Then (X, cl^*) is also a finite matroidal space.

Proof. Suppose that (X, cl) is a finite matroidal space. cl^* is the dual pre-closure operator of cl on X, so (X, cl^*) is a space. Let $F \subseteq X$ be finite. Since (X, cl) is a finite matroidal space, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on F. By Theorems 12 and 151, $\mathrm{cl}^*_{\restriction F} = (\mathrm{cl}._F)^*$ and $\mathrm{cl}^*._F = \left(\mathrm{cl}_{\restriction F}\right)^*$ are matroidal closure operators on F. Hence (X, cl^*) is a finite matroidal space. □

Definition 305. Let (X, cl) be a finite matroidal space. The finite matroidal space (X, cl^*), defined in Theorem 304, is called the dual of finite matroidal space (X, cl) and denoted by $(X, \mathrm{cl})^* = (X, \mathrm{cl}^*)$.

Theorem 306. *Let (X, cl) be a finite matroidal space. Then*

$$(X, cl^*)^* = (X, cl).$$

Proof. It is an immediate consequence of Theorem 11. □

In Theorem 304, we saw that finite matroidal spaces work nicely with duality. We will see that they also work nicely with restrictions and contractions.

Theorem 307. *Let* (X, cl) *be a finite matroidal space and* $Y \subseteq X$. *Then* $\left(Y, cl_{\restriction Y}\right)$ *and* $(Y, cl._{Y})$ *are finite matroidal spaces.*

Proof. Suppose that (X, cl) be a finite matroidal space and $Y \subseteq X$. So, (X, cl) is a space and then $\left(Y, \mathrm{cl}_{\restriction Y}\right)$ and $(Y, \mathrm{cl}._{Y})$ are spaces. Let $F \subseteq Y$ be finite. By Theorem 13, we have

$$\left(\mathrm{cl}_{\restriction Y}\right)_{\restriction F} = \mathrm{cl}_{\restriction F}$$
$$(\mathrm{cl}._{Y})._{F} = \mathrm{cl}._{F}$$

Since $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._{F}$ are matroidal closure operators on F, then $\left(\mathrm{cl}_{\restriction Y}\right)_{\restriction F}$ and $(\mathrm{cl}._{Y})._{F}$ are matroidal closure operators on F. It remains to show that $\left(\mathrm{cl}_{\restriction Y}\right)._{F}$ and $(\mathrm{cl}._{Y})_{\restriction F}$ are matroidal closure operators on F. Let $A \subseteq F$ and $x \in F \setminus A$. Then

$$x \in \left(\mathrm{cl}_{\restriction Y}\right)._{F}(A)$$

if and only if

$$x \in \left(\mathrm{cl}_{\setminus(X \setminus Y)}\right)_{/(Y \setminus F)}(A)$$

if and only if

$$x \in \mathrm{cl}_{\setminus(X \setminus Y)/(Y \setminus F)}(A)$$

if and only if (by Theorem 15)

$$x \in \mathrm{cl}_{/(Y \setminus F)\setminus(X \setminus Y)}(A)$$

if and only if

$$x \in \left(\mathrm{cl}_{/(Y \setminus F)}\right)_{\setminus(X \setminus Y)}(A)$$

if and only if

$$x \in (\mathrm{cl}._{F})_{\restriction F}(A)$$

if and only if

$$x \in \mathrm{cl}._{F}(A)$$

Therefore, $\left(\mathrm{cl}_{\restriction Y}\right)._{F} = \mathrm{cl}._{F}$. Since $\mathrm{cl}._{F}$ is a matroidal closure operator on F, then $\left(\mathrm{cl}_{\restriction Y}\right)._{F}$ is a matroidal closure operator on F. Now, by Theorem 304, (X, cl^*) is a finite matroidal space. Then $\left(\mathrm{cl}^*_{\restriction Y}\right)._{F}$ is a matroidal closure operator on F. By Theorem 151,

$$\left(\left(\mathrm{cl}^*_{\restriction Y}\right)._{F}\right)^* = (\mathrm{cl}._{Y})_{\restriction F}$$

is also a matroidal closure operator on F. Then $\left(Y, \mathrm{cl}_{\restriction Y}\right)$ and $(Y, \mathrm{cl}._Y)$ are finite matroidal spaces. □

Remark 308. Basis and circuits may not exist in finite matroidal spaces.

Proof. Let X be an infinite set. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

Then cl is a closure operator on X. So, (X, cl) is a space. Let $F \subseteq X$ be finite. Thus,

$$\mathrm{cl}_{\restriction F}(A) = A \text{ for each } A \subseteq F.$$
$$\mathrm{cl}._F(A) = F \text{ for each } A \subseteq F.$$

Then $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on F. Therefore, (X, cl) is a finite matroidal space. The collection $\mathscr{I}$ of all independent sets induced by cl is

$$\begin{aligned} \mathscr{I} &= \{A \subseteq X : x \notin \mathrm{cl}(A \setminus \{x\}) \text{ for each } x \in A\} \\ &= \{A \subseteq X : A \text{ is finite}\}. \end{aligned}$$

Therefore, $\mathscr{I}$ has no maximal elements, and hence (X, cl) has no basis. Now, the collection $\mathscr{P}(X) \setminus \mathscr{I}$ of all dependent sets is

$$\begin{aligned} \mathscr{P}(X) \setminus \mathscr{I} &= \{A \subseteq X : A \notin \mathscr{I}\} \\ &= \{A \subseteq X : A \text{ is infinite}\}. \end{aligned}$$

Thus, $\mathscr{P}(X) \setminus \mathscr{I}$ has no minimal elements, and hence (X, cl) has no circuits. □

Now, we need to compare between finite matroidal spaces and the old spaces such as matroids, pre-independence spaces [34] and exchange systems [8]. These comparisons show that finite matroidal spaces are new and different.

Theorem 309. *If $(X, \mathscr{I})$ is a matroid, then (X, cl) is a finite matroidal space where cl is the matroidal closure operator on X induces $\mathscr{I}$.*

Proof. Let $(X, \mathscr{I})$ be a matroid and cl be the matroidal closure operators on X induces $\mathscr{I}$. Then (X, cl) is a space. Now, let $F \subseteq X$ be finite. By Theorem 156, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$

are matroidal closure operators on every finite $F \subseteq X$. Hence (X, cl) is a finite matroidal space. □

Example 310. Any vector space over the field $\mathbb{F}$ is a finite matroidal space.

Example 311. The uniform matroids, finite cycle matroids, bond matroids, algebraic cycle matroids and nibble matroids are finite matroidal spaces.

Remark 312. If (X, cl) is a finite matroidal space, then $(X, \mathscr{I})$ may not be a matroid where $\mathscr{I}$ is the simplicial complex on X induced by cl.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topological closure operator cl on X. In Example 301, we have shown that (X, cl) is a finite matroidal space. Now, the collection $\mathscr{I}$ of independent sets induced by cl is

$$\begin{aligned} \mathscr{I} &= \{A \subseteq X : x \notin \mathrm{cl}(A \setminus \{x\}) \text{ for each } x \in A\} \\ &= \{A \subseteq X : 0 \notin A \text{ or } A \text{ is finite}\}. \end{aligned}$$

Take $Y = X$ and $A = \{0\}$. So, $A \subseteq Y$ and $A \in \mathscr{I}$. But for each $B \in \mathscr{I}$ with $A \subseteq B \subseteq Y$, we have B is not maximal. Thus, $\mathscr{I}$ does not have the maximality property on X. Therefore, $(X, \mathscr{I})$ is not a matroid. □

Definition 313. Let X be any set and $\mathscr{I} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. Then $(X, \mathscr{I})$ is called a pre-independence space (or pi-space) if and only if

(I1) $\varnothing \in \mathscr{I}$.

(I2) If $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.

(I3') If $I_1, I_2 \in \mathscr{I}$ are finite and $|I_1| \lneqq |I_2|$, there is $x \in I_2 \setminus I_1$ such that $I_1 \cup \{x\} \in \mathscr{I}$.

Example 314. Let $X = \mathbb{R}$ and $\mathscr{I} = \{A \subseteq X : A \text{ is countable}\}$. Clearly, the axioms (I1), (I2) and (I3') hold. Then $(X, \mathscr{I})$ is a pre-independence space.

Example 315. Let $X = \mathbb{N}$ and

$$\begin{aligned} \mathscr{I} = &\{A \subseteq X : |A| \leq 1\} \\ &\cup \{A \subseteq X : |A| = 2 \text{ and } A \text{ does not contain consecutive numbers}\}. \end{aligned}$$

Clearly, (I1) and (I2) are hold. Take $I_1 = \{2\}$ and $I_2 = \{1,3\}$. Then $I_1, I_2 \in \mathscr{I}$ are finite with $|I_1| \lneqq |I_2|$ but

$$I_1 \cup \{1\} = \{1,2\} \notin \mathscr{I}$$
$$I_1 \cup \{3\} = \{2,3\} \notin \mathscr{I}$$

Thus, $\mathscr{I}$ does not satisfy (I3'). Then $(X, \mathscr{I})$ is not a pre-independence space.

Theorem 316. *If (X, cl) is a finite matroidal space, then $(X, \mathscr{I})$ is a pre-independence space where $\mathscr{I}$ is the simplicial complex on X induced by cl.*

Proof. Suppose that (X, cl) is a finite matroidal space. Let $\mathscr{I}$ be the simplicial complex on X induced by cl. Thus, $\mathscr{I}$ satisfies (I1) and (I2). It remains to show that $\mathscr{I}$ satisfies (I3'). Let $I_1, I_2 \in \mathscr{I}$ be finite such that $|I_1| \lneqq |I_2|$. Then $I_1 \cup I_2$ is finite. Since (X, cl) is a finite matroidal space, $\mathrm{cl}_{\restriction I_1 \cup I_2}$ is a matroidal closure operator on $I_1 \cup I_2$. By Theorem 131, $\mathrm{cl}_{\restriction I_1 \cup I_2}$ induces the finite matroid

$$\mathscr{I}_{\restriction I_1 \cup I_2} = \{I \subseteq I_1 \cup I_2 : I \in \mathscr{I}\}.$$

So, $I_1, I_2 \in \mathscr{I}_{\restriction I_1 \cup I_2}$. Thus, there is $x \in I_2 \setminus I_1$ such that $I_1 \cup \{x\} \in \mathscr{I}_{\restriction I_1 \cup I_2}$. This implies that $I_1 \cup \{x\} \in \mathscr{I}$. Hence $(X, \mathscr{I})$ is a pre-independence space. □

Remark 317. If $(X, \mathscr{I})$ is a pre-independence space, then (X, cl) is not necessarily a finite matroidal space where cl is the pre-closure operator on X induced by $\mathscr{I}$.

Proof. Let $X = \mathbb{N}$ and

$$\mathscr{B} = \{X \setminus \{1\}, X \setminus \{2,3\}, X \setminus \{4,5,6\}, X \setminus \{7,8,9,10\}, \ldots\}.$$

Let

$$\mathscr{I} = \{I \subseteq X : \text{ there is } B \in \mathscr{B} \text{ such that } I \subseteq B\}.$$

Then $\mathscr{I}$ clearly satisfies (I1) and (I2). Also, $\mathscr{I}$ satisfies (I3') since all finite sets of X are in $\mathscr{I}$. Therefore, $(X, \mathscr{I})$ is a pre-independence space. Now, the pre-closure operator cl on X induced by $\mathscr{I}$ is

$$\mathrm{cl}(A) = A \cup \{x \in X \setminus A : \text{ there is } I \subseteq A \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\}$$

for all $A \subseteq X$. Let $F = \{1, 2\}$. We want to show that $\mathrm{cl.}_F$ is not a matroidal closure operator on F. We know

$$\mathrm{cl.}_F(A) = \{x \in F : x \in \mathrm{cl}(A \cup (X \setminus F))\}$$

Thus,

$$\begin{aligned}\mathrm{cl.}_F(\varnothing) &= \{x \in F : x \in \mathrm{cl}(\varnothing \cup (X \setminus F))\}\\ &= \{x \in F : x \in \mathrm{cl}(X \setminus \{1,2\})\}\\ &= \{x \in F : x \in X \setminus \{2\}\} = \{1\}\end{aligned}$$

Also,

$$\begin{aligned}\mathrm{cl.}_F(\{1\}) &= \{x \in F : x \in \mathrm{cl}(\{1\} \cup (X \setminus F))\}\\ &= \{x \in F : x \in \mathrm{cl}(X \setminus \{2\})\}\\ &= \{x \in F : x \in X\} = F\end{aligned}$$

Therefore,

$$\mathrm{cl.}_F(\mathrm{cl.}_F(\varnothing)) = \mathrm{cl.}_F(\{1\}) = F$$
$$\mathrm{cl.}_F(\varnothing) = \{1\}$$

Then $\mathrm{cl.}_F$ is not a closure operator on F, and hence $\mathrm{cl.}_F$ is not a matroidal closure operator on F. This leads to that (X, cl) is not a finite matroidal space. □

Definition 318. A pre-independence space $(X, \mathscr{I})$ is called an independence space if $\mathscr{I}$ satisfies the following condition

(I4) If $A \subseteq X$ and for each finite $A' \subseteq A$ we have $A' \in \mathscr{I}$, then $A \in \mathscr{I}$.

So, the class of all independence spaces is the same as the class of all finitary matroids. By Theorem 309 and Remark 312, every independence space is a finite matroidal space but the converse is not true.

Definition 319. Let X be any set and $\mathscr{I} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. Then $\mathscr{I}$ has the maximal condition if and only if for each $I \in \mathscr{I}$, there is a maximal element $B \in \mathscr{I}$ such that $I \subseteq B$.

Definition 320. An mpi-space is a pre-independence space $(X, \mathscr{I})$ provided that $\mathscr{I}$ has the maximal condition.

Remark 321. If (X, cl) is a finite matroidal space, then $(X, \mathscr{I})$ does not have to be an mpi-space where $\mathscr{I}$ is the simplicial complex on X induced by cl.

Proof. Let $X = \mathbb{N}$ and define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

In Remark 308, we proved that (X, cl) is a finite matroidal space. Now, the simplicial complex $\mathscr{I}$ on X induced by cl is

$$\begin{aligned} \mathscr{I} &= \{A \subseteq X : x \notin \mathrm{cl}(A \setminus \{x\}) \text{ for each } x \in A\} \\ &= \{A \subseteq X : A \text{ is finite}\}. \end{aligned}$$

So, $(X, \mathscr{I})$ is a pre-independence space. Consider $\{1\} \in \mathscr{I}$. There is no maximal element $B \in \mathscr{I}$ such that $\{1\} \subseteq B$. Thus, $\mathscr{I}$ has no the maximal condition. Hence $(X, \mathscr{I})$ is not an mpi-space. □

Remark 322. If $(X, \mathscr{I})$ is an mpi-space, then (X, cl) is not necessarily a finite matroidal space where cl is the pre-closure operator on X induced by $\mathscr{I}$.

Proof. Let $X = \mathbb{N}$ and

$$\mathscr{B} = \{X \setminus \{1\}, X \setminus \{2,3\}, X \setminus \{4,5,6\}, X \setminus \{7,8,9,10\}, \ldots\}.$$

Let

$$\mathscr{I} = \{I \subseteq X : \text{ there is } B \in \mathscr{B} \text{ such that } I \subseteq B\}.$$

The pre-independence space $(X, \mathscr{I})$ is obviously an mpi-space. In Remark 317, it has proven that (X, cl) is not a finite matroidal space where cl is the pre-closure operator on X induced by $\mathscr{I}$. □

Definition 323. Let X be any set and $\mathscr{I} \subseteq \mathscr{P}(X)$ be a collection of subsets of X. Then $(X, \mathscr{I})$ is called an exchange system if and only if

(I1) $\varnothing \in \mathscr{I}$.

(I2) If $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.

(I3”) Let $Y \subseteq X$ and B_1, B_2 be maximal in $\mathscr{I}_{\restriction Y}$. If $x \in B_1 \setminus B_2$, then there is $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ and $(B_2 \setminus \{y\}) \cup \{x\}$ are maximal in $\mathscr{I}_{\restriction Y}$.

Remark 324. Let (X, cl) be a finite matroidal space. Then $(X, \mathscr{I})$ does not have to be an exchange system where $\mathscr{I}$ is the simplicial complex on X induced by cl.

Proof. Let $X = X_1 \cup X_2$ where X_1, X_2 are disjoint infinite sets. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup X_1 & \text{if } A \cap X_1 \text{ is finite and } A \cap X_2 \text{ is infinite} \\ A \cup X_2 & \text{if } A \cap X_2 \text{ is finite and } A \cap X_1 \text{ is infinite} \\ X & \text{if } A \cap X_1 \text{ and } A \cap X_2 \text{ are infinite} \end{cases}$$

for each $A \subseteq X$. Then cl is a pre-closure (not a closure) operator on X. So, (X, cl) is a space. Let $F \subseteq X$ be finite. Thus,

$$\mathrm{cl}_{\restriction F}(A) = A \text{ for each } A \subseteq F.$$
$$\mathrm{cl}_{\cdot F}(A) = F \text{ for each } A \subseteq F.$$

Then $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}_{\cdot F}$ are matroidal closure operators on F. Therefore, (X, cl) is a finite matroidal space. Now, cl induces the simplicial complex $\mathscr{I}$ as follows

$$\begin{aligned} \mathscr{I} &= \{A \subseteq X : x \notin \mathrm{cl}(A \setminus \{x\}) \text{ for each } x \in A\} \\ &= \mathscr{P}(X_1) \cup \mathscr{P}(X_2) \cup \{A \subseteq X : A \text{ is finite}\}. \end{aligned}$$

Then X_1, X_2 are maximal in $\mathscr{I}$. Pick $x_1 \in X_1$. Since $X_1 \cap X_2 = \varnothing$, then $x_1 \in X_1 \setminus X_2$. But for all $x_2 \in X_2 \setminus X_1$, we have

$$(X_1 \setminus \{x_1\}) \cup \{x_2\} \notin \mathscr{I}.$$

Therefore, $\mathscr{I}$ does not satisfies (I3”). Hence $(X, \mathscr{I})$ is not an exchange system. □

In Theorem 174, Higgs proved that we only get an algebraic cycle matroid of a graph G if and only if G does not contain a subgraph isomorphic to a subdivision of the Bean graph, see Figure 2.2.1. The following theorem shows that this condition is not needed to get a finite matroidal space.

Theorem 325. *Let G be a graph, $\mathscr{C}$ be the collection of all algebraic circuit of G and $X =$*

$E(G)$. *Define* $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ *by*

$$cl(A) = A \cup \{x \in X \setminus A : \text{ there is } C \in \mathscr{C} \text{ such that } x \in C \subseteq A \cup \{x\}\}$$

for all $A \subseteq X$. *Then* (X, cl) *is a finite matroidal space.*

Proof. cl is a pre-closure (not a closure) operator on X. Then (X, cl) is a space. If G does not contain a subgraph isomorphic to a subdivision of the Bean graph. By Higgs Theorem 174, cl is a matroidal closure operator on X and hence, by Theorem 309, (X, cl) is a finite matroidal space. Assume G does contain a subgraph isomorphic to a subdivision of the Bean graph. Let $F \subseteq X$ be finite. So, F is an edge set of a locally finite graph. By Corollary 177, $\mathrm{cl}_{\restriction F}$ is a matroidal closure operator on F. So, we just need to show that $\mathrm{cl}._F$ is a matroidal closure operator on F. In Higgs Theorem 174, it has shown in general that cl has the exchange property on X. By Theorem 106, $\mathrm{cl}._F$ has the exchange property on F. It remains to show that $\mathrm{cl}._F$ is a closure operator on F. Suppose, by way of contradiction, that there is $A \subseteq F$ and $x \in F \setminus A$ such that

$$x \in \mathrm{cl}._F(\mathrm{cl}._F(A)) \setminus \mathrm{cl}._F(A).$$

Then

$$x \in \mathrm{cl}(\mathrm{cl}(A \cup (X \setminus F))) \setminus \mathrm{cl}(A \cup (X \setminus F)).$$

This happens only when $F \setminus A$ is infinite which contradicts that F is finite. Thus, $\mathrm{cl}._F$ is a closure operators on F, and hence $\mathrm{cl}._F$ is a matroidal closure operator on F. Then (X, cl) is a finite matroidal space. □

3.2 Matrological Spaces

As we saw, the pre-closure operator of a finite matroidal space behaves as a matroidal closure operator on the finite subsets. A matrological space is defined as a finite matroidal space in addition to an axiom that makes the pre-closure operator of the finite matroidal space behave as a topological closure operator on the infinite subsets. Two important key theorems in our research that tell when a matroid or topological space is a matrological space will be presented in this section.

Definition 326. A finite matroidal space (X, cl) is called matrological if and only if for all

$A, B \subseteq X$ and all $x \in X$ if

$$x \in \mathrm{cl}(A \cup B) \setminus [\mathrm{cl}(A) \cup \mathrm{cl}(B)],$$

then there is a finite $F \subseteq A \cup B$ such that $x \in \mathrm{cl}(F)$.

Example 327. Let X be any set and define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = A \text{ for all } A \subseteq X.$$

cl is a matroidal closure operator on X. By Theorem 309, (X, cl) is a finite matroidal space. For all $A, B \subseteq X$, we have

$$\mathrm{cl}(A \cup B) = A \cup B = \mathrm{cl}(A) \cup \mathrm{cl}(B).$$

Then (X, cl) is a matrological space.

The following example is slightly different from the one that we have in Remark 324.

Example 328. Let $X = X_1 \cup X_2$ where X_1 and X_2 are disjoint and infinite. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup X_2 & \text{if } A \cap X_1 \text{ is finite and } A \cap X_2 \text{ is infinite} \\ A \cup X_1 & \text{if } A \cap X_2 \text{ is finite and } A \cap X_1 \text{ is infinite} \\ X & \text{if } A \cap X_1 \text{ and } A \cap X_2 \text{ are infinite} \end{cases}$$

for each $A \subseteq X$. Note that cl is a closure operator on X. So, (X, cl) is a space. Let $F \subseteq X$ be finite. Thus,

$$\mathrm{cl}_{\restriction F}(A) = A \text{ for each } A \subseteq F.$$
$$\mathrm{cl}._F(A) = F \text{ for each } A \subseteq F.$$

Then $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on F. Therefore, (X, cl) is a finite matroidal space. We want to show that (X, cl) is a matrological space. It suffices to show that

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for each } A, B \subseteq X.$$

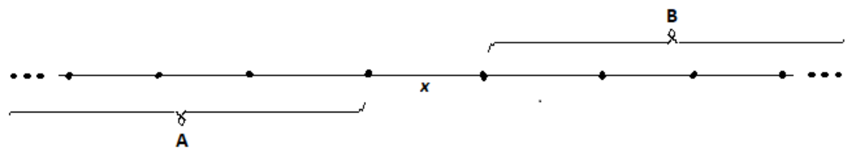

Figure 3.2.1: One Double Ray.

By (CL2), we know

$$\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B) \text{ for each } A, B \subseteq X.$$

Now, let $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ such that $x \in \mathrm{cl}(A \cup B)$. So, $A \cup B$ is not finite. Thus, $(A \cup B) \cap X_1$ or $(A \cup B) \cap X_2$ is infinite or both. Therefore, $A \cap X_1$ or $A \cap X_2$ is infinite or both or $B \cap X_1$ or $B \cap X_2$ is infinite or both. So, $x \in \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Thus,

$$\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for each } A, B \subseteq X.$$

Thus,

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for each } A, B \subseteq X.$$

Therefore, (X, cl) is a matrological space.

In Theorem 309, we proved that every matroid is a finite matroidal space.

Remark 329. If $(X, \mathscr{I})$ is a matroid, (X, cl) may not be a matrological space where cl is the matroidal closure operators on X induces$\mathscr{I}$.

Proof. Consider the One Double Ray G as shown in Figure 3.2.1. Let $X = E(G)$ and cl be the algebraic cycle matroidal closure operator on X. Take $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ as in Figure 3.2.1. Then

$$x \in \mathrm{cl}(A \cup B) \setminus (\mathrm{cl}(A) \cup \mathrm{cl}(B)).$$

But $x \notin \mathrm{cl}(F)$ for each finite $F \subseteq A \cup B$. Therefore, (X, cl) is not a matrological space. □

Note that the matroid $(X, \mathscr{I})$ in Remark 329 is not finitary, but it is cofinitary and nearly finitary [6] where cl is the matroidal closure operator on X induces $\mathscr{I}$. In general, we have the following important key theorem in our research that says the only matroids that are matrological spaces are the finitary matroids.

Theorem 330. *Let $(X, \mathscr{I})$ be a matroid. Then $(X, \mathscr{I})$ is a finitary matroid if and only if (X, cl) is a matrological space where cl is the matroidal closure operators on X induced by $\mathscr{I}$.*

Proof. Suppose that $(X, \mathscr{I})$ be a finitary matroid and cl be the matroidal closure operator on X induces $\mathscr{I}$. By Theorem 309, (X, cl) is a finite matroidal space. Now, let $A, B \subseteq X$ and $x \in X$ be such that

$$x \in \mathrm{cl}(A \cup B) \setminus (\mathrm{cl}(A) \cup \mathrm{cl}(B)).$$

Since cl is a finitary matroidal closure operator on X, there is a finite set $F \subseteq A \cup B$ such that $x \in \mathrm{cl}(F)$. Thus, (X, cl) is a matrological space.

Now, assume that (X, cl) is a matrological space. We want to show that $(X, \mathscr{I})$ is a finitary matroid. Suppose, by way of contradiction, that $(X, \mathscr{I})$ is not a finitary matroid. So, there is an infinite circuit $C \in \mathscr{C}$ where $\mathscr{C}$ is the collection of all circuits of the matroid $(X, \mathscr{I})$. Let $x \in C$ and let

$$P = \{A_0, B_0\} \subseteq \mathscr{P}(C \setminus \{x\})$$

be a partition[1] of $C \setminus \{x\}$. We get

$$x \in C \subseteq (A_0 \cup B_0) \cup \{x\}$$

and hence $x \in \mathrm{cl}(A_0 \cup B_0)$. If there is $C_0 \in \mathscr{C}$ such that

$$x \in C_0 \subseteq A_0 \cup \{x\},$$

then $C_0 \subseteq C$ which is contradiction. Then $x \notin \mathrm{cl}(A_0)$. Similarly, $x \notin \mathrm{cl}(B_0)$. So, we have

$$x \in \mathrm{cl}(A_0 \cup B_0) \setminus (\mathrm{cl}(A_0) \cup \mathrm{cl}(B_0)).$$

Since (X, cl) is a matrological space, there is a finite set $F \subseteq A_0 \cup B_0$ such that

$$x \in \mathrm{cl}(F).$$

Thus, there is a finite circuit $C' \in \mathscr{C}$ such that

$$x \in C' \subseteq F \cup \{x\}.$$

This implies that $C' \subseteq C$, contradiction. Thus, cl is a finitary matroidal closure operator on X and hence $(X, \mathscr{I})$ is a finitary matroid. $\square$

[1] Let X be a set and $P \subseteq \mathscr{P}(X)$ with $P \neq \varnothing$. Then P is a partition of X if and only if $\varnothing \notin P$, $\bigcup_{A \in P} A = X$ and $A \cap B \neq \varnothing$ for each distinct $A, B \in P$.

Example 331. Any vector space over the field $\mathbb{F}$ is a matrological space.

Example 332. The finite cycle matroids are matrological spaces but the bond matroids may not be matrological spaces.

Example 333. For each $n \in \mathbb{N}$, the uniform matroid of rank n is a matrological space but the uniform matroid of co-rank n might not be a matrological space.

We know that every matrological space is a finite matroidal space. The following theorem shows that there are finite matroidal spaces that are not matrological spaces.

Remark 334. If (X, cl) is a finite matroidal space, then (X, cl) might not be a matrological space.

Proof. Let G be the One Double Ray as shown in Figure 3.2.1. Let $X = E(G)$ and cl be the algebraic cycle matroidal closure operator on X. By Theorem 309, (X, cl) is a finite matroidal space. But, in the proof of Remark 329, we showed that (X, cl) is not a matrological space. □

If the ground set is finite, then matroids, finite matroidal spaces and matrological spaces are the same.

Theorem 335. *Let X be a finite set and cl be a pre-closure operator on X. Then the following are equivalent:*

1. *cl is a matroidal closure operator on X.*

2. *(X, cl) is a finite matroidal space.*

3. *(X, cl) is a matrological space.*

Proof. (1)⇒(2) Let cl be the matroidal closure operators on X. By Theorem 309, (X, cl) is a finite matroidal space. (2)⇒(3) Let (X, cl) be a finite matroidal space. Now, let $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ be such that

$$x \in \mathrm{cl}(A \cup B) \setminus (\mathrm{cl}(A) \cup \mathrm{cl}(B)).$$

Consider $F = A \cup B$. Thus F is finite and $x \in \mathrm{cl}(F)$. Therefore, (X, cl) is a matrological space. (3)⇒(1) Let (X, cl) be a matrological space. Thus, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on each finite $F \subseteq X$. Take $F = X$. Then

$$\mathrm{cl} = \mathrm{cl}_{\restriction F} = \mathrm{cl}._F.$$

Therefore, cl is the matroidal closure operators on X. □

Example 336. Let $X = \{1,2,3\}$ and $\mathscr{I} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{2,3\}\}$. Then the pre-closure cl induced by $\mathscr{I}$ is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A = \varnothing \\ \{1,2\} & \text{if } A = \{2\} \\ \{1,3\} & \text{if } A = \{3\} \\ X & \text{for all } A \notin \{\varnothing, \{2\}, \{3\}\}. \end{cases}$$

In Remark 117, we proved that cl is not a closure operator on X. Then cl is not a matroidal closure on X. By Theorem 335, (X, cl) is neither a finite matroidal space nor a matrological space.

Example 337. Consider the symmetric group $\langle S_4, \circ \rangle$. Let cl be the algebraic closure operator on S_4. By remark 180, cl is not a matroidal closure operators on S_4. By Theorem 335, (S_4, cl) is neither a finite matroidal space nor a matrological space.

The following is another important key theorem in our research that says the only topological spaces that are matrological spaces are those in which the topological closure operators have the exchange property.

Theorem 338. *Let X be a topological space and cl be the topological closure operator of X. Then cl has the exchange property on X if and only if (X, cl) is a matrological space.*

Proof. Suppose that cl has the exchange property on X. By Theorem 106, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}_{\cdot F}$ have the exchange property on every finite $F \subseteq X$. Since cl is a closure operator on X, then, by Theorem 5, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}_{\cdot F}$ are closure operators on every finite $F \subseteq X$. Therefore, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}_{\cdot F}$ are matroidal closure operators on every finite $F \subseteq X$. Since cl is a topological closure operator on X, we have

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for all } A, B \subseteq X.$$

Hence (X, cl) is a matrological space.

Now, suppose that (X, cl) is a matrological space. Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

Since cl is a topological closure operator on X, we have

$$y \in [\mathrm{cl}(A) \cup \mathrm{cl}(\{x\})] \setminus \mathrm{cl}(A).$$

It follows that $y \in \mathrm{cl}(\{x\})$. Now, consider $F = \{x, y\}$. Then

$$y \in \mathrm{cl}_{\restriction F}(\varnothing \cup \{x\}) \setminus \mathrm{cl}_{\restriction F}(\varnothing).$$

Since $\mathrm{cl}_{\restriction F}$ is a matroidal closure operator on the finite set $F \subseteq X$, then $\mathrm{cl}_{\restriction F}$ has the exchange property on F. So, we get

$$x \in \mathrm{cl}_{\restriction F}(\varnothing \cup \{y\})$$

and hence

$$x \in \mathrm{cl}(\varnothing \cup \{y\}) = \mathrm{cl}(\{y\}).$$

Thus,

$$x \in \mathrm{cl}(A) \cup \mathrm{cl}(\{y\}) = \mathrm{cl}(A \cup \{y\}).$$

Then cl has the exchange property on X. □

Example 339. Let X be an indiscrete topological space and cl be the topological closure operator on X. Thus,

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ X & \text{if } A \neq \varnothing \end{cases} \text{ for each } A \subseteq X.$$

Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that $y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$. This implies that $A = \varnothing$. Thus,

$$x \in \mathrm{cl}(A \cup \{y\}) = \mathrm{cl}(\{y\}) = X.$$

Then cl has the exchange property on X. By Theorem 338, (X, cl) is a matrological space.

Example 340. Take a discrete topology on a set X. The topological closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ is defined by

$$\mathrm{cl}(A) = A \text{ for each } A \subseteq X.$$

Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

So,

$$y \in (A \cup \{x\}) \setminus A.$$

Thus, $x = y$ which is impossible. Hence cl has the exchange property on X. By Theorem 338, (X, cl) is a matrological space.

Example 341. Let X be any set and $B \subseteq X$. Define a topological closure operator cl : $\mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ A \cup B & \text{if } A \neq \varnothing \end{cases} \text{ for each } A \subseteq X.$$

Take $x \notin B$ and $y \in B$. So,

$$\mathrm{cl}(\{x\}) = \{x\} \cup B$$
$$\mathrm{cl}(\{y\}) = B.$$

Therefore, we have $x, y \in X$ are distinct such that $y \in \mathrm{cl}(\varnothing \cup \{x\}) \setminus \mathrm{cl}(\varnothing)$, but $x \notin \mathrm{cl}(\varnothing \cup \{y\})$. Then cl does not have the exchange property on X. By Theorem 338, (X, cl) is not a matrological space.

Remark 342. A T_0-space may not be matrological.

Proof. Consider $X = \{1, 2\}$ with the Sierpinski topology $\tau = \{\varnothing, \{2\}, X\}$. Thus, X is T_0. Let cl be the topological closure operator on X and take $A = \varnothing$ and $1, 2 \in X \setminus A$. Then

$$\mathrm{cl}(A) = \varnothing$$
$$\mathrm{cl}(\{1\}) = \{1\}$$
$$\mathrm{cl}(\{2\}) = X.$$

Thus, $1 \in \mathrm{cl}(A \cup \{2\}) \setminus \mathrm{cl}(A)$ but $2 \notin \mathrm{cl}(A \cup \{1\})$. Then cl does not have the exchange property on X. By Theorem 338, (X, cl) is not a matrological space. □

Corollary 343. *Any T_1-space is matrological.*

Proof. Let cl be the topological closure operator on a T_1-topological space X. By Corollary 51, there is an open neighborhood base $\mathscr{N}$ that induces cl. Now, let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that $y \notin \mathrm{cl}(A)$. We claim that $y \notin \mathrm{cl}(A \cup \{x\})$. Suppose, by way of contradiction, that $y \in \mathrm{cl}(A \cup \{x\})$. Then for each $U \in \mathscr{N}(y)$, we have $U \cap (A \cup \{x\}) \neq \varnothing$. Since $y \notin \mathrm{cl}(A)$, there exists $U_0 \in \mathscr{N}(y)$ such that $U_0 \cap A = \varnothing$. It follows that $x \in U_0$, otherwise $y \notin \mathrm{cl}(A \cup \{x\})$. By Theorem 237, $\{x\}$ is closed in X. Thus, $V_0 = X \setminus \{x\} \in \mathscr{N}(y)$. It follows that $U_0 \cap V_0 \in \mathscr{N}(y)$ and $(U_0 \cap V_0) \cap (A \cup \{x\}) = \varnothing$. This contradicts that $y \in \mathrm{cl}(A \cup \{x\})$. Therefore, it is impossible to have $y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$. Hence cl has the exchange property on X. By Theorem 338, (X, cl) is a matrological space. □

Example 344. Any cofinite topological space is matrological.

Corollary 345. *Any Hausdorff space is matrological.*

Proof. This is an immediate consequence of Theorem 241 and Corollary 343. □

Example 346. The standard topology on $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is T_2. So, (X, cl) is a matrological space where cl is the topological closure operator on X.

Example 347. Any metric space is matrological.

Example 348. The product topology on $X = \prod_{\alpha \in A} \mathbb{R}$ is Hausdorff. Therefore, (X, cl) is a matrological space where cl is the topological closure operator on X.

Corollary 349. *Any regular space is matrological.*

Proof. Let cl be the topological closure operator on a regular topological space X. By Corollary 51, there is an open neighborhood base $\mathscr{N}$ that induces cl. Now, let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that $y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$. Then for each $U \in \mathscr{N}(y)$ we have $U \cap (A \cup \{x\}) \neq \varnothing$, and there exists $U_0 \in \mathscr{N}(y)$ such that $U_0 \cap A = \varnothing$. It follows that $x \in U_0$, otherwise $y \notin \mathrm{cl}(A \cup \{x\})$. Suppose, by way of contradiction, that $x \notin \mathrm{cl}(A \cup \{y\})$, which is closed in X. By regularity, there are two disjoint open sets V_0, W_0 in X such that $x \in V_0$ and $\mathrm{cl}(A \cup \{y\}) \subseteq W_0$. But $U_0 \cap W_0 \in \mathscr{N}(y)$ and $(U_0 \cap V_0) \cap (A \cup \{x\}) = \varnothing$. Thus, $y \notin \mathrm{cl}(A \cup \{x\})$ which is a contradiction. Therefore, $x \in \mathrm{cl}(A \cup \{y\})$, and hence cl has the exchange property on X. By Theorem 338, (X, cl) is a matrological space. □

Corollary 350. *Any T_3-space is matrological.*

Proof. This is true from the definition of T_3-spaces, see Definition 247. □

Corollary 351. *Any completely regular space is matrological.*

Proof. This is obvious from Theorem 251. □

Corollary 352. *Any Tychonoff space is matrological.*

Proof. This is true from the definition of Tychonoff spaces, see Definition 252. □

Remark 353. A normal space does not have to be matrological.

Example 354. Let X be a set with at least 3 elements and $x_0 \in X$. Define a topology

$$\tau = X \cup \{U \subseteq X : x_0 \notin U\}.$$

Note that τ is the Sierpinski topology for X when X has only two elements. Thus, X is normal. The family of closed sets of X is

$$\mathscr{F} = \{\varnothing\} \cup \{A \subseteq X : x_0 \in A\}.$$

So, the topological closure operator cl on X is

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A = \varnothing \\ A \cup \{x_0\} & \text{if } A \neq \varnothing \end{cases} \text{ for each } A \subseteq X.$$

Now, take $A = \varnothing$ and $x, x_0 \in X \setminus A$ where $x \neq x_0$. Then

$$\mathrm{cl}(A) = \varnothing$$
$$\mathrm{cl}(\{x\}) = \{x, x_0\}$$
$$\mathrm{cl}(\{x_0\}) = \{x_0\}.$$

Thus, $x_0 \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A)$ but $x \notin \mathrm{cl}(A \cup \{x_0\})$. Then cl does not have the exchange property on X. By Theorem 338, (X, cl) is not a matrological space.

Note that the space X in the last remark is normal and T_0 and neither T_1, T_2 nor regular. In the next example, we have a normal space that is neither T_0, T_1, T_2 nor regular.

Example 355. Let $X = (0, 1)$ and

$$\tau = \{\varnothing, X\} \cup \left\{U_n = \left(0, 1 - \frac{1}{n}\right) : n \in \mathbb{N} \setminus \{1\}\right\}.$$

Then τ is a normal topology for X that is neither T_0, T_1, T_2 nor regular. The collection of closed sets of X is

$$\mathscr{F} = \{\varnothing, X\} \cup \left\{A_n = \left[1 - \frac{1}{n}, 1\right) : n \in \mathbb{N} \setminus \{1\}\right\}.$$

This example does not satisfy the exchange property on X because

$$\begin{aligned} \mathrm{cl}(\varnothing) &= \varnothing \\ \mathrm{cl}\left(\left\{\frac{1}{3}\right\}\right) &= X \\ \mathrm{cl}\left(\left\{\frac{1}{2}\right\}\right) &= \left[\frac{1}{2}, 1\right) \end{aligned}$$

where cl is the topological closure operator on X. Thus, $\frac{1}{2} \in \mathrm{cl}\left(\varnothing \cup \left\{\frac{1}{3}\right\}\right) \setminus \mathrm{cl}(\varnothing)$ but $\frac{1}{3} \notin \mathrm{cl}\left(\varnothing \cup \left\{\frac{1}{2}\right\}\right)$. By Theorem 338, (X, cl) is not a matrological space.

In Theorems 304 and 307, we saw that finite matroidal spaces work nicely with duality, restrictions and contractions. Now, we will see whether matrological spaces work nicely with duality, restrictions and contractions.

Theorem 356. *Let (X, cl) be a matrological space and $Y \subseteq X$. Then $\left(Y, cl_{\restriction Y}\right)$ and $(Y, cl._Y)$ are matrological spaces.*

Proof. Suppose that (X, cl) be a matrological space and $Y \subseteq X$. Then (X, cl) is a finite matroidal space. By Theorem 307, $\left(Y, \mathrm{cl}_{\restriction Y}\right)$ and $(Y, \mathrm{cl}._Y)$ are finite matroidal spaces. Let $A, B \subseteq Y$ and $x \in Y \setminus (A \cup B)$ be such that

$$x \in \mathrm{cl}_{\restriction Y}(A \cup B) \setminus \left[\mathrm{cl}_{\restriction Y}(A) \cup \mathrm{cl}_{\restriction Y}(B)\right].$$

So,

$$x \in \mathrm{cl}(A \cup B) \setminus [\mathrm{cl}(A) \cup \mathrm{cl}(B)].$$

Since (X, cl) is a matrological space, there is a finite set $F \subseteq A \cup B$ such that $x \in \mathrm{cl}(F)$. It follows that $x \in \mathrm{cl}_{\restriction Y}(F)$.

Now, let $A, B \subseteq Y$ and $x \in Y \setminus (A \cup B)$ be such that

$$x \in \mathrm{cl}._Y(A \cup B) \setminus [\mathrm{cl}._Y(A) \cup \mathrm{cl}._Y(B)].$$

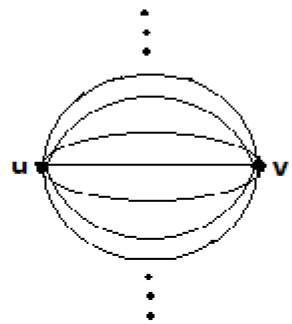

Figure 3.2.2: Illustration for Remark 357.

Thus,

$$x \in \mathrm{cl}((A \cup B) \cup (X \setminus Y)) \setminus [\mathrm{cl}(A \cup (X \setminus Y)) \cup \mathrm{cl}(B \cup (X \setminus Y))].$$

Therefore,

$$x \in \mathrm{cl}((A \cup (X \setminus Y)) \cup (B \cup (X \setminus Y))) \setminus [\mathrm{cl}(A \cup (X \setminus Y)) \cup \mathrm{cl}(B \cup (X \setminus Y))].$$

Since (X, cl) is a matrological space, there is a finite set

$$F \subseteq (A \cup (X \setminus Y)) \cup (B \cup (X \setminus Y)) = (A \cup B) \cup (X \setminus Y)$$

such that $x \in \mathrm{cl}(F)$. Take $F' = F \setminus (X \setminus Y)$. It implies that $F' \subseteq A \cup B$ is a finite set. Since $x \in \mathrm{cl}(F)$ and $F \subseteq F' \cup (X \setminus Y)$, we have

$$x \in \mathrm{cl}\left(F' \cup (X \setminus Y)\right).$$

It follows that $x \in \mathrm{cl}._Y(F')$. Hence $\left(Y, \mathrm{cl}_{\restriction Y}\right)$ and $(Y, \mathrm{cl}._Y)$ are matrological spaces. □

Remark 357. The dual of a matrological space is not necessarily a matrological space.

Example 358. Let G be a graph with two vertices u, v and infinite edges between u and v, see Figure 3.2.2. Let $X = E(G)$. Consider the finite cycle matroid on X. So, $(X, \mathscr{I})$ is a finitary matroid where $\mathscr{I}$ is induced by the matroidal closure

$$\mathrm{cl}(A) = A \cup \{x \in X \setminus A : \text{ there is } C \in \mathscr{C} \text{ such that } x \in C \subseteq A \cup \{x\}\}$$

for all $A \subseteq X$ where

$$\mathscr{C} = \{C \subseteq E(G) : |C| = 2\}$$

is the collection of all circuits of the matroid $(X, \mathscr{I})$. By Theorem 330, (X, cl) is a matrological space. The bond matroid $(X, \mathscr{I}^*)$ is the dual matroid of $(X, \mathscr{I})$. Then $(X, \mathscr{I}^*)$ has only one circuit $C^* = E(G)$ which is infinite. Therefore, $(X, \mathscr{I}^*)$ is not a finitary matroid. By Theorem 330, (X, cl^*) is not a matrological space where cl^* is the matroidal closure on X that induces $\mathscr{I}^*$.

Example 359. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topology. So, the topological closure operator on X is $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

X is a T_1-topological space. By Corollary 343, (X, cl) is a matrological space. By Theorem 6, the dual of standard topological closure on X is $\mathrm{cl}^* : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$\mathrm{cl}^*(A) = \begin{cases} X \setminus \{0\} & \text{if } 0 \notin A \text{ and } X \setminus A \text{ is infinite} \\ X & \text{if } 0 \in A \text{ or } X \setminus A \text{ is finite} \end{cases} \quad \text{for each } A \subseteq X.$$

In Remark 301, we showed that (X, cl^*) is a finite matroidal space. But take $A = \left\{\frac{1}{2n} : n \in \mathbb{N}\right\}$ and $B = \left\{\frac{1}{2n+1} : n \in \mathbb{N}\right\}$. So, we have

$$0 \in \mathrm{cl}^*(A \cup B) \setminus [\mathrm{cl}^*(A) \cup \mathrm{cl}^*(B)]$$

but for each finite $F \subseteq A \cup B$, we have $0 \notin \mathrm{cl}^*(F)$.

Remark 360. A matrological space might not have basis and circuits.

Proof. Let X be an infinite set. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

Then X is a cofinite topological space which is T_1. By Corollary 343, (X, cl) is a matrological space. In Remark 308, we showed that (X, cl) has no basis or circuits. □

In Section 3.1, we compared between finite matroidal spaces and the old spaces such as matroids, pre-independence spaces and exchange systems. Now, we compare them with matrological spaces to show that matrological spaces are new and different.

Remark 361. If (X, cl) is a matrological space, then $(X, \mathscr{I})$ may not be a matroid where $\mathscr{I}$ is the simplicial complex on X induced by cl.

Proof. Consider $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the standard topological closure operator cl on X. X is a T_1-topological space. By Corollary 343, (X, cl) is a matrological space. Now, the collection $\mathscr{I}$ of independent sets induced by cl is

$$\begin{aligned}\mathscr{I} &= \{A \subseteq X : x \notin \mathrm{cl}(A \setminus \{x\}) \text{ for each } x \in A\} \\ &= \{A \subseteq X : 0 \notin A \text{ or } A \text{ is finite}\}.\end{aligned}$$

In Remark 312 we showed that $(X, \mathscr{I})$ is not a matroid. □

Theorem 362. *If (X, cl) is a matrological space, then $(X, \mathscr{I})$ is a pre-independence space where $\mathscr{I}$ is the simplicial complex on X induced by cl.*

Proof. Every matrological space is a finite matroidal space. By Theorem 316, $(X, \mathscr{I})$ is a pre-independence space. □

Remark 363. If $(X, \mathscr{I})$ is a pre-independence space, then (X, cl) is not necessarily a matrological space where cl is the pre-closure operator on X induced by $\mathscr{I}$.

Proof. Let $X = \mathbb{N}$ and

$$\mathscr{B} = \{X \setminus \{1\}, X \setminus \{2,3\}, X \setminus \{4,5,6\}, X \setminus \{7,8,9,10\}, \ldots\}.$$

Let

$$\mathscr{I} = \{I \subseteq X : \text{ there is } B \in \mathscr{B} \text{ such that } I \subseteq B\}.$$

In Remark 317, we proved that $(X, \mathscr{I})$ is a pre-independence space. Now, the pre-closure operator cl on X induced by $\mathscr{I}$ is

$$\mathrm{cl}(A) = A \cup \{x \in X \setminus A : \text{ there is } I \subseteq A \text{ such that } I \in \mathscr{I} \text{ and } I \cup \{x\} \notin \mathscr{I}\}$$

for all $A \subseteq X$. In Remark 317, we showed that (X, cl) is not a finite matroidal space. Thus, (X, cl) is not a matrological space. □

Remark 364. If (X, cl) is a matrological space, then $(X, \mathscr{I})$ does not have to be an mpi-space where $\mathscr{I}$ is the simplicial complex on X induced by cl.

Proof. Let $X=\mathbb{N}$ and define $\mathrm{cl}:\mathscr{P}(X)\to\mathscr{P}(X)$ by

$$\mathrm{cl}(A)=\begin{cases}A & \text{if } A \text{ is finite}\\ X & \text{if } A \text{ is infinite}\end{cases}\quad \text{for each } A\subseteq X.$$

X is a T_1-topological space. By Corollary 343, (X,cl) is a matrological space. Now, the simplicial complex $\mathscr{I}$ on X induced by cl is

$$\begin{aligned}\mathscr{I}&=\{A\subseteq X : x\notin \mathrm{cl}(A\setminus\{x\}) \text{ for each } x\in A\}\\ &=\{A\subseteq X : A \text{ is finite}\}.\end{aligned}$$

In Remark 321, we proved that $(X,\mathscr{I})$ is not an mpi-space. □

Remark 365. If $(X,\mathscr{I})$ is an mpi-space, then (X,cl) is not necessarily a matrological space where cl is the pre-closure operator on X induced by $\mathscr{I}$.

Proof. Let $X=\mathbb{N}$ and

$$\mathscr{B}=\{X\setminus\{1\},X\setminus\{2,3\},X\setminus\{4,5,6\},X\setminus\{7,8,9,10\},\ldots\}.$$

Let

$$\mathscr{I}=\{I\subseteq X: \text{ there is } B\in\mathscr{B} \text{ such that } I\subseteq B\}.$$

The pre-independence space $(X,\mathscr{I})$ is obviously an mpi-space. In Remark 317, it has proven that (X,cl) is not a finite matroidal space where cl is the pre-closure operator on X induced by $\mathscr{I}$. Thus, (X,cl) is not a matrological space □

Remark 366. Let (X,cl) be a matrological space. Then $(X,\mathscr{I})$ does not have to be an exchange system where $\mathscr{I}$ is the simplicial complex on X induced by cl.

Proof. Let $X=X_1\cup X_2$ where X_1 and X_2 are disjoint infinite sets. Define $\mathrm{cl}:\mathscr{P}(X)\to\mathscr{P}(X)$ by

$$\mathrm{cl}(A)=\begin{cases}A & \text{if } A \text{ is finite}\\ A\cup X_1 & \text{if } A\cap X_1 \text{ is finite and } A\cap X_2 \text{ is infinite}\\ A\cup X_2 & \text{if } A\cap X_2 \text{ is finite and } A\cap X_1 \text{ is infinite}\\ X & \text{if } A\cap X_1 \text{ and } A\cap X_2 \text{ are infinite}\end{cases}$$

for each $A \subseteq X$. In Remark 324, we proved that (X, cl) is a finite matroidal space. We want to prove that (X, cl) is a matrological space. It suffices to show that

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for each } A, B \subseteq X.$$

By (CL2), we know

$$\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B) \text{ for each } A, B \subseteq X.$$

Now, let $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ such that $x \in \mathrm{cl}(A \cup B)$. So, $A \cup B$ is not finite. Thus, $(A \cup B) \cap X_1$ or $(A \cup B) \cap X_2$ is infinite or both. Therefore, $A \cap X_1$ or $A \cap X_2$ is infinite or both or $B \cap X_1$ or $B \cap X_2$ is infinite or both. So, $x \in \mathrm{cl}(A) \cup \mathrm{cl}(B)$. Thus,

$$\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for each } A, B \subseteq X.$$

Thus,

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B) \text{ for each } A, B \subseteq X.$$

Therefore, (X, cl) is a matrological space. Now, cl induces the simplicial complex

$$\begin{aligned} \mathscr{I} &= \{A \subseteq X : x \notin \mathrm{cl}(A \setminus \{x\}) \text{ for each } x \in A\} \\ &= \mathscr{P}(X_1) \cup \mathscr{P}(X_2) \cup \{A \subseteq X : A \text{ is finite}\}. \end{aligned}$$

In Remark 324, we showed that $(X, \mathscr{I})$ is not an exchange system. □

Chapter 4

Common Closure Operators

As we know, any topological vector space over the field $\mathbb{F}$ is a set with a linear structure and a topological structure combined together by the topological continuity of the vector space operations. We know that every vector space is a finitary matroid and we saw in the key theorem (Theorem 330) that any finitary matroid is a matrological space. We also know that the topology of any topological vector space is always regular (Corollary 287), and we can see from Theorem 338 and Corollary 349 that the topological closure operator of any regular topological space has the exchange property. The other key theorem (Theorem 338) shows that any topological space in which the topological closure operator has the exchange property is a matrological space.

Based on all of the above, our strategy for generalizing the concept of topological vector spaces starts with replacing the vector space with a finitary matroid and the regular topology with a topology in which the topological closure operator has the exchange property. We then combine the matroidal closure operator of the finitary matroid and the topological closure operator that has the exchange property into a single closure operator called the common closure operator. After that, we investigate when the common closure operator on the given set produces a finite matroidal space or a matrological space.

Definition 367. Let cl' and cl'' be two pre-closure operators on a set X. If $C \subseteq X$ is cl'-closed and cl''-closed, we say that C is a common closed set in X.

Theorem 368. *Let cl' and cl'' be two pre-closure operators on a set X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$\begin{aligned} cl(A) &= \{x \in X : \textit{for each common closed set } C \textit{ with } A \subseteq C, \textit{ we have } x \in C\} \\ &= \bigcap \{C \subseteq X : C \textit{ is a common closed set with } A \subseteq C\} \end{aligned}$$

for each $A \subseteq X$. Then cl is a closure operator on X.

Proof. We want to show that cl satisfies (CL1), (CL2) and (CL3). Let $A \subseteq X$. Let $x \in A$ and C be a common closed set with $A \subseteq C$. Then $x \in C$. Thus, $x \in \mathrm{cl}(A)$. Therefore, $A \subseteq \mathrm{cl}(A)$ and hence cl satisfies (CL1). Let $A \subseteq B \subseteq X$ and $x \in \mathrm{cl}(A)$. So, $x \in C$ for each common closed set C with $A \subseteq C$. From (CL1), we have $B \subseteq \mathrm{cl}(B)$. Then $A \subseteq \mathrm{cl}(B)$. But $\mathrm{cl}(B)$ itself is a common closed set in X. Thus, $x \in \mathrm{cl}(B)$. Therefore $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ and hence cl satisfies (CL2). Let $A \subseteq X$. From (CL1), we have $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$. Now, let $x \in \mathrm{cl}(\mathrm{cl}(A))$. Thus, $x \in C$ for each common closed set C with $\mathrm{cl}(A) \subseteq C$. But $\mathrm{cl}(A)$ itself is a common closed set in X. Thus, $x \in \mathrm{cl}(A)$. Therefore, $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ and hence cl satisfies (CL3). Therefore, cl is a closure operator on X. □

It is clear that $\mathrm{cl}(A)$ is the smallest common closed set containing A. Hence $\mathrm{cl}'(A) \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}''(A) \subseteq \mathrm{cl}(A)$ for all $A \subseteq X$.

Definition 369. The closure operator cl on a set X defined in Theorem 368 is called the common closure operator on X and denoted by $\mathrm{cl} := \mathrm{cl}' \oplus \mathrm{cl}''$.

Now, we will investigate when combining a matroidal closure operator and a topological closure operator with the exchange property on a given set produces a finite matroidal space or a matrological space.

Remark 370. Let X be any set, cl' be a matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Then (X, cl) might not be a finite matroidal space.

Example 371. Consider the graph G as shown in Figure 4.0.1. Let $X = E(G)$ and take $x \in X$ as indicated in Figure 4.0.1. Consider the finite cycle matroid and the Uncountable Fort Space on X, see Example 206. Let cl' be the matroidal closure operator on X and cl'' be the topological closure operator on X. So,

$$\mathrm{cl}''(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{x\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for each } A \subseteq X.$$

Thus, cl' is a finitary matroidal closure operator on X and cl'' is a T_1(regular)-topological closure operator which has the exchange property on X. Now, define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$

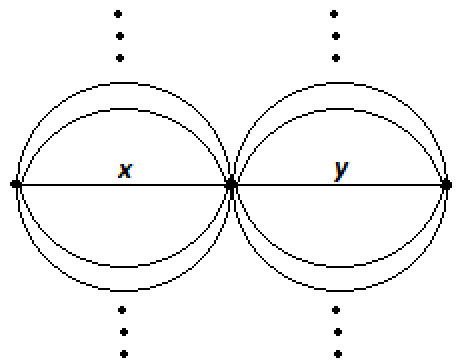

Figure 4.0.1: Illustration for Example 371.

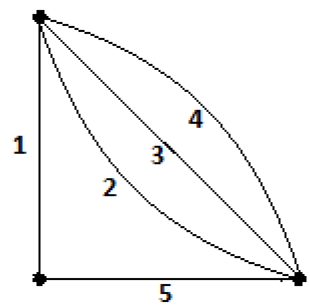

Figure 4.0.2: Illustration for Example 372.

by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Consider $Y = \{x, y\} \subseteq X$ where y is chosen as in Figure 4.0.1. Let $A = \varnothing$. Then $x, y \in Y \setminus A$ and

$$\mathrm{cl}_{\restriction Y}(\varnothing) = \varnothing$$
$$\mathrm{cl}_{\restriction Y}(\{x\}) = \{x\}$$
$$\mathrm{cl}_{\restriction Y}(\{y\}) = \{x, y\}.$$

So, $x \in \mathrm{cl}_{\restriction Y}(\varnothing \cup \{y\}) \setminus \mathrm{cl}_{\restriction Y}(\varnothing)$ but $y \notin \mathrm{cl}_{\restriction Y}(\varnothing \cup \{x\})$. Therefore, $\mathrm{cl}_{\restriction Y}$ does not have the exchange property on Y. Hence (X, cl) is not a finite matroidal space.

Example 372. Let $X = E(G)$ where G is the graph shown in Figure 4.0.2 Let

$$\tau = \{\varnothing, \{1,2\}, \{3,4\}, \{5\}, \{1,2,3,4\}, \{1,2,5\}, \{3,4,5\}, X\}.$$

Then X is a regular (not T_1) topological space. Let cl' be the finite cycle matroidal closure operator on X and cl'' be the topological closure operator which has the exchange property on X. Now, define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

We have

$$\begin{aligned} \mathrm{cl}(\varnothing) &= \varnothing \\ \mathrm{cl}(\{1\}) &= X \\ \mathrm{cl}(\{5\}) &= \{5\}. \end{aligned}$$

So, $x \in \mathrm{cl}(\varnothing \cup \{y\}) \setminus \mathrm{cl}(\varnothing)$ but $y \notin \mathrm{cl}(\varnothing \cup \{x\})$. Therefore, cl does not have the exchange property on X. Hence (X, cl) is not a finite matroidal space.

Remark 370 failed to produce a finite matroidal space when we combined a matroidal closure operator and a topological closure operator with the exchange property. So, we need to add more conditions to get a finite matroidal space.

Note that in Example 371 we have

$$\{x, y\} = \mathrm{cl}_{\restriction Y}(\{y\}) \neq \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(\{y\}) = \{y\},$$

and we also have

$$\mathrm{cl}(\{y\}) \neq \mathrm{cl}'(\{y\}) \cup \mathrm{cl}''(\{y\}).$$

In Example 372, if we take $Y = \{1,3,4,5\}$, then

$$Y = \mathrm{cl}_{\restriction Y}(\{1\}) \neq \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(\{1\}) = \{1\}.$$

Also, note that

$$X = \mathrm{cl}(\{1\}) \neq \mathrm{cl}'(\{1\}) \cup \mathrm{cl}''(\{1\}) = \{1\} \cup \{1,2\} = \{1,2\}.$$

Lemma 373. *Let cl' and cl'' be two closure operators on a set X. Define a closure operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \textit{ for all } A \subseteq X.$$

Then

$$cl_{\restriction Y} = cl'_{\restriction Y} \oplus cl''_{\restriction Y} \textit{ for all } Y \subseteq X$$

if and only if

$$cl(A) = cl'(A) \cup cl''(A) \text{ for all } A \subseteq X.$$

Proof. Assume that $\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}$ for each $Y \subseteq X$. Since $\mathrm{cl}'(A) \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}''(A) \subseteq \mathrm{cl}(A)$ for all $A \subseteq X$, then

$$\mathrm{cl}'(A) \cup \mathrm{cl}''(A) \subseteq \mathrm{cl}(A) \text{ for all } A \subseteq X.$$

Let $A \subseteq X$ and $x \in X \setminus A$ be such that $x \in \mathrm{cl}(A)$. Suppose, by way of contradiction, that $x \notin \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$. So, $\mathrm{cl}'(A) \neq \mathrm{cl}''(A)$ since otherwise $\mathrm{cl}(A) = \mathrm{cl}'(A) = \mathrm{cl}''(A)$. Take[1] $Y = X \setminus \left[\mathrm{cl}'(A) \bigtriangleup \mathrm{cl}''(A)\right]$. Then $A \subseteq Y$ and

$$\mathrm{cl}'_{\restriction Y}(A) = \mathrm{cl}''_{\restriction Y}(A) = \mathrm{cl}'(A) \cap \mathrm{cl}''(A).$$

So,

$$x \notin \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(A) = \mathrm{cl}'(A) \cap \mathrm{cl}''(A).$$

But $x \in \mathrm{cl}_{\restriction Y}(A)$. Thus, $\mathrm{cl}_{\restriction Y} \neq \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}$ which is a contradiction. Therefore, $x \in \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$, and hence

$$\mathrm{cl}(A) \subseteq \mathrm{cl}'(A) \cup \mathrm{cl}''(A) \text{ for all } A \subseteq X.$$

It follows that

$$\mathrm{cl}(A) = \mathrm{cl}'(A) \cup \mathrm{cl}''(A) \text{ for all } A \subseteq X.$$

Now, assume that $\mathrm{cl}(A) = \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$ for each $A \subseteq X$. Suppose, by way of contradiction, that there is $Y \subseteq X$ such that $\mathrm{cl}_{\restriction Y} \neq \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}$. So, there are $A \subseteq Y$ and $y \in Y \setminus A$ such that

$$y \in \mathrm{cl}_{\restriction Y}(A) \setminus \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(A).$$

Let $D = \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(A)$. So, $\mathrm{cl}'_{\restriction Y}(D) = \mathrm{cl}''_{\restriction Y}(D) = D$. Thus, $y \notin \mathrm{cl}'_{\restriction Y}(D) \cup \mathrm{cl}''_{\restriction Y}(D)$, and hence $y \notin \mathrm{cl}'_{\restriction Y}(A) \cup \mathrm{cl}''_{\restriction Y}(A)$. Therefore, $y \in \mathrm{cl}(A) \setminus \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$. This contradicts that $\mathrm{cl}(A) = \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$ for all $A \subseteq X$. Then

$$\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y} \text{ for each } Y \subseteq X.$$

□

Theorem 374. *Let X be any set, cl' be a matroidal closure operator on X and cl'' be a*

[1] The symmetric difference of subsets A and B of a set X is $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$.

topological closure operator with the exchange property on X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

If $cl_{\restriction Y} = cl'_{\restriction Y} \oplus cl''_{\restriction Y}$ for each $Y \subseteq X$, then (X, cl) is a finite matroidal space.

Proof. cl is a closure operator on X, so (X, cl) is a space. By Theorem 5, $\text{cl}_{\restriction Y}$ and $\text{cl}._Y$ are closure operators on every $Y \subseteq X$, in particular $\text{cl}_{\restriction F}$ and $\text{cl}._F$ are closure operators on every finite $F \subseteq X$. Now, we need to show that $\text{cl}_{\restriction F}$ and $\text{cl}._F$ have the exchange property on every finite $F \subseteq X$. It suffices to show that cl has the exchange property on X. Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \text{cl}(A \cup \{x\}) \setminus \text{cl}(A).$$

By Lemma 373, we have

$$y \in \text{cl}'(A \cup \{x\}) \cup \text{cl}''(A \cup \{x\}) \setminus \text{cl}'(A) \cup \text{cl}''(A).$$

cl' and cl'' have the exchange property on X, we have $x \in \text{cl}'(A \cup \{y\}) \cup \text{cl}''(A \cup \{y\})$. Thus, $x \in \text{cl}(A \cup \{y\})$. Therefore, cl has the exchange property on X. By Theorem 106, $\text{cl}_{\restriction Y}$ and $\text{cl}._Y$ have the exchange property on every $Y \subseteq X$. Then $\text{cl}_{\restriction Y}$ and $\text{cl}._Y$ are matroidal closure operators on every $Y \subseteq X$. Hence (X, cl) is a finite matroidal space. □

Corollary 375. *Let X be any set, cl' be a matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

If $cl(A) = cl'(A) \cup cl''(A)$ for each $A \subseteq X$, then (X, cl) is a finite matroidal space.

Proof. This is an immediate consequence of Lemma 373. □

The conditions we have in Theorem 374 and Corollary 375 are sufficient to produce a finite matroidal space when we combine a matroidal closure operator and a topological closure operator with the exchange property, but they are not sufficient to produce a matrological space.

Remark 376. Let X be any set, cl' be a matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $\text{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$

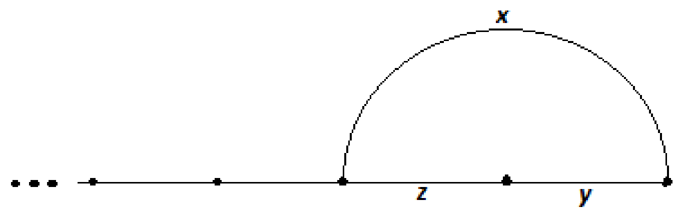

Figure 4.0.3: Illustration for Remark 377.

by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

If $\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}$ for each $Y \subseteq X$, then (X, cl) is not necessarily a matrological space.

Proof. Let $X = E(G)$ where G is the One Double Ray G see Figure 3.2.1. Let cl' be the algebraic cycle matroidal closure operator on X and cl'' be the discrete topological closure operator on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Since $\mathrm{cl}''(A) = A$ for all $A \subseteq X$, then $\mathrm{cl} = \mathrm{cl}'$. Note that $\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y} = \mathrm{cl}'_{\restriction Y}$ for each $Y \subseteq X$. But it has been shown in Remark 329 that (X, cl) is not a matrological space. □

Note that, in the proof of Remark 376, cl' is not a finitary matroidal closure operator on X.

Remark 377. Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Then (X, cl) may not be a matrological space.

Proof. Consider the graph G as in Figure 4.0.3. Let $X = E(G)$ and take $x \in X$ as indicated in Figure 4.0.3. Let cl' be the finite cycle matroidal closure operator on X and cl'' be the T_1-topological closure operator on X defined by

$$\mathrm{cl}''(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{x\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

Thus, cl' is a finitary matroidal closure operator on X and cl'' is a topological closure operator with the exchange property on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Now, take $A = X \setminus \{x, y, z\}$ and $B = \{z\}$ where y, z are chosen as in Figure 4.0.3. Then

$$\begin{aligned} \mathrm{cl}(A) &= A \cup \{x\} \\ \mathrm{cl}(B) &= B \\ \mathrm{cl}(A \cup B) &= X \end{aligned}$$

Thus, $y \in \mathrm{cl}(A \cup B) \setminus (\mathrm{cl}(A) \cup \mathrm{cl}(B))$. But $y \notin \mathrm{cl}(F)$ for all finite $F \subseteq A \cup B$. Therefore, (X, cl) is not a matrological space. □

Again, note that if we take $Y = X \setminus \{x\} \subseteq X$ and $A = X \setminus \{x, y\}$, we get

$$y \in \mathrm{cl}_{\restriction Y}(A) \text{ and } y \notin \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(A).$$

Thus,

$$\mathrm{cl}_{\restriction Y}(A) \neq \left(\mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}\right)(A)$$

Theorem 378. *Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \textit{ for all } A \subseteq X.$$

If $cl_{\restriction Y} = cl'_{\restriction Y} \oplus cl''_{\restriction Y}$ for each $Y \subseteq X$, then (X, cl) is a matrological space.

Proof. By Theorem 374, (X, cl) is a finite matroidal space. Now, let $A, B \subseteq X$ and $x \in X \setminus (A \cup B)$ be such that

$$x \in \mathrm{cl}(A \cup B) \setminus (\mathrm{cl}(A) \cup \mathrm{cl}(B)).$$

By Lemma 373, we get

$$x \in \left[\mathrm{cl}'(A \cup B) \cup \mathrm{cl}''(A \cup B)\right] \setminus \left[\left(\mathrm{cl}'(A) \cup \mathrm{cl}''(A)\right) \cup \left(\mathrm{cl}'(B) \cup \mathrm{cl}''(B)\right)\right].$$

Since $\mathrm{cl}''(A \cup B) = \mathrm{cl}''(A) \cup \mathrm{cl}''(B)$, then

$$x \in \mathrm{cl}'(A \cup B) \setminus \left(\mathrm{cl}'(A) \cup \mathrm{cl}'(B)\right).$$

Since $\left(X, \mathrm{cl}'\right)$ is a matrological space, there is a finite set $F \subseteq A \cup B$ such that

$$x \in \mathrm{cl}'(F) \subseteq \mathrm{cl}(F).$$

Then (X, cl) is a matrological space. □

Corollary 379. *Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

If $cl(A) = cl'(A) \cup cl''(A)$ for each $A \subseteq X$, then (X, cl) is a matrological space.

Proof. This is an immediate consequence of Lemma 373. □

We have seen that the conditions in Theorem 378 and Corollary 379 are sufficient to produce a matrological space. In the following remark, we check whether these conditions are necessary or not.

Remark 380. Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

If (X, cl) is a matrological space, then the condition $\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}$ for each $Y \subseteq X$ might not hold.

Proof. Consider the vector space $X = \mathbb{R}$ over the field $\mathbb{R}$ with the indiscrete topology. X is a topological vector space over $\mathbb{R}$. Let cl' be the linear closure operator on X and cl'' be the topological closure operator on X. So, cl' is a finitary matroidal closure operator on X and cl'' is a topological closure operator with the exchange property on X. Define the common closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Then

$$\mathrm{cl}(A) = X \text{ for all } A \subseteq X.$$

Thus, (X,cl) is a matrological space but we have

$$\mathrm{cl}(\varnothing) = X$$
$$\mathrm{cl}'(\varnothing) \cup \mathrm{cl}''(\varnothing) = \{0\} \cup \varnothing = \{0\}.$$

So,

$$\mathrm{cl}(\varnothing) \neq \mathrm{cl}'(\varnothing) \cup \mathrm{cl}''(\varnothing).$$

By Lemma 373, the condition $\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y}$ for each $Y \subseteq X$ does not hold. □

The following example shows that the exchange property of cl'' is an important condition.

Example 381. Let $X = \{1,2\}$ and cl' be the discrete matroidal closure operator on X, which is a finitary matroidal closure operator on X. Consider the Sierpinski topology

$$\tau = \{\varnothing, \{2\}, X\}$$

for X. Thus, the topological closure operator cl'' on X is defined by

$$\mathrm{cl}''(\varnothing) = \varnothing$$
$$\mathrm{cl}''(\{1\}) = \{1\}$$
$$\mathrm{cl}''(\{2\}) = \mathrm{cl}''(X) = X.$$

In Remark 342, we proved that cl'' does not have the exchange property on X. Now, define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Since $\mathrm{cl}'(A) = A$ for all $A \subseteq X$, then $\mathrm{cl} = \mathrm{cl}''$. Note that $\mathrm{cl}_{\restriction Y} = \mathrm{cl}'_{\restriction Y} \oplus \mathrm{cl}''_{\restriction Y} = \mathrm{cl}''_{\restriction Y}$ for each $Y \subseteq X$. In Remark 342, we proved that (X,cl) is not a matrological space.

As we saw in the proof of Remark 380, the common closure operator cl of an indiscrete topological vector space with more than one element does not satisfy the conditions in Theorem 378 and Corollary 379. By Theorem 291, in any topological vector space, the common closure operator cl is the composition of the topological closure operator cl'' and the linear closure operator cl'. This motivates us to generalize this to common closure operators that combine finitary matroidal closure operators and topological closure operators with the exchange property.

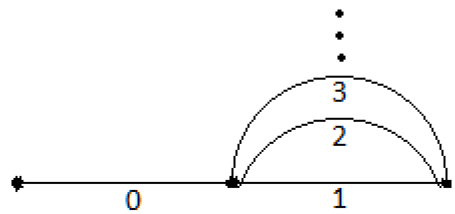

Figure 4.0.4: Illustration for Example 383.

Remark 382. Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

If $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$, then (X, cl) may not be a finite matroidal space or a matrological space.

Example 383. Consider the graph G in Figure 4.0.4. Let $X = E(G)$, cl' be the finite cycle matroidal closure operator on X and cl'' be the topological closure operator on X defined by

$$\mathrm{cl}''(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \text{ for all } A \subseteq X.$$

So, cl' is a finitary matroidal closure operator on X and cl'' is a topological closure operator with the exchange property on X. Define the common closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Therefore,

$$\mathrm{cl}(A) = \begin{cases} A & \text{if } A = \varnothing \text{ or } A = \{0\} \\ X & \text{otherwise} \end{cases} \text{ for all } A \subseteq X.$$

Thus,

$$\mathrm{cl}(A) = \mathrm{cl}''\left(\mathrm{cl}'(A)\right) \text{ for all } A \subseteq X.$$

Take $F = \{0, 1\}$. Then

$$\begin{aligned} \mathrm{cl}_{\restriction F}(\varnothing) &= \varnothing \\ \mathrm{cl}_{\restriction F}(\varnothing \cup \{1\}) &= \{0, 1\} \\ \mathrm{cl}_{\restriction F}(\varnothing \cup \{0\}) &= \{0\}. \end{aligned}$$

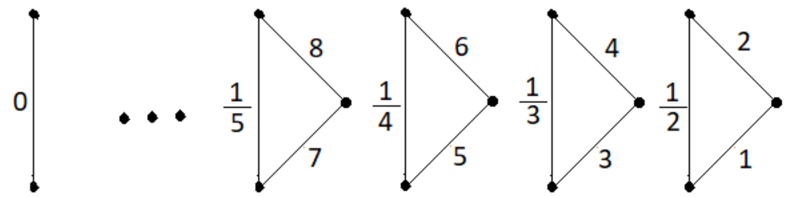

Figure 4.0.5: Illustration for Example 384.

So, $0 \in \mathrm{cl}_{\restriction F}(\varnothing \cup \{1\}) \setminus \mathrm{cl}_{\restriction F}(\varnothing)$ but $1 \notin \mathrm{cl}_{\restriction F}(\varnothing \cup \{0\})$. Thus, $\mathrm{cl}_{\restriction F}$ does not have the exchange property on F. Hence (X, cl) is not a finite matroidal space and thus not a matrological space.

In the following example, (X, cl) is a finite matroidal space but not a matrological space.

Example 384. Consider the graph G in Figure 4.0.5. Let $X = E(G)$, cl' be the finite cycle matroidal closure operator on X and cl'' be the topological closure operator on X defined by

$$\mathrm{cl}''(A) = \begin{cases} A & \text{if } A \text{ does not contain an infinite subset of } \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \\ A \cup \{0\} & \text{if } A \text{ contains an infinite subset of } \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \end{cases}$$

for all $A \subseteq X$. So, cl' is a finitary matroidal closure operator on X and cl'' is a topological closure operator with the exchange property on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Therefore,

$$\mathrm{cl}(A) = \begin{cases} \mathrm{cl}'(A) & \text{if } \mathrm{cl}'(A) \text{ does not contain an infinite subset of } \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \\ \mathrm{cl}'(A) \cup \{0\} & \text{if } \mathrm{cl}'(A) \text{ contains an infinite subset of } \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \end{cases}$$

for all $A \subseteq X$. Thus,

$$\mathrm{cl}(A) = \mathrm{cl}''\left(\mathrm{cl}'(A)\right) \text{ for all } A \subseteq X.$$

Note that cl has the exchange property on X. So, (X, cl) is a finite matroidal space. Now, consider the sets

$$A_0 = \{n \in \mathbb{N} : n \text{ is odd}\}$$
$$B_0 = \{m \in \mathbb{N} : m \text{ is even}\}.$$

So, we have $A_0, B_0 \subseteq X$ and $0 \in X \setminus A_0 \cup B_0$ such that

$$0 \in \mathrm{cl}(A_0 \cup B_0) \setminus [\mathrm{cl}(A_0) \cup \mathrm{cl}(B_0)].$$

Now, let $F \subseteq A_0 \cup B_0$ be finite. Thus, $\mathrm{cl}'(F)$ is finite and $0 \notin \mathrm{cl}'(F)$. Therefore, $\mathrm{cl}(F)$ is also finite and $0 \notin \mathrm{cl}(F)$. Hence (X, cl) is not a matrological space.

Note that in Example 383, we have $\mathrm{cl}'(\{1\}) \setminus \mathrm{cl}''(\{1\})$ is infinite and in Example 384, we have $\mathrm{cl}'(A_0 \cup B_0) \setminus \mathrm{cl}''(A_0 \cup B_0)$ is infinite. So, in the following theorem, we add an additional condition to the Remark 382.

Theorem 385. *Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define a closure operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

If $cl = cl'' \circ cl'$ and $cl'(A) \setminus cl''(A)$ is finite for each $A \subseteq X$, then (X, cl) is a matrological space.

Proof. cl is a closure operator on X, so (X, cl) is a space. By Theorem 5, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}_{.F}$ are closure operators on every finite $F \subseteq X$. Now, we need to show that $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}_{.F}$ have the exchange property on each finite $F \subseteq X$. It suffices to show that cl has the exchange property on X. Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

By the condition $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$, we get

$$y \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right) \setminus \mathrm{cl}''\left(\mathrm{cl}'(A)\right).$$

We want to show that

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right) = \mathrm{cl}(A \cup \{y\}).$$

If $y \in \mathrm{cl}'(A \cup \{x\}) \cup \mathrm{cl}''(A \cup \{x\})$. By the exchange property of cl' and cl'' on X, we get

$$x \in \mathrm{cl}'(A \cup \{y\}) \cup \mathrm{cl}''(A \cup \{y\}).$$

By (CL1) and (CL2) on cl'', we have

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right).$$

If $y \notin \mathrm{cl}'(A \cup \{x\}) \cup \mathrm{cl}''(A \cup \{x\})$.

Case 1: If $y \in \mathrm{cl}''\left(\mathrm{cl}'(A) \cup \{x\}\right)$. Since cl'' is a topological closure operator on X, then

$$y \in \mathrm{cl}''\left(\mathrm{cl}'(A)\right) \cup \mathrm{cl}''(\{x\}).$$

Since $y \notin \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$, then $y \in \mathrm{cl}''(\{x\}) = \mathrm{cl}''(\varnothing \cup \{x\})$. By using the exchange property of cl'', we get

$$x \in \mathrm{cl}''(\varnothing \cup \{y\}) = \mathrm{cl}''(\{y\}).$$

By (CL2) on cl'', we have

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right).$$

Case 2: If $y \notin \mathrm{cl}''\left(\mathrm{cl}'(A) \cup \{x\}\right)$. Suppose, by way of contradiction, that

$$x \notin \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right).$$

Since $y \notin \mathrm{cl}'(A \cup \{x\}) \cup \mathrm{cl}''(A \cup \{x\})$, we have

$$\mathrm{cl}'(A \cup \{x\}) \nsubseteq \left[\mathrm{cl}''(A \cup \{x\}) \cup \mathrm{cl}''\left(\mathrm{cl}'(A)\right)\right],$$

otherwise $y \notin \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right)$ or $y \in \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$ which are contradictions. Let

$$D = \mathrm{cl}'(A \cup \{x\}) \setminus \left[\mathrm{cl}''(A \cup \{x\}) \cup \mathrm{cl}''\left(\mathrm{cl}'(A)\right)\right].$$

Clearly, from the hypothesis, D is a finite set. Note that $D \neq \varnothing$ and for each $U \in \mathcal{N}''(y)$, we have

$$U \cap D \neq \varnothing, \tag{4.0.1}$$

since otherwise $y \notin \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right)$, contradiction where $\mathcal{N}''$ is the open neighborhood base on X that induces cl'', see Corollary 51. Now, we need to the following claim.

Claim 386. $D \cap \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right) = \varnothing$.

Proof. Suppose, by way of contradiction, that

$$D \cap \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right) \neq \varnothing.$$

Then there is $w \in D \cap \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right)$. Since $D \subseteq \mathrm{cl}'(A \cup \{x\})$, there is a finite circuit C

under cl' such that

$$w \in C \subseteq (A \cup \{x\}) \cup \{w\}.$$

We must have $x \in C$, otherwise $w \in \mathrm{cl}'(A)$ which leads to $w \in \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$ contradicting $w \in D$. So, $\mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right)$ is not cl'-closed which contradicts the condition $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$. Therefore,

$$D \cap \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right) = \varnothing.$$

□

By (CL2) on cl'', the Claim 386 leads to $D \cap \mathrm{cl}''(A \cup \{y\}) = \varnothing$. Thus

$$z \notin \mathrm{cl}''(A \cup \{y\}) \text{ for each } z \in D.$$

By (CL2) on cl'', we get

$$z \notin \mathrm{cl}''(\{y\}) \text{ for each } z \in D.$$

Using the exchange property of cl'' on X, we get

$$y \notin \mathrm{cl}''(\{z\}) \text{ for each } z \in D.$$

So, for each $z \in D$ there is $U_z \in \mathcal{N}''(y)$ such that

$$U_z \cap \{z\} = \varnothing.$$

Since, by the hypothesis, D is finite, then $\bigcap_{z \in D} U_z$ is open under cl'' containing y and does intersect with D. Therefore, there is $V_0 \in \mathcal{N}''(y)$ such that

$$V_0 \cap D = \varnothing$$

which contradicts (4.0.1). Thus,

$$y \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right).$$

Therefore, cl has the exchange property on X. By Theorem 106, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ have the exchange property on every finite $F \subseteq X$. Then $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on every finite $F \subseteq X$. Hence (X, cl) is a finite matroidal space.

Now, let $A,B \subseteq X$ and $x \in X$ be such that

$$x \in \mathrm{cl}(A \cup B) \setminus [\mathrm{cl}(A) \cup \mathrm{cl}(B)].$$

Then

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup B)\right) \setminus \left[\mathrm{cl}''\left(\mathrm{cl}'(A)\right) \cup \mathrm{cl}''\left(\mathrm{cl}'(B)\right)\right].$$

Since cl'' is a topological closure on X, we have

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup B)\right) \setminus \left[\mathrm{cl}''\left(\mathrm{cl}'(A) \cup \mathrm{cl}'(B)\right)\right].$$

If $x \in \mathrm{cl}'(A \cup B) \cup \mathrm{cl}''(A \cup B)$. Since cl'' is a topological closure operator on X, then $x \in \mathrm{cl}'(A \cup B) \setminus \mathrm{cl}''(A \cup B)$, since otherwise $x \in \mathrm{cl}''\left(\mathrm{cl}'(A)\right) \cup \mathrm{cl}''\left(\mathrm{cl}'(B)\right)$ which is a contradiction. Since cl' is a finitary matroidal closure operator on X, there is a finite set $F \subseteq A \cup B$ such that

$$x \in \mathrm{cl}'(F).$$

By (CL1) on cl'', we get

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(F)\right) = \mathrm{cl}(F).$$

If $x \notin \mathrm{cl}'(A \cup B) \cup \mathrm{cl}''(A \cup B)$. Let $D = \mathrm{cl}'(A \cup B) \setminus \left[\mathrm{cl}''\left(\mathrm{cl}'(A) \cup \mathrm{cl}'(B)\right)\right]$. Clearly, from the hypothesis, D is a finite set. Note that $D \neq \varnothing$ and for each $U \in \mathcal{N}''(y)$, we have

$$U \cap D \neq \varnothing, \tag{4.0.2}$$

since otherwise $x \notin \mathrm{cl}''\left(\mathrm{cl}'(A \cup B)\right)$, contradiction.

Claim 387. There is $z \in D$ such that for each $U \in \mathcal{N}''(x)$, we have

$$U \cap \{z\} \neq \varnothing.$$

Proof. Suppose, by way of contradiction, that for each $z \in D$ there is $U_z \in \mathcal{N}''(x)$ such that

$$U_z \cap \{z\} = \varnothing.$$

Since D is finite, then $\bigcap_{z \in D} U_z$ is open under cl'' containing x and does not intersect with D. So, Therefore, there is $V_0 \in \mathcal{N}''(x)$ such that

$$V_0 \cap D = \varnothing$$

which contradicts (4.0.2). $\square$

Since $z \in D \subseteq \mathrm{cl}'(A \cup B)$, there is a finite circuit C under cl' such that

$$z \in C \subseteq (A \cup B) \cup \{z\}.$$

Take $F = C \setminus \{z\}$. So, $F \subseteq A \cup B$ is finite and $z \in \mathrm{cl}'(F)$. Thus,

$$\{z\} \subseteq \mathrm{cl}'(F).$$

By Claim 387, for each $U \in \mathcal{N}''(x)$, we have

$$U \cap \mathrm{cl}'(F) \neq \varnothing.$$

Therefore,

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(F)\right) = \mathrm{cl}(F).$$

Then (X, cl) is a matrological space. $\square$

As a consequence of Theorem 385, if the whole set is finite, we will get a finite matroid.

Corollary 388. *Let X be a finite set, cl' be a matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

If $cl = cl'' \circ cl'$, then $(X, \mathscr{I})$ is a matroid where cl is the matroidal closure operator on X induces $\mathscr{I}$.

Proof. X is a finite set, cl' is a finitary matroidal closure operator on X and $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$. So, by Theorem 385, (X, cl) is a matrological space, and by Theorem 335, cl is a matroidal closure operator on X. So, then $(X, \mathscr{I})$ is a matroid. $\square$

The proof of the exchange property in Theorem 385 leads to the conclusion that the contraction of cl on every finite set has the exchange property.

Corollary 389. *Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define a closure operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

If cl $= \mathrm{cl}'' \circ \mathrm{cl}'$, then $\mathrm{cl}._F$ has the exchange property on every finite $F \subseteq X$.

Proof. Let $F \subseteq X$ be a finite set. Let $A \subseteq F$ and $x, y \in F \setminus A$ be distinct such that

$$y \in \mathrm{cl}._F(A \cup \{x\}) \setminus \mathrm{cl}._F(A).$$

Thus,

$$y \in \mathrm{cl}(A \cup \{x\} \cup (X \setminus F)) \setminus \mathrm{cl}(A \cup (X \setminus F)).$$

By the condition $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$, we get

$$y \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\} \cup (X \setminus F))\right) \setminus \mathrm{cl}''\left(\mathrm{cl}'(A \cup (X \setminus F))\right).$$

Since $F \setminus A$ is finite, then $\mathrm{cl}'(A \cup (X \setminus F)) \setminus \mathrm{cl}''(A \cup (X \setminus F))$ must be finite. By Theorem 385, we get

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\} \cup (X \setminus F))\right).$$

By the condition $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$, we get

$$x \in \mathrm{cl}(A \cup \{y\} \cup (X \setminus F)).$$

Therefore,

$$x \in \mathrm{cl}._F(A \cup \{y\}).$$

Hence $\mathrm{cl}._F$ has the exchange property on F. □

The conditions in Theorem 385 are sufficient but not necessary.

Remark 390. Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define a closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

If (X, cl) is a matrological space, then one of the conditions $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$ or $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$ might fail.

Example 391. Consider the graph G as in Figure 4.0.6. Let $X = E(G)$, cl' be the finite cycle matroidal closure operator on X and cl'' be the topological closure operator on X

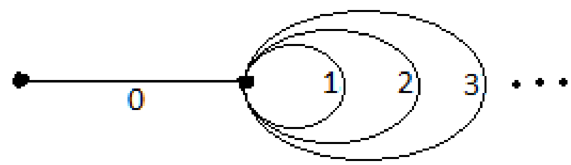

Figure 4.0.6: Illustration for Example 391.

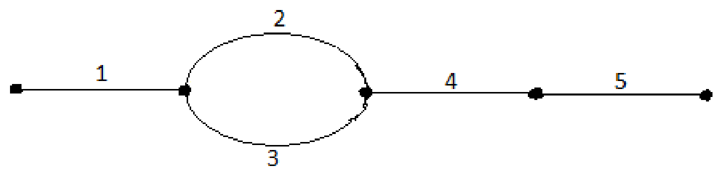

Figure 4.0.7: Illustration for Example 392.

defined by

$$\mathrm{cl}''(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{0\} & \text{if } A \text{ is infinite} \end{cases} \quad \text{for all } A \subseteq X.$$

So, cl' is a finitary matroidal closure operator on X and cl'' is a topological closure operator has the exchange property on X. Define the common closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Therefore,

$$\mathrm{cl}(A) = X \text{ for each } A \subseteq X.$$

Thus, (X, cl) is a matrological space. Note that

$$\mathrm{cl}(A) = \mathrm{cl}''\left(\mathrm{cl}'(A)\right) \text{ for all } A \subseteq X.$$

But $\mathrm{cl}'(\{1\}) \setminus \mathrm{cl}''(\{1\})$ is infinite.

Example 392. Let $X = E(G)$ where G is the graph shown in Figure 4.0.7. Take a regular topology τ on X as follows

$$\tau = \{\varnothing, \{1,2\}, \{3,4\}, \{5\}, \{1,2,3,4\}, \{1,2,5\}, \{3,4,5\}, X\}.$$

Let cl' be the finite cycle matroidal closure operator on X and cl'' be the topological closure operator on X. So, cl' is a finitary matroidal closure operator on X and cl'' has the exchange

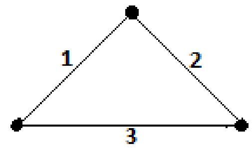

Figure 4.0.8: The C^3 graph.

property on X. Now, define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Therefore,

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{1,2,3,4\} & \text{if } A \subseteq \{1,2,3,4\} \\ \{5\} & \text{if } A = \{5\} \\ X & \text{otherwise} \end{cases} \text{ for all } A \subseteq X.$$

(X, cl) is a matrological space. Note that $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for all $A \subseteq X$ but

$$\mathrm{cl}(\{1\}) = \{1,2,3,4\}$$
$$\mathrm{cl}''\left(\mathrm{cl}'(\{1\})\right) = \{1,2\}.$$

Hence

$$\mathrm{cl}(\{1\}) \neq \mathrm{cl}''\left(\mathrm{cl}'(\{1\})\right).$$

In Theorem 385, the common closure operator cl is the composition of the topological closure operator cl'' with the linear closure operator cl'. What about if the common closure operator cl is the composition of the linear closure operator cl' with the topological closure operator cl''?

Remark 393. Let X be any set, cl' be a finitary matroidal closure operator on X and cl'' be a topological closure operator with the exchange property on X. Define a closure operator $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

If $\mathrm{cl} = \mathrm{cl}' \circ \mathrm{cl}''$ and $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$, then (X, cl) may not be a matrological space.

Proof. Consider the C^3 graph as shown in Figure 4.0.8. Let $X = E(G)$ and take a regular topology τ on X as follows

$$\tau = \{\varnothing, \{1,2\}, \{3\}, X\}.$$

Let cl' be the finite cycle matroidal closure operator on X which is a finitary matroidal closure operator on X and cl'' be the topological closure operator which has the exchange property on X . Now, define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Therefore,

$$\mathrm{cl}(A) = \begin{cases} \varnothing & \text{if } A = \varnothing \\ \{3\} & \text{if } A = \{3\} \\ X & \text{otherwise} \end{cases} \text{ for all } A \subseteq X.$$

Note that $\mathrm{cl} = \mathrm{cl}' \circ \mathrm{cl}''$ and $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$ but

$$3 \in \mathrm{cl}(\varnothing \cup \{1\}) \setminus \mathrm{cl}(\varnothing)$$

and

$$1 \notin \mathrm{cl}(\varnothing \cup \{3\}).$$

Thus, cl does not have the exchange property on X, and therefore (X, cl) is not a matrological space. □

Chapter 5

Relations between Topological Vector Spaces and the New Spaces

The main research point of this thesis is to find relationships of topological vector spaces with both finite matroidal spaces and matrological spaces. In this chapter, we will use the definition of topological vector spaces to find out which space is a generalization of topological vector spaces.

5.1 Relation between Topological Vector Spaces and Finite Matroidal Spaces

The following theorem shows that every topological vector space is a finite matroidal space. So, finite matroidal spaces generalize topological vector spaces.

Theorem 394. *Let X be a topological vector space over $\mathbb{F}$. Let cl' be the linear closure operator on X and cl'' be the topological closure operator on X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

Then (X, cl) is a finite matroidal space.

Proof. By Theorem 368, cl is a closure operator on X. So, (X, cl) is a space. Now, we need to show that $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are matroidal closure operators on each finite $F \subseteq X$. By Theorem 5, $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ are closure operators on each finite $F \subseteq X$. So, we only need to show that $\mathrm{cl}_{\restriction F}$ and $\mathrm{cl}._F$ have the exchange property on each finite $F \subseteq X$. It suffices, by

Theorem 106, to show that cl has the exchange property on X. Let $A \subseteq X$ and $x, y \in X \setminus A$ be distinct such that

$$y \in \mathrm{cl}(A \cup \{x\}) \setminus \mathrm{cl}(A).$$

By Theorem 291,

$$\mathrm{cl}(A) = \mathrm{cl}''\left(\mathrm{cl}'(A)\right) \text{ for each } A \subseteq X.$$

This means that $\mathrm{cl}(A)$ is the smallest common closed vector subspace of X containing A for each $A \subseteq X$. So, we have

$$y \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right) \setminus \mathrm{cl}''\left(\mathrm{cl}'(A)\right).$$

We want to show that $x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right)$. Note that $x, y \notin \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$. Let $M = \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$, $N_1 = \{rx : r \in \mathbb{F}\}$ and $N_2 = \{ry : r \in \mathbb{F}\}$. Then M is a closed vector subspace of X and N_1, N_2 are one-dimensional vector subspaces of X. By Theorem 293, $M + N_1$ and $M + N_2$ are closed vector subspaces of X. We claim that

$$M + N_1 = \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right)$$
$$M + N_2 = \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right).$$

Since $M + N_1$ is the smallest vector subspace of X containing both M and N_1, then

$$M + N_1 \subseteq \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right).$$

It is clear that

$$A \cup \{x\} \subseteq M + N_1.$$

By (CL2), we get

$$\mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right) \subseteq \mathrm{cl}''\left(\mathrm{cl}'(M + N_1)\right) = M + N_1.$$

Thus,

$$M + N_1 = \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right).$$

Similarly,

$$M + N_2 = \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right).$$

Now, since $y \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{x\})\right) \setminus \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$, then $y \in M + N_1$ and $y \notin M$. Thus,

$$y = m + rx$$

for some $m \in M$ and $r \in \mathbb{F}$ with $r \neq 0$. Solving for x, we get

$$x = -\frac{1}{r}m + \frac{1}{r}y.$$

Since $M = \mathrm{cl}''\left(\mathrm{cl}'(A)\right)$ is a vector subspace of X, we have $x \in M + N_2 = \mathrm{cl}''\left(\mathrm{cl}'(A \cup \{y\})\right)$. Thus, cl has the exchange property on X. Hence (X, cl) is a finite matroidal space. $\square$

5.2 Relations between Topological Vector Spaces and Matrological Spaces

The following theorem shows that every finite-dimensional topological vector space is a matrological space.

Theorem 395. *Let X be a finite-dimensional topological vector space over $\mathbb{F}$. Let cl' be the linear closure operator on X and cl'' be the topological closure operator on X. Define cl : $\mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

Then (X, cl) is a matrological space.

Proof. By Theorem 394, (X, cl) is a finite matroidal space. Now, let $A, B \subseteq X$ and $x \in X$ be such that

$$x \in \mathrm{cl}(A \cup B) \setminus [\mathrm{cl}(A) \cup \mathrm{cl}(B)].$$

We want to find a finite $F \subseteq A \cup B$ such that $x \in \mathrm{cl}(F)$. By Theorem 291, we have

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(A \cup B)\right) \setminus \left[\mathrm{cl}''\left(\mathrm{cl}'(A)\right) \cup \mathrm{cl}''\left(\mathrm{cl}'(B)\right)\right].$$

Let $F \subseteq A \cup B$ be a spanning set of $\mathrm{cl}'(A \cup B)$. Since X is finite-dimensional, F must be finite. So, we have

$$\mathrm{cl}'(F) = \mathrm{cl}'(A \cup B).$$

Therefore,

$$x \in \mathrm{cl}''\left(\mathrm{cl}'(F)\right) = \mathrm{cl}(F).$$

Hence (X, cl) is a matrological space. $\square$

Corollary 396. *Let X be a locally compact Hausdorff topological vector space over $\mathbb{F}$. Let cl' be the linear closure operator on X and cl'' be the topological closure operator on X. Define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ by*

$$cl(A) = \left(cl' \oplus cl''\right)(A) \text{ for all } A \subseteq X.$$

Then (X, cl) is a matrological space.

Proof. This is an immediate consequence of Theorems 294 and 395. $\square$

The following remark introduces a counterexample that shows there is an infinite-dimensional topological vector space over $\mathbb{F}$ that is not a matrological space. This example was constructed using an example in [17].

Remark 397. Let X be a topological vector space over $\mathbb{F}$. Let cl' be the linear closure operator on X and cl'' be the topological closure operator on X. Define $\mathrm{cl} : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\mathrm{cl}(A) = \left(\mathrm{cl}' \oplus \mathrm{cl}''\right)(A) \text{ for all } A \subseteq X.$$

Then (X, cl) might not be a matrological space.

Proof. Consider the Hilbert space $\langle \ell_2, \|\cdot\|_2 \rangle$ which is the set of all sequences $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{F}$ with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and the inner product on ℓ_2 is defined by

$$\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \sum_{n=1}^{\infty} \overline{x_n} y_n.$$

By Theorem 281, $\langle \ell_2, \|\cdot\|_2 \rangle$ is a Hausdorff topological vector space over $\mathbb{F}$. Let cl' be the linear closure operator on ℓ_2 and cl'' be the topological closure operator on ℓ_2. Consider

$$Y_1 = \left\{(x_n)_{n\in\mathbb{N}} \in \langle \ell_2, \|\cdot\|_2 \rangle : x_n = 0 \text{ for } n \text{ is odd}\right\}$$
$$Y_2 = \left\{(x_n)_{n\in\mathbb{N}} \in \langle \ell_2, \|\cdot\|_2 \rangle : x_{2n} = n x_{2n-1} \text{ for } n \geq 1\right\}$$

and $x_0 = \left(1, 0, \frac{1}{2}.0, \frac{1}{3}, 0, \ldots\right) \in \langle \ell_2, \|\cdot\|_2 \rangle$. Y_1 and Y_2 are closed vector subspaces of $\langle \ell_2, \|\cdot\|_2 \rangle$. By Theorem 291, we have

$$\mathrm{cl}(Y_1) = \mathrm{cl}''\left(\mathrm{cl}'(Y_1)\right) = Y_1$$
$$\mathrm{cl}(Y_2) = \mathrm{cl}''\left(\mathrm{cl}'(Y_2)\right) = Y_2.$$

By Theorems 74 and 77, $\mathrm{cl}'(Y_1 \cup Y_2) = Y_1 \oplus Y_2$. We claim that $Y_1 \oplus Y_2$ is dense in $\langle \ell_2, \|\cdot\|_2 \rangle$. To prove the claim, it suffices to show that the normed space $\langle c_{00}, \|\cdot\|_2 \rangle \subseteq Y_1 \oplus Y_2$, see Example 269. Let $x = (x_1, x_2, x_3, x_4, \ldots, x_k, 0, 0, \ldots) \in \langle c_{00}, \|\cdot\|_2 \rangle$ where $k \in \mathbb{N}$. The vectors

$$a = \left(0, x_2 - x_1, 0, x_4 - 2x_3, \ldots, 0, x_k - \frac{k}{2}x_{k-1}, 0, 0, \ldots\right) \in Y_1,$$
$$b = \left(x_1, x_1, x_3, 2x_3, \ldots, x_{k-1}, \frac{k}{2}x_{k-1}, 0, 0, \ldots\right) \in Y_2$$

are unique such that $x = a + b$. Then $x \in Y_1 \oplus Y_2$. Hence $\langle c_{00}, \|\cdot\|_2 \rangle \subseteq Y_1 \oplus Y_2$, and therefore $Y_1 \oplus Y_2$ is dense in $\langle \ell_2, \|\cdot\|_2 \rangle$. Thus,

$$\langle \ell_2, \|\cdot\|_2 \rangle = \mathrm{cl}''(Y_1 \oplus Y_2) = \mathrm{cl}''\left(\mathrm{cl}'(Y_1 \cup Y_2)\right).$$

So, we have

$$x_0 = \left(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\right) \in \mathrm{cl}''\left(\mathrm{cl}'(Y_1 \cup Y_2)\right) \setminus \left[\mathrm{cl}''\left(\mathrm{cl}'(Y_1)\right) \cup \mathrm{cl}''\left(\mathrm{cl}'(Y_2)\right)\right].$$

Let $F \subseteq Y_1 \cup Y_2$ be finite. So, $\mathrm{cl}'(F)$ is a finite-dimensional vector subspace of X. By Theorems 281 and 292, $\mathrm{cl}'(F)$ is closed. Now,

$$x_0 = \left(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\right) \notin Y_1 \oplus Y_2 = \mathrm{cl}'(Y_1 \cup Y_2).$$

By (CL2),

$$x_0 = \left(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\right) \notin \mathrm{cl}'(F) = \mathrm{cl}''\left(\mathrm{cl}'(F)\right).$$

Hence $(\langle \ell_2, \|\cdot\|_2 \rangle, \mathrm{cl})$ is not a matrological space where $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$. □

So, matrological spaces generalize finite-dimensional topological vector spaces but do not generalize topological vector spaces in general.

Chapter 6

Conclusions

In this research, finite matroidal spaces and matrological spaces were defined to generalize the known concept of a topological vector space. After establishing the basic knowledge about finite matroidal spaces and matrological spaces and studying their properties, the following conclusions have been achieved:

1. The dual of a finite matroidal space is also a finite matroidal space.
2. The restrictions and contractions of a finite matroidal space are finite matroidal spaces.
3. Basis and circuits may not exist in a finite matroidal space.
4. Every matroid is a finite matroidal space.
5. There is a finite matroidal space that is not a matroid.
6. Higgs proved that we only get the algebraic cycle matroid of a graph G if G does not contain a subgraph isomorphic to a subdivision of the Bean graph. Either G contains or does not contain a subgraph isomorphic to a subdivision of the Bean graph, we get a finite matroidal space.
7. Finite matroidal spaces differ from old spaces.
8. There is a matroid that is not a matrological space.
9. A matroid is a matrological space if and only if it is finitary.
10. Every matrological space is a finite matroidal space, but there is a finite matroidal space that is not a matrological space.

11. If the whole set is finite, then finite matroidal spaces, matrological spaces and matroids are all the same.

12. A topological space is matrological if and only if its topological closure operator has the exchange property.

13. The dual of a matrological space is not necessarily a matrological space.

14. The restrictions and contractions of a matrological space are matrological spaces.

15. A matrological space might not have basis and circuits.

16. If we combine a matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have $\mathrm{cl}(A) = \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$ for each $A \subseteq X$, then (X, cl) is a finite matroidal space not necessarily a matrological space.

17. If we combine a finitary matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have $\mathrm{cl}(A) = \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$ for each $A \subseteq X$, then (X, cl) is a matrological space.

18. If we combine a finitary matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have (X, cl) is a matrological space (hence it is a finite matroidal space), then the condition $\mathrm{cl}(A) = \mathrm{cl}'(A) \cup \mathrm{cl}''(A)$ for each $A \subseteq X$ might not hold.

19. If we combine a finitary matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$ and $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$, then (X, cl) is a matrological space.

20. If X is a finite set and we combine a matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$, then (X, cl) is a matroid.

21. If we combine a finitary matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have (X, cl) is a matrological space, then one of the conditions $\mathrm{cl} = \mathrm{cl}'' \circ \mathrm{cl}'$ or $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$ might fail.

22. If we combine a finitary matroidal closure operator cl' and a topological closure operator cl'' with the exchange property into a common closure operator cl on a set X and we have $\mathrm{cl} = \mathrm{cl}' \circ \mathrm{cl}''$ and $\mathrm{cl}'(A) \setminus \mathrm{cl}''(A)$ is finite for each $A \subseteq X$, then (X, cl) might not be a matrological space.

23. Topological vector spaces are finite matroidal spaces.

24. Any finite-dimensional topological vector space is a matrological space.

25. Any locally compact Hausdorff topological vector space is a matrological space.

26. There is an infinite-dimensional topological vector space that is not a matrological space.

Bibliography

[1] E. Aigner-Horev, J. Carmesin, and J.-O. Frohlich; Infinite Matroid Union, 2011, arXiv:1111.0602.

[2] E. Aigner-Horev, J. Carmesin, and J.-O. Frohlich; On the Intersection of Infinite Matroids, 2012, arXiv:1111.0606.

[3] S. Axler; Linear Algebra Done Right, 3rd Edition, Springer, (2015).

[4] V. Bogachev; O. Smolyanov; Topological Vector Spaces and their Applications, Springer (2017).

[5] F. S. Botelho; Functional Analysis, Calculus of Variations and Numerical Methods for Models in Physics and Engineering (1st ed.). CRC Press. (2020). https://doi.org/10.1201/9780429343315.

[6] N. Bowler and J. Carmesin, Matroids with an Infinite Circuit-cocircuit Intersection, J. Comb. Theory, Ser. B 107 (2014), no. 0, 78-91, arXiv:1202.3406.

[7] R. A, Brualdi; Comments on bases in dependence structures, Bulletin of the Australian Mathematical Society. 1 (2): 161–167. doi:10.1017/S000497270004140X. ISSN 1755-1633.

[8] R. A, Brualdi, E. B. Scrimger; Exchange Systems, matchings and transversals, J. Combinatorial Theory 5 (1968) 244-257.

[9] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, and P. Wollan, Axioms for Infinite Matroids, Advances in Mathematics 239 (2013), 18-46, arXiv:1003.3919.

[10] J. B. Conway, A Course in Functional Analysis, 2nd Edition, Springer-Verlag, (1990).

[11] R. Diestel, Graph Theory, 5th Edition, Springer-Verlag, (2016).

[12] S. H. Friedberg, A. J. Insel, and L. E. Spence; Linear Algebra, 5th Edition, Pearson, (2019).

[13] D.A. Higgs, Equicardinality of Bases in B-matroids, Canadian Mathematical Bulletin, 12:861- 862, (1969).

[14] D. A. Higgs, Infinite Graphs and Matroids, Recent Progress in Combinatorics, Proceedings Third Waterloo Conference on Combinatorics, Academic Press (1969), 245-253.

[15] D. A. Higgs, Matroids and Duality, Colloq. Math. 20 (1969), 215-220.

[16] N. Kashyap; E. Soljanin; P. Vontobel; Applications of Matroid Theory and Combinatorial Optimization to Information and Coding Theory. (2009).

[17] S.M. Khaleelulla; Counterexamples in Topological Vector Spaces (Lecture Notes in Mathematics, 936), Springer-Verlag, (1982).

[18] V. Klee, The Greedy Algorithm for Finitary and Cofinitary Matroids, Combinatorics, Proceeding of symposia in pure mathematics, Volume XIX, AMS, (1971), 137-152.

[19] B. Mendelson, Introduction to Topology, Dover Publications Inc., Third Edition, (1990).

[20] L. Mirsky; Transversal Theory: An Account of Some Aspects of Combinatorial Mathematics, Academic Press Inc., Vol. 75, 1st Edition, (1971).

[21] J. R. Munkres; Topology, 2nd Edition, (2000).

[22] L. Narici; E. Beckenstein; Topological Vector Spaces. Pure and applied mathematics (Second ed.). Boca Raton, FL: CRC Press. ISBN 978-1584888666. OCLC 144216834. (2011).

[23] D. L. Neel; N. A. Neudauer; Matroids You Have Know. Mathematics Magazine. 82 (1): 26–41. doi:10.4169/193009809x469020. (2009).

[24] J. Oxley; Infinite matroids, Proc. London Math. Soc. 37 (1978), no. 3, 259-272.

[25] J. Oxley. Infinite Matroids. In N. White, editor, Matroid Applications, volume 40 of Encycl. Math. Appl., pages 73-90. Cambridge University Press, (1992).

[26] J. Oxley, Matroid theory, 2nd Edition, Oxford University Press (2006).

[27] R. Rado, Abstract linear dependence, Colloq. Math. 14 (1966), 257–264.

[28] M. Reed; B. Simon; Functional Analysis (Methods of Modern Mathematical Physics (Volume 1)), Academic Press, (1980).

[29] A. P. Robertson, W. Robertson, Topological Vector Spaces, 2nd Edition, (1964).

[30] W. Rudin, Functional Analysis, 2nd Edition, McGraw-Hill, (1973).

[31] W. Sierpinski, General Topology, Oxford University Press, (1956).

[32] L. A. Steen, J. Seebach Jr, Counterexamples in Topology; Dover Publications Inc., (1978).

[33] W.T. Tutte, Introduction to the Theory of Matroids, Elsevier, (1971).

[34] D. J. A. Welsh, Matroid Theory, Dover Publications Inc., (1976).

[35] D. B. West, Introduction to Graph Theory, 2nd Edition, Prentice-Hall, Inc., (2001).

[36] N. White, Matroid Applications, Cambridge University Press, (1992).

[37] N. White, Theory of Matroids, 1st Edition, Cambridge University Press, (2008).

[38] H. Whitney, On the abstract properties of linear dependence, Amer. J. Math. 57 (1935) no. 3, 509–533.

[39] S. Willard, General Topology, 1st Edition, Addison-Wesley Company, (1970).

[40] J. Wojciechowski, Introduction to Infinite Combinatorics Notes, Fall 2017.

List of Figures