Exploring the difference hierarchies on μ-calculus and arithmetic---From the point of view of Gale-Stewart games

著者	Kawakami Pacheco Leonardo
number	99
学位授与機関	Tohoku University
学位授与番号	理博第3440号
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Exploring the difference hierarchies on μ-calculus and arithmetic: from the point of view of Gale–Stewart games (Summary of PhD thesis)

C0SD1005 Leonardo Kawakami Pacheco

In this thesis, we study two problems related to difference hierarchies. The difference hierarchy for a point class Γ classifies the Boolean combinations of sets in Γ by their complexity. Gale–Stewart games play essential roles in both problems.

In the first part of this thesis, we study the μ-calculus' alternation hierarchy over various semantics. The μ-calculus is obtained by adding least and greatest fixed point operators to modal logic. In general, it is much more expressive than modal logic. While modal logic only allows us to express 'local' properties, the μ-calculus allows us to express 'global' properties. For example, if we use fixed-points, we can write a formula expressing that some statement is common knowledge; this is not possible in modal logic without fixed points. One can also think of fixed-point formulas as abbreviations for infinitary formulas.

The μ-calculus' alternation hierarchy classifies its formulas by how many interdependent fixed-point operators appear in a given formula. This measure is called alternation depth. Bradfield [2] showed that the alternation hierarchy is strict, that is, for all $n \in \mathbb{N}$, there is a μ-formula with alternation depth n which is not equivalent to any μ-formula with a smaller alternation depth. This may not happen if we modify the semantics.

The μ-formulas are usually interpreted over Kripke models, labelled directed graphs. Alberucci and Facchini [1] showed that, if we restrict the μ-calculus to transitive frames, the alternation hierarchy collapses to its alternation-free fragment; that is, every μ-formula is equivalent to a formula with no entangled fixed-point operator. Similarly, they showed that over equivalence relations, the alternation hierarchy collapses to modal logic; that is, every μ-formula is equivalent to a modal formula without fixed-point operators.

We refine Alberucci and Facchini's proof to show that the alternation hierarchy collapses to modal logic in bigger classes of frames.

Theorem 1. *Let $\mathcal{F}$ be a class of Kripke frames. Suppose there is n such that, for all frame $F = \langle W, R \rangle \in \mathcal{F}$ and for all sequence $w_0 R^* w_1 R^* \cdots R^* w_n$, there is $i < j \leq n$ such that $w_i R = w_j R$. The μ-calculus' alternation hierarchy collapses to modal logic over $\mathcal{F}$.*

We use this characterization to study the epistemic logics S4.2, S4.3, S4.3.2, and S4.4.

Theorem 2. *The alternation hierarchy collapses to its alternation-free fragment over* S4.2 *and* S4.3*. Furthermore, the alternation hierarchy does not collapse to modal logic over* S4.2 *and* S4.3*.*

Theorem 3. *The alternation hierarchy collapses to modal logic over* S4.3.2*,* S4.4*, and* KD45*.*

We define degrees of ignorance using the μ-calculus.

We show that different logics imply the possibility of a different number of degrees of ignorance. Afterwards, we study the collapse to alternative semantics for modal logic.

Theorem 4. *The alternation hierarchy collapses to modal logic over non-normal equivalence relations.*

Theorem 5. *The graded μ-calculus' alternation hierarchy collapses to graded modal logic over equivalence relations.*

Theorem 6. *Over models of* IS5*, every μ-formula is equivalent to a modal formula.*

On the other hand, the alternation hierarchy is strict on multimodal μ-calculus over equivalence relations.

Theorem 7. *Therefore the alternation hierarchy is strict over multimodal* S5*.*

We also show that current proofs of the collapse do not work on the non-monotone μ-calculus.

Furthermore, we show that the alternation hierarchy collapses to its alternation-free fragment over weakly transitive frames and related classes.

Theorem 8. *The alternation hierarchy collapses to the alternation-free fragment over weakly transitive frames.*

Theorem 9. *Given $n \in \omega$, the alternation hierarchy collapses to the alternation-free fragment over frames frames where $\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^n(\bot)$.*

We then use a finite model property to extend the collapse to derivative topological semantics.

Theorem 10. *The alternation hierarchy collapses to its alternation-free fragment on derivative topological semantics.*

Here, we interpret μ-formulas over topologies and interpret the $\Diamond$ modality as the Cantor derivative.

In the second part of this thesis, we study the connection between Gale–Stewart games and reflection principles in second-order arithmetic. In the Gale–Stewart game with payoff $A \subseteq \omega^\omega$, two players alternate picking natural numbers to build an infinite sequence α; the first player wins the game iff $\alpha \in A$. Gale–Stewart games have been studied in reverse mathematics since its beginning and are central to descriptive set theory. Sets definable by the μ-calculus are exactly the winning regions of Gale–Stewart games whose payoffs are Boolean combinations of Σ^0_2 sets.

We study (syntactical) reflection principles of the form Π^1_n-$\mathsf{Ref}(\Gamma)$ stating that every Π^1_n-formula provable in Γ is true. These reflection principles can be thought as strengthenings of the consistency of Γ. Heinatsch and Möllerfeld showed that a formalized version of the μ-calculus is equivalent to the determinacy of Boolean combinations of Σ^0_2 sets. In turn, Kołodziejczyk and Michalewski [3] used this result to prove that the determinacy of Boolean combinations of Σ^0_2 sets is equivalent to the reflection principle $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_2\text{-}\mathsf{CA}_0)$.

Now, the alternation-free fragment of the μ-calculus defines the winning regions of Gale–Stewart games whose payoffs are Boolean combinations of Σ^0_1 sets. Furthermore, the formalized alternation-free μ-calculus on second-order arithmetic is closely related to $\Pi^1_1\text{-}\mathsf{CA}_0$. This fact suggests a variation for the result above: the determinacy of Boolean combinations of Σ^0_1 sets is equivalent to the reflection principle $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0)$.

Theorem 11. *Over* ACA_0*:*

1. ACA_0', $\forall n.(\Sigma^0_1)_n\text{-}\mathsf{Det}$, $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0)$, *and* $\Pi^1_1\text{-}\mathsf{CA}_0'$ *are pairwise equivalent.*
2. $\forall n.(\Sigma^0_1)_n\text{-}\mathsf{Det}^*$, $\Pi^1_2\text{-}\mathsf{Ref}(\mathsf{ACA}_0)$, *and* ACA_0' *are pairwise equivalent.*
3. $\forall n.(\Sigma^0_2)_n\text{-}\mathsf{Det}$, $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_2\text{-}\mathsf{CA}_0)$, *and* $\forall n.[\Sigma^1_1]^n\text{-}\mathsf{ID}$ *are pairwise equivalent.*

We prove these results using finite sequences of coded β-models of arbitrary length.

References

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[2] Julian C. Bradfield. "The Modal Mu-Calculus Alternation Hierarchy Is Strict." In: *Theoretical Computer Science* 195.2 (1998), pp. 133–153.

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別　紙

論文審査の結果の要旨

本博士論文では，二人無限完全情報ゲームにおける必勝戦略の存在を主張する「決定性公理」の視点を軸として，

I.　知識の伝播を理論計算機科学の立場で分析する「様相論理」

II.　数学基礎論の立場から公理体系の強さ・無矛盾性を表現する「反映原理」

の二つのテーマが扱われている．決定性公理は通常の集合論で証明不可能ではある一方，実数の任意の部分集合のルベーグ可測性を保証するなど非常に強力な公理であり，これを適切な強さに制限して扱うことで，多様な問題の分析が行われている．

Ｉのテーマは主として論文の 3-5 章で，知識の伝播を表現する様相論理式のクリプキモデル(伝播を表現する半順序構造)における真偽を弱い決定性公理により適切に決定する手法を一般化し，共通知を表現する知識伝播の不動点演算子を導入した様相論理体系の表現力の分析が行われている．特に Alberucci/Facchini のアイデアを多様な条件下に一般化できることを示したことで，これまで特定の形状のクリプキモデルにおいてのみ知られていた様相不動点演算子階層の崩壊定理/非崩壊性を様々な条件下で示している．

Ⅱのテーマは論文の 8 章で主に扱われており，決定性公理の階層と集合存在公理の反映原理の階層間の等価性を明らかにした．特に Kołodziejczyk/Michalewski によるΠ^1_2-内包公理のΠ^1_3-反映原理とΣ^0_2-交代階層の決定性公理の等価性に着目し，この結果をより良い条件下で再証明すると共に同種の定理が他の交代階層でも成立することが示されている．この方向の決定性公理の研究は直近でも多くの研究者が注目しており，より高いボレル階層での決定性公理の強さを証明論的に分析するウイーン工科大学の研究プロジェクトが進められており，今後 Pacheco 氏もこれに参画して更なる研究の進展を目指していく．

またＩ,Ⅱの研究は深いところで結びついており，実はμ計算で表現される集合の複雑さはその真偽を決定するゲームの決定性の強さに依存している．これについては７章で分析が行われている，

いずれの研究成果も集合論・証明論・理論計算機科学の巧みに組み合わせ，既知の手法の鍵を見抜いて一般化する洞察力が見事で，研究の注目度・今後の発展性も高い．以上のとおり論文提出者は自立して研究活動を行うに必要な高度の研究能力と学識を有することを示している．したがって，Pacheco 氏提出の博士論文は，博士（理学）の学位論文として合格と認める．