Exploring the difference hierarchies on μ-calculus and arithmetic---From the point of view of Gale-Stewart games

EXPLORING THE DIFFERENCE HIERARCHIES ON μ-CALCULUS AND ARITHMETIC

FROM THE POINT OF VIEW OF GALE–STEWART GAMES

A Thesis presented by

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Abstract

In this thesis, we study two problems related to difference hierarchies. The difference hierarchy for a point class Γ classifies the Boolean combinations of sets in Γ by their complexity. Gale–Stewart games play essential roles in both problems.

In the first part of this thesis, we study the μ-calculus' alternation hierarchy over various semantics. The μ-calculus is obtained by adding least and greatest fixed-point operators to modal logic. In general, it is much more expressive than modal logic. While modal logic only allows us to express 'local' properties, the μ-calculus allows us to express 'global' properties. For example, if we use fixed-points, we can write a formula expressing that some statement is common knowledge; this is not possible in modal logic without fixed-points. One can also think of fixed-point formulas as abbreviations for infinitary formulas.

The μ-calculus' alternation hierarchy classifies its formulas by how many interdependent fixed-point operators appear in a given formula. This measure is called alternation depth. Bradfield[1] showed that the alternation hierarchy is strict, that is, for all $n \in \mathbb{N}$, there is a μ-formula with alternation depth n which is not equivalent to any μ-formula with a smaller alternation depth. This may not happen if we modify the semantics.

The μ-formulas are usually interpreted over Kripke models, labeled directed graphs. Alberucci and Facchini[2] showed that, if we restrict the μ-calculus to transitive frames, the alternation hierarchy collapses to its alternation-free fragment; that is, every μ-formula is equivalent to a formula with no entangled fixed-point operator. Similarly, they showed that over equivalence relations, the alternation hierarchy collapses to modal logic; that is, every μ-formula is equivalent to a modal formula without fixed-point operators.

We refine Alberucci and Facchini's proof to show that the alternation hierarchy collapses to modal logic in bigger classes of frames. We use this characterization to study various epistemic logics. We define degrees of ignorance and show that different logics imply the possibility of a different number of degrees of ignorance.

Afterwards, we study the collapse to alternative semantics for modal logic. We show that, on graded semantics, constructive semantics and modal logic with impossible worlds, the alternation hierarchy collapses to modal logic over equivalence relations. On the other hand, the alternation hierarchy is strict on multimodal μ-calculus over equivalence relations. We also show that current proofs of the collapse do not work on the non-monotone μ-calculus.

[1] J.C. Bradfield. *"The modal mu-calculus alternation hierarchy is strict"*. In: Theoretical Computer Science 195.2 (1998), pp. 133– 153.

[2] L. Alberucci and A. Facchini. *"The modal μ-calculus hierarchy over restricted classes of transition systems"*. In: The Journal of Symbolic Logic 74.4 (2009), pp. 1367–1400.

Furthermore, we show that the alternation hierarchy collapses to its alternation-free fragment over weakly transitive frames. We then use a finite model property to extend the collapse to derivative topological semantics. Here, we interpret μ-formulas over topologies and interpret the $\Diamond$ modality as the Cantor derivative. At last, we show that the weak alternation hierarchy is strict over transitive frames. The weak alternation hierarchy classifies alternation-free μ-formulas by how many nested quantifiers they contain.

In the second part of this thesis, we study the connection between Gale–Stewart games and reflection principles in second-order arithmetic. In the Gale–Stewart game with payoff $A \subseteq \omega^\omega$, two players alternate picking natural numbers to build an infinite sequence α; the first player wins the game iff $\alpha \in A$. Gale–Stewart games have been studied in reverse mathematics since its beginning and are central to descriptive set theory. Sets definable by the μ-calculus are exactly the winning regions of Gale–Stewart games whose payoffs are Boolean combinations of Σ^0_2 sets.

We study (syntactical) reflection principles of the form $\Pi^1_n\text{-}\mathsf{Ref}(\Gamma)$ stating that every Π^1_n-formula provable in Γ is true. These reflection principles can be thought as strengthenings of the consistency of Γ. Heinatsch and Möllerfeld showed that a formalized version of the μ-calculus is equivalent to the determinacy of Boolean combinations of Σ^0_2 sets. In turn, Kołodziejczyk and Michalewski[3] used this result to prove that the determinacy of Boolean combinations of Σ^0_2 sets is equivalent to the reflection principle $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_2\text{-}\mathsf{CA}_0)$.

Now, the alternation-free fragment of the μ-calculus defines the winning regions of Gale–Stewart games whose payoffs are Boolean combinations of Σ^0_1 sets. Furthermore, the formalized alternation-free μ-calculus on second-order arithmetic is closely related to $\Pi^1_1\text{-}\mathsf{CA}_0$. This fact suggests a variation for the result above: the determinacy of Boolean combinations of Σ^0_1 sets is equivalent to the reflection principle $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0)$. We prove this result using finite sequences of coded β-models of arbitrary length.

We also use the methods above to give a new proof of Kołodziejczyk and Michalewski's result. We also modify it to prove that the determinacy of Boolean combinations of Σ^0_1 sets of Cantor space is equivalent to the reflection principle $\Pi^1_2\text{-}\mathsf{Ref}(\mathsf{ACA}_0)$.

[3] L. A. Kołodziejczyk and H. Michalewski. *"How unprovable is Rabin's decidability theorem?* In: 2016 31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). July 2016, pp. 1–10.

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Chapter 1

Introduction

We study two questions related to difference hierarchies. First, we study the collapse of μ-calculus' alternation hierarchy over various semantics. Then, we study the connection between determinacy axioms and reflection principles in second-order arithmetic.

Difference hierarchies classify the complexity of boolean combinations of sets. For example, we consider the difference hierarchy for open sets. The first level of the difference hierarchy consists of the open sets themselves; the second level consists of differences $A \setminus B$ of two open sets; the third level consists of differences $A \setminus (B \setminus C)$ of three open sets; and so on. The boolean combinations of open sets are the sets obtained by finite conjunction, finite disjunction, and complementation starting from the open sets. Every boolean combination of open sets can be written as a difference of open sets using application of de Morgan's rule.

We will focus on difference hierarchies for Σ^0_1 and Σ^0_2 sets of the Baire space. In this setting, there is a deep connection between determinacy of Gale–Stewart games, the modal μ-calculus and reflection principles.

Fix a set X and set $A \subseteq X^\omega$. In the Gale–Stewart game $\mathcal{G}(A)$, two players—I and II—alternate picking elements of X to form a sequence $\alpha \in X^\omega$. Such a sequence is called a run. We call A the payoff of $\mathcal{G}(A)$. Player I wins a run α of $\mathcal{G}(A)$ iff $\alpha \in A$; II wins otherwise. We say $\mathcal{G}(A)$ is determined iff one of the players has a winning strategy. The axiom of determinacy states that all games are determined. While the axiom of determinacy contradicts the axiom of choice, weaker versions of determinacy are compatible with choice. In this thesis, we will study only cases where $X = \{0, 1\}$ and $X = \omega$.

After a chain of results by Gale and Stewart [GS53], Wolfe [Wol55], Davis [Dav64], and Paris [Par72], Martin [Mar75] proved Borel determinacy over ZFC—all games whose payoffs are Borel sets are determined. These results on determinacy have also been formalized in the setting of second-order arithmetic. This leads us to talk about reverse mathematics.

In reverse mathematics, we want to classify the logical strength of theorems of ordinary mathematics. Determinacy axioms have been a mainstay of reverse mathematics since its beginning. Steel [Ste77] proved that the determinacy of open sets is equivalent to the axiom system known as ATR_0. Tanaka [Tan90] proved that the determinacy of differences of open sets is equivalent to Π^1_1-CA_0. He also proved that Π^1_1-TR_0 is equivalent to the determinacy of Δ^0_2 sets—this proof depends on the transfinite levels of the difference hierarchy. In [Tan91], Tanaka proved that the determinacy of Σ^0_2-sets is equivalent to Σ^1_1-MI. Then, MedSalem and Tanaka [MT07] proved that the determinacy for differences of n many Σ^0_2 sets is equivalent to $[\Sigma^1_1]^n$-ID. Therefore, the difference hierarchies for Σ^0_1 and Σ^0_2 induce a hierarchy of determinacy axioms in second-order arithmetic.

Kołodziejczyk and Michalewski [KM16] proved that the determinacy of differences of arbitrarily many Σ^0_2 sets is equivalent to the reflection principle for Π^1_3-formulas provable from Π^1_2-CA_0. This reflection principle states that all Π^1_3-formulas provable in Π^1_2-CA_0 are true. Their proof depended on a result by Heinatsch and Möllerfeld [HM10] relating the determinacy axiom above to a formalized version of the μ-calculus.

The μ-calculus is obtained by adding least and greatest fixed-point operators to modal logic. It was first studied by Kozen [Koz83]. Modal logic extends propositional logic with modal operators $\Box$ and $\Diamond$. Given a formula φ, $\Box\varphi$ is read as "φ is necessary" and $\Diamond\varphi$ is read as "φ is possible". The precise meaning of "necessary" and "possible" will depend on the semantics under consideration.

The μ-calculus is much more expressive than modal logic. The fixed-point operators μ and ν allow us to describe properties which otherwise would require infinitary modal formulas. We give an example from epistemic logic—the modal logic of knowledge. Read $\Box\varphi$ as "everyone knows that φ". Then φ is common knowledge iff φ is true, $\Box\varphi$ is true, $\Box\Box\varphi$ is true, and so on. While common knowledge of φ cannot be defined by a finitary modal formula, it can be defined by the μ-formula $\nu X.\varphi \wedge \Box X$.

Modal and μ-formulas are interpreted over Kripke models—labeled directed graphs. We interpret the nodes as possible worlds, each world being a possible state of affairs. Relation between the worlds are described by the edges of the graph. While we can consider the semantics over all Kripke models, it is also common to consider restricted classes of graphs. These restrictions are natural in modal logic, and represent different interpretations for necessity and possibility. For example, if we restrict ourselves to reflexive Kripke models, then necessity implies truth and truth implies possibility.

The μ-calculus' alternation hierarchy classifies the μ-formulas according to the entanglement of its fixed-point operators. For example, the fixed-point operators in the μ-formula $\nu X.(\mu Y.P \vee \Diamond Y) \wedge \Box X$ are not entangled. On the other hand, the fixed-point operators in $\nu X \mu Y((P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y))$ are entangled. In the second formula, the valuation of the inner fixed-point operator μY depends on the outer operator νX, as the variable X is in the scope of μY.

Bradfield, Duparc and Quickert [BDQ05; BDQ10] proved that the μ-calculus defines the winning regions of Gale–Stewart games whose payoffs are differences of Σ^0_2 sets. A μ-formula is alternation-free iff its least and greatest fixed-point operators are not entangled. The author, Li and Tanaka [PLT22] proved that the alternation-free μ-calculus defines the winning regions of Gale–Stewart games whose payoffs are differences of Σ^0_1 sets. These proofs also describe a close relation between the levels of the difference hierarchies for Σ^0_1 and Σ^0_2 sets and the levels of the μ-calculus' alternation hierarchy.

We now turn our focus on the μ-calculus' alternation hierarchy. The alternation depth of a μ-formula φ is a natural number measuring the entanglement between least and greatest fixed-point operators in φ. Bradfield [Bra98b] proved that the alternation hierarchy is strict over all Kripke models. That is, for all natural number n, there is a formula with alternation depth $n + 1$ which is not equivalent to any formula with alternation depth n.

Alberucci and Facchini [AF09b] proved that the alternation hierarchy collapses to its alternation-free fragment over transitive Kripke models—every μ-formula is equivalent to a formula with alternation depth 1. They also proved that the alternation hierarchy collapses to modal logic over equivalence relations—every μ-formula is equivalent to a modal formula without fixed-point operators. The alternation hierarchy has also been studied in other traditional classes of models, but there are still few results outside these classes.

In the proofs relating to the alternation hierarchy's strictness or collapse, we use evaluation games. As their name indicates, evaluation games are used to evaluate μ-formulas. They are also essential to understand the semantics of the μ-calculus. Evaluation games are included in a special class of Gale–Stewart games called parity games. Formulas defining the winning region of parity games are witnesses for the alternation hierarchy's strictness over all models. Evaluation games are also used to prove the collapse to modal logic over equivalence relations. They also give a clearer view of the collapse to the alternation-free fragment over transitive frames.

Outline of this thesis

This thesis consists of two parts.

Part I is a contribution to modal logic. We study the μ-calculus' alternation hierarchy on various settings. We extend the existing results on the alternation hierarchy in two directions. First, we extend the collapse results to new classes of Kripke models. Second, we show that the collapse to modal logic also holds over some alternative semantics. In this direction, we also show that the alternation hierarchy is strict over multimodal Kripke models where all the accessibility relations are equivalence relations. This is work towards expanding the knowledge from where the alternation hierarchy collapses to why the collapses happen.

Part II is a contribution to reverse mathematics in second-order arithmetic. We study the relation between determinacy axioms and reflection principles. Our work builds on existing work on the reverse mathematics of determinacy. We show that the determinacy of boolean combination of Σ^0_1 and Σ^0_2 sets is closely related to reflection principles. While reflection principles are well studied in first-order arithmetic, not much is known about them in the setting of second-order arithmetic.

Part I

Chapter 2 contains preliminary definitions and results for Part I. We first review basic definitions of modal logic. We then define the μ-calculus and its alternation hierarchy. We also define the μ-calculus game semantics and parity games.

In Chapter 3, we extend the proof of the collapse to modal logic to more general classes of models. We apply our results to the modal logics S4.2, S4.3, S4.3.2, and S4.4. These logics are studied in epistemic logic. We show that the alternation hierarchy collapses to the alternation-free fragment over S4.2 and S4.3, and to modal logic over S4.3.2 and S4.4. We use the results on the collapse to study the difference between these logics with respect to degrees of ignorance.

In Chapter 4, we study the alternation hierarchy over equivalence relations using alternative semantics. We show that the alternation hierarchy collapses to modal logic over non-normal, graded and intuitionistic semantics. We then show that the alternation hierarchy is strict over multimodal semantics. At last, we show that our proof of the collapse does not work for the inflationary μ-calculus. This chapter depends on Section 3.1.

In Chapter 5, we study the alternation hierarchy over weakly transitive frames (and related classes of frames). We show that the alternation hierarchy collapses to its alternation-free fragment over weakly transitive frames. We use this result to show that the alternation hierarchy collapses to its alternation-free fragment over derivative topological semantics.

Part II

Chapter 6 contains preliminary definitions and theorems for Part II. We first review some basic definitions used for reverse mathematics in second-order arithmetic. In the last three sections, we define reflection principles, determinacy axioms, and axioms for inductive definitions.

In Chapter 7, we study the μ-arithmetic, a logic obtained by adding least and greatest fixed-points to first-order arithmetic. We define the μ-arithmetic and explain its relation to the μ-calculus. We also relate the alternation-free versions of μ-arithmetic and μ-calculus. We then comment on the relation between μ-arithmetic and Gale–Steward games. At last, we define a version of the μ-arithmetic formalized in second-order arithmetic and comment on results relating it to determinacy axioms and reflection principles. This chapter depends on Sections 2.2 and 2.3.

In Chapter 8, we prove various theorems relating determinacy axioms and reflection principles. We improve Kołodziejczyk and Michalewski's result by showing that, over ACA_0, the determinacy of differences of arbitrarily many Σ^0_2 sets is equivalent to the reflection principle for Π^1_3-formulas provable on Π^1_2-CA_0. We also prove that the determinacy of differences of arbitrarily many Σ^0_1 sets of Baire space is equivalent to the reflection principle for Π^1_3-formulas provable from Π^1_1-CA_0; and that the determinacy of differences of arbitrarily many Σ^0_1 sets of Cantor space is equivalent to the reflection principle for Π^1_2-formulas provable from ACA_0.

Part I

Characterizing the collapse of the alternation hierarchy

Chapter 2

The μ-calculus

In this chapter we define the μ-calculus, its semantics and present its alternation hierarchy. In section 2.1, we review some basic concepts of modal logic. In section 2.2, we define the μ-calculus' syntax and semantics; we also work out some examples. In section 2.3, we define the μ-calculus' alternation hierarchy and comment on existing results. In section 2.4, we define game semantics for the μ-calculus, and explain its relation to parity games.

2.1 Modal logic

In this section, we sketch the syntax and semantics of modal logic, while commenting on some key properties and applications. This is meant as a warm-up for the μ-calculus. While basic modal logic is essential for this thesis, it is not the main focus. I leave here a few recommendations for the reader who wants more. The standard textbook for modal logic is Blackburn *et al.* [BdV01]. The Handbook of Modal Logic [BvW07] is also a good reference, in particular its first article [Bv07]. A more recent textbook with pointers to many applications is van Benthem [van10]. Chellas [Che80] is a little older, it has many exercises. Also see Priest [Pri08] for alternative semantics for modal logic.

Modal logic is obtained by adding the modal operators $\Box$ (read as "box") and $\Diamond$ (read as "diamond") to propositional logic. One way to read the formulas $\Box\varphi$ and $\Diamond\varphi$ is, respectively, "φ is necessary" and "φ is possible". There are other alternative readings, some of which we will see later. In Section 3.4, we study epistemic logic, and read $\Box\varphi$ as "the agent knows that φ is true". In Section 5.3, we study topological semantics and read $\Diamond\varphi$ as "the Cantor derivative of φ" in topological semantics. Yet more interpretations of the modalities are in Chapter 4.

THE MODAL FORMULAS. Fix a set Prop of propositional symbols. The modal language $\mathcal{L}_{\mathrm{ML}}$ contains the symbols in Prop; the logical constants $\bot$ and $\top$; the logical connectives $\neg$, $\wedge$, $\vee$, $\to$; and the modal operators $\Box$ and $\Diamond$. The modal formulas are defined by the following grammar:

$$\varphi := P \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi,$$

where $P \in \mathrm{Prop}$. We use parenthesis to disambiguate formulas, when necessary. We sometimes use $\triangle$ to denote either $\Box$ or $\Diamond$.

KRIPKE MODELS. The standard semantics for modal logics is given by Kripke models. A Kripke model is a triple $M = \langle W, R, V\rangle$ consisting of:

- W, a non-empty set;
- $R \subset W \times W$, a binary relation on W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$, a function from propositional symbols to subsets of W.

The elements of W are called possible worlds, and W itself is called the set of possible worlds. R is called the accessibility relation; when wRv we say that v is accessible from w. V is called the valuation function, and assigns to each propositional symbol P the set of worlds where P is true. If $w \in W$, then we call (M, w) a pointed Kripke model. The set of worlds accessible from w is denoted by $wR := \{w' \in W \mid wRw'\}$. Denote the transitive closure of R by R^*.

Example 1. The following triples are Kripke models:

- $M_0 = \langle W_0, R_0, V_0\rangle$ with $W_0 = \{w, u, v\}$, $R_0 = \{\langle w, u\rangle, \langle w, v\rangle, \langle v, w\rangle\}$, and $V_0(P) = \{u, v\}$.
- $M_1 = \langle W_1, R_1, V_1\rangle$ with $W_1 = \{r, s, t\}$, $R_1 = \{\langle r, t\rangle, \langle t, r\rangle, \langle s, s\rangle\}$, and $V_1(P) = \{s, t\}$.

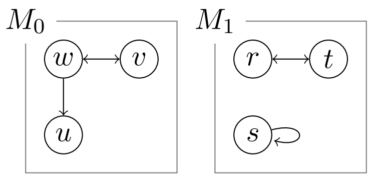


Figure 2.1: The models M_0 and M_1 from Example 1.

Kripke models are also known as transition systems. The elements of W and R are called states and transitions, respectively. Yet another name for Kripke models is labeled directed graphs. In this case the elements of W and R are called nodes and edges, respectively; and V is called a labeling function on W.

KRIPKE SEMANTICS. Fix a Kripke model $M = \langle W, R, V\rangle$. We define the valuation of modal formulas by induction on the structure of the formulas:

- $\|P\|^M = V(P)$;
- $\|\bot\|^M = \emptyset$;
- $\|\top\|^M = W$;
- $\|\neg\varphi\|^M = W \setminus \|\varphi\|^M$;
- $\|\varphi \wedge \psi\|^M = \|\varphi\|^M \cap \|\psi\|^M$;
- $\|\varphi \vee \psi\|^M = \|\varphi\|^M \cup \|\psi\|^M$;
- $\|\varphi \to \psi\|^M = \|\neg\varphi\|^M \cup \|\psi\|^M$;

- $\|\Box\varphi\|^M = \{w \in W \mid \forall v.wRv \to v \in \|\varphi\|^M\}$; and
- $\|\Diamond\varphi\|^M = \{w \in W \mid \exists v.wRv \wedge v \in \|\varphi\|^M\}$;

Given $w \in W$, we write $M, w \models \varphi$ when $w \in \|\varphi\|^M$. When $M, w \models \varphi$ we say that φ is true at w, or that (M, w) satisfies φ. Define $M \models \varphi$ to hold iff $M, w \models \varphi$ for all $w \in W$. We say φ is valid iff $M \models \varphi$ holds for all Kripke models M. If φ is valid, we write $\models \varphi$. If L is a set of modal formulas, then $M \models \mathsf{L}$ iff $M \models \varphi$ for all $\varphi \in \mathsf{L}$.

Example 2. Consider the models from Example 1.

- Let $M_0 = \langle W_0, R_0, V_0\rangle$ with $W_0 = \{w, u, v\}$, $R_0 = \{\langle w, u\rangle, \langle w, v\rangle, \langle v, w\rangle\}$, and $V_0(P) = \{u, v\}$. Then
  - $\|\Box P\|^{M_0} = \{w, u\}$,
  - $\|\Diamond P\|^{M_0} = \{w\}$,
  - $\|\Diamond\Box P\|^{M_0} = \{v\}$, and
  - $\|\Box\Diamond P\|^{M_0} = \{u, v\}$.
- $M_1 = \langle W_1, R_1, V_1\rangle$ with $W_1 = \{r, s, t\}$, $R_1 = \{\langle r, t\rangle, \langle t, r\rangle, \langle s, s\rangle\}$, and $V_1(P) = \{s, t\}$. Then
  - $\|\Box P\|^{M_1} = \{r, s\}$,
  - $\|\Diamond P\|^{M_1} = \{r, s\}$,
  - $\|\Diamond\Box P\|^{M_1} = \{s, t\}$, and
  - $\|\Box\Diamond P\|^{M_1} = \{s, t\}$.

Do note that some of the symbols in $\mathcal{L}_{\mathrm{ML}}$ are superfluous. For example, we may work with only $\neg$, $\wedge$ and $\Box$, as

- $M, w \models \bot$ iff $M, w \models P \wedge \neg P$;
- $M, w \models \top$ iff $M, w \models P \vee \neg P$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \neg(\neg\varphi \wedge \neg\psi)$;
- $M, w \models \varphi \to \psi$ iff $M, w \models \neg\varphi \vee \psi$; and
- $M, w \models \Diamond\varphi$ iff $M, w \models \neg\Box\neg\varphi$.

As the restricted language is as expressive as the full modal language, we use whichever is most convenient in a given moment.

FRAME CORRESPONDENCE. A Kripke frame F is a pair $\langle W, R\rangle$ consisting of a set of possible worlds and an accessibility relation. That is, a Kripke frame is a Kripke model without a valuation function. Alternatively, a Kripke model is a Kripke frame with a valuation function. In case if $M = \langle W, R, V\rangle$ and $F = \langle W, R\rangle$, we say that the model M extends the frame F. We say a Kripke frame F satisfies the formula φ iff $M \models \varphi$ for all model M extending F. If F satisfies φ, we write $F \models \varphi$.

Any formula φ defines a class of frames $\{F \mid F \models \varphi\}$. Some axioms and the class of frames they define are listed in Table 2.1.

Example 3. The formula $\Box P \to \Box\Box P$ defines the class of transitive frames. That is, for any frame $F = \langle W, R\rangle$, $F \models \Box P \to \Box\Box P$ iff R is a transitive relation on W.

First, suppose that F is transitive. Let $M = \langle W, R, V\rangle$ be a model extending F and $w \in W$ be such that $M, w \models \Box P$. Let v be a world accessible from w, then any world u accessible from v is also accessible from w. Since $M, w \models \Box P$, then $M, u \models P$. Since this holds for arbitrary u, $M, v \models \Box P$. Similarly, $M, w \models \Box\Box P$ and $M \models \Box\Box P$. Since this holds for any M extending F, we conclude $F \models \Box P \to \Box\Box P$.

Now, suppose $F \models \Box P \to \Box\Box P$. Let $w, v, u \in W$ be such that wRv and vRu. Let M be obtained by adding to F the valuation $V(P) = \{w' \mid wRw'\}$. Then $M, w \models \Box P$. By our hypothesis on F, $M, w \models \Box\Box P$. Therefore $M, v \models \Box P$ and $M, u \models P$. By the definition of V, we have that $u \in \|P\|^M = \{w' \mid wRw'\}$. That is, wRu. We conclude that F is transitive.

Name	Axiom	Property	Frame condition
4	$\Box P \to \Box\Box P$	transitive	$\forall w, v, u.wRv \wedge vRu \to wRu$
T	$\Box P \to P$	reflexive	$\forall w.wRw$
D	$\Box P \to \Diamond P$	serial	$\forall w \exists v.wRv$
5	$\Diamond P \to \Box\Diamond P$	euclidean	$\forall w, v, u.wRv \wedge wRu \to vRu$
B	$P \to \Box\Diamond P$	symmetric	$\forall w, v.wRv \to vRw$
.2	$\Diamond\Box P \to \Box\Diamond P$	convergent	$\forall w, v, u.wRv \wedge wRu \to (\exists s.vRs \wedge uRs)$
.3	$\Box(\Box P \to Q) \vee \Box(\Box Q \to P)$	weakly connected	$wRv \wedge wRu \to v = u \vee vRu \vee uRv$
.3.2	$(\Diamond P \wedge \Diamond\Box Q) \to \Box(\Diamond P \vee Q)$	semi-euclidean	$wRv \wedge wRu \wedge \neg uRw \to vRu$
.4	$(P \wedge \Diamond\Box P) \to \Box P$	—	$wRv \wedge w \neq u \wedge wRu \to vRu$
$w4$	$\Diamond\Diamond P \to P \vee \Diamond P$	weakly transitive	$\forall w, v, u.wRv \wedge vRu \to wRu \vee w = u$

Table 2.1: Some standard axioms and the frame properties they define.

PROOF SYSTEMS FOR MODAL LOGIC. In this paragraph, we consider Hilbert-style proof systems for normal modal logics.

Define the axiom

$$K := \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi,$$

and the inference rules

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi} \quad \text{and} \quad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}.$$

A normal modal logic is a set of formulas which includes all propositional tautologies and all instances of K, and is closed under **Nec** and **MP**. The modal logic K is the closure under **Nec** and **MP** of the set of all propositional tautologies and all instances of K. K is the smallest normal modal logic. Given a modal logic L, we also write $\vdash_{\mathsf{L}} \varphi$ when $\varphi \in \mathsf{L}$.

Other modal logics can be obtained by adding other axioms to K. For example, the logic $\mathsf{K4}$ is obtained by adding the axiom $\Box\varphi \to \Box\Box\varphi$ (and taking the closure by **Nec** and **MP**). Table 2.2 shows some of the logics obtainable using axioms from Table 2.1. Note that some of these logics can be obtained in more than one way. For example, $\mathsf{S5}$ can also be obtained by adding D, 4 and B to K.

A modal logic L is strongly complete with respect to a class of frames defined by L iff $\vdash_{\mathsf{L}} \varphi$ is equivalent to $\mathsf{L} \models \varphi$. All the modal logics in Table 2.2 are strongly

Logic	Axioms
D	$\mathsf{K} + D$
T	$\mathsf{K} + T$
K4	$\mathsf{K} + 4$
S4	$\mathsf{K} + T + 4$
S5	$\mathsf{K} + T + 5$
B	$\mathsf{K} + B$
S4.2	$\mathsf{K} + T + 4 + .2$
S4.3	$\mathsf{K} + T + 4 + .3$
S4.3.2	$\mathsf{K} + T + 4 + .3.2$
S4.4	$\mathsf{K} + T + 4 + .4$
wK4	$\mathsf{K} + T + w4$

Table 2.2: Some normal modal logics.

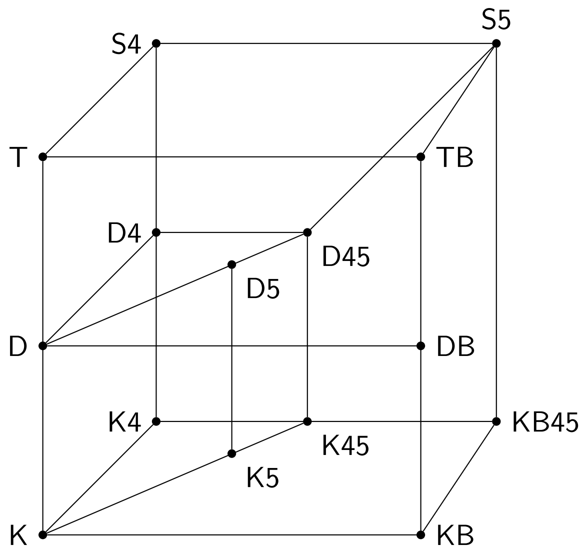


Figure 2.2: The modal cube. Adapted from [Gar21].

complete with respect to the class of frames they respectively define. Here $\mathsf{L} \models \varphi$ is an abbreviation of, for all Kripke model M, $M \models \mathsf{L}$ implies $M \models \varphi$.

Example 4. K is strongly complete for all Kripke frames, that is, $\vdash_{\mathsf{K}} \varphi$ iff $M \models \varphi$ for all Kripke model M. This is proved using the canonical Kripke model $M_{\mathsf{K}} = \langle W^{\mathsf{K}}, R^{\mathsf{K}}, V^{\mathsf{K}} \rangle$ for K, where:

- W^{K} consists of all maximal consistent extensions of K;
- $\Gamma R^{\mathsf{K}} \Delta$ iff $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$, for all modal formula φ; and
- $\Gamma \in V^{\mathsf{K}}(P)$ iff $P \in \Gamma$.

Remember, Γ is a maximal consistent extension of K iff, for all formula φ, exactly one of $\varphi \in \Gamma$ and $\neg\varphi \in \Gamma$ hold.

The Truth Lemma for K states that, for all $\Gamma \in W_x$,

$$M_{\mathsf{K}}, \Gamma \models \varphi \text{ iff } \varphi \in \Gamma.$$

Now, if $\vdash_{\mathsf{K}} \varphi$ then φ is valid. If $\not\vdash_{\mathsf{K}} \varphi$, then there is a maximal consistent extension Γ of K such that $\neg\varphi \in \Gamma$. By the Truth Lemma, $M_{\mathsf{K}}, \Gamma \models \neg\varphi$ and so $M_{\mathsf{K}}, \Gamma \not\models \varphi$. The Truth Lemma can be proved by induction on formulas; see Chapter 4 of [BdV01] for a detailed proof.

CANONICAL LOGICS. For a given modal logic L, define the canonical model $M_{\mathsf{L}} = \langle W^{\mathsf{L}}, R^{\mathsf{L}}, V^{\mathsf{L}} \rangle$ as we defined M_{K} above, but the elements of W^{L} are now complete extensions of L. If L is one of the logics on Table 2.1 and M_{L} is its canonical model, then $M_{\mathsf{L}} \models \mathsf{L}$, with the exception of GL. For example, M_{S4} is a transitive and reflexive model.

The logics L where $M_{\mathsf{L}} \models \mathsf{L}$ are called canonical logics. If L is canonical, then we can prove its completeness with respect to the class of frames it defines the same way we did for K. Do note that there exist non-canonical logics.

Example 5. Let GL be the logic obtained by adding Löb's Axiom $\Box(\Box P \to P) \to \Box P$ to K. GL is not both sound and strongly complete with respect to any class of frames, and hence not canonical. For a proof, see Theorem 4.43 of [BdV01].

STANDARD TRANSLATION. We can think of any Kripke model $M = \langle W, R, V \rangle$ as a first-order structure M_{fo} with domain W, a binary relation symbol R and a predicate symbol P for each propositional symbols $P \in \mathrm{Prop}$. We inductively translate modal formulas to first-order formulas as follows:

- $\mathrm{ST}_x(P) := P(x)$;
- $\mathrm{ST}_x(\neg\varphi) := \neg\mathrm{ST}_x(\varphi)$;
- $\mathrm{ST}_x(\varphi \wedge \psi) := \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi)$;
- $\mathrm{ST}_x(\varphi \vee \psi) := \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi)$;
- $\mathrm{ST}_x(\varphi \to \psi) := \mathrm{ST}_x(\varphi) \to \mathrm{ST}_x(\psi)$;
- $\mathrm{ST}_x(\Box\varphi) := \forall y.xRy \to \mathrm{ST}_y(\varphi)$; and
- $\mathrm{ST}_x(\Diamond\varphi) := \exists y.xRy \wedge \mathrm{ST}_y(\varphi)$;

where x and y are first-order variables. We have that

$$M, w \models \varphi \text{ iff } M_{\text{fo}} \models \mathrm{ST}_w(\varphi),$$

for any Kripke model M.

We can use the standard translation to transfer to modal logic results from first-order logic. We are interested particularly in the compactness theorem for modal logic. It states that any set of modal formulas is satisfiable iff all of its finite subsets are. Furthermore, the compactness theorem also holds for the modal logics in Table

2.2. For example, the compactness theorem for K4 is: Γ is satisfiable by a transitive model iff all of its finite subsets are satisfied by transitive models.

Do note that the standard translation can only be used to transfer results from first-order logic to modal logic, but not the reverse way. For example, modal logic is decidable, but first-order logic is not decidable.

SAHLQVIST CORRESPONDENCE. Sahlqvist [Sah75] proved a result showing that many modal formulas have a computable first-order correspondent. We follow here the presentation of the Sahlqvist Theorems given in [vBH12].

We define the modal Sahlqvist formulas as follows: any formula without negations is a Sahlqvist formula; any formula of the form $\neg\Box\cdots\Box P$ is a Sahlqvist formula; and, if φ and ψ are Sahlqvist formulas, then $\varphi \vee \psi$ and $\Box\varphi$ are Sahlqvist formulas.

Theorem 1 (Sahlqvist Correspondence Theorem). *For any Sahlqvist formula φ there is a first-order sentence χ_φ, the frame correspondent of φ, such that χ_φ is true on a Kripke frame iff φ is valid in that frame. Furthermore, χ_φ can be computed from φ.*

Theorem 2 (Sahlqvist Completeness Theorem). *For any Sahlqvist formula φ, the modal logic obtained by adding φ to K is complete for the class of frames defined by φ.*

Note that Löb's axiom $\Box(\Box P \to P) \to \Box P$ has no first-order correspondent, so it is not equivalent to any Sahlqvist formula.

There are also formulas which are not Sahlqvist but have first-order correspondent. For example, the conjunction

$$(\Box\Diamond P \to \Diamond\Box P) \wedge (\Diamond\Diamond P \to \Diamond P)$$

has a first order correspondent, but is not equivalent to any Sahlqvist formula. See Example 3.57 of [BdV01] for details. Furthermore, Chagrova's Theorem states that it is undecidable whether an arbitrary modal formula has a first-order correspondent.

BISIMULATIONS. We use bisimulations to compare Kripke models. While bisimulations are weaker than isomorphisms, they are enough to preserve modal truth. A bisimulation between $M_0 = \langle W_0, R_0, V_0\rangle$ and $M_1 = \langle W_1, R_1, V_1\rangle$ is a non-empty relation $B \subseteq W_0 \times W_1$ such that

- if $v_0 B v_1$, then $M_0, v_0 \models P$ iff $M_1, v_1 \models P$, for all $P \in \mathrm{Prop}$;
- if $v_0 R_0 v_0'$ and $v_0 B v_1$, then there is v_1' such that $v_1 R_1 v_1'$ and $v_0' B v_1'$; and
- if $v_1 R_1 v_1'$ and $v_0 B v_1$, then there is v_0' such that $v_0 R_0 v_0'$ and $v_0' B v_1'$.

Now, let (M, w) and (N, v) be pointed models. (M, w) and (N, v) are bisimilar iff wBv and there is a bisimulation B between M and N. If (M, w) and (N, v) are bisimilar, then

$$M, w \models \varphi \text{ iff } N, v \models \varphi,$$

for all modal formula φ. Not only does bisimulations preserve the truth value of modal formulas; but modal logic is the biggest fragment of first-order logic where bisimulations preserve truth values:

Theorem 3 (Van Benthem's Characterization Theorem). *A first-order formula is equivalent to a modal formula iff it is invariant under bisimulations.*

Example 6. The following Kripke models are presented on Figure 2.3.

- $M_0 = \{\{w, v\}, \{\langle w, v\rangle, \langle v, w\rangle\}, V_0(P) = \emptyset\}$;
- $M_1 = \{\{u\}, \{\langle u, u\rangle\}, V_1(P) = \emptyset\}$;
- $M_2 = \{\{w_i \mid i \in \omega\}, \{\langle w_i, w_{i+1}\rangle \mid i \in \omega\}, V_2(P) = \emptyset\}$; and
- $M_3 = \{\{s, t\}, \{\langle s, s\rangle, \langle s, t\rangle\}, V_3(P) = \emptyset\}$.

The models M_0, M_1, and M_2 are pairwise bisimilar. The models M_3 and M_4 are not bisimilar to any of the other four models.

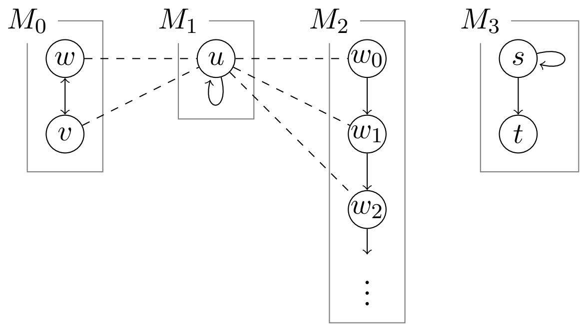


Figure 2.3: The models M_0, M_1, M_2, and M_3 from Example 6. The dotted lines describe bisimulations between M_0 and M_1, and between M_1 and M_2.

TREE MODEL PROPERTY. Let φ be a modal formula satisfied by a pointed model (M, w). We can find a tree-like model M_t by unfolding (M, w): start at the root w and whenever we can access some world v, we move to a fresh copy v' of v. For an example, see Figure 2.4. As (M, w) and (M_t, w) are bisimilar, $M_t, w \models \varphi$. Therefore if φ is satisfiable, then it is satisfiable by a tree model.

FINITE MODEL PROPERTY. Again, let (M, w) be a pointed model and φ be a formula such that $M, w \models \varphi$. We define the filtrated model $M_f = \langle W_f, R_f, V_f\rangle$. We will have M_f is finite and $M_f, [w] \models \varphi$. Let $v \sim v'$ iff, for all subformula ψ of φ, $M, v \models \psi \iff M, v' \models \psi$. Denote the equivalence class of $\sim$ containing v by $[v]$. Take W_f to be the set of equivalence classes of $\sim$, that is, $W_f = \{[v] \mid v \in W\}$. Let $[v]R_f[u]$ iff there are v', u' such that $v \sim v'$, $u \sim u'$ and $v'Ru'$. Let $[v] \in V_f(P)$ iff there is v' such that $v \sim v'$ and $v \in V(P)$. For all subformula ψ of φ, we can show that

$$M, v \models \psi \text{ iff } M_f, [v] \models \psi.$$

In particular, $M_f, [w] \models \varphi$. Furthermore, if φ has n subformulas, then the size of M_f is at most 2^n. Therefore any satisfiable formula is satisfiable by a finite model.

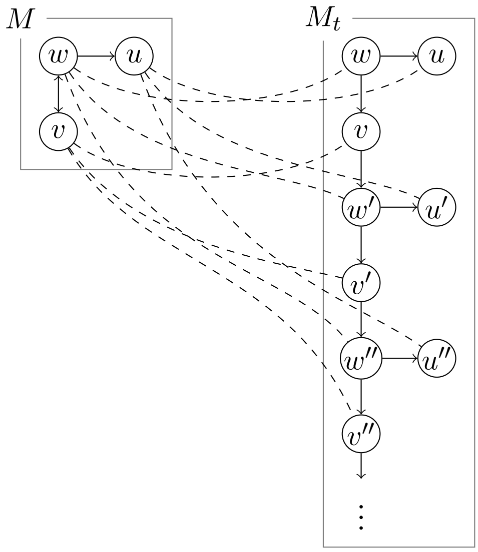


Figure 2.4: A pointed model (M, w) and its unfolding (M_t, w). The dotted lines describe the bisimulation $B = \{\langle w, w_i\rangle, \langle v, v_i\rangle, \langle u, u_i\rangle \mid i \in \omega\}$ between M and M_t.

GAME SEMANTICS. In this paragraph, we define the evaluation games for modal logic. Let (M, w) be a pointed Kripke model and φ be a modal formula, the evaluation game $\mathcal{G}(M, w \models \varphi)$ has two players: V (verifier), who wants to show that $M, w \models \varphi$; and R (refuter), who wants to show that $M, w \not\models \varphi$. We show that V wins the evaluation game $\mathcal{G}(M, w \models \varphi)$ iff $M, w \models \varphi$; that is, game semantics and Kripke semantics are equivalent. Game semantics will play a key role in our proofs for the μ-calculus. Evaluation games can also be defined for many other logics, see [HV19; Vää11].

The game positions are pairs $\langle v, \psi\rangle$ where v is in W and ψ is a subformula of φ. The game starts at the position $\langle w, \varphi\rangle$. The players advance in the game graph as follows:

- at $\langle v, \psi_0 \wedge \psi_1\rangle$, R chooses one of $\langle v, \psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \psi_0 \vee \psi_1\rangle$, V chooses one of $\langle v, \psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \psi_0 \to \psi_1\rangle$, V chooses one of $\langle v, \neg\psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \Box\psi\rangle$, R chooses $\langle v', \psi\rangle$ with vRv';
- at $\langle v, \Diamond\psi\rangle$, V chooses $\langle v', \psi\rangle$ with vRv'; and
- at $\langle v, \neg\psi\rangle$, the players move to $\langle v, \psi\rangle$ and exchange roles.

At a node $\langle v, P\rangle$, V wins iff $w \in V(P)$ and R wins iff $w \notin V(P)$. Note that every evaluation game for a modal formula φ is finite, indeed, the game length is bounded by the length of φ.

Proposition 4. *Let $M = \langle W, R, V\rangle$ be a Kripke model and φ be a modal formula, then:*

$$\text{V } \textit{wins } \mathcal{G}(M, w \models \varphi) \textit{ iff } M, w \models \varphi, \textit{ and } \text{R } \textit{wins } \mathcal{G}(M, w \models \varphi) \textit{ iff } M, w \not\models \varphi.$$

Proof. Fix a Kripke model $M = \langle W, R, V \rangle$. We proceed by structural induction on modal formulas. We do three representative cases.

The base case is the simplest: V wins $\mathcal{G}(M, w \models P)$ iff $M, w \models P$, by the definition of evaluation games.

Suppose that φ is $\psi \vee \theta$ and that V wins $\mathcal{G}(M, w \models \varphi)$ via the strategy σ. Then $\sigma(\langle w, \varphi \rangle)$ is either $\langle w, \psi \rangle$ or $\langle w, \theta \rangle$. Without loss of generality, suppose it is $\langle w, \psi \rangle$. Then V wins $\mathcal{G}(M, w \models \psi)$ via σ. By the induction hypothesis, $M, w \models \psi$ and so $M, w \models \psi \vee \theta$. Now, if $M, w \models \varphi$ then $M, w \models \psi$ or $M, w \models \theta$. Without loss of generality, suppose $M, w \models \psi$. Let σ' be a winning strategy for V in $\mathcal{G}(M, w \models \psi)$. If we define $\sigma(\langle w, \varphi \rangle) := \langle w, \psi \rangle$ and σ equal to σ' on other positions, then σ is a winning strategy for V in $\mathcal{G}(M, w \models \varphi)$.

Suppose that φ is $\Box\psi$ and that V wins $\mathcal{G}(M, w \models \varphi)$ via the strategy σ. For any w' such that wRw', V wins $\mathcal{G}(M, w' \models \psi)$ via σ. Therefore $M, w \models \varphi$. If $M, w \models \varphi$, then for all w' such that wRw', $M, w' \models \psi$. By the induction hypothesis, there is a winning strategy $\sigma^{w'}$ for V on $\mathcal{G}(M, w' \models \psi)$ for each w'. On $\mathcal{G}(M, w \models \varphi)$, define σ by having V play $\sigma_{w'}$ after the first move $\langle w', \psi \rangle$ by R. $\square$

MULTIMODAL LOGIC. One can also consider modal logics with multiple box and diamond modalities. Let I be a set of labels. Define the multimodal logic formulas by the grammar:

$$\varphi := P \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box_a\varphi \mid \Diamond_a\varphi,$$

where P is a proposition symbol and a is a label in I. It is also common to denote $\Box_a$ and $\Diamond_a$ by $[a]$ and $\langle a \rangle$.

Kripke models for multimodal logics have multiple accessibility relations, one for each label in I. Formally, a Kripke model M is a tuple $\langle W, \{R_a\}_{a \in I}, V \rangle$. We define the semantics for propositional and logical symbols as in the unimodal case. The semantics for the modal symbols are analogous, given by:

- $M, w \models \Box_a\varphi$ iff for all v, wR_av implies $M, v \models \varphi$; and
- $M, w \models \Diamond_a\varphi$ iff there is v such that wR_av and $M, v \models \varphi$.

The results stated above for unimodal logics also hold for multimodal logics.

EPISTEMIC LOGIC. Fix a set G of labels. We interpret each $a \in G$ as an (epistemic) agent. We write K_a for $\Box_a$. Read $K_a\varphi$ as "the agent a knows that φ". We can thus model knowledge in modal logic. Common axioms for knowledge are:

$$\begin{aligned} T :& K_a\varphi \to \varphi, \\ 4 :& K_a\varphi \to K_aK_a\varphi, \text{ and} \\ 5 :& \neg K_a\varphi \to K_a\neg K_a\varphi. \end{aligned}$$

The first one states that knowledge is correct. The last two ones are known as introspection axioms, and state that an agent knows what they know and what they don't know, respectively.

One can add a belief modality B_a for each agent. Read $B_a\varphi$ as "the agent a believes that φ". Common axioms for belief are

$$\begin{aligned} D :&\neg B_a\bot, \\ 4 :&B_a\varphi \to B_aB_a\varphi, \text{ and} \\ 5 :&\neg B_a\varphi \to B_a\neg B_a\varphi. \end{aligned}$$

While we required that knowledge is true, we only require that belief is consistent.

We consider two extensions of epistemic logic in this thesis. The first one is the ignorance modality:

$$I_a\varphi := \neg K_a\varphi \wedge \neg K_a\neg\varphi,$$

first defined by van der Hoek and Lomuscio [vL04]. The second one is the common knowledge modality:

$$C\varphi := \varphi \wedge E\varphi \wedge EE\varphi \wedge \cdots,$$

where E is the "everyone knows" modality. Given a finite group of agents G, $E\varphi$ is defined as

$$E\varphi := \bigwedge_{a\in G} K_a\varphi.$$

Note that common knowledge cannot be expressed by a finitary formula using only the knowledge modality.

MODEL CHECKING. We think of Kripke models as transition systems. We use transition systems $T = \langle W, \{R_a\}_{a\in I}, V\rangle$ to model the execution of programs. Call the elements of W states. Each $a \in I$ represent a routine, and sR_at means that when we run the routine a at the state s, the system may go to state t. This process may be nondeterministic, that is, there may be $t \neq t'$ such that sR_at and sR_at'.

We interpret $T, s \models \Box_a\varphi$ to mean that, after the execution of a at s, the system goes to a state where φ holds; we interpret $T, s \models \Diamond_a\varphi$ to mean that, after the execution of a at s, the system may go to a state where φ holds. The model checking problem is to decide whether $M, w \models \varphi$, given a Kripke model M, a world w, and a formula φ.

For model checking, modal logic is quite limited. We will show that the μ-calculus describes properties not captured by modal logic. Indeed, the modal μ-calculus was first defined by Kozen [Koz83], extending a model checking logic called PDL. See also [BW18; Cla+18].

TOPOLOGICAL SEMANTICS. We can also interpret modal formulas over topological spaces. A topological model is a triple $\mathcal{X} = \langle W, \tau, V\rangle$ where $\langle W, \tau\rangle$ is a topological space and V is a valuation function on W.

In Chapter 5, we study derivative topological semantics—where $\|\Diamond\varphi\|^{\mathcal{X}}$ is the Cantor derivative of $\|\varphi\|^{\mathcal{X}}$. That is, $w \in \|\Diamond\varphi\|^{\mathcal{X}}$ iff w is a limit point of $\|\varphi\|^{\mathcal{X}}$. Derivative semantics is complete for the logic wK4.

An alternative topological semantics is obtained by defining $\|\Box\varphi\|^{\mathcal{X}}$ as the interior of $\|\varphi\|^{\mathcal{X}}$. Interior semantics is complete for the logic S4. Note that derivative semantics is more expressive than interior semantics: the interior of a set A is definable as A minus the Cantor derivative of its complement $(W \setminus A)'$. Both derivative semantics and interior semantics were first defined by Tarski and McKinsey [MT44]. See also [vB07]

OTHER INTERPRETATIONS FOR MODAL LOGIC. Modal logic has many interpretations. We list here some of the applications we do not study in this thesis: alethic logic, for pure necessity and possibility; deontic logic, for obligations; time logic, with modalities for both past and future; and public announcement logic, an epistemic logic where we allow truths to be announced to all agents. See also [Gar21; van10].

2.2 Basic definitions

THE μ-FORMULAS. Fix a set of propositional symbols Prop and a set of variable symbols Var. The language $\mathcal{L}_\mu$ of the μ-calculus utilizes the symbols in Prop and Var along with logical symbols $\neg$, $\wedge$ and $\vee$; the modal operators $\Box$ and $\Diamond$; and the fixed-point operators μ and ν. The operators μ and ν are called least and greatest fixed-point operators, respectively.

The μ-formulas of the μ-calculus are defined by the following grammar:

$$\varphi := P \mid \neg P \mid X \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

where $P \in \mathrm{Prop}$ and $X \in \mathrm{Var}$.

Fix a variable symbol X and a formula φ. Let η denote a fixed-point operator μ or ν. An occurrence of X in φ is in the scope of a fixed-point operator η iff the occurrence is in a subformula $\eta X.\psi$ of φ. An occurrence of X in φ is bound iff it is in the scope of some fixed-point operator. An occurrence of X in φ is free iff it is not bound. A formula without free variables is called a sentence. We write $\varphi(X)$ to specify the free occurrences of X in φ (with the possibility that there is no occurrence). Let φ be a μ-formula and X be a variable bounded in φ. We say X is a μ-variable iff it is bounded by a μ-operator; X is a ν-variable iff it is bounded by a ν-operator.

Note that the negation symbol is only allowed before propositional symbols. We can define the negation of formulas by the following recursive rules:

- $\neg\neg\varphi := \varphi$;
- $\neg(\varphi \wedge \psi) := \neg\varphi \vee \neg\psi$;
- $\neg(\varphi \vee \psi) := \neg\varphi \wedge \neg\psi$;
- $\neg(\Box\varphi) := \Diamond\neg\varphi$;
- $\neg(\Diamond\varphi) := \Box\neg\varphi$;
- $\neg\mu X.\varphi(X) := \nu X.\neg\varphi(\neg X)$; and
- $\neg\nu X.\varphi(X) := \mu X.\neg\varphi(\neg X)$.

With this definition, the negation of a sentence is still a sentence.

We may allow the negation of arbitrary formulas if we put restrictions on the formulas we can apply fixed-point operators. Consider the following grammar:

$$\varphi := P \mid X \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \mu X.\varphi,$$

where $P \in \mathrm{Prop}$, $X \in \mathrm{Var}$, and X is positive in φ. Say X is positive in φ when each free occurrence of X in φ is in the scope of an even number of negation symbols (possibly in the scope of none). The restriction of $\eta X.\varphi$ to formulas where X is positive is necessary so that the operator $\Gamma_{\varphi(X)} := \|\varphi(X)\|^M$ is monotone on any Kripke model M. We could also allow implication in our formulas. In case we do so, X is positive in $\varphi \to \psi$ iff it is positive in ψ and not positive in φ.

We will usually assume that μ-formulas are well-named. A formula φ is well-named when:

- every fixed-point operator ηX in φ binds exact one occurrence of X;
- every variable which occurs in φ occurs only once; and
- if the fixed-point operator ηX occurs in φ then there is a modality $\triangle$ which is in the scope of ηX and X is in the scope of $\triangle$.

Proposition 5. *Every μ-formula φ is equivalent to a well-named μ-formula* $\mathrm{wn}(\varphi)$.

We tacitly suppose that all formulas below are well-named.

KRIPKE SEMANTICS. As in the semantics for modal logic, a Kripke model is a triple $M = \langle W, R, V\rangle$ consisting of:

- W, a non-empty set;
- $R \subset W \times W$, a binary relation on W; and
- $V : \mathrm{Prop} \to \mathcal{P}(W)$, a function from propositional symbols to subsets of W.

The elements of W are called possible worlds, and W itself is called the set of possible worlds. R is called the accessibility relation; when wRv we say that v is accessible from w. V is called the valuation function, and assigns to each propositional symbol P the set of worlds where P is true. If $w \in W$, we call (M, w) a pointed Kripke model.

Kripke models are also called labeled directed graphs and transition systems. The elements of W are also called nodes and states. These alternative names come from distinct applications of modal logic.

Fix a Kripke model $M = \langle W, R, V\rangle$ and $w \in W$. Define the operator $\Gamma_{\varphi(X)} : \mathcal{P}(W) \to \mathcal{P}(W)$ by

$$X \mapsto \|\varphi(X)\|.$$

$\Gamma_{\varphi(X)}$ is a monotone operator, that is, if $X \subseteq Y \in \mathcal{P}(W)$ then $\Gamma_{\varphi(X)}(X) \subseteq \Gamma_{\varphi(X)}(Y)$. The Knaster–Tarski Theorem implies that $\Gamma_{\varphi(X)}$ has least and greatest fixed-points.

Theorem 6 (Knaster, Tarski, see [AN01])**.** *Given a set S and a function monotone with respect to the set inclusion $f : \mathcal{P}(S) \to \mathcal{P}(S)$, then f has a least fixed-point* lfp_f *and greatest fixed-point* gfp_f *such that:*

- $\mathrm{lfp}_f = \cap\{S' \subseteq S \mid f(S') \subseteq S'\}$; *and*
- $\mathrm{gfp}_f = \cup\{S' \subseteq S \mid f(S') \supseteq S'\}$.

Let $M = \langle W, R, V\rangle$ be a Kripke model, A be a subset of W, and $\varphi(X)$ be a μ-formula where X is free. Define the augmented model $M[X \mapsto A]$ by setting $V(X) := A$.

We define the valuation of formulas on $M = \langle W, R, V\rangle$ by induction on the structure of the formulas:

- $\|P\|^M = V(P)$;
- $\|X\|^{M[X\mapsto A]} = A$;
- $\|\bot\|^M = \emptyset$;
- $\|\top\|^M = W$;
- $\|\neg P\|^M = W \setminus \|P\|^M$;
- $\|\varphi \wedge \psi\|^M = \|\varphi\|^M \cap \|\psi\|^M$;
- $\|\varphi \vee \psi\|^M = \|\varphi\|^M \cup \|\psi\|^M$;
- $\|\Box\varphi\|^M = \{w \in W \mid \forall v.wRv \to v \in \|\varphi\|^M\}$;
- $\|\Diamond\varphi\|^M = \{w \in W \mid \exists v.wRv \wedge v \in \|\varphi\|^M\}$;
- $\|\mu X.\varphi(X)\|^M$ is the least fixed-point of $\Gamma_{\varphi(X)}$; and
- $\|\nu X.\varphi(X)\|^M$ is the greatest fixed-point of $\Gamma_{\varphi(X)}$.

We also write $w \in \|\varphi\|^M$ as $M, w \models \varphi$. When $M, w \models \varphi$ we say that φ is true at w, or that (M, w) satisfies φ. Define $M \models \varphi$ to hold iff $M, w \models \varphi$ for all $w \in W$. Furthermore $\models \varphi$ holds iff $M \models \varphi$ holds for all Kripke models M.

We can also define the valuations of fixed-point formulas using the expressions given by the Knaster–Tarski theorem:

- $\|\mu X.\varphi\|^M = \bigcap\{W' \subseteq W \mid \|\varphi\|^{M[X:=W']} \subseteq W'\}$; and
- $\|\nu X.\varphi\|^M = \bigcup\{W' \subseteq W \mid \|\varphi\|^{M[X:=W']} \supseteq W'\}$.

Example 7. Fix a Kripke model $M = \langle W, R, V\rangle$.

- $M, w \models \mu X.P \vee \Diamond X$ iff there is a world reachable in finitely many steps from w where P holds.
- $w \in \|\nu X.P \wedge \Box X$ iff P holds in all worlds reachable in finitely many steps from w.
- $M, w \models \mu X.P \vee \Box X$ iff P eventually holds in all paths starting from w.
- $M, w \models \mu X.(\nu Y.P \wedge \Box Y) \vee \Diamond X$ iff there is a world v reachable in finitely many steps from w such that P holds in all worlds reachable in finitely many steps from v.

- $M, w \models \nu X \mu Y.(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)$ iff there is a path starting at w where eventually P always holds.
- $M, w \models \mu X.\Box X$ iff there is no infinite path starting from w.
- $M, w \models \nu X.\Diamond X$ iff there is an infinite path starting from w.

NEGATION OF FORMULAS. We can inductively define the negation of μ-formulas by:

- $\neg(\neg\varphi) := \varphi$;
- $\neg(\varphi \wedge \psi) := \neg\varphi \vee \neg\psi$;
- $\neg(\varphi \vee \psi) := \neg\varphi \wedge \neg\psi$;
- $\neg(\Box\varphi) := \Diamond\neg\varphi$;
- $\neg(\Diamond\varphi) := \Box\neg\varphi$;
- $\neg\mu X.\varphi := \nu X.\neg\varphi[X/\neg X]$; and
- $\neg\nu X.\varphi := \mu X.\neg\varphi[X/\neg X]$.

Note that on the negation of formulas of the form $\mu X.\varphi$ and $\nu X.\varphi$ the negation on the variable X is eliminated when we push the other negation inside φ. So X is still positive and the obtained formulas are well-defined.

Now, there are other ways to define $\mathcal{L}_\mu$ are superfluous. The following equivalences suggest some options we have:

- $M, w \models \bot$ iff $M, w \models P \wedge \neg P$;
- $M, w \models \top$ iff $M, w \models P \vee \neg P$;
- $M, w \models \neg\varphi$ iff $M, w \models \varphi \to \bot$;
- $M, w \models \varphi \vee \psi$ iff $M, w \models \neg(\neg\varphi \wedge \neg\psi)$;
- $M, w \models \varphi \to \psi$ iff $M, w \models \neg\varphi \vee \psi$;
- $M, w \models \Diamond\varphi$ iff $M, w \models \neg\Box\neg\varphi$; and
- $M, w \models \mu X.\varphi(X)$ iff $M, w \models \neg\nu X.\neg\varphi(\neg X)$.

APPROXIMANTS. Let $\varphi(X)$ be a μ-formula and M be a Kripke model. We now define the approximants of the fixed-points $\mu X.\varphi$ and $\nu X.\varphi$. By ordinal recursion, define the approximants of the least fixed-point $\mu X.\varphi$ to be

- $\mu^0 X.\varphi := \emptyset$;
- $\mu^{\alpha+1} X.\varphi := \|\varphi(X)\|^{[X \mapsto \mu^\alpha X.\varphi]}$;
- $\mu^\lambda X.\varphi := \bigcup_{\alpha<\lambda} \mu^\alpha X.\varphi$, for λ limit; and
- $\mu^\infty X.\varphi := \bigcup_{\alpha\in\mathrm{Ord}} \mu^\alpha X.\varphi$;

and the approximants of the greatest fixed-point $\nu X.\varphi$ to be

- $\nu^0 X.\varphi := W$;
- $\nu^{\alpha+1} X.\varphi := \|\varphi(X)\|^{[X \mapsto \nu^\alpha X.\varphi]}$;
- $\nu^\lambda X.\varphi := \bigcap_{\alpha<\lambda} \nu^\alpha X.\varphi$, for λ limit; and
- $\nu^\infty X.\varphi := \bigcap_{\alpha\in\mathrm{Ord}} \nu^\alpha X.\varphi$.

Proposition 7. *Let $M = \langle W, R, V\rangle$ be a Kripke model. There are ordinals α_{lfp} and α_{gfp} such that:*

- $\mu^{\alpha_{\mathrm{lfp}}} X.\varphi = \mu^\infty X.\varphi = \|\mu X.\varphi\|^M$*; and*
- $\nu^{\alpha_{\mathrm{gfp}}} X.\varphi = \nu^\infty X.\varphi = \|\nu X.\varphi\|^M$.

In this thesis, we will be mainly interested in the finite approximants $\eta^n X.\varphi$ of μ and ν operators. Approximants will play important roles in the collapses of the alternation hierarchy. Define $\varphi^0(X) := X$ and $\varphi^{n+1}(X) := \varphi(\varphi^n(X))$. By induction, $\|\varphi^n(\bot)\|^M = \mu^n X.\varphi$ and $\|\varphi^n(\top)\|^M = \nu^n X.\varphi$ for all $n \in \omega$ and Kripke model M.

PROOF SYSTEMS FOR μ-CALCULUS. A logic $\mu\mathsf{K}$ for modal μ-calculus was proposed by Kozen [Koz83]. It consists of the axioms

$$\begin{aligned} \mathrm{K} &:= \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi, \\ \mathrm{FP} &:= \nu X.\varphi \to \varphi(\nu X.\varphi), \end{aligned}$$

and the inference rules

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi}, \qquad (\mathbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi}, \quad (\mathbf{Ind})\ \frac{\varphi \to \psi(\varphi)}{\varphi \to \nu X.\psi}.$$

That is $\mu\mathsf{K}$ consists of the modal logic K with axioms for the fixed-point formulas. Kozen proved the completeness of his proof system for a fragment of the μ-calculus. Walukiewicz [Wal95] showed the completeness for the full μ-calculus. Note that decidability [Wal95] and the finite model property [Koz88] also holds for the μ-calculus.

Given a modal logic L, we can define the logic $\mu\mathsf{L}$ by adding the axiom FP and the inference rule **Ind** to L. In general, the proof of completeness, decidability and finite model property for $\mu\mathsf{L}$ is not easy. Indeed, some of these properties may fail. For example, the finite model property fails for $\mu\mathsf{B}$—the μ-calculus over symmetric frames [DL13].

BISIMULATIONS AND THE CHARACTERIZATION THEOREM. Recall that bisimulation between $M_0 = \langle W_0, R_0, V_0\rangle$ and $M_1 = \langle W_1, R_1, V_1\rangle$ is a non-empty relation $B \subseteq W_0 \times W_1$ such that: $v_0 B v_1$ implies $M_0, v_0 \models P$ iff $M_1, v_1 \models P$, for all $P \in \mathrm{Prop}$; $v_0 R_0 v_0'$ and $v_0 B v_1$ imply that there is v_1' such that $v_1 R_1 v_1'$ and $v_0' B v_1'$; $v_1 R_1 v_1'$ and $v_0 B v_1$ imply that there is v_0' such that $v_0 R_0 v_0'$ and $v_0' B v_1'$. If (M, w) and (N, v) are pointed models, then (M, w) and (N, v) are bisimilar iff there is a bisimulation B between M and N and wBv.

We saw that bisimulations preserve truth values for modal formulas. The same happens for μ-formulas: if (M, w) and (N, v) are bisimilar and φ is a μ-formula, then

$$M, w \models \varphi \text{ iff } N, v \models \varphi.$$

Similar to modal logic being the biggest fragment of first-order logic invariant under bisimulations; μ-calculus is the biggest fragment of monadic second-order logic which is invariant under bisimulations [BW18]. Monadic second-order logic is an extension of the first-order logic with set variables, the membership predicate, and quantification using set variables.

STANDARD TRANSLATION. We can define a standard translation for the μ-calculus to first-order logic with fixed-points. Let $\varphi(x, X)$ be a first-order formula with X positive in φ and M be a Kripke model. Define $\Gamma_{\varphi(X)}(X) := \{w \in M \mid M \models \varphi(w, X)\}$. Then $\|\mu x X.\varphi\|^M$ is the least fixed-point of $\Gamma_{\varphi(X)}$ and $\|\nu x X.\varphi\|^M$ is the greatest fixed-point of $\Gamma_{\varphi(X)}$. Let

- $\mathrm{ST}_x(\mu X.\varphi) := \mu x X.\mathrm{ST}_x(\varphi)$;
- $\mathrm{ST}_x(\nu X.\varphi) := \nu x X.\mathrm{ST}_x(\varphi)$;

and $\mathrm{ST}_x(\varphi)$ be the same as in modal logic for other formulas. All μ-formula φ is modally equivalent to $\mathrm{ST}_x(\varphi)$.

FRAME CORRESPONDENCE. As in modal logic, where some formulas had first-order correspondents; in some μ-sentences have correspondents in first-order logic with fixed-points. We follow here the presentation from van Benthem *et al.* [vBH12].

We define the PIA formulas[1] as follows: any atom is a PIA formula; any variable is a PIA formula; if φ and ψ are PIA formulas, so are $\varphi \wedge \psi$, $\Box\varphi$ and $\nu X.\varphi$; and if ψ is PIA and φ is a positive μ-sentence, $\varphi \to \psi$ is a PIA formula.

We can now define the Sahlqvist μ-formulas as follows: Any positive μ-sentence is a Sahlqvist μ-formula; any negated PIA sentence is a Sahlqvist μ-formula; any variable is a Sahlqvist μ-formula; if φ and ψ are Sahlqvist μ-formulas, so are $\varphi \wedge \psi$, $\Box\varphi$ and $\nu X.\varphi$; and if φ and ψ are Sahlqvist μ-formulas and, if one of them is not a sentence then the other is a positive sentence, then $\varphi \vee \psi$ is a Sahlqvist μ-formula. A Sahlqvist μ-sentence is a Sahlqvist μ-formula without free variables.

Theorem 8 (Sahlqvist Correspondence Theorem for the μ-calculus)**.** *Any Sahlqvist μ-sentence φ has a correspondent χ_φ in first-order logic with fixed-points. That is, χ_φ is true in a frame F iff φ is true in that same frame. The correspondent χ_φ is computable from φ.*

Now, Löb's axiom $L := \Box(\Box P \to P) \to \Box P$ is Sahlqvist μ-sentence, and so has a correspondent in first-order logic with fixed-points: $F \models L$ iff F is transitive and reverse well-ordered. McKinsey's axiom $\Box\Diamond P \to \Diamond\Box P$ still does not have a Sahlqvist correspondent.

[1] PIA stands for "positive implies atomic".

2.3 The alternation hierarchy

One measure of complexity for μ-formulas is their alternation depth. It gives rise to the μ-calculus' alternation hierarchy. In general, the alternation hierarchy is strict; that is, for every $n \in \omega$, there is a formula with alternation depth n which is not equivalent to any formula with alternation depth lower than n. On the other hand, if we restrict the μ-calculus to some class of frames, the alternation hierarchy may collapse. That is, there is $n \in \omega$ such that every formula is equivalent to a formula with alternation depth less or equal than n.

We define the alternation hierarchy as follows:

- $\Sigma_0^\mu = \Pi_0^\mu$ consists of all the formulas without fixed-point operators;
- Σ_{n+1}^μ is the closure of $\Sigma_n^\mu \cup \Pi_n^\mu$ under conjunction, disjunction, $\Box$, $\Diamond$, μ, and the substitution: if $\varphi(X) \in \Sigma_{n+1}^\mu$, $\psi \in \Sigma_{n+1}^\mu$ and the variable X does not appear free in the scope of some fixed-point operator ηY for any variables Y free in ψ, then $\varphi(\psi) \in \Sigma_{n+1}^\mu$;
- Π_{n+1}^μ is the closure of $\Sigma_n^\mu \cup \Pi_n^\mu$ under conjunction, disjunction, $\Box$, $\Diamond$, ν, and the substitution: if $\varphi(X) \in \Pi_{n+1}^\mu$, $\psi \in \Pi_{n+1}^\mu$ and the variable X does not appear free in the scope of some fixed-point operator ηY for any variables Y free in ψ, then $\varphi(\psi) \in \Pi_{n+1}^\mu$; and
- $\Delta_n^\mu = \Sigma_n^\mu \cap \Pi_n^\mu$.

The alternation hierarchy is also called the Niwiński alternation hierarchy [Niw86]. Two alternative formulations are:

- the Emerson–Lei alternation hierarchy [EL86], where the substitution rule requires that ψ is a sentence; and
- the simple alternation hierarchy, where the substitution rule is omitted.

ALTERNATION DEPTH. An equivalent way to classify formulas by their complexity is the alternation depth. We follow here the definition given in [BW18]. The dependency order on bound variables of a μ-formula φ is the smallest partial order such that $X \leq_\varphi Y$ if X occurs free in $\eta Y.\psi \in \mathrm{Sub}(\varphi)$. The alternation depth of a μ-variable X in formula φ is the maximal length of a chain $X_1 \leq_\varphi \cdots \leq_\varphi X_n$ where $X = X_1$, variables $X_1, X_3, \ldots$ are μ-variables and variables $X_2, X_4, \ldots$ are ν-variables. The alternation depth of ν-variables is defined similarly. The alternation depth of formula φ is the maximum of the alternation depths of variables bound in φ, or zero if there are no fixed-points.

THE ALTERNATION-FREE FRAGMENT. The alternation-free fragment of the alternation hierarchy is $\Delta_2^\mu = \Sigma_2^\mu \cap \Pi_2^\mu$. Alternatively, a formula φ is alternation-free iff it has alternation depth no larger than one. Alternation-free formulas allow only simpler fixed-point operators, where the evaluation of subformulas does not depend on the evaluation of some bigger formula.

Example 8. Consider the following formulas:

- $\mu X.(\nu Y.P \wedge \Box Y) \vee \Diamond X$ is alternation-free.
- $\nu X \mu.Y(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)$ is not alternation-free.
- The winning region formula for parity games

$$W'_n = \eta X_n \cdots \mu X_1 \nu X_0 \bigvee_{i \leq n} (P_\exists \wedge P_i \wedge \Diamond X_i) \vee \bigvee_{i \leq n} (P_\forall \wedge P_i \wedge \Box X_i)$$

are not alternation-free formula, whenever $n \geq 2$. The formula W'_1 is an alternation-free formula.

COLLAPSING THE ALTERNATION HIERARCHY. If we change the class of frames we evaluate formulas on, the alternation hierarchy may be strict or collapse to some fragment. Fix a class of frames $\mathcal{F}$ to evaluate the μ-formulas:

- if, for all n, there is $\varphi \in \Sigma^\mu_{n+1}$ which is not equivalent to any formula $\psi \in \Sigma^\mu_n \cup \Pi^\mu_n$ over $\mathcal{F}$; then the alternation hierarchy is strict.
- if, for all μ-formula φ, there is an alternation-free formula ψ equivalent to φ over $\mathcal{F}$; then the alternation hierarchy collapses to the alternation-free fragment.
- if, for all μ-formula φ, there is a modal formula ψ equivalent to φ over $\mathcal{F}$; then the alternation hierarchy collapses to modal logic.

When we say the alternation hierarchy collapses to the alternation-free fragment over a class of frames $\mathcal{F}$, we also implicitly mean that it does not collapse to modal logic over $\mathcal{F}$. The next theorem lists the existing results on the collapse of the alternation hierarchy:

Theorem 9. *The alternation hierarchy is strict over:*

- *all frames [Bra98b; Bra98a];*
- *recursive frames [Bra98b]; trees [Len96];*
- *binary trees [Arn99];*
- *reflexive frames [AF09b]; and*
- *symmetric frames [DL13; DL15].*

The alternation hierarchy collapses to its alternation-free fragment over:

- *finite directed acyclic graphs [Mat02];*
- *transitive frames [AF09b; DL10; DO05];*
- *transitive frames with feedback vertex set of bounded size [GKL14];*
- *ω-regular languages [Kai95];*
- *visibly pushdown ω-languages [GKL14];*

- *weakly transitive frames [PT22];*
- S4.2 *frames [PT22]; and*
- S4.3 *frames [PT22].*

The alternation hierarchy collapses to modal logic over:

- *equivalence relations [AF09b; DO05];*
- S4.3.2 *frames [PT22]; and*
- S4.4 *frames [PT22].*

We will prove the collapse for weakly transitive frames in Section 5.2; and prove the collapse for frames of S4.2, S4.3, S4.3.2, and S4.4 in Section 3.4.

THE WEAK ALTERNATION HIERARCHY. We also define a hierarchy on the alternation-free fragment. This hierarchy is called the weak alternation hierarchy. It is defined as follows

- $\Sigma_0^{W\mu} = \Pi_0^{W\mu}$ consists of all the formulas without fixed-point operators;
- $\Sigma_{n+1}^{W\mu}$ is the closure of $\Sigma_n^{W\mu} \cup \Pi_n^{W\mu}$ under conjunction, disjunction, $\Box$, $\Diamond$, and the substitution: if $\varphi(X) \in \Sigma_1^{\mu}$ and $\psi \in \Sigma_{n+1}^{W\mu}$ is closed, then $\varphi(\psi) \in \Sigma_{n+1}^{W\mu}$;
- $\Pi_{n+1}^{W\mu}$ is the closure of $\Sigma_n^{W\mu} \cup \Pi_n^{W\mu}$ under conjunction, disjunction, $\Box$, $\Diamond$, and the substitution: if $\varphi(X) \in \Pi_1^{\mu}$ and $\psi \in \Pi_{n+1}^{W\mu}$ is closed, then $\varphi(\psi) \in \Pi_{n+1}^{W\mu}$; and
- $\Delta_n^{W\mu} = \Sigma_n^{W\mu} \cap \Pi_n^{W\mu}$.

The weak alternation hierarchy is strict over all Kripke frames [PLT22], that is, for all n, there is a formula $\varphi \in \Sigma_{n+1}^{\mathrm{W}\mu} \cup \Pi_{n+1}^{\mathrm{W}\mu}$ which is not equivalent to any $\psi \in \Sigma_n^{\mathrm{W}\mu} \cup \Pi_n^{\mathrm{W}\mu}$. The author, Li and Tanaka showed the strictness of the weak alternation hierarchy using formulas describing the winning region of weak parity games.

THE VARIABLE HIERARCHY. One can also measure the complexity of μ-formulas by counting how many variables they use. Fix the set of variables $\mathrm{Var} := \{X_i \mid i \in \omega\}$. A μ-formula φ is in $\mathcal{L}_\mu[n]$ iff all variables occurring in φ are in $\{X_i \mid i < n\}$. That is, φ is in $\mathcal{L}_\mu[n]$ iff it uses up to n distinct variables, up to renaming of variables. All alternation-free formulas are equivalent to formulas in $\mathcal{L}_\mu[1]$, and *vice-versa*.

Example 9. Consider the following formulas:

- $\mu X.(\nu Y.P \wedge \Box Y) \vee \Diamond X$ is in $\mathcal{L}_\mu[1]$.
- $\nu X \mu.Y(P \wedge \Diamond X) \vee (\neg P \wedge \Diamond Y)$ is in $\mathcal{L}_\mu[2]$.
- The winning region formula for parity games

$$W'_n = \eta X_n \cdots \mu X_1 \nu X_0 \bigvee_{i \leq n} (P_\exists \wedge P_i \wedge \Diamond X_i) \vee \bigvee_{i \leq n} (P_\forall \wedge P_i \wedge \Box X_i)$$

  in $\mathcal{L}_\mu[n]$. Berwanger [BGL07] showed that, for all n, W'_n can be expressed by an in $\mathcal{L}_\mu[2]$-formula.

Berwanger [Ber03] proved that the same does not happen for $\mathcal{L}_\mu[2]$:

Theorem 10. *The alternation hierarchy restricted to $\mathcal{L}_\mu[n]$ is strict, for $n \geq 2$. That is, for all k, there is a formula in $(\Sigma^\mu_{k+1} \cup \Pi^\mu_{k+1}) \cap \mathcal{L}_\mu[n]$ which is not equivalent to any formula in $(\Sigma^\mu_k \cup \Pi^\mu_k) \cap \mathcal{L}_\mu[n]$.*

Furthermore, Berwanger [BGL07] showed that more variables increase the μ-calculus' expressiveness:

Theorem 11. *The variable hierarchy is strict. That is, for all n, there is a formula in $\mathcal{L}_\mu[n+1]$ which is not equivalent to any formula in $\mathcal{L}_\mu[n]$.*

2.4 Game semantics and parity games

GAME SEMANTICS. Let $M = \langle W, R, V\rangle$ be a Kripke model, $w \in W$, and φ be a well-named μ-formula. We define an evaluation game $\mathcal{G}(M, w \models \varphi)$ to decide whether $M, w \models \varphi$. We require formulas to be well-named to simplify the description of the game semantics. As every formula is equivalent to a well-named formula, there is no loss.

The game $\mathcal{G}(M, w \models \varphi)$ has two players: verifier, who wants to show that $M, w \models \varphi$; and refuter, who wants to show that $M, w \not\models \varphi$. We denote verifier by V and refuter by R. The game positions are pairs $\langle v, \psi\rangle$ where v is in W and ψ is a subformula of φ. The game starts at $\langle w, \varphi\rangle$. The players advance in the game graph as follows:

- at $\langle v, \psi_0 \wedge \psi_1\rangle$, R chooses one of $\langle v, \psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \psi_0 \vee \psi_1\rangle$, V chooses one of $\langle v, \psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \Box\psi\rangle$, R chooses $\langle v', \psi\rangle$ with vRv';
- at $\langle v, \Diamond\psi\rangle$, V chooses $\langle v', \psi\rangle$ with vRv';
- at $\langle v, \eta X.\psi\rangle$, the players move to $\langle v, \psi\rangle$; and
- at $\langle v, X\rangle$, the players move to $\langle v, \eta X.\psi\rangle$ where $\eta X.\psi$ is a subformula of φ.

When the players go from a position labeled $\langle v, X\rangle$ to a position labeled $\langle v, \eta X.\psi\rangle$, we say the variable X was regenerated.

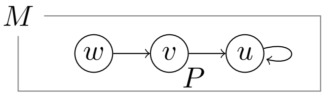
At a position $\langle v, P\rangle$, V wins iff $w \in V(P)$ and R wins iff $w \notin V(P)$. Similarly, at $\langle v, \neg P\rangle$, V wins iff $w \notin V(P)$ and R wins iff $w \in V(P)$. In an infinite run of an evaluation game, V wins iff the outermost $\eta X.\psi$ which appears infinitely often in the run is of the form $\nu X.\psi$.

Example 10. Let $M = \langle \{w, v, u\}, \{\langle w, v\rangle, \langle v, u\rangle, \langle u, u\rangle\}, V(P) = \{v\}\rangle$. Then V wins the evaluation game $\mathcal{G}(M, w \models \mu X.P \vee \Diamond X)$. V's winning strategy σ is to choose $\langle w, \Diamond X\rangle$ at $\langle w, P \vee \Diamond X\rangle$, and $\langle v, P\rangle$ at $\langle v, P\rangle$. The player R never has any move available. Figure 2.5 pictures M, the evaluation game $\mathcal{G}(M, w \models \mu X.P \vee \Diamond X)$ and σ.

Game semantics and Kripke semantics are equivalent:

Table 2.3: Rules of evaluation games for modal μ-calculus.

Verifier		Refuter	
Position	Admissible moves	Position	Admissible moves
$\langle w, \psi_1 \vee \psi_2 \rangle$	$\{\langle w, \psi_1 \rangle, \langle \psi_2 \rangle\}$	$\langle w, \psi_1 \wedge \psi_2 \rangle$	$\{\langle w, \psi_1 \rangle, \langle w, \psi_2 \rangle\}$
$\langle w, \Diamond\psi \rangle$	$\{\langle v, \psi \rangle \mid \langle w, v \rangle \in R\}$	$\langle w, \Box\psi \rangle$	$\{\langle v, \psi \rangle \mid \langle w, v \rangle \in R\}$
$\langle w, P \rangle$ and $w \notin V(P)$	$\emptyset$	$\langle w, P \rangle$ and $w \in V(P)$	$\emptyset$
$\langle w, \neg P \rangle$ and $w \in V(P)$	$\emptyset$	$\langle w, \neg P \rangle$ and $w \notin V(P)$	$\emptyset$
$\langle w, \mu X.\psi_X \rangle$	$\{\langle w, \mu X.\psi_X \rangle\}$	$\langle w, \nu X.\psi_X \rangle$	$\{\langle w, \nu X.\psi_X \rangle\}$
$\langle w, X \rangle$	$\{\langle w, \psi_X \rangle\}$	$\langle w, X \rangle$	$\{\langle w, \psi_X \rangle\}$



$\mathcal{G}(M, w \models \mu X.P \vee \Diamond X)$

$$\begin{tikzcd}
\langle w, \mu X.P \vee \Diamond X \rangle \arrow[r] & \langle w, P \vee \Diamond X \rangle \arrow[r] \arrow[d] & \langle w, \Diamond X \rangle \arrow[d] \\
 & \langle w, P \rangle & \langle v, X \rangle \arrow[d] \\
\langle v, \Diamond X \rangle \arrow[d] & \langle v, P \vee \Diamond X \rangle \arrow[l] \arrow[d] & \langle v, \mu X.P \vee \Diamond X \rangle \arrow[l] \\
\langle u, X \rangle \arrow[d] & \langle v, P \rangle & \\
\langle u, \mu X.P \vee \Diamond X \rangle \arrow[r] & \langle u, P \vee \Diamond X \rangle \arrow[r] \arrow[d] & \langle u, \Diamond X \rangle \arrow[d] \\
 & \langle u, P \rangle & \langle u, X \rangle \arrow[d] \\
\cdots & \langle u, P \vee \Diamond X \rangle \arrow[l] \arrow[d] & \langle u, \mu X.P \vee \Diamond X \rangle \arrow[l] \\
 & \langle u, P \rangle &
\end{tikzcd}$$

Figure 2.5: The model M of Example 10 and the evaluation game $\mathcal{G}(M, w \models \mu X.P \vee \Diamond X)$. The thick arrows describe V's winning strategy.

Theorem 12. *Let $M = \langle W, R, V \rangle$ be a Kripke model, $w \in W$ and φ be a μ-sentence, then*

$$\mathsf{V} \textit{ wins } \mathcal{G}(M, w \models \varphi) \textit{ iff } M, w \models \varphi, \textit{ and } \mathsf{R} \textit{ wins } \mathcal{G}(M, w \models \varphi) \textit{ iff } M, w \not\models \varphi.$$

Proof. We follow the proof from [Ong15].

Let $M = \langle W, R, V \rangle$ be a Kripke model, $w \in W$ be a world and φ be a well-named μ-formula. Suppose $M, w \models \varphi$.

We list the fixed-point subformulas of φ in decreasing order:

$$\eta_0 X_0.\chi_0, \eta_1 X_1.\chi_1, \ldots, \eta_{n-1} X_{n-1}.\chi_{n-1}.$$

That is, if $i \leq j$ then either $\eta_i X_i.\chi_i$ and $\eta_j X_j.\chi_j$ are uncomparable, or $\eta_j X_j.\chi_j \in \mathrm{Sub}(\eta_i X_i.\chi_i)$. We modify M so that when evaluating subformulas of φ where some

X_i is free, the evaluation agrees with the interpretation of the fixed-points in φ. For $i < n$, let

$$M_0 := M; \text{ and}$$
$$M_{i+1} := M_i[X_i \mapsto \|\eta_i X_i.\chi_i\|^{M_i}].$$

Given $v \in W$ and $\psi \in \mathrm{Sub}(\varphi)$, we say $\langle v, \psi\rangle$ is a true position of the evaluation game iff $M_n, w \models \psi$. In particular $\langle w, \varphi\rangle$ is a true position.

Now, list the fixed-point formulas starting with a μ-operator in decreasing order:

$$\mu_0 Y_0.\theta_0, \mu_1 Y_1.\theta_1, \dots, \mu_{n-1} Y_{n-1}.\theta_{m-1}.$$

Note that these are the fixed-points that verifier does not want to regenerate infinitely often in a play. We define refinements of the M_i, where we interpret the μ-variables by their approximants. Let $\alpha = \langle \alpha_0, \dots, \alpha_{m-1}\rangle$ be a sequence of m-many ordinals. We call such sequences μ-signatures. We order the μ-signatures by the lexicographical order. Given two signatures α and β, we say $\alpha =_k \beta$ iff $\alpha_l = \beta_l$ for all $l < k$.

Given a μ-signature α, define

$$M_0^\alpha := M;$$
$$M_{i+1}^\alpha := M_i[X_i \mapsto \|\nu_i X_i.\chi_i\|^{M_i}] \text{ if } X_i \text{ is a } \nu\text{-variable; and}$$
$$M_{i+1}^\alpha := M_i[X_i \mapsto \|\mu_i^\alpha X_i.\chi_i\|^{M_i}] \text{ if } X_i \text{ is a } \mu\text{-variable.}$$

By the well-ordering of the ordinals, if $\langle v, \psi\rangle$ is a true position then there is a least signature α such that $M_n^\alpha, w \models \psi$. Call this least signature $\mathrm{sig}_\mu(v, \psi)$.

When moving through the evaluation game $\mathcal{G}(M, w \models \varphi)$, the signatures of the positions mostly decrease:

- $\mathrm{sig}_\mu(v, \psi_0 \vee \psi_1) = \mathrm{sig}_\mu(v, \psi_i)$ for $i \in \{0, 1\}$;
- $\mathrm{sig}_\mu(v, \psi_0 \wedge \psi_1) \leq \mathrm{sig}_\mu(v, \psi_i)$ for $i \in \{0, 1\}$;
- $\mathrm{sig}_\mu(v, \Diamond\psi) = \mathrm{sig}_\mu(v', \psi)$ for some v' such that vRv';
- $\mathrm{sig}_\mu(v, \Box\psi) \geq \mathrm{sig}_\mu(v', \psi)$ for all v' such that vRv';
- if X_i is a ν-variable, $\mathrm{sig}_\mu(v, \nu X_i.\chi_i) = \mathrm{sig}_\mu(v, X_i) = \mathrm{sig}_\mu(v, \chi_i)$; and
- if X_i is a μ-variable, $\mathrm{sig}_\mu(v, \nu X_i.\chi_i) =_{i-1} \mathrm{sig}_\mu(v, X_i) =_{i-1} \mathrm{sig}_\mu(v, \chi_i)$ and $\mathrm{sig}_\mu(v, X_j)(j) > \mathrm{sig}_\mu(v, \chi_j)(j)$.

We can now define a winning strategy σ for V in $\mathcal{G}(M, w \models \varphi)$. At a true position $\langle v, \psi_0 \vee \psi_1\rangle$ choose $\langle v, \psi_i\rangle$ such that ψ_i is a true position. If both $\langle v, \psi_i\rangle$ are true positions, choose the one with least μ-signature. At a true position $\langle v, \Diamond\psi\rangle$ choose $\langle v', \psi\rangle$ such that vRv' and $\langle v', \psi\rangle$ has the smallest possible signature. At fixed-point formulas and variables, there is only one choice of move. On false positions, V's choice does not matter.

Now consider a play of $\mathcal{G}(M, w \models \varphi)$ where V uses σ. Note that if the players are in a true position, R can only move to other true positions. Therefore, if V uses σ

in a play, then all the positions are true positions. In particular, the last position in a finite play is true. So V wins finite plays.

Now suppose ρ is an infinite play where V uses σ. For a contradiction, suppose the outermost infinitely often repeated variable is the μ-variable X_i. The decrease of μ-signatures implies the existence of an infinite descending sequence of ordinals. This is a contradiction. Therefore, the outermost infinitely occurring variable in an infinite play is a μ-variable (and the μ-signatures are eventually constant). Therefore σ is winning for infinite plays too. We conclude V wins $\mathcal{G}(M, w \models \varphi)$.

Now, if $M, w \not\models \varphi$, we can build a winning strategy τ for R similarly. The difference is that we consider false positions and ν-signatures. □

PARITY GAMES. A parity game is a tuple $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$. We suppose $V_\exists$ and $V_\forall$ are disjoint sets of vertices; $E \subseteq (V_\exists \cup V_\forall)^2$ is a set of edges; and $\Omega : V_\exists \cup V_\forall \to n$ is a priority function, for some $n \in \omega$. While playing $\mathcal{P}$, two players—$\exists$ and $\forall$—move a token in the graph $\langle V_\exists \cup V_\forall, E\rangle$. A play is the resulting path on the graph. We say a parity game is weak iff Ω is non-increasing.

In any given moment, player $\exists$ chooses the next vertex when the token is in some element of $V_\exists$; and player $\forall$ chooses the next vertex when the token is in some element of $V_\forall$. If a player cannot move, then the other player wins. In an infinite play ρ, the winner is determined by the following parity condition: $\exists$ wins ρ iff the largest priority which appears infinitely often in ρ is even; otherwise, $\forall$ wins ρ. $\exists$ wins the parity game $\mathcal{P}$ iff $\exists$ has a winning strategy; a winning strategy for $\exists$ is a function σ from $V_\exists$ to $V_\exists \cup V_\forall$, where, if $\exists$ follows σ, all resulting plays are winning for them. Similarly, $\forall$ wins $\mathcal{P}$ iff $\forall$ has a winning strategy. Parity games are Borel Gale–Stewart games. By the Borel Determinacy, one of the players has a winning strategy.

Example 11. Figure 2.6 pictures a parity game where $\exists$ wins in finite many plays. In any infinite play of this parity game, 1 appears infinitely often, so $\forall$ wins. Note the resemblance to the evaluation game from Example 10.

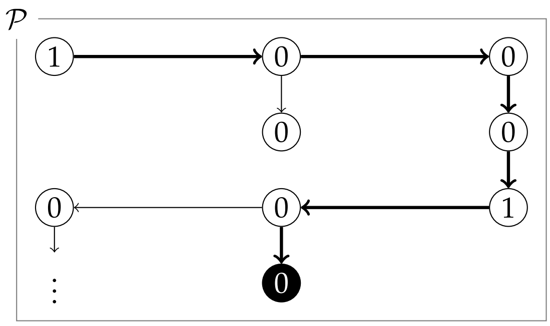


Figure 2.6: The parity game $\mathcal{P}$ from Example 11. Black nodes are owned by $\forall$. The thick arrows describe $\exists$'s winning strategy.

EVALUATION GAMES AS PARITY GAMES. Given an evaluation game $\mathcal{G}(M, w \models \varphi)$, we can define an equivalent parity game $\mathcal{G}^P(M, w \models \varphi)$; both games are played on the same graph and V wins $\mathcal{G}(M, w \models \varphi)$ iff $\exists$ wins $\mathcal{G}'$.

Let $\langle V, E\rangle$ be the game graph of $\mathcal{G}(M, w \models \varphi)$. We now define $\mathcal{G}(M, w \models \varphi) = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$. Let $V_\exists$ be the set of vertices owned by V in $\mathcal{G}(M, w \models \varphi)$; and $V_\forall$ be the set of vertices owned by R in $\mathcal{G}(M, w \models \varphi)$. We let the set E of edges be as in the graph of $\mathcal{G}(M, w \models \varphi)$. Remember that each vertex $v \in V$ is of the form $\langle w, \psi\rangle$, where ψ is a subformula of φ and $w \in W$. Define

$$\Omega(\langle \psi, w\rangle) = \begin{cases} 2i + \varepsilon & \text{if } \psi = \mu X.\psi' \in \Sigma^\mu_{2i+\varepsilon} \setminus \Pi^\mu_{2i+\varepsilon}, \\ 2i & \text{if } \psi = \nu X.\psi' \in \Pi^\mu_{2i+\varepsilon} \setminus \Sigma^\mu_{2i+\varepsilon}, \\ 0 & \text{otherwise;} \end{cases}$$

with $\varepsilon \in \{0, 1\}$. It is straightforward to show that, given a run $\rho \in W^\omega$, the outermost operator appearing infinitely often in ρ is a ν-operator iff the greatest parity appearing infinitely often is even. Therefore, V wins $\mathcal{G}(M, w \models \varphi)$ starting at an edge iff V wins $\mathcal{G}'$ starting at w. In the following, we identify $\mathcal{G}(M, w \models \varphi)$ and $\mathcal{G}'$.

Example 12. If we rewrite the evaluation game $\mathcal{G}(M, w \models \varphi)$ of Figure 2.5, we get the parity game in Figure 2.6.

PARITY GAMES AS KRIPKE MODELS. Let $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ be a parity game. We define a Kripke model $\mathcal{P}^{\mathrm{K}} = \langle W, R, V\rangle$ by

- $W = V_\exists \cup V_\forall$;
- $R = E$;
- $\|P_\exists\| = V_\exists$, $\|P_\forall\| = V_\forall$ and $\|P_n\| = \Omega^{-1}(n)$, for all $n \in \omega$.

Given $n \in \omega$, define:

$$W'_n = \eta X_n \cdots \mu X_1 \nu X_0 \bigvee_{i \leq n} (P_\exists \wedge P_i \wedge \Diamond X_i) \vee \bigvee_{i \leq n} (\neg P_\forall \wedge P_i \wedge \Box X_i).$$

Bradfield [Bra98a] showed that the W'_n define the winning regions of player $\exists$ in a parity game using priorities up to n. That is, V wins $\mathcal{P}$ starting at v iff $\mathcal{P}^{\mathrm{K}}, v \models W'_n$, whenever $v \in V_\exists \cup V_\forall$ and $\max\{\Omega(v) \mid v \in V_\exists \cup V_\forall\} \leq n$. Again, we identify a parity game with its correspondent Kripke model.

In [Bra98a], Bradfield constructs a (recursive) parity game $\mathcal{G}$ such that, for all $n \in \omega$, $\|W'_n\|^{\mathcal{G}}$ is a strict Σ^μ_n, that is, there is no formula $\varphi \in \Sigma^\mu_{n-1} \cup \Pi^\mu_{n-1}$ such that $\|W'_n\|^{\mathcal{G}} = \|\varphi\|^{\mathcal{G}}$. Bradfield shows this by proving that $\|W'_n\|^{\mathcal{G}}$ is a strict arithmetic-Σ^μ_n set. The strictness of the μ-arithmetic alternation hierarchy was shown by Lubarsky [Lub93].

An alternative proof of the strictness can be found in Alberucci [Alb02], who uses alternating tree automata. The author, Li and Tanaka [PLT22] adapted this proof to show the strictness of the weak alternation hierarchy of alternation-free formulas, using the μ-calculus and its game semantics.

Chapter 3

The collapse to modal logic on Kripke semantics

In this chapter, we characterize some classes of frames on which the μ-calculus collapses to modal logic. We use this result to study the μ-calculus' alternation hierarchy on the logics S4.2, S4.3, S4.3.2, and S4.4. We show that the alternation hierarchy collapses to its alternation-free fragment on frames of S4.2 and S4.3; and that the alternation hierarchy collapses to modal logic over frames of S4.3.2 and S4.4. These logics have been also studied from the point of view of epistemic logic. We study the effect on degrees of ignorance implied by the collapse of the alternation hierarchy. We also show that our approach is not viable for degrees of doubt. The results on this chapter are joint work with Kazuyuki Tanaka.

3.1 Warm-up: collapse on S5

THE MODAL LOGIC S5. We define S5 as the closure under **Nec** and **MP** of the set of all propositional tautologies and all instances of the axioms K, T, 4, and 5. This logic is commonly studied in epistemic logic [Fag+03; van10]. S5 also axiomatizes Leibniz's logic of necessity, where φ is necessary iff it is true in *all* possible worlds.

If a frame $F = \langle W, R\rangle$ satisfies S5, then R is an equivalence relation, that is, R is reflexive, transitive and symmetric. In this case, we say F itself is an equivalence relation. We can further show:

Proposition 13. S5 *is complete over equivalence relations.*

From the completeness over equivalence relations, it follows that every string of modal operators is equivalent to its last modal operator: $\triangle \cdots \triangle\Box\varphi$ is equivalent to $\Box\varphi$, and $\triangle \cdots \triangle\Diamond\varphi$ is equivalent to $\Diamond\varphi$. A similar idea can be used to show the following:

Lemma 14 (Alberucci, Facchini [AF09b]). *Let $M = \langle W, R, V\rangle$ be a transitive Kripke model, w' be a member of the strongly connected component of w, φ be a μ-formula, and $\triangle \in \{\Box, \Diamond\}$. Then $w \in \|\triangle\varphi\|^M$ iff $w' \in \|\triangle\varphi\|^M$.*

Proof. Suppose w and w' are in the same connected component, that is, there are $m, n \in \omega$ such that $wR^m w'$ and $w'R^n w$. By the transitivity of R, it follows that wRw' and $w'Rw$.

Now, suppose $w \models \Box\varphi$, then $v \models \varphi$ for all v accessible from w. As wRw', if v' is accessible from w', then v' is also accessible from w. Therefore $w'Rv'$ implies $v' \models \varphi$. And so $w' \models \Box\varphi$. We can do the same to show that $w' \models \Box\varphi$ implies $w \models \Box\varphi$. Also, the proof that $w' \models \Diamond\varphi$ is equivalent to $w \models \Diamond\varphi$ is similar. $\square$

THE COLLAPSE. Alberucci and Facchini [AF09b] showed that the alternation hierarchy collapses to modal logic over frames of S5. We do a slight modification on their proof. The aim of this modification is to make the proof easier to understand, but we do explain the original proof later.

Given a transitive model $M = \langle W, R, V\rangle$, we suppose that, for a world $w \in W$, $M, w \models \varphi(\varphi(\top))$ and $M, w \not\models \varphi(\varphi(\varphi(\top)))$, and get to a contradiction. In our proof, the players V and R simultaneously play the evaluation games $\mathcal{G}(M, w \models \varphi(\varphi(\top)))$ and $\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top))))$. In the former, V uses their winning strategy σ, and, in the later, V plays moves analogous to the ones indicated by σ. Similarly, R uses his winning strategy τ on the later and a modified τ on the former.

For example, let us start the games above on the positions $\langle v, \theta \vee \Box\psi(\top)\rangle$ and $\langle v, \theta \vee \Box\psi(\varphi(\top))\rangle$. Suppose V moves to $\langle v, \Box\psi(\top)\rangle$ on the first game, then they move to $\langle v, \Box\psi(\varphi(\top))\rangle$ on the second game. Now, if R moves to $\langle v', \psi(\varphi(\top))\rangle$ on the second game, they move to $\langle v', \psi(\top)\rangle$ on the first game. We illustrate these moves in Figure 3.1.

$$
\begin{array}{c}
\mathcal{G}(M, w \models \varphi(\varphi(\top))) \\
\vdots \\
\downarrow \\
\langle v, \theta \vee \Box\psi(\top)\rangle \\
\downarrow \\
\langle v, \Box\psi(\top)\rangle \\
\downarrow \\
\langle v', \psi(\top)\rangle \\
\downarrow \\
\vdots
\end{array}
\qquad
\begin{array}{c}
\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top)))) \\
\vdots \\
\downarrow \\
\langle v, \theta \vee \Box\psi(\varphi(\top))\rangle \\
\downarrow \\
\langle v, \Box\psi(\varphi(\top))\rangle \\
\downarrow \\
\langle v', \psi(\varphi(\top))\rangle \\
\downarrow \\
\vdots
\end{array}
$$

Figure 3.1: Simultaneous plays of the evaluation games $\mathcal{G}(M, w \models \varphi(\varphi(\top)))$ and $\mathcal{G}(M, w \models \varphi(\varphi(\varphi(\top))))$.

If the players continue like this, we will eventually reach a contradiction, either stating that $M, v \models P$ and $M, v \not\models P$ for some v, or violating Lemma 14. And so $w \in W$, $M, w \models \varphi(\varphi(\top))$ is equivalent to $M, w \models \varphi(\varphi(\varphi(\top)))$.

Lemma 15 is the key result for the collapse of the alternation hierarchy over equivalence relations. We define $\varphi^0(X) := X$ and $\varphi^{n+1}(X) := \varphi(\varphi^n(X))$. This notation will be helpful when generalizing the Lemma 15 to other frame classes and semantics.

Lemma 15 (Alberucci, Facchini [AF09b]). *Let $M = \langle W, R, V \rangle$ be a Kripke model where R is an equivalence relation, and $\nu X.\varphi$ be a well-named μ-formula. Then*

$$\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M \text{ and } \|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M.$$

Proof. We first show that $\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M$. Let $M = \langle W, R, V \rangle$ be a Kripke model where R is an equivalence relation, and $\nu X.\varphi$ is a well-named μ-formula. Therefore X has a unique occurrence in φ and is in the scope of some modal operator. Let α and β be formulas such that φ is of the form $\alpha(\triangle\beta(X))$ with $\triangle \in \{\Box, \Diamond\}$.

We show that $\nu X.\varphi$ is equivalent to $\varphi^2(\top)$. As X is positive in $\varphi(X)$, we have that $\|\varphi^3(\top)\|^M \subseteq \|\varphi^2(\top)\|^M$. So we need only to show that $\|\varphi^2(\top)\|^M \subseteq \|\varphi^3(\top)\|^M$.

For a contradiction, suppose that $w \in \|\varphi^2(\top)\|^M$ and $w \notin \|\varphi^3(\top)\|^M$. Then V has a winning strategy σ for the evaluation game $\mathcal{G}_2 = \mathcal{G}(M, w \models \varphi^2(\top))$; and R has a winning strategy τ for the evaluation game $\mathcal{G}_3 = \mathcal{G}(M, w \models \varphi^3(\top))$ We use σ and τ to define strategies σ' for V in $\mathcal{G}_3$ and τ' for R in $\mathcal{G}_2$.

Suppose the players are in positions $\langle v, \psi(\top)\rangle$ in $\mathcal{G}_2$ and $\langle v, \psi(\varphi(\top))\rangle$ in $\mathcal{G}_3$. Both have the same owner, that is, either it is V's turn in both games, or it is R's turn in both games. Suppose it is V's turn; in $\mathcal{G}_2$, they play $\sigma(\langle v, \psi(\top)\rangle) = \langle v', \psi'(\top)\rangle$ using their existing strategy, and $\sigma'(\langle v, \psi(\varphi(\top))\rangle) := \langle v', \psi'(\varphi(\top))\rangle$ in $\mathcal{G}_3$. Similarly, if it is R's turn, they play $\tau(\langle v, \psi(\varphi(\top))\rangle) = \langle v', \psi'(\varphi(\top))\rangle$ in $\mathcal{G}_3$ using their existing strategy, and play $\tau'(\langle v, \psi(\top)\rangle) := \langle v', \psi'(\top)\rangle$ in $\mathcal{G}_2$.

The players continue both games following the strategies described above until they get to a position of the form $\langle v, P\rangle$ (or $\langle v, \neg P\rangle$) in both games; or they get to positions of the form $\langle w'', \triangle\beta(\top)\rangle$ in $\mathcal{G}_2$ and $\langle w'', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_3$.

Case 1. Suppose they players are in a position $\langle v, P\rangle$ in both games. As σ is winning for V in $\mathcal{G}_2$, $v \in \|P\|^M$. As τ is winning for R in $\mathcal{G}_3$, $v \notin \|P\|^M$. And so we have a contradiction. A similar contradiction is reached if they are in a position $\langle v, \neg P\rangle$.

Case 2. Suppose the players are in positions of the form $\langle w'', \triangle\beta(\top)\rangle$ in $\mathcal{G}_2$ and $\langle w'', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_3$. As τ is a winning strategy for R in $\mathcal{G}_3$, $w'' \notin \|\triangle\beta(\varphi(\top))\|^M$. Previously, the players must have been through some a position $\langle w', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_2$. As σ is a winning strategy for V in $\mathcal{G}_2$, $w' \in \|\triangle\beta(\varphi(\top))\|^M$ By the definition of the game semantics, $w'R^*w''$. As R is an equivalence relation, $w'Rw''$ and $w''Rw'$. By Lemma 14, $w' \in \|\triangle\beta(\varphi(\top))\|^M$ iff $w'' \in \|\triangle\beta(\varphi(\top))\|^M$; and we have our contradiction.

Either way, we conclude that $\|\varphi^2(\top)\|^M \subseteq \|\varphi^3(\top)\|^M$.

We could do as above and show that $\|\varphi^3(\bot)\|^M \subseteq \|\varphi^2(\bot)\|^M$, but we prove $\|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M$ by a direct calculation:

$$\mu X.\varphi \equiv \neg\nu X.\neg\varphi(\neg X) \equiv \neg(\neg\varphi(\neg\neg\varphi(\neg\top))) \equiv \varphi(\varphi(\bot)).$$

The first equivalence follows by an alternative definition of $\mu X.\varphi$, the second by the first half of Lemma 15, and the third by negation cancelling. $\Box$

Alberucci and Facchini's original proof constructs a winning strategy σ' for V in $\mathcal{G}(M, w \models \varphi^3(\top))$ using the winning strategy σ for V in $\mathcal{G}(M, w \models \varphi^2(\top))$, by first emulating σ and then using Lemma 14 to find a new winning strategy when necessary.

Theorem 16 (Alberucci, Facchini [AF09b]). *The alternation hierarchy collapses to modal logic over equivalence relations.*

Proof. We argue by structural induction on μ-formulas. First, some of the easy cases. P is equivalent to a modal formula, as it is a modal formula. If φ and ψ are equivalent to μ-formulas φ' and ψ' then $\varphi \wedge \psi$ is equivalent to $\varphi' \wedge \psi'$. If φ is equivalent to the modal formula φ', then $\Box\varphi$ is equivalent to $\Box\varphi'$.

Now, the interesting cases. Suppose φ is equivalent to a modal formula φ'. Then $\nu X.\varphi$ is equivalent to $\nu X.\varphi'$. By Lemma 15, $\nu X.\varphi'$ is equivalent to $\varphi'(\varphi'(\top))$, which is a modal formula. Similarly, $\mu X.\varphi'$ is equivalent to $\varphi'(\varphi'(\bot))$ by Lemma 15.

Therefore every μ-formula is equivalent to a modal formula over equivalence relations. ☐

Alternatively, given a μ-formula, one could also repeatedly substitute its fixed-point operators by iterations of the respective subformulas.

3.2 Warm-up: collapse on S4.3.2

In this section, we prove the collapse of the alternation hierarchy on frames of S4.3.2; showing how to generalize the collapse to modal logic over equivalence relations to bigger classes of frames. Our proof uses game semantics, and follows roughly the same idea of the section above.

Our objective in this section is not to study S4.3.2 in specific, but to analyze a simple generalization of the collapse of the alternation hierarchy to modal logic over equivalence relations. Namely, we prove:

Theorem 17 (P., Tanaka [PT22]). *The alternation hierarchy collapses to modal logic over* S4.3.2.

We will further generalize it in the next section.

THE MODAL LOGIC S4.3.2. The modal logic S4.3.2 is obtained by adding the axiom

$$(\Diamond P \wedge \Diamond\Box Q) \to \Box(\Diamond P \vee Q)$$

to S4. The logic S4.3.2 is complete for reflexive and transitive frames which can be decomposed into two equivalence classes. This logic was first studied by Zeman [Zem68].

FRAMES OF S4.3.2. We first generalize Lemma 15. In an equivalence relation $F = \langle W, R\rangle$, we had that wRv implies vRw. In S4.3.2 frames, we get a similar property:

Lemma 18. *Let $F = \langle W, R\rangle$ be a Kripke frame. If F satisfies* S4.3.2, *then*

$$wRv \wedge vRu \to vRw \vee uRv.$$

Proof. Remember that S4.3.2 is obtained by adding to S4 the axiom

$$(\Diamond P \wedge \Diamond\Box Q) \to \Box(\Diamond P \vee Q).$$

So any S4.3.2 is reflexive and transitive.

Suppose that F satisfies S4.3.2 and that $wRvRu$ holds. Define a model M over F by taking $V(P) := \{w\}$ and $V(Q) := \{u' \mid uRu'\}$. Since wRw, $M, w \models \Diamond P$. Since $M, u \models \Box Q$, $M, w \models \Diamond\Box Q$. So $M, w \models \Box(\Diamond P \vee Q)$. In particular, $M, v \models \Diamond P \vee Q$. If $M, v \models \Diamond P$, then vRw; if $M, v \models Q$, then uRv. $\Box$

Alternatively, one can characterize the S4.3.2 frames as follows:[1]

Lemma 19. *If $F = \langle W, R\rangle$ satisfies* S4.3.2, *then we can decompose W into disjoint sets* Ini *and* Fin *such that:*

1. xRx' *for all* $x, x' \in \mathrm{Ini}$*;*
2. yRy' *for all* $y, y' \in \mathrm{Fin}$*; and*
3. $xRy \wedge \neg yRx$ *for all* $x \in \mathrm{Ini}$*,* $y \in \mathrm{Fin}$.

Proof. First, as $F = \langle W, R\rangle$ satisfies S4.3.2, F is transitive and reflexive. Now, we fix $w \in W$ and suppose $W = \{w' \mid wRw'\}$, as the evaluation of any formula on w depends only on worlds accessible from w.

Define $\mathrm{Ini} := \{v \mid vRw\}$. By the transitivity of R, xRx' for all $x, x' \in \mathrm{Ini}$. Define $\mathrm{Fin} := W \setminus \mathrm{Ini}$. By the definition of Ini and Fin, $\mathrm{Ini} \cup \mathrm{Fin} = W$ and $\mathrm{Ini} \cap \mathrm{Fin} = \emptyset$ and $\neg yRx$ for all $x \in \mathrm{Ini}$ and $y \in \mathrm{Fin}$. By our supposition on W, xRy for all $x \in \mathrm{Ini}$ and $y \in \mathrm{Fin}$.

If $\mathrm{Fin} = \emptyset$, we have nothing to do. Otherwise, let $v, u \in \mathrm{Fin}$. Define a model M over F by taking $V(P) := \{w\}$ and $V(Q) := \{v' \mid vRv'\}$. Then $M, w \models \Diamond P \wedge \Diamond\Box Q$, and so $M, w \models \Box(\Diamond P \vee Q)$. In particular $M, u \models \Diamond P \vee Q$. As $M, u \models \Diamond P$ implies uRw and $u \notin \mathrm{Ini}$, we must have $M, u \models Q$. That is, vRu. We conclude that yRy' for all $y, y' \in \mathrm{Fin}$. $\Box$

THE COLLAPSE TO MODAL LOGIC. While Lemma 18 is weaker than what we have on equivalence relations, it is good enough to prove a version of Lemma 15 for S4.3.2 frames:

Lemma 20. *If $M = \langle W, R, V\rangle$ is a Kripke model where $F = \langle W, R\rangle$ satisfies* S4.3.2*, and $\eta X.\varphi$ is a well-named μ-formula, then*

$$\|\nu X.\varphi\|^M = \|\varphi^3(\top)\|^M \text{ and } \|\mu X.\varphi\|^M = \|\varphi^3(\bot)\|^M.$$

Proof. Let $M = \langle W, R, V\rangle$ be a Kripke model where $F = \langle W, R\rangle$ satisfies S4.3.2, and $\nu X.\varphi$ is a well-named μ-formula. We suppose that φ is of the form $\alpha(\triangle\beta(X))$ with $\triangle \in \{\Box, \Diamond\}$.

We show that $\nu X.\varphi$ is equivalent to $\varphi^3(\top)$. As X is positive in $\varphi(X)$, we have that $\|\varphi^4(\top)\|^M \subseteq \|\varphi^3(\top)\|^M$. So we need only to show that $\|\varphi^3(\top)\|^M \subseteq \|\varphi^4(\top)\|^M$.

For a contradiction, suppose that $w \in \|\varphi^3(\top)\|^M$ and $w \notin \|\varphi^4(\top)\|^M$. Then V has a winning strategy σ for the evaluation game $\mathcal{G}_3 = \mathcal{G}(M, w \models \varphi^3(\top))$; and R has

[1] This characterization is originally by Zeman, building on his semantics for S4.4—where Ini has at most one element. See [Zem71; Zem72]. Semantics for S4.4 get the cooler name though: Zeman calls it "end of the world semantics".

a winning strategy τ for the evaluation game $\mathcal{G}_4 = \mathcal{G}(M, w \models \varphi^4(\top))$ We use σ and τ to define strategies σ' for V in $\mathcal{G}_4$ and τ' for R in $\mathcal{G}_3$.

Suppose the players are in positions $\langle v, \psi(\top)\rangle$ in $\mathcal{G}_3$ and $\langle v, \psi(\varphi(\top))\rangle$ in $\mathcal{G}_4$, both owned by the same player; that is, either it is V's turn in both games, or it is R's turn in both games. Suppose it is V's turn; in $\mathcal{G}_3$, they play $\sigma(\langle v, \psi(\top)\rangle) = \langle v', \psi'(\top)\rangle$ using their existing strategy, and $\sigma'(\langle v, \psi(\varphi(\top))\rangle) := \langle v', \psi'(\varphi(\top))\rangle$ in $\mathcal{G}_4$. Similarly, if it is R's turn, they play $\tau(\langle v, \psi(\varphi(\top))\rangle) = \langle v', \psi'(\varphi(\top))\rangle$ in $\mathcal{G}_4$ using their existing strategy, and play $\tau'(\langle v, \psi(\top)\rangle) := \langle v', \psi'(\top)\rangle$ in $\mathcal{G}_3$.

The players continue both games following the strategies described above until they get to a position of the form $\langle v, P\rangle$ (or $\langle v, \neg P\rangle$) in both games; or they get to positions of the form $\langle w''', \psi(\top)\rangle$ in $\mathcal{G}_3$ and $\langle w''', \psi(\varphi(\top))\rangle$ in $\mathcal{G}_4$.

Case 1. Suppose they players are in a position $\langle v, P\rangle$ in both games. As σ is winning for V in $\mathcal{G}_3$, $v \in \|P\|^M$. As τ is winning for R in $\mathcal{G}_4$, $v \notin \|P\|^M$. And so we have a contradiction. A similar contradiction is reached if they are in a position $\langle v, \neg P\rangle$.

Case 2. Suppose the players are in positions of the form $\langle w''', \triangle\beta(\top)\rangle$ in $\mathcal{G}_3$ and $\langle w''', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_4$. Previously, the players must have been through some a positions $\langle w', \triangle\beta(\varphi^2(\top))\rangle$ and $\langle w'', \triangle\beta(\varphi((\top))\rangle$ in $\mathcal{G}_3$; and positions $\langle w', \triangle\beta(\varphi^3(\top))\rangle$ and $\langle w'', \triangle\beta(\varphi^2(\top))\rangle$ in $\mathcal{G}_4$. As the frame F is transitive, $w'Rw''$ and $w''Rw'''$. By Lemma 18, either $w''Rw'$ or $w'''Rw''$. Remember that V played $\mathcal{G}_3$ with their winning strategy σ and R played $\mathcal{G}_4$ with their winning strategy τ If $w''Rw'$, as the players had been in the position $\langle w', \triangle\beta(\varphi^2(\top))\rangle$ in $\mathcal{G}_3$ and in the position $\langle w'', \triangle\beta(\varphi^2(\top))\rangle$ in $\mathcal{G}_4$, therefore $w' \in \|\triangle\beta(\varphi^2(\top))\|^M$ and $w'' \notin \|\triangle\beta(\varphi^2(\top))\|^M$. This contradicts Lemma 14. Similarly if $w'''Rw''$, as the players had been in the position $\langle w'', \triangle\beta(\varphi((\top))\rangle$ in $\mathcal{G}_3$ and in position $\langle w''', \triangle\beta(\varphi(\top))\rangle$, we again contradict Lemma 14.

Either way, we conclude that $\|\varphi^3(\top)\|^M \subseteq \|\varphi^4(\top)\|^M$. □

We prove Theorem 17 as we proved Theorem 16:

Proof of Theorem 17. We argue by structural induction on μ-formulas. We only prove the interesting cases. Suppose $\varphi(X)$ is equivalent to a modal formula ψ. Then $\nu X.\varphi$ is equivalent to $\nu X.\psi$. By Lemma 20, $\nu X.\psi$ is equivalent to $\psi^3(\top)$, which is a modal formula. Similarly, $\mu X.\psi$ is equivalent to $\psi^3(\bot)$. Therefore every μ-formula is equivalent to a modal formula over frames which satisfy S4.3.2. □

3.3 Generalizing the collapse to modal logic

In this section we generalize the proofs in Sections 3.1 and 3.2 to bigger classes of frames.

Let $M = \langle W, R, V\rangle$ be a Kripke model and $w \in W$. Denote the set of worlds accessible from w by $wR := \{w' \in W \mid wRw'\}$. Denote the transitive closure of R by R^*.

Theorem 21. *Let $\mathcal{F}$ be a class of Kripke frames. Suppose there is n such that, for all frame $F = \langle W, R\rangle \in \mathcal{F}$ and for all sequence $w_0 R^* w_1 R^* \cdots R^* w_n$, there is $i < j \leq n$ such that $w_i R = w_j R$. The μ-calculus' alternation hierarchy collapses to modal logic over $\mathcal{F}$.*

The lemma below follows from the definition of the semantics for the modalities $\Box$ and $\Diamond$:

Lemma 22. *Let $M = \langle W, R, V\rangle$ be a Kripke model, $w, w' \in W$, and $wR = w'R$. If φ be a μ-formula and $\triangle \in \{\Box, \Diamond\}$, then $w \in \|\triangle\varphi\|^M$ iff $w' \in \|\triangle\varphi\|^M$.*

We now prove the Lemma 23 as we proved Lemmas 15 and 20:

Lemma 23. *Let $M = \langle M, R, V\rangle$ be a Kripke model. Suppose there is n such that, for all sequence $w_0R^*w_1R^* \cdots R^*w_n$, there is $i < j \leq n$ such that $w_iR = w_jR$. If $\eta X.\varphi$ is well-named, then*

$$\|\nu X.\varphi\|^M = \|\varphi^{n+1}(\top)\|^M \text{ and } \|\mu X.\varphi\|^M = \|\varphi^{n+1}(\bot)\|^M.$$

Proof. Suppose $\nu X.\varphi$ is be a well-named μ-formula. We suppose that φ is of the form $\alpha(\triangle\beta(X))$ with $\triangle \in \{\Box, \Diamond\}$.

We show that $\nu X.\varphi$ is equivalent to $\varphi^{n+1}(\top)$. Since X is positive in $\varphi(X)$, a straight induction argument show that $\|\varphi^{n+2}(\top)\|^M \subseteq \|\varphi^{n+1}(\top)\|^M$. So we need only to show that $\|\varphi^{n+1}(\top)\|^M \subseteq \|\varphi^{n+2}(\top)\|^M$.

For a contradiction, suppose that $w \in \|\varphi^{n+1}(\top)\|^M$ and $w \notin \|\varphi^{n+2}(\top)\|^M$. Then V has a winning strategy σ for the evaluation game $\mathcal{G}_{n+1} = \mathcal{G}(M, w \models \varphi^{n+1}(\top))$; and R has a winning strategy τ for the evaluation game $\mathcal{G}_{n+2} = \mathcal{G}(M, w \models \varphi^{n+2}(\top))$ We use σ and τ to define strategies σ' for V in $\mathcal{G}_{n+2}$ and τ' for R in $\mathcal{G}_{n+1}$.

Suppose the players are in positions $\langle v, \psi(\top)\rangle$ in $\mathcal{G}_{n+1}$ and $\langle v, \psi(\varphi(\top))\rangle$ in $\mathcal{G}_{n+2}$. Both have the same owner, that is, either it is V's turn in both games, or it is R's turn in both games. Suppose it is V's turn and they play $\sigma(\langle v, \psi(\top)\rangle) = \langle v', \psi'(\top)\rangle$ in $\mathcal{G}_{n+1}$. Then set $\sigma'(\langle v, \psi(\varphi(\top))\rangle) := \langle v', \psi'(\varphi(\top))\rangle$ in $\mathcal{G}_{n+2}$. Similarly, if it is R's turn and they play $\tau(\langle v, \psi(\varphi(\top))\rangle) = \langle v', \psi'(\varphi(\top))\rangle$ in $\mathcal{G}_{n+2}$, then they play $\tau'(\langle v, \psi(\top)\rangle) := \langle v', \psi'(\top)\rangle$ in $\mathcal{G}_{n+1}$.

The players continue both games following the strategies described above until they get to a position of the form $\langle v, P\rangle$ (or $\langle v, \neg P\rangle$) in both games; or they get to positions of the form $\langle w_n, \triangle\beta(\top)\rangle$ in $\mathcal{G}_{n+1}$ and $\langle w_n, \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_{n+2}$.

Case 1. Suppose they players are in a position $\langle v, P\rangle$ in both games. As σ is winning for V in $\mathcal{G}_{n+1}$, $v \in \|P\|^M$. As τ is winning for R in $\mathcal{G}_{n+2}$, $v \notin \|P\|^M$. And so we have a contradiction. A similar contradiction is reached if they are in a position $\langle v, \neg P\rangle$.

Case 2. Suppose the players are in positions of the form $\langle w_n, \triangle\beta(\top)\rangle$ in $\mathcal{G}_{n+1}$ and $\langle w_n, \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_{n+2}$. As φ is well named, there must be worlds $w_0, \ldots, w_{n-1}$ such that the players have been through positions $\langle w_i, \triangle\beta(\varphi^{n-i}(\top))\rangle$ in $\mathcal{G}_{n+1}$ and $\langle w_i, \triangle\beta(\varphi^{n-i+1}(\top))\rangle$ in $\mathcal{G}_{n+2}$, for all $i \in \{1, \ldots n-1\}$. By the hypothesis on F, there must be $i < j \leq n$ such that $w_iR = w_jR$, as $w_0R^*w_1R^* \cdots R^*w_n$. We represent a partial play of $\mathcal{G}_{n+1}$ and $\mathcal{G}_{n+2}$ in Figure 3.2.

Since V is following a winning strategy in $\mathcal{G}_{n+1}$, $M, w_i \models \triangle\beta(\varphi^{n-i}(\top))$ and $M, w_j \models \triangle\beta(\varphi^{n-j}(\top))$ hold. As $w_iR = w_jR$, Lemma 22 implies $M, w_i \models \triangle\beta(\varphi^{n-j}(\top))$. By the positivity of X in $\triangle\beta(X)$,

$$M, w_i \models \triangle\beta(\varphi^k(\top)) \text{ for all } k = n-i, \ldots, n-j.$$

Similarly, using the fact that R is using a winning strategy in $\mathcal{G}_{n+2}$,

$$M, w_i \not\models \triangle\beta(\varphi^k(\top)) \text{ for all } k = n-i+1, \ldots, n-j+1.$$

For $k_0 = n - i + 1$, this means that

$$M, w_i \models \triangle\beta(\varphi^{k_0}(\top)) \text{ and } M, w_i \not\models \triangle\beta(\varphi^{k_0}(\top)).$$

$$
\begin{array}{c}
\mathcal{G}_{n+1} \\
\vdots \\
\downarrow \\
\langle w_i, \triangle\beta(\varphi^{n-i}(\top))\rangle \\
\downarrow \\
\vdots \\
\downarrow \\
\langle w_j, \triangle\beta(\varphi^{n-j}(\top))\rangle \\
\downarrow \\
\vdots
\end{array}
\qquad
\begin{array}{c}
\mathcal{G}_{n+2} \\
\vdots \\
\downarrow \\
\langle w_i, \triangle\beta(\varphi^{n-i+1}(\top))\rangle \\
\downarrow \\
\vdots \\
\downarrow \\
\langle w_j, \triangle\beta(\varphi^{n-j+1}(\top))\rangle \\
\downarrow \\
\vdots
\end{array}
$$

Figure 3.2: Simultaneous runs of the games $\mathcal{G}_{n+1}$ and $\mathcal{G}_{n+2}$ of Lemma 23.

And so we have a contradiction.

As both cases result in contradictions, we conclude that $\|\varphi^{n+1}(\top)\| \subseteq \|\varphi^{n+2}(\top)\|^M$. $\|\mu X.\varphi\| = \|\varphi^{n+1}(\bot)\|^M$ now follows by a direct calculation:

$$\mu X.\varphi \equiv \neg\nu X.\neg\varphi(\neg X) \equiv \neg((\neg\varphi\neg)^{n+1}(\top)) \equiv \varphi^{n+1}(\bot).$$

The first equivalence follows by an alternative definition of $\mu X.\varphi$, the second by the first half of this proof, and the third by negation cancelling. $\square$

Proof of Theorem 21. We argue by structural induction on μ-formulas. Again, we only prove the interesting cases. Suppose φ is equivalent to a modal formula ψ. Then $\nu X.\varphi$ is equivalent to $\nu X.\psi$. By Lemma 23, $\nu X.\psi$ is equivalent to $\psi^{n+1}(\top)$, which is a modal formula. Similarly, $\mu X.\psi$ is equivalent to $\psi^{n+1}(\bot)$. Therefore every μ-formula is equivalent to a modal formula over frames of $\mathcal{F}$. $\square$

A QUESTION. A frame $F = \langle W, R\rangle$ is reverse well-founded iff there is no infinite sequence $\{w_i\}_{i\in\omega}$ with $w_i R w_{i+1}$ for all $n \in \omega$. Alberucci and Facchini [AF09b] proved that the alternation hierarchy collapses on transitive and reverse well-founded frames. These are the frames of the modal logic GL. Using Lemmas 63 and 64 from Chapter 5, we can prove a generalization of their theorem:

Theorem 24. *Fix $n \in \omega$. Then, over reverse well-founded frames which satisfy $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$, the alternation hierarchy collapses to modal logic.*

Proof. Let $\nu X.\varphi$ be well-named. Either X is weakly universal or existential in $\nu X.\varphi$. That is, either X in the scope of some $\Box$ modality or only in the scope of $\Diamond$ modalities. If X is weakly universal, then $\nu X.\varphi$ is equivalent to $\varphi^{n+1}(\top)$.

Suppose X is existential.[2] The formulas $\mu X.\varphi$ and $\nu X.\varphi$ are equivalent, as there is no infinite plays in the evaluation games $M, w \models \mu X.\varphi$ and $M, w \models \mu X.\varphi$. Therefore these two games are equivalent. Now, X is weakly existential in $\mu X.\varphi$, which is equivalent to $\varphi^{n+1}(\bot)$ by Lemma 64.

Therefore, over reverse well-founded frames which satisfy $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$, we can eliminate all fixed-point operators in any formula. $\square$

[2] X is *existential* iff X is not in the scope of any $\Box$ in ψ, that is, X is only in the scope of diamonds.

The class of reverse well-founded frames where $\Diamond\mu X.\varphi(X)$ are equivalent $\Diamond\varphi^n(\bot)$ contains frames not considered in Theorem 21.

Question 1. *Are the proofs of collapses to modal logic in Theorems 21 and 24 sufficient to capture all the collapses to modal logic?*

3.4 Degrees of ignorance in epistemic logic

In this section, we analyze the meaning of some μ-formulas from the point of view of Epistemic Logic. The formulas we consider describe degrees of ignorance. We argue that logics "closer" to S4.2 allow greater degrees of ignorance compared to logics "closer" to S5.

A LOGIC FOR KNOWLEDGE. In this section, we write K for $\Box$ and $\hat{K}$ for $\Diamond$, and consider a second box modality B, for belief. We will work with only one agent, so we read $K\varphi$ as "the agent knows that φ is true", $\hat{K}\varphi$ as "the agent considers φ (epistemically) possible" and $B\varphi$ as "the agent believes that φ' is true'. We will also consider conditional belief $B^{\psi}\varphi$, read as "the agent believes that φ is true, given ψ".

Before going into details, we reiterate that $K\varphi$ is read as "the agent knows that φ". The conception of knowledge we study is different from "the agent knows *whether* φ is true"—which can be formalized in our logic as $K\varphi \vee K\neg\varphi$. It is also different from knowledge *how*, knowledge *about*, *etc*.

We follow Stalnaker [Sta06] and Aucher [Auc14]. The modality K satisfies S4. Remember, the axioms of S4 are:

- $K := K(\varphi \to \psi) \to K\varphi \to K\psi$,[3]
- $T := K\varphi \to \varphi$; and
- $4 := K\varphi \to KK\varphi$.

T means that if the agent knows that φ, then φ is indeed true. 4 means that if the agent knows that φ, then they also know that they know φ. That is, the agent has privileged access to their knowledge.

The axiom K and the necessitation rule **Nec** are instances of the so called logical omniscience. K and **Nec** implies that the agent knows about all the logical consequences of their knowledge. For example, suppose that the agent knows the axioms of ZFC. Then logical omniscience implies they know Cohen's proof of the independence of the continuum hypothesis. We will see some ways to evade logical omniscience in Chapter 4.

The belief modality B satisfies KD45:

- $K := \Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$,
- $D := \neg B\bot$;
- $4 := B\varphi \to BB\varphi$; and
- $5 := \neg B\varphi \to B\neg B\varphi$.

[3] We use K for both the axiom and the modality.

K and 4 are as in knowledge. D is a weakening of T; we do not require beliefs to be true, only that they are consistent. 5 is analogous to 4, implying that if the agent does not believe something, then they also believe that they do not believe it. It allows the agent to access negative facts about their beliefs, while 4 only allows access for positive facts.

The axioms above only describe how knowledge and belief behave independently. We add the three interaction axioms below to our logic:

- $K\varphi \to B\varphi$;
- $B\varphi \to KB\varphi$; and
- $\neg B\varphi \to K\neg B\varphi$.

These axioms imply that: if an agent knows something, they must also believe it; if they believe something, they must know about their belief; and if they do not believe something, they must know about their disbelief. This conception of belief is called strong belief [Auc14]—in contrast to weak belief, where $B_w\varphi$ iff the agent considers probability that φ holds to be greater that $1/2$. If K and B satisfy these axioms, then $B\varphi$ can be defined as $\hat{K}K\varphi$, so we can assume we have only the modality K. Furthermore, the assumptions we have on K and B imply that K satisfies $.2$: $\hat{K}KP \to K\hat{K}P$. Proof of these result can be found in the appendices of [Auc14].

We suppose conditional belief satisfies the following properties:

- $B^\psi\psi$;
- $B^\psi\varphi_0 \wedge B^\psi\varphi_1 \to B^\psi(\varphi_0 \wedge \varphi_1)$;
- $B^{\psi_0}\varphi \wedge B^{\psi_1}\varphi \to B^{\psi_0 \vee \psi_1}\varphi$;
- $B^\psi\varphi \wedge B^\psi\chi \to B^{\psi\wedge\varphi}\chi$;
- if $\psi \leftrightarrow \psi'$ then $B^\psi\varphi \leftrightarrow B^{\psi'}\varphi$; and
- if $\varphi \leftrightarrow \varphi'$ then $B^\psi\varphi \leftrightarrow B^\psi\varphi'$.

This logical system is usually called P.

SOME EPISTEMIC LOGICS. We briefly review the systems of epistemic logic we will study.

The basic logic we use for knowledge is S4.2. This logic is Lenzen's [Len78] and Stalnaker's [Sta06] logic of knowledge. It also axiomatizes Voorbraak's logic of justified knowledge [Voo93]. If we suppose K satisfies S4.2 and define $B\varphi :\leftrightarrow \hat{K}K\varphi$, then B satisfies KD45 and the interaction axioms above hold.

S4.3 is obtained by adding the axiom

$$K(KP \to Q) \vee K(KQ \to P)$$

to S4.2. Aucher [Auc14] shows that .3 is a consequence of the following interaction axioms for knowledge and conditional belief:

- $K\varphi \to B^\psi\varphi$;

- $B^{\psi}\varphi \to KB^{\psi}\varphi$; and
- $\neg B^{\psi}\varphi \to K(\hat{K}\psi \to \neg B^{\psi}\varphi)$.

S4.3 is Lehrer and Paxon's [LP69] logic of knowledge, with knowledge being undefeated justified true belief. It is also van der Hoek's logic of knowledge in [van93].

S4.3.2 is obtained by adding the axiom .3.2 defined by $(\hat{K}P \wedge \hat{K}\Box Q) \to K(\hat{K}P \vee Q)$ to S4.2. Aucher[Auc14] shows that .3.2 is a consequence of the interaction axiom

$$(K\varphi \to K\psi) \wedge B(K\varphi \to K\psi) \to K(K\varphi \to K\psi).$$

It is also the consequence of the interaction axioms for knowledge and conditional belief which implied .3 above and:

$$B\neg\psi \to (B^{\psi}\varphi \to K(\psi \to \varphi)).$$

S4.3.2 appear in non-monotonic contexts [ST92]. It has also been studied in Lenzen [Len78].

S4.4 is the logic of knowledge as true belief. It is obtained by adding the interaction axiom:

$$K\varphi \leftrightarrow \varphi \wedge B\varphi.$$

It can also be obtained by adding $(P \wedge \hat{K}KP) \to KP$ to S4.2. Aucher says its the logic of knowledge considered by Kutschera in [Kut76].

S5 is the standard logic for multi-agent epistemic logic. It is obtained by adding the axiom $\neg K\varphi \to K\neg K\varphi$ to S4.2. In S5, belief collapses to knowledge. That is, $K\varphi$ iff $B\varphi$. It is a quite uninteresting logic for epistemic logic with only one agent, but it is much better with multiple agents—we study what happens to the μ-calculus on multimodal S5 in Section 4.4.

THE ALTERNATION HIERARCHY ON EPISTEMIC LOGICS. We now use Theorem 21 to study the epistemic logics we described above. We first show that, as S4.3 is an extension of S4.2, the later logic has fewer frames. We then show that the alternation hierarchy collapses to the alternation-free fragment over frames of S4.3.

Proposition 25. *Every* S4.3 *frame is also an* S4.2 *frame.*

Theorem 26. *The alternation hierarchy collapses to its alternation-free fragment over* S4.2 *and* S4.3*. Furthermore, the alternation hierarchy does not collapse to modal logic over* S4.2 *and* S4.3*.*

Proof. As frames of S4.2 and S4.3 are transitive, it is enough to show that the alternation hierarchy collapses to its alternation-free fragment over transitive frames. This result is already established [AF09b; DL10].

Now, we prove that the non-collapse to modal logic for S4.3 implies the non-collapse for S4.2. Let $\varphi(X) := \Diamond(P\wedge\Diamond(\neg P\wedge X))$. We show $\nu X.\varphi(X)$ is not equivalent to any modal formula. For a contradiction, suppose $\nu X.\varphi(X)$ is equivalent to a modal formula φ'. Given a model $M = \langle W, R, V\rangle$ over an S4.3 frame $F = \langle W, R\rangle$ and $w_0 \in W$, $\nu X.\varphi$ holds on w_0 iff there is a sequence $\{w_i\}_{i\in\omega}$ such that w_iRw_{i+1}, $w_{2i+1} \in \|P\|$ and $w_{2i+2} \in \|\neg P\|$. For $n \in \omega$, define a model $M_n = \langle W_n, R_n, V_n\rangle$ by

- $W_n = \{w_1, w_2, \ldots, w_{2n}\}$;

- $R_n = \{\langle w_i, w_j\rangle \mid 0 \le w_i \le w_j \le 2n-1\}$; and
- $V_n(P) = \{w_i \mid i \text{ is odd}\}$.

M_n has a path of length $2n$ with P and $\neg P$ appearing alternately, but no such path is infinite; that is, $M_n, w_1 \models \varphi^n(\top) \wedge \neg\varphi'$.

Define $\mathsf{L} = \mathsf{S4.3} + \neg\varphi' + \{\top, \varphi^1, \varphi^2, \dots\}$. Note that the compactness theorem holds for S4.3, as its frames are first-order definable. As each finite subtheory of L is satisfied by some M_n, L has a model with an S4.3 frame. S4.3 also has the finite model property, and so there is a finite model (M_f, w_f) of L with S4.3 frame. An application of the Pigeonhole Principle implies that $\nu X.\varphi$ holds on (M_f, w_f), as M_f is finite and has paths of arbitrary length starting from w_f where P and $\neg P$ hold alternately. Therefore (M_f, w_f) satisfies both $\nu X.\varphi$ and $\neg\nu X.\varphi$, a contradiction. Therefore $\nu X.\varphi$ is not equivalent to any modal formula over frames of S4.3. $\square$

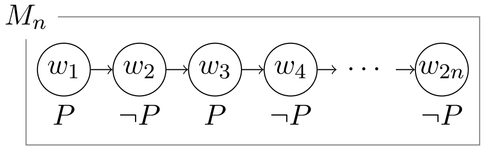


Figure 3.3: Models M_n used to show the alternation hierarchy does not collapse to modal logic over S4.3.2 frames. We omit the reflexive and transitive arrows.

We now characterize S4.4 frames:

Lemma 27. *If $F = \langle W, R\rangle$ satisfies* S4.4*, then we can decompose W into sets* Ini *and* Fin *such that:*

- Ini *has at most one element;*
- Fin *is not empty;*
- xRx' *for all* $x, x' \in \mathrm{Ini}$*;*
- yRy' *for all* $y, y' \in \mathrm{Fin}$*; and*
- $xRy \wedge \neg yRx$ *for all* $x \in \mathrm{Ini}$, $y \in \mathrm{Fin}$.

Proof. Since S4.4 is a strengthening of S4.3.2, every S4.4 frame F is also an S4.3.2 frame. So we can decompose F into sets Ini and Fin as above, and need only to check if $|\mathrm{Ini}| \le 1$.

Suppose $w, v \in \mathrm{Ini}$. Let $M = \langle W, R, V\rangle$ be defined by setting $V(P) = \mathrm{Fin} \cup \{w\}$. Then $M, w \models P \wedge \hat{K}KP$ all elements of Fin satisfy KP. By the axiom $.4$, $M, w \models KP$. As wRv, $M, v \models P$. Therefore $v = w$. $\square$

Frames of KD45 may fail reflexivity, but they are quite similar to S4.4 frames. The next lemma implies that we can apply Theorem 21 to KD45 frames.

Lemma 28. *If $F = \langle W, R\rangle$ satisfies* KD45*, then we can decompose W into sets* Ini *and* Fin *such that:*

- Ini *has at most one element;*
- Fin *is not empty;*
- $\neg xRx$ *for all* $x \in \mathrm{Ini}$*;*
- yRy' *for all* $y, y' \in \mathrm{Fin}$*; and*
- $xRy \wedge \neg yRx$ *for all* $x \in \mathrm{Ini}$, $y \in \mathrm{Fin}$.

Theorem 29. *The alternation hierarchy collapses to modal logic over* S4.3.2, S4.4, *and* KD45.

Proof. Let $F = \langle W, R\rangle$ be an S4.3.2 frame. By Lemma 19, we suppose W can be decomposed into two disjoint sets Ini and Fin such that $x \in \mathrm{Ini}$ and $y \in W$ imply $\langle x, y\rangle \in R$; $x, y \in \mathrm{Fin}$ imply $\langle x, y\rangle \in R$; and there is no $\langle x, y\rangle \in R$ with $x \in \mathrm{Fin}$ and $y \in \mathrm{Ini}$. Therefore, if wR^*vR^*u, then either $wR = vR$ or $vR = uR$. Theorem 21 implies the alternation hierarchy collapses to modal logic over S4.3.2 frames.

Now, every S4.4 frame is an S4.3.2 frame where $|\mathrm{Ini}| \leq 1$. And so the alternation hierarchy collapses to modal logic over S4.4 frames. Similarly, a KD45 is almost an S4.4, but we assume R is not reflexive on the world in Ini. □

IGNORANCE. Ignorance is a traditional concept in epistemology, already discussed by Plato in his *Theaetetus*. It is also an active research topic within epistemic logic [Car+21; Fan21; Rv21]; Peels and Blaauw [PB16] compile many recent papers on ignorance.

Van der Hoek and Lomuscio [vL04] defined the ignorance modality:

$$I\varphi :\leftrightarrow \neg K\varphi \wedge \neg K\neg\varphi.$$

We read $I\varphi$ as "the agent is ignorant *whether* φ holds". Fine [Fin18] studied high-order versions of the ignorance modality: $I^1\varphi := I\varphi$, $I^{n+1}\varphi := I(I^n\varphi)$. Fine showed that even second-order ignorance is unobtainable on S4 frames: $\neg KI^n\varphi$ is valid on all S4 frames for any formula φ and $n \geq 2$. We will develop below another method of distinguishing types of ignorance.

One can also think about the ignorance modality $I\varphi$ as the epistemic version of the contingency modality $\nabla\varphi := \neg\Box\varphi \wedge \neg\Box\neg\varphi$. See Fan *et al.* [FWD15] for the connection between ignorance and contingency.

DEGREES OF IGNORANCE. Fix a μ-sentence φ, we define formulas α^n_φ for $n \in \omega\cup\{\infty\}$. We read α^n_φ as "the agent has nth degree ignorance whether φ". Define:

- $\alpha_\varphi(X) := \hat{K}(\varphi \wedge X) \wedge \hat{K}(\neg\varphi \wedge X)$;
- $\alpha^1_\varphi := \alpha_\varphi(\top)$;
- $\alpha^{n+1}_\varphi := \alpha_\varphi(\alpha^n_\varphi)$; and
- $\alpha^\infty_\varphi := \nu X.\alpha_\varphi(X)$.

The monotonicity of α_φ implies that any degree of ignorance implies the weaker degrees of ignorance. That is, if $i \leq j$, then $\alpha_\varphi^j \to \alpha_\varphi^i$. In particular, α_φ^1 is equivalent to $I\varphi$, so we may say that our α_φ^i are refinements of van der Hoek and Lomuscio's ignorance modality.

Over S4.2 and S4.3, the formulas

$$\alpha_\varphi^\infty, \alpha_\varphi^1, \alpha_\varphi^2, \alpha_\varphi^3, \ldots$$

are pairwise disjoint. We now look at formulas of the form $\alpha_\varphi^n \wedge \neg\alpha_\varphi^{n+1}$. The formula $\alpha_\varphi^1 \wedge \neg\alpha_\varphi^2$ is equivalent to the agent having a false belief and, in all accessible worlds, believing that they know φ. Similarly, $\alpha_\varphi^2 \wedge \neg\alpha_\varphi^3$ holds when the agent has a true belief but thinks it is possible to have a false belief. We can generalize this to other $\alpha_\varphi^i \wedge \neg\alpha_\varphi^{i+1}$, and express higher degrees of self-doubt.

When the alternation hierarchy collapses to modal logic, we do not have infinitely many degrees: α_φ^∞ is equivalent to a modal formula. Over S4.3.2 and S4.4, α_φ^2 is equivalent to α_φ^∞, so we have only two degrees of ignorance: α_φ^1, the agent has a false belief; and α_φ^2, the agent has no belief. In S5, α_φ^1 is equivalent to α_φ^∞ and we have only one degree of ignorance; here, $B\varphi$ is equivalent to $K\varphi$ and the agent cannot have wrong beliefs.

In settings other than S5, having no belief implies a high degree of ignorance, but it is possible to have a high degree of ignorance and a belief at the same time, that is: for any $1 \leq n < \infty$, $\neg(B\varphi \vee B\neg\varphi)$ implies α_φ^∞; and $\alpha_\varphi^\infty \wedge B\varphi$ and $\alpha_\varphi^\infty \wedge B\neg\varphi$ are satisfiable. In S5, ignorance and lack of belief are equivalent.

Stalnaker's [Sta06] criticism of S4.3 and S4.3.2 is that in both systems false belief can deny knowledge the agent may be justified in having: in S4.3, a false belief denies *some* knowledge; in S4.3.2, a false belief denies *all* non-trivial knowledge. From our point of view, S4.3.2 allows us to express only a few degrees of ignorance, while S4.3 allows us to express infinitely many degrees.

DEGREES OF DOUBT. One can also do a similar analysis to belief and doubt. Olsson and Proietti [OP16] defined a doubt modality D by $D\varphi := \neg B\varphi \wedge \neg B\neg\varphi$. We can then define degrees of doubt by substituting belief for knowledge in the definition of degrees of ignorance. Formally, fix φ and define:

- $\delta_\varphi(X) := \hat{B}(\varphi \wedge X) \wedge \hat{B}(\neg\varphi \wedge X)$;
- $\delta_\varphi^1 := \delta_\varphi(\top)$;
- $\delta_\varphi^{n+1} := \delta_\varphi(\delta_\varphi^n)$; and
- $\delta_\varphi^\infty := \nu X.\delta_\varphi(X)$.

A short argument shows that δ_φ^1 and δ_φ^∞ are equivalent over frames of KD45. Therefore we can only define one degree of doubt in our framework.

Our analysis of degrees of ignorance and disbelief contrasts with existing discussions of degrees of knowledge and belief. Olsson and Proietti [OP16] argue that belief and doubt have many degrees but knowledge has only one. They also argue that ignorance has only one degree. Hetherington [Het01] argues that knowledge also has many degrees.

Chapter 4

The alternation hierarchy on variations of S5

In this chapter, we study the μ-calculus' alternation hierarchy over frames of variants of S5. We begin by studying non-normal semantics, where the necessitation rule $\varphi \vdash \Box\varphi$ does not hold. We then take a look at graded semantics; here we have formulas such as $\Diamond^{>1}\varphi$, stating the existence of more than one accessible world satisfying φ. Next, we study intuitionistic semantics, where the law of the excluded middle does not holds. On these semantics, the alternation hierarchy is going to collapse to modal logic. These are all semantics considered in the context of epistemic logic. Variations of Theorem 21 also hold for these semantics; we omit them for simplicity's sake.

Another semantics studied in epistemic logic is multimodal semantics, where we have more than one pair of $\Box$ and $\Diamond$ modalities. Here, the alternation hierarchy is strict. We show its strictness using parity games and their winning region formulas. Multimodal semantics is also used in model checking, a context where the μ-calculus first appeared.

At last, we study inflationary semantics, where we allow the use of fixed-point operators on non-positive variables. We show that the direct adaptation of the proof of the collapse to modal logic is not enough to show the collapse with inflationary semantics. We contrast this situation with the case over GL frames.

4.1 Non-normal modal logics

NON-NORMAL WORLDS. The modal logics we studied in chapters 2 and 3 were all *normal* modal logics, that is, they satisfied the necessitation rule

$$(\mathbf{Nec})\ \frac{\varphi}{\Box\varphi}.$$

Every Kripke frame validates **Nec**. Therefore, to study logics where **Nec** doesn't hold, we need to modify our semantics.

We now consider Kripke models with non-normal worlds. In a non-normal world, everything is possible and nothing is necessary. Non-normal were first considered by Kripke in [Kri65] as models to Lewis logics S2 and S3 [LL59].[1] Non-normal worlds are also known as impossible worlds. In epistemic logic, the necessitation rule is a form of logical omniscience: if some statement is true, then every agent knows that it is true.

For more information on non-normal worlds, see [Pri08] and [BJ22].

SEMANTICS. Non-normal Kripke models are tuples $W = \langle W, N, R, V\rangle$ where W is a set of worlds, $R \subseteq W \times W$ is an accessibility relation, and V is a valuation on W, as in usual Kripke models; the set $N \subseteq W$ is the set of normal worlds. On normal worlds, we interpret $\Box\varphi$ and $\Diamond\varphi$ as usual; in non-normal worlds, we interpret $\Box\varphi$ and $\Diamond\varphi$ respectively as false and true. That is, we define:

- $\|\Box\varphi\|^M = \{w \in N \mid \forall v.wRv \to v \in \|\varphi\|^M\}$; and
- $\|\Diamond\varphi\|^M = \{w \in N \mid \exists v.wRv \wedge v \in \|\varphi\|^M\} \cup (W \setminus N)$.

The defining conditions for the other formulas are unchanged from normal Kripke models.

A non-normal Kripke model $M = \langle W, N, R, V\rangle$ satisfies a formula φ iff $M, w \models \varphi$ for all $w \in N$. If M satisfies φ, we write $M \models \varphi$. We define non-normal Kripke frames as triples $F = \langle W, N, R\rangle$. We say that F satisfies a formula φ iff for all M extending F, $M \models \varphi$. We write $F \models \varphi$ when F satisfies φ.

Note that if $F \models T + 4 + 5$, then $R \restriction N \times N$ is an equivalence relation. One can also suppose $R \subseteq N \times N$, as the valuation of formulas $\triangle\varphi$ on non-normal worlds does not depend on the accessibility relation.

GAME SEMANTICS. We can adapt the μ-calculus' standard game semantics for Kripke models with non-normal worlds.[2] As the definition of Kripke semantics for non-normal worlds suggests, we need only to modify the positions of the forms $\langle w, \Box\varphi\rangle$ and $\langle w, \Diamond\varphi\rangle$.

Let $M = \langle W, N, R, V\rangle$ be a non-normal Kripke model, $w \in W$ and φ be a μ-formula. We define the non-normal evaluation game $\mathcal{G}(M, w \models \varphi)$. On a normal world w, we proceed as in the usual game semantics, that is, on a vertex $\langle w, \triangle\varphi\rangle$ either V or R choose a vertex w' accessible from w and move to $\langle w', \varphi\rangle$. If w is non-normal, we treat $\Box\varphi$ and $\Diamond\varphi$ as the propositions $\bot$ and $\top$, respectively. So R wins at $\langle w, \Box\varphi\rangle$ and V wins at $\langle w, \Diamond\varphi\rangle$. Table 4.1 describe the possible moves for the players in a non-normal evaluation game.

Theorem 30. *Let $M = \langle W, N, R, V\rangle$ be a non-normal Kripke model, $w \in W$ and φ be a μ-formula. Then*

$$M, w \models \varphi \text{ iff V wins } \mathcal{G}(M, w \models \varphi), \text{ and}$$
$$M, w \not\models \varphi \text{ iff R wins } \mathcal{G}(M, w \models \varphi).$$

Proof. The proof is the same as the proof of Theorem 12. On normal worlds, there is no difference; on non-normal worlds, formulas of the forms $\Box\varphi$ and $\Diamond\varphi$ are essentially the same as $\bot$ and $\top$. $\Box$

[1] Models S2 are reflexive and models for S3 are reflexive and transitive. Equivalence relations are models for the non-Lewis logic S3.5. Semantics for S1 were obtained by Cresswell in [Cre95].

[2] As far as I know, this game semantics do not appear in the literature.

Table 4.1: Rules of evaluation games for non-normal μ-calculus.

Verifier		Refuter	
Position	Admissible moves	Position	Admissible moves
$\langle w, \psi_1 \vee \psi_2\rangle$	$\{\langle w, \psi_1\rangle, \langle w, \psi_2\rangle\}$	$\langle w, \psi_1 \wedge \psi_2\rangle$	$\{\langle w, \psi_1\rangle, \langle w, \psi_2\rangle\}$
$\langle w, \Diamond\psi\rangle$ and $w \in N$	$\{\langle v, \psi\rangle \mid \langle w, v\rangle \in R\}$	$\langle w, \Box\psi\rangle$ and $w \in N$	$\{\langle v, \psi\rangle \mid \langle w, v\rangle \in R\}$
$\langle w, \Box\psi\rangle$ and $w \notin N$	$\emptyset$	$\langle w, \Diamond\psi\rangle$ and $w \notin N$	$\emptyset$
$\langle w, P\rangle$ and $w \notin V(P)$	$\emptyset$	$\langle w, P\rangle$ and $w \in V(P)$	$\emptyset$
$\langle w, \neg P\rangle$ and $w \in V(P)$	$\emptyset$	$\langle w, \neg P\rangle$ and $w \notin V(P)$	$\emptyset$
$\langle w, \mu X.\psi_X\rangle$	$\{\langle w, \mu X.\psi_X\rangle\}$	$\langle w, \nu X.\psi_X\rangle$	$\{\langle w, \nu X.\psi_X\rangle\}$
$\langle w, X\rangle$	$\{\langle w, \psi_X\rangle\}$	$\langle w, X\rangle$	$\{\langle w, \psi_X\rangle\}$

COLLAPSE TO MODAL LOGIC. We can show that the alternation hierarchy collapses to modal logic over non-normal equivalence relations. We need only slight modifications to the proof of Theorem 16.

As in transitive normal models, if two worlds are in the same strongly connected component, then they satisfy the same formulas of the form $\triangle\varphi$:

Lemma 31. *Let $M = \langle W, N, R, V\rangle$ be a transitive non-normal Kripke model, and $w, w' \in N$, and $wRw'Rw$. If φ be a μ-formula, and $\triangle \in \{\Box, \Diamond\}$, then $w \in \|\triangle\varphi\|^M$ iff $w' \in \|\triangle\varphi\|^M$.*

Using Lemma 31, we can prove a non-normal version of Lemma 15. The proof is essentially the same, but we need to take care of the case where the play end in a position $\langle w, \psi\rangle$ with $w \notin N$.

Lemma 32. *If $M = \langle W, N, R, V\rangle$ is a Kripke model where R is an equivalence relation, and $\nu X.\varphi$ is a well-named μ-formula, then $\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M$ and $\|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M$.*

Proof. Let $M = \langle W, N, R, V\rangle$ be a Kripke model where R is an equivalence relation. Suppose $\nu X.\varphi$ is a well-named μ-formula of the form $\alpha(\triangle\beta(X))$, with $\triangle \in \{\Box, \Diamond\}$. We show that $\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M$.

Remember that $\varphi^0(X) := X$ and $\varphi^{n+1}(X) := \varphi(\varphi^n(X))$. As in Lemma 15, we suppose that $w \in \|\varphi^2(\top)\|^M$ and $w \notin \|\varphi^3(\top)\|^M$ to get a contradiction. Again, the two player V and R will play simultaneously the games $\mathcal{G}(M, w \models \varphi^2(\top))$ and $\mathcal{G}(M, w \models \varphi^3(\top))$. In the former game, V uses their winning strategy σ, and, in the latter, V plays moves analogous to the ones they did σ. Similarly, R uses his winning strategy τ on the latter game, and a strategy analogous to τ on the former game.

The players continue both games until

1. they get to a position of the form $\langle v, P\rangle$ (or $\langle v, \neg P\rangle$) in both games;

2. they get to positions of the form $\langle v, \triangle\beta(\top)\rangle$ in $\mathcal{G}_2$ and $\langle v, \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_3$, with $v \in N$; or

3. they get to positions of the form $\langle v, \triangle\psi(\top)\rangle$ in $\mathcal{G}_2$ and $\langle v, \triangle\psi(\varphi(\top))\rangle$ in $\mathcal{G}_3$ with $v \notin N$.

Case 1 and 2 are proved as in Lemma 15, with the use of Lemma 14 replaced by Lemma 31. We need only to consider Case 3: the players are in positions $\langle v, \triangle\psi(\top)\rangle$ in $\mathcal{G}_2$ and $\langle v, \triangle\psi(\varphi(\top))\rangle$ in $\mathcal{G}_3$ with $v \notin N$. Since σ is winning for V in $\mathcal{G}_2$ and τ is winning for R in $\mathcal{G}_3$, $M, v \models \triangle\psi(\top)$ and $M, v \models \triangle\psi(\varphi(\top))$. As v is non-normal, if $\triangle$ is $\Box$, then both must be false; and if $\triangle$ is $\Diamond$, then both must be true. Either way, we get a contradiction.

As all three cases end in contradictions, we can conclude that $\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M$.

Again, we prove $\|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M$ by a direct calculation:

$$\mu X.\varphi \equiv \neg\nu X.\neg\varphi(\neg X) \equiv \neg(\neg\varphi(\neg\neg\varphi(\neg\top))) \equiv \varphi(\varphi(\bot)). \qquad \Box$$

The proof of the alternation hierarchy's collapse to modal logic over non-normal equivalence relations is now the same as proof for normal equivalence relations, but using Lemma 32:

Theorem 33. *The alternation hierarchy collapses to modal logic over non-normal equivalence relations.*

NEIGHBORHOOD MODELS. Another way to evade the necessitation rule is to use neighborhood models. A neighborhood model is a tuple $M = \langle W, \mathcal{N}, V\rangle$, where W is a set of possible worlds, $\mathcal{N} : W \to \mathcal{P}(\mathcal{P}(W))$ is a neighborhood function, and V is a valuation function. Given $w \in W$, the elements of $\mathcal{N}(w)$ are called the neighborhoods of w.

As in the case of non-normal worlds, to define the semantics for neighborhood models we need only to define the valuation of formulas $\Box\varphi$ and $\Diamond\varphi$:

- $w \models \Box\varphi$ iff $\|\varphi\|^M \in \mathcal{N}(w)$; and
- $w \models \Diamond\varphi$ iff $\|\neg\varphi\|^M \notin \mathcal{N}(w)$.

For more on neighborhood models, see [Che80] and [Pac17]. Neighborhood semantics is also known as Scott–Montague semantics.

Neighborhood models are more general than non-normal Kripke models.

Proposition 34. *Every non-normal Kripke model $M = \langle W, N, R, V\rangle$ is equivalent to a neighborhood model $M_{\text{nbhd}} = \langle W, \mathcal{N}, V\rangle$.*

Proof. We only need to define the neighborhood function $\mathcal{N}$. The set of worlds and valuation of M_{nbhd} are the same as those of M. Let $w \in W$. If $w \in N$, define $wR = \{w' \mid wRw\}$ and $N(w) = \{S \subseteq W \mid wR \subseteq S\}$. If $w \notin N$, define $N(w) = \emptyset$. Then $M, w \models \varphi$ iff $M', w \models \varphi$, for all formula φ. $\Box$

The following examples show a non-normal Kripke model along with its equivalent neighborhood model, and a neighborhood model which is not equivalent to any non-normal Kripke model.

Example 13. Let $M = \langle W, N, R, V\rangle$ be a non-normal Kripke model with

- $W = \{w, v, u\}$;

- $N = \{w, u\}$;
- $R = \{\langle w, v\rangle, \langle w, u\rangle, \langle u, v\rangle\}$; and
- valuation $V(P) = \emptyset$ for all proposition P.

The model M is modally equivalent to the neighborhood $M_{\text{nbhd}} = \langle W, \mathcal{N}, V\rangle$ with neighborhood function defined by

$$\mathcal{N}(w) = \{\{v, u\}, \{w, v, u\}\}; \text{ and}$$
$$\mathcal{N}(v) = \mathcal{N}(u) = \{\{v\}, \{v, u\}, \{w, v\}, \{w, v, u\}\}.$$

M and M_{nbhd} are represented in Figure 4.1.

Example 14. Let $N = \langle W, \mathcal{N}, V\rangle$ be a neighborhood model with:

- $W = \{w, v, u\}$;
- $\mathcal{N}(w) = \{\{w\}, \{w, v, u\}\}$;
- $\mathcal{N}(v) = \{\emptyset\}$, $\mathcal{N}(u) = \emptyset$; and
- $V(P) = \{w\}$, $V(Q) = \{w, v\}$.

There is no non-normal Kripke equivalent to N.

Here, we have that $N, w \models \Box(P \to Q)$ and $N, w \models \Box P$, but $N, w \not\models \Box Q$. Furthermore, $N, v \not\models \Box\top$ and $N, v \models \Box\bot$. N is represented in Figure 4.1.

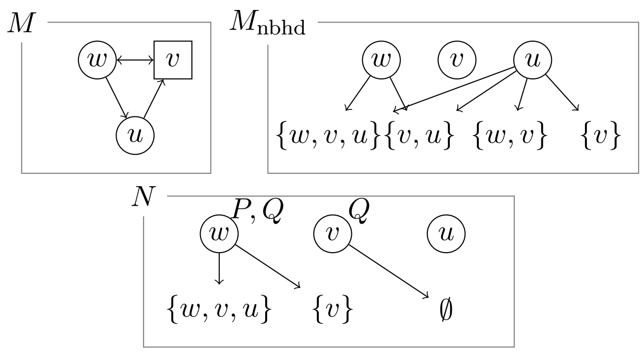


Figure 4.1: The non-normal model M and the equivalent neighborhood model M_{nbhd} from Example 13; along with the neighborhood model N from Example 14.

The neighborhood models obtained from non-normal Kripke models are regular neighborhood models, that is, they satisfy the axiom

$$(\mathbf{Df}\Diamond)\ \Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$$

and validate the inference rule

$$(\mathbf{Reg})\ \frac{\varphi \wedge \psi \to \theta}{\Box\varphi \wedge \Box\psi \to \Box\theta}.$$

If $M = \langle W, \mathcal{N} \rangle$ is a regular neighborhood frame, then the following properties hold: if $A \subseteq B \subseteq W$ and $A \in \mathcal{N}(w)$, then $B \in \mathcal{N}(w)$; and, if $A, B \in \mathcal{N}(w)$, then $A \cap B \in \mathcal{N}(w)$.

On a model extending regular neighborhood frame, if $\varphi(X)$ is a μ-formula, and X is positive in $\varphi(X)$, then the operator $\Gamma_{\varphi(X)}$ is monotone. Therefore the μ-calculus is well-defined on regular neighborhood model. But the μ-calculus can also be extended to a greater class of frames. We say a neighborhood frame is monotone iff it satisfies the axiom $\mathbf{Df}\Diamond$ and validates the inference rule

$$(\mathbf{Mon})\ \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}.$$

On a monotone frame, if $A \subseteq B \subseteq W$ and $A \in \mathcal{N}(w)$ then $B \in \mathcal{N}(w)$.

Again, on a model extending neighborhood frame, if $\varphi(X)$ is a μ-formula, and X is positive in $\varphi(X)$, then the operator $\Gamma_{\varphi(X)}$ is monotone. Therefore the μ-calculus is also well-defined on monotone neighborhood frames. As in normal Kripke models, modal axioms define classes of Kripke frames.

Proposition 35. *Let $F = \langle W, \mathcal{N} \rangle$ be a neighborhood frame. Then*

- *F satisfies T iff $w \in \bigcap N \mid N \in \mathcal{N}(w)$, for all $w \in F$;*
- *F satisfies 4 iff $X \in \mathcal{N}(w)$ implies $\{v \mid X \in \mathcal{N}(v) \in \mathcal{N}(w)$; and*
- *F satisfies 5 iff $X \notin \mathcal{N}(w)$ implies $\{v \mid X \notin \mathcal{N}(v) \in \mathcal{N}(w)$.*

We have the following question:

Question 2. *Does the μ-calculus' alternation hierarchy collapse to modal logic over monotone neighborhood frames which satisfy the axioms T, 4 and 5?*

4.2 Graded modal logics

We generalize the modalities $\Box$ and $\Diamond$ to modalities $\Box^{\leq n}$ and $\Diamond^{>n}$, for $n \in \omega$. The formula $\Box^{\leq n}\varphi$ holds at a world w iff there are at most n pairwise different worlds accessible from w where φ fails. A formula $\Box^{>n}\varphi$ holds at a world w iff there are more than n pairwise different worlds accessible from w where φ holds. We follow [van92] in our treatment of graded modalities.

BASIC DEFINITIONS. The graded μ-formulas are defined by the grammar

$$\varphi := P \mid \neg P \mid X \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box^{\leq n}\varphi \mid \Diamond^{>n}\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

$\Box^{\leq n}$ and $\Diamond^{>n}$ are dual. That is $\Box^{\leq n}\varphi$ is equivalent to $\neg\Diamond^{>n}\neg\varphi$. As in the μ-calculus, every graded μ-formula is equivalent to a well-named graded μ-formula.

Kripke semantics for graded modalities are similar to Kripke semantics for the standard μ-calculus. We use the same Kripke models $M = \langle W, R, V \rangle$, with

- $w \in \|\Box^{\leq n}\varphi\|^M$ iff $|\{v \in W \mid wRv$ and $v \in \|\varphi\|^M\}| \leq n$; and
- $w \in \|\Diamond\varphi\|^M$ iff $|\{v \in W \mid wRv$ and $v \in \|\varphi\|^M\}| > n$.

As each variable occurring in a graded μ-formula φ is positive, the semantics for $\mu X.\varphi$ and $\nu X.\varphi$ as fixed-points is well-defined.

Proposition 36. *Graded modal logic is stronger than modal logic.*

Proof. Consider the Kripke models $M = \langle \{w_0\}, \{\langle w_0, w_0\rangle\}, V(P) = \{w_0\}\rangle$ and $N = \langle \{v_0, v_1\}, \{\langle v_0, v_0\rangle, \langle v_0, v_1\rangle, \langle v_1, v_0\rangle, \langle v_1, v_1\rangle\}, V(P) = \{v_0, v_1\}\rangle$. The models M and N are bisimilar via $B = \{\langle w_0, v_0\rangle, \langle w_0, v_1\rangle\}$.

Note that $M, w_0 \not\models \Diamond^{>1}P$ and $N, v_0 \models \Diamond^{>1}P$. If $\Diamond^{>1}P$ was equivalent to a modal formula, then we would have $M, w \models \Diamond^{>1}\varphi$, as bisimulations preserve the truth of modal formulas. This is impossible. $\square$

GAME SEMANTICS. We modify the μ-calculus' game semantics to obtain a game semantics with graded modalities. Intuitively, on a position $\langle w, \Box^{\leq n}\varphi\rangle$, R needs to choose $n+1$ worlds accessible from w where (R thinks) φ fails; to refute R's choice, V needs only to show that φ holds in one of these worlds. We embed V's choice into the evaluation game, and so we do not need to play $n+1$ simultaneous games, only one game. The case for a position $\langle w, \Diamond^{>n}\varphi\rangle$ is analogous: V picks $n+1$ worlds, and R picks one to show that φ fails.

Formally, when the players are in a position $\langle w, \Diamond^{>n}\varphi\rangle$, V chooses a list of $n+1$ pairwise different accessible worlds $\langle w_0, \dots, w_n\rangle$ and move to $\langle\langle w_0, \dots, w_n\rangle, \varphi\rangle$. R chooses one of the w_i, and moves to $\langle w_i, \varphi\rangle$. When the players are in a position $\langle w, \Box^n\varphi\rangle$, R chooses a list of pairwise different accessible worlds $\langle w_1, \dots, w_n\rangle$, moves to $\langle [w_0, \dots, w_n], \varphi\rangle$. V then chooses one of the w_i and moves to $\langle w_i, \varphi\rangle$. To distinguish V's lists and R's lists, we denote the list of worlds chosen by R by $[w_1, \dots, w_n]$, using square brackets.

The game semantics is equivalent to the standard semantics:

Theorem 37. *Let $M = \langle W, R, V\rangle$ be a Kripke model, $w \in W$ and φ be a graded μ-formula. Then*

$$M, w \models \varphi \textit{ iff } \mathsf{V} \textit{ wins } \mathcal{G}(M, w \models \varphi), \textit{ and}$$
$$M, w \not\models \varphi \textit{ iff } \mathsf{R} \textit{ wins } \mathcal{G}(M, w \models \varphi).$$

Proof. Similar to the proof of Theorem 12. $\square$

COLLAPSE TO MODAL LOGIC. As in standard μ-calculus, the evaluation of formulas of the form $\Box^{\leq}\varphi$ (or $\Diamond^{>n}\varphi$) is the same on all worlds in a same equivalence class.

Lemma 38. *Let $M = \langle W, R, V\rangle$ be a Kripke model where R is an equivalence class, and worlds $w, w' \in N$. If φ is a graded μ-formula and $\triangle \in \{\Box^{\leq n}, \Diamond^{>n}\}$, then*

$$w \in \|\triangle\varphi\|^M \textit{ iff } w' \in \|\triangle\varphi\|^M.$$

Proof. Suppose $M = \langle W, R, V\rangle$ is a Kripke model and R is an equivalence relation. Fix two worlds w and w' such that wRw' Suppose $w \in \|\Diamond^{>n}\|^M$. There are pairwise distinct $w_0, \dots, w_n$ such that wRw_i and $w_i \in \|\varphi\|^M$, for all $i \leq n$. Since R is an equivalence relation, $w'Rw_i$ for all $i \leq n$. Therefore $w' \in \|\Diamond^{>n}\varphi\|^M$ too. The proof that $w' \in \|\Diamond^{>n}\varphi\|^M$ implies $w \in \|\Diamond^{>n}\varphi\|^M$ is symmetric.

Table 4.2: Rules of evaluation games for the graded modal μ-calculus.

Verifier

Position	Admissible moves
$\langle w, \psi_1 \vee \psi_2 \rangle$	$\{\langle w, \psi_1 \rangle, \langle w, \psi_2 \rangle\}$
$\langle w, \Diamond^n \psi \rangle$ and $w \in N$	$\{\langle\langle w_0, \dots w_n \rangle, \psi\rangle \mid \langle w, w_i \rangle \in R$ for all $i \leq n$ and $i \neq j$ implies $w_i \neq w_j\}$
$\langle [w_0, \dots w_n], \psi \rangle$	$\{\langle w_i, \psi \rangle \mid i = 1, \dots, n\}$
$\langle w, P \rangle$ and $w \notin V(P)$	$\emptyset$
$\langle w, \neg P \rangle$ and $w \in V(P)$	$\emptyset$
$\langle w, \mu X.\psi_X \rangle$	$\{\langle w, \psi_X \rangle\}$
$\langle w, X \rangle$	$\{\langle w, \mu X.\psi_X \rangle\}$

Refuter

Position	Admissible moves
$\langle w, \psi_1 \wedge \psi_2 \rangle$	$\{\langle w, \psi_1 \rangle, \langle w, \psi_2 \rangle\}$
$\langle w, \Box^n \psi \rangle$ and $w \in N$	$\{\langle [w_0, \dots w_n], \psi\rangle \mid \langle w, w_i \rangle \in R$ for all $i \leq n$ and $i \neq j$ implies $w_i \neq w_j\}$
$\langle\langle w_0, \dots w_n \rangle, \psi \rangle$	$\{\langle w_i, \psi \rangle \mid i = 1, \dots, n\}$
$\langle w, P \rangle$ and $w \in V(P)$	$\emptyset$
$\langle w, \neg P \rangle$ and $w \notin V(P)$	$\emptyset$
$\langle w, \nu X.\psi_X \rangle$	$\{\langle w, \psi_X \rangle\}$
$\langle w, X \rangle$	$\{\langle w, \nu X.\psi_X \rangle\}$

Now, suppose $w \in \|\Box^{\leq n}\varphi\|^M$. As R is an equivalence relation, the worlds accessible from w' are the same as the worlds accessible from w. Therefore there are at most n worlds accessible from w' where φ fail. That is, $w' \in \|\Box^{\leq n}\varphi\|^M$. $\square$

We can then show the key lemma for the graded μ-calculus:

Lemma 39. *If $M = \langle W, R, V\rangle$ is a Kripke model where R is an equivalence relation, and $\nu X.\varphi$ is a well-named graded μ-formula, then*

$$\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M \text{ and } \|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M.$$

Proof. Let $M = \langle W, R, V\rangle$ be a Kripke model where R is an equivalence relation. We suppose $\nu X.\varphi$ is a well-named μ-formula of the form $\alpha(\triangle\beta(X))$, with $\triangle \in \{\Box, \Diamond\}$. We show that $\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M$.

Remember that $\varphi^0(X) := X$ and $\varphi^{n+1}(X) := \varphi(\varphi^n(X))$. As in Lemma 15, we suppose that $w \in \|\varphi^2(\top)\|^M$ and $w \notin \|\varphi^3(\top)\|^M$ to get a contradiction. The two player V and R will play simultaneously the games $\mathcal{G}_2 = \mathcal{G}(M, w \models \varphi^2(\top))$ and $\mathcal{G}_3 = \mathcal{G}(M, w \models \varphi^3(\top))$. In the former game, V uses their winning strategy σ, and, in the later, V plays moves analogous to the ones they did σ. Similarly, R uses his winning strategy τ on the later game, and a strategy analogous to τ on the former game. For example, if V plays $\sigma(\langle v, \Diamond^{>n}\psi(\top)\rangle) = \langle\langle v_0, \dots, v_n\rangle, \psi(\top)\rangle$, then they play $\langle\langle v_0, \dots, v_n\rangle, \psi(\varphi(\top))\rangle$ in $\mathcal{G}_3$.

The players continue both games until they get to a position of the form $\langle v, P\rangle$ (or $\langle v, \neg P\rangle$) in both games—which implies $v \in V(P)$ and $v \notin V(P)$—or they get to positions of the form $\langle v, \triangle\beta(\top)\rangle$ in $\mathcal{G}_2$ and $\langle v, \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_3$—which implies $v \in \|\triangle\beta(\varphi(\top))\|^M$ and $v \notin \|\triangle\beta(\varphi(\top))\|^M$. Either case gives a contradiction as in Lemma 15.

Yet again, we prove $\|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M$ by a direct calculation:

$$\mu X.\varphi \equiv \neg\nu X.\neg\varphi(\neg X) \equiv \neg(\neg\varphi(\neg\neg\varphi(\neg\top))) \equiv \varphi(\varphi(\bot)). \qquad \square$$

Theorem 40. *The graded μ-calculus' alternation hierarchy collapses to graded modal logic over equivalence relations.*

Proof. We argue by structural induction on μ-formulas. Yet again, we only prove the interesting cases.

Suppose φ is equivalent to the graded modal formula ψ. Then $\Box^{\leq n}\varphi$ is equivalent to $\Box^{\leq n}\psi$ and $\Diamond^{>n}\varphi$ is equivalent to $\Diamond^{>n}\psi$, for all $n \in \omega$.

Similarly, $\nu X.\varphi$ is equivalent to $\nu X.\psi$. By Lemma 39, $\nu X.\psi$ is equivalent to $\psi^2(\top)$, which is a graded modal formula. The same argument shows that $\mu X.\psi$ is equivalent to $\psi^2(\bot)$.

Therefore every graded μ-formula is equivalent to a graded modal formula over equivalence relations. $\square$

4.3 Intuitionistic modal logic

We now consider IS5, an intuitionistic variant of S5. Over IS5, the excluded middle fails and $\Box$ is not the dual of $\Diamond$. That is, there are models where $P \vee \neg P$ fail and worlds where $\Box P \leftrightarrow \neg\Diamond\neg P$ fails. To study IS5, we need to modify our Kripke frames. We follow [Sim94].

INTUITIONISTIC FRAMES. We can use Kripke frames to give semantics to intuitionistic logic. If $F = \langle W, \preceq\rangle$ is an intuitionistic frame, then $\preceq$ is a transitive and reflexive relation. On Kripke models for intuitionistic logic, we have an additional requirement: if $M = \langle W, \preceq, V\rangle$ is an intuitionistic model, wRv and $w \in V(P)$, then $v \in V(P)$.

The intuitionistic *propositional* formulas are given by the following grammar:

$$\varphi := P \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

Sometimes different symbols are used to differentiate classical and intuitionistic versions of negation and implication. As we will not use both in our formulas, we use the same symbols.

Over an intuitionistic model $M = \langle W, \preceq, V\rangle$, we define the valuation of formulas as follows:

- $\|P\|^M = V(P)$;
- $\|\bot\|^M = \emptyset$;
- $\|\top\|^M = W$;
- $\|\neg\varphi\|^M = \{w \mid \forall v.\text{if } w \preceq v \text{ then } v \notin \|\varphi\|^M\}$;
- $\|\varphi \wedge \psi\|^M = \|\psi\|^M \cap \|\psi\|^M$;

- $\|\varphi \vee \psi\|^M = \|\psi\|^M \cup \|\psi\|^M$; and
- $\|\varphi \to \psi\|^M = \{w \mid \forall v.\text{if } w \preceq v \text{ then } v \in \|\varphi\|^M \text{ implies } v \in \|\psi\|^M\}$.

One can also define $\neg\varphi$ as the abbreviation of $\varphi \to \bot$.

One can think of the relation $\preceq$ as relating the amount of information contained in worlds. In a intuitionistic Kripke model M, if $w \preceq v$ and $M, w \models \varphi$ then $M, v \models \varphi$. The proof is done by structural induction on formulas.

Example 15. The law of excluded middle does not hold on intuitionistic semantics. Consider the model $M = \langle W, \preceq, V\rangle$ defined by $W = \{w, v, u\}$, $R = \{\langle w, v\rangle, \langle w, u\rangle\}$ and $V(P) = \{u\}$. Then

- $M, w \not\models P$ and $M, w \not\models \neg P$.
- $v \models \neg P$.
- $u \models P$.

INTUITIONISTIC S5. The intuitionistic S5 (denoted by IS5) consists of all the intuitionistic tautologies; the inference rules

$$(\textbf{Nec})\ \frac{\varphi}{\Box\varphi} \qquad \text{and} \qquad (\textbf{MP})\ \frac{\varphi \quad \varphi \to \psi}{\psi};$$

and the axioms

- $K := \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \wedge \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$,
- $T := \Box\varphi \to \varphi \wedge \varphi \to \Diamond\varphi$,
- $4 := \Box\varphi \to \Box\Box\varphi \wedge \Diamond\Diamond\varphi \to \Diamond\varphi$,
- $5 := \Diamond\varphi \to \Box\Diamond\varphi \wedge \Diamond\Box\varphi \to \Box\varphi$,
- $FS := (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$,
- $DP := \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$,
- $N := \neg\Diamond\bot$.

BI-RELATIONAL FRAMES FOR IS5. We consider bi-relational Kripke frames $F = \langle W, \preceq, \equiv\rangle$ where $\langle W, \preceq\rangle$ is an intuitionistic frame, $\langle W, \equiv\rangle$ is an S5 frame, and the following confluence conditions are satisfied:

- $F = \langle W, \preceq, \equiv\rangle$ is forward confluent: $w \preceq w'$ and $w \equiv v$ imply there is v' such that $v \preceq v'$ and $w' \equiv v'$.
- $F = \langle W, \preceq, \equiv\rangle$ is backward confluent: $w \equiv v \preceq v'$ implies there is w' such that $w \preceq w' \equiv v'$.

$$\begin{tikzcd} w \arrow[r, "\preceq"] \arrow[d, "\equiv"'] & w' \arrow[d, dashed, "\equiv"] \\ v \arrow[r, dashed, "\preceq"'] & v' \end{tikzcd} \qquad \begin{tikzcd} w \arrow[r, dashed, "\preceq"] \arrow[d, "\equiv"'] & w' \arrow[d, dashed, "\equiv"] \\ v \arrow[r, "\preceq"'] & v' \end{tikzcd}$$

Figure 4.2: Schematics for forward and backward confluence.

The two confluence conditions are illustrated in Figure 4.2. We call such frames IS5 frames.A bi-relational model is a tuple $\langle W, \preceq, \equiv, V\rangle$ where $\langle W, \preceq, \equiv\rangle$ is an IS5 frame, V is a valuation, and if wRv and $w \in V(P)$, then $v \in V(P)$.

Let $M = \langle W, \preceq, \equiv, V\rangle$ be an IS5 frame. Then $\Box\varphi$ holds at w iff, for all w' $\preceq$-accessible from w and for all w'' $\equiv$-accessible from w', φ holds at w''. Similarly, $\Diamond\varphi$ holds at w iff, for all w' $\preceq$-accessible from w and there w'' $\equiv$-accessible from w' such that φ holds at w''. That is,

- $\|\Box\varphi\|^M = \{w \mid \forall v \succeq w \forall u \equiv v.u \in \|\varphi\|^M\}$; and
- $\|\Diamond\varphi\|^M = \{w \mid \forall v \succeq w \exists u \equiv v.u \in \|\varphi\|^M\}$.

Theorem 41 (Ono [Ono77], Fischer Servi [Fis78])**.** IS5 *is complete over* IS5 *frames.*

INTUITIONISTIC μ-FORMULAS. In the standard μ-calculus, we supposed every formula is in negative normal form. We cannot do the same on intuitionistic semantics. For example, $\neg\Diamond\neg\varphi$ is not equivalent to $\Box\varphi$ over intuitionistic semantics. Let X be a variable symbol, then:

- X is positive and negative in P;
- X is positive in X ;
- if $Y \neq X$, X is positive and negative in Y ;
- if X is positive (negative) in φ, then X is negative (positive) in φ;
- if X is positive (negative) in φ and ψ, then X is positive (negative) in $\varphi \wedge \psi$, $\varphi \vee \psi$, $\Box\varphi$, and $\Diamond\varphi$;
- if X is positive (negative) in φ, then X is negative (positive) in $\varphi \to \psi$;
- if X is positive (negative) in ψ, then X is positive (negative) in $\varphi \to \psi$;
- X is not free in $\eta X.\varphi$.

The intuitionistic μ-formulas are defined by the following grammar:

$$\varphi := P \mid X \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

where $\eta X.\varphi$ is defined iff X is positive in φ. Positiveness will guarantee that the operators $\Gamma_{\varphi(X)}$ are monotone.

Proposition 42. *Fix a bi-relational model $M = \langle W, \preceq, \equiv, V\rangle$ and sets of worlds $A \subseteq B \subseteq W$. If X is positive in φ, then $\|\varphi(A)\|^M \subseteq \|\varphi(B)\|^M$. Symmetrically, if X is negative in φ, then $\|\varphi(B)\|^M \subseteq \|\varphi(A)\|^M$.*

Proof. We prove the proposition above by structural induction on μ-formulas. We will only prove a few representative cases.

The cases of formulas of the form P, X, Y, $\varphi \wedge \psi$, and $\varphi \vee \psi$, the proposition follows by direct calculations. For example, suppose X is positive in $\varphi \vee \psi$. Then X is positive in φ and in ψ. Therefore:

$$\begin{aligned} \|(\varphi \vee \psi)(A)\|^M &= \|\varphi(A)\|^M \cup \|\psi(A)\|^M \\ &\subseteq \|\varphi(B)\|^M \cup \|\psi(b)\|^M = \|(\varphi \vee \psi)(B)\|^M. \end{aligned}$$

The case for formulas of the form $\eta X.\varphi$ is trivial, as X is not free in $\eta X.\varphi$.

The proof for formulas of the form $\neg\varphi$ or $\varphi \to \psi$ is a little bit more complex. Suppose X is positive in $\varphi \to \psi$, then X is positive in ψ and negative in φ. Therefore:

$$\begin{aligned} w \in \|(\varphi \to \psi)(A)\|^M &\iff \forall v \succeq w. v \in \|\varphi(A)\|^M \text{ implies } \|\psi(A)\|^M \\ &\implies \forall v \succeq w. v \in \|\varphi(B)\|^M \text{ implies } \|\psi(B)\|^M \\ &\iff w \in \|(\varphi \to \psi)(B)\|^M. \end{aligned}$$

Now, we prove the proposition for formulas are of the form $\Box\varphi$ or $\Diamond\varphi$. Suppose X is positive in $\Box\varphi$, then X is positive in φ. Therefore:

$$\begin{aligned} w \in \|\Box\varphi(A)\|^M &\iff \forall v \succeq w \forall u \equiv v. u \in \|\varphi(A)\|^M \\ &\implies \forall v \succeq w \forall u \equiv v. u \in \|\varphi(B)\|^M \\ &\iff w \in \|\Box\varphi(B)\|^M. \end{aligned}$$ ☐

GAME SEMANTICS FOR INTUITIONISTIC μ-CALCULUS. We can also define game semantics for intuitionistic μ-calculus. We now need to consider negation of formulas, as we cannot assume formulas are in the negation normal form.

Fix a bi-relational model $M = \langle W, \preceq, \equiv, V\rangle$, a world $w \in W$, and a μ-formula φ. The evaluation game $\mathcal{G}(M, w \models \varphi)$ has two players. We call them I and II. The two players will alternate the roles of Verifier and Refuter. The games begin at the state $\langle w, \varphi\rangle$, with I in the role of V and II in the role of R.

At a position of the form $\langle v, \neg\psi\rangle$, R chooses $v' \succeq v$ and challenges V to show that $M, v' \not\models \varphi$; that is, the game goes to the position $\langle v', \psi\rangle$ and the players exchange roles. Positions of the form $\langle v, \psi \to \theta\rangle$ is similar. In this case, R chooses $v' \succeq v$ and V chooses whether to show that $M, v' \not\models \psi$ or $M, v' \models \theta$; in case V chooses $\langle v', \psi\rangle$, the players exchange roles.

On a position of the form $\langle v, \Box\psi\rangle$, R chooses v' and v'' such that $v \preceq v' \equiv v''$, and then the game goes to the position $\langle v'', \psi\rangle$. On a position of the form $\langle v, \Diamond\psi\rangle$, R chooses $v' \succeq v$, V chooses $v'' \equiv v'$, and then the game goes to the position $\langle v'', \psi\rangle$.

Theorem 43. *Let $M = \langle W, \preceq, \equiv, V\rangle$ be an* IS5 *model, $w \in W$ and φ be a μ-formula. Then*

I *wins* $\mathcal{G}(M, w \models \varphi)$ *iff* $M, w \models \varphi$, *and*
II *wins* $\mathcal{G}(M, w \models \varphi)$ *iff* $M, w \not\models \varphi$.

Table 4.3: Rules of evaluation games for the intuitionistic modal μ-calculus.

Verifier

Position	Admissible moves
$\langle w, \psi_1 \vee \psi_2 \rangle$	$\{\langle w, \psi_1 \rangle, \langle w, \psi_2 \rangle\}$
$\langle w, \psi_0 ? \psi_1 \rangle$	$\{\langle w, \psi_0 \rangle$ and exchange roles, $\langle w, \psi_1 \rangle\}$
$\langle [w], \psi \rangle$	$\{\langle v, \psi \rangle \mid w \equiv v\}$
$\langle w, P \rangle$ and $w \notin V(P)$	$\emptyset$
$\langle w, \mu X.\psi_X \rangle$	$\{\langle w, \psi_X \rangle\}$
$\langle w, X \rangle$	$\{\langle w, \mu X.\psi_X \rangle\}$

Refuter

Position	Admissible moves
$\langle w, \psi_1 \wedge \psi_2 \rangle$	$\{\langle w, \psi_1 \rangle, \langle w, \psi_2 \rangle\}$
$\langle w, \neg\psi \rangle$	$\{\langle v, \psi \rangle \mid w \preceq w\}$ and exchange roles
$\langle w, \psi_1 \to \psi_2 \rangle$	$\{\langle v, \psi_0 ? \psi_1 \rangle \mid w \preceq w\}$
$\langle w, \Diamond\psi \rangle$	$\{\langle \langle v \rangle, \psi \rangle \mid w \preceq v\}$
$\langle w, \Box\psi \rangle$	$\{\langle [v], \psi \rangle \mid w \preceq v\}$
$\langle \langle w \rangle, \psi \rangle$	$\{\langle v, \psi \rangle \mid w \equiv v\}$
$\langle w, P \rangle$ and $w \in V(P)$	$\emptyset$
$\langle w, \nu X.\psi_X \rangle$	$\{\langle w, \psi_X \rangle\}$
$\langle w, X \rangle$	$\{\langle w, \nu X.\psi_X \rangle\}$

Proof. Similar to the proof of Theorem 12. Here, instead of players being verifier and refuter, the players alternate the roles of verifier and refuter.

We define signatures as we did in Theorem 12, but we consider I-signatures and II-signatures tracking the variables which each player does not want to appear infinitely often. We can do this since, given φ and $\eta X.\psi \in \mathrm{Sub}(\varphi)$, a player will always have the same role in a position of the form $\langle v, \psi \rangle$. $\square$

COLLAPSE TO MODAL LOGIC. Yet again, for the third time in this chapter, we generalize the proof of Lemma 15. We cannot prove Lemma 14 in intuitionistic semantics, but we get a good enough lemma:

Lemma 44. *If $M = \langle W, \preceq, \equiv, V \rangle$ is a bi-relational model, then $\preceq;\equiv$ is transitive.*

Proof. Suppose $w \preceq w' \equiv v \preceq v' \equiv u$. By backward confluence, there is u' such that $w' \preceq u' \equiv v'$. By the transitivity of $\preceq$ and $\equiv$, $w \preceq u' \equiv u$. $\square$

Lemma 45. *Let $M = \langle W, \preceq, \equiv, V \rangle$ be a bi-relational model, φ be a μ-formula, and ρ be a play of the evaluation game $\mathcal{G}(M, w \models \psi)$. For all $i < \mathrm{len}(\rho)$, if $\rho_i = \langle v, \psi \rangle$ and $\rho_{i+1} = \langle v', \psi' \rangle$, then $v \preceq;\equiv v'$. Therefore if $i < j$ and $\rho_i = \langle v, \psi \rangle$ and $\rho_j = \langle v', \psi' \rangle$, then $v \preceq;\equiv v'$.*

Lemma 46. *Let $M = \langle W, \preceq, \equiv, V\rangle$ be a bi-relational model and $w \preceq;\equiv w'$. Then*

$$M, w \models \triangle\varphi \text{ implies } M, w' \models \triangle\varphi,$$

where $\triangle \in \{\Box, \Diamond\}$.

Proof. Suppose $w \preceq;\equiv w'$ and $M, w \models \Diamond\varphi$. For all $v \succeq w$, there is $u \equiv v$ such that $M, u \models \varphi$. Let v, v' be such that $w \preceq v \equiv w' \preceq v'$. By downward confluence, there is u such that $v \preceq u \equiv v'$. By the transitivity of $\preceq$, $w \preceq u$. So there is $u' \succeq w$ such that $u \equiv u'$ and $M, u' \models \varphi$. As $v' \equiv u \equiv u'$, $v' \equiv u'$. So for all $v' \succeq w'$ there is $u' \equiv v'$ such that $M, u' \models \varphi$. That is, $M, w' \models \Diamond\varphi$. $\Box$

Lemma 47. *Let $M = \langle W, \preceq, \equiv, V\rangle$ be a bi-relational model and φ be a formula where X is positive. Then*

$$\|\mu X.\varphi\|^M = \|\varphi^2(\top)\|^M \text{ and } \|\nu X.\varphi\|^M = \|\varphi^2(\bot)\|^M.$$

Proof. We first show that $\|\nu X.\varphi\|^M = \|\varphi^2(\top)\|^M$. Let $M = \langle W, R, V\rangle$ be a Kripke model where R is an equivalence relation, and $\nu X.\varphi$ is a well-named μ-formula. We can also suppose that φ is of the form $\alpha(\triangle\beta(X))$ with $\triangle \in \{\Box, \Diamond\}$.

We show that $\nu X.\varphi$ is equivalent to $\varphi^2(\top)$. As X is positive in $\varphi(X)$, we have that $\|\varphi^3(\top)\|^M \subseteq \|\varphi^2(\top)\|^M$. So we need only to show that $\|\varphi^2(\top)\|^M \subseteq \|\varphi^3(\top)\|^M$.

For a contradiction, suppose that $w \in \|\varphi^2(\top)\|^M$ and $w \notin \|\varphi^3(\top)\|^M$. Then I has a winning strategy σ for the evaluation game $\mathcal{G}_2 = \mathcal{G}(M, w \models \varphi^2(\top))$; and II has a winning strategy τ for the evaluation game $\mathcal{G}_3 = \mathcal{G}(M, w \models \varphi^3(\top))$. We use σ and τ to define strategies σ' for I in $\mathcal{G}_3$ and τ' for II in $\mathcal{G}_2$. Remember that I starts on the role of V and II starts on the role of R.

Yet again, we have the players use analogous strategies on both games. Suppose the players are in positions $\langle v, \psi(\top)\rangle$ in $\mathcal{G}_2$ and $\langle v, \psi(\varphi(\top))\rangle$ in $\mathcal{G}_3$. Both positions have the same owner, in the same role. That is, if I's turn in some game, it is I's turn in both games; and the owner's role is V in some game, their role is V in both games. For example, suppose I is playing the role of R and the players are in positions $\langle v, \neg\psi(\top)\rangle$ and $\langle v, \neg\psi(\varphi(\top))\rangle$ in $\mathcal{G}_2$ and $\mathcal{G}_3$. If I plays $\sigma(\langle v, \neg\psi(\top)\rangle) = \langle v', \psi(\top)\rangle$ in $\mathcal{G}_2$, they play $\langle v, \psi(\varphi(\top))\rangle$ in $\mathcal{G}_3$. After there moves, I is playing the role of V in both games.

The players continue both games following the strategies described above until they get to a position of the form $\langle v, P\rangle$ in both games; or they get to positions of the form $\langle w'', \triangle\beta(\top)\rangle$ in $\mathcal{G}_2$ and $\langle w'', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_3$.

Case 1. Suppose the players are in a position $\langle v, P\rangle$ in both games. Without loss of generality, suppose I is V and II is R. As σ is winning for I in $\mathcal{G}_2$, $v \in \|P\|^M$. As τ is winning for II in $\mathcal{G}_3$, $v \notin \|P\|^M$. And so we have a contradiction. A similar contradiction is reached if I is R and II is V.

Case 2. Suppose the players are in positions of the form $\langle w'', \triangle\beta(\top)\rangle$ in $\mathcal{G}_2$ and $\langle w'', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_3$. Without loss of generality, suppose I is V and II is R. As τ is a winning strategy for II in $\mathcal{G}_3$, $w'' \notin \|\triangle\beta(\varphi(\top))\|^M$. Previously, the players must have been through some a position $\langle w', \triangle\beta(\varphi(\top))\rangle$ in $\mathcal{G}_2$. As σ is a winning strategy for I in $\mathcal{G}_2$, $w' \in \|\triangle\beta(\varphi(\top))\|^M$. By Lemma 45, $w' \preceq;\equiv w''$. By Lemma 44, $w'' \in \|\triangle\beta(\varphi(\top))\|^M$ since $w' \in \|\triangle\beta(\varphi(\top))\|^M$. We have our contradiction.

Either way, we conclude that $\|\varphi^2(\top)\|^M \subseteq \|\varphi^3(\top)\|^M$. And so $\|\nu X.\varphi\|^M = \|\varphi^2(\bot)\|^M$.

In intuitionistic semantics, we cannot prove $\|\mu X.\varphi\|^M = \|\varphi^2(\bot)\|^M$ by a direct calculation as before, as the proof of Lemma 15 required the use of the law of excluded middle. But we can prove it directly.

First, $\|\varphi^2(\bot)\|^M \subseteq \|\mu X.\varphi\|^M$ holds as X is positive in $\varphi(X)$. If we suppose there is w such that $w \in \|\mu X.\varphi\|^M$ and $w \notin \|\varphi^2(\bot)\|^M$, we get a similar contradiction. $\square$

Theorem 48. *Over models of* IS5*, every μ-formula is equivalent to a modal formula.*

Proof. Yet one more time, we argue by structural induction on μ-formulas. We only prove the interesting cases.

Suppose φ is equivalent to the modal formula ψ. Then $\Box\varphi$ is equivalent to $\Box\psi$ and $\Diamond\varphi$ is equivalent to $\Diamond\psi$.

Similarly, $\nu X.\varphi$ is equivalent to $\nu X.\psi$. By Lemma 47, $\nu X.\psi$ is equivalent to $\psi^2(\top)$, which is a modal formula. The same argument shows that $\mu X.\psi$ is equivalent to $\psi^2(\bot)$.

Therefore every μ-formula is equivalent to a modal formula over models of IS5. $\square$

4.4 Multimodal semantics

Consider the modal μ-calculus with two modalities $\Box_0$ and $\Box_1$, both satisfying S5. We prove how to show the alternation hierarchy is strict in this setting via evaluation games. We restrict ourselves to two modalities, as this will be enough to show the strictness of the alternation hierarchy on multimodal S5. We also describe how the methods used on S5 frames can be generalized for other frame classes.

FUSION LOGICS. Let L_1 and L_2 be modal logics with disjoint sets of modal operators. The fusion $\mathsf{L}_1 \otimes \mathsf{L}_2$ is the smallest modal logic which contains L_1 and L_2. We study frames of S5 $\otimes$ S5 in this section. S5 $\otimes$ S5 is also known as $\mathsf{S5}_2$. In general, if L is a modal logic, define L_n by $\underbrace{\mathsf{L} \otimes \cdots \otimes \mathsf{L}}_{n \text{ times}}$. For more on fusion logics, see [CC20; Kur07].

EPISTEMIC LOGIC. G is a finite group of agents. For each $a \in G$, let $K_a\varphi$ mean "the agent a knows that φ is true". Define "everyone knows" modality by

$$E\varphi := \bigwedge_{a \in G} K_a\varphi.$$

Then we define the common knowledge modality by

$$C\varphi := \nu X.\varphi \wedge E\varphi.$$

For any φ, $C\varphi$ is not equivalent to any modal fomula, over frames where the modalities K_a satisfy S5.

MULTIMODAL FORMULAS. Fix a nonempty set of labels Λ. The multimodal μ-formulas are defined by the grammar below:

$$\varphi := P \mid \neg P \mid X \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box_0\varphi \mid \Diamond_i\varphi \mid \Box_i\varphi \mid \mu X.\varphi \mid \nu X.\varphi,$$

where P are propositional symbols, X are variable symbols, and $i \in \Lambda$. We will focus on the case $\Lambda = \{0, 1\}$.

MULTIMODAL SEMANTICS. Fix a nonempty set of labels Λ. A Kripke model is a tuple $M = \langle W, \{R_i\}_{i\in\Lambda}, V\rangle$ consisting of a set W of possible worlds, an accessibility relation R_i for each label $i \in \Lambda$, and a valuation function V. Semantics are defined as in the standard μ-calculus, with the valuation of $\Box_i$ and $\Diamond_i$ depending on the relation R_i:

- $\|P\|^M = V(P)$;
- $\|X\|^{M[X\mapsto A]} = A$;
- $\|\bot\|^M = \emptyset$;
- $\|\top\|^M = W$;
- $\|\neg P\|^M = W \setminus \|P\|^M$;
- $\|\varphi \wedge \psi\|^M = \|\psi\|^M \cap \|\psi\|^M$;
- $\|\varphi \vee \psi\|^M = \|\psi\|^M \cup \|\psi\|^M$;
- $\|\Box_i\varphi\|^M = \{w \in W \mid \forall v.wR_iv \to v \in \|\varphi\|^M\}$, for $i \in \Lambda$;
- $\|\Diamond_i\varphi\|^M = \{w \in W \mid \exists v.wR_iv \wedge v \in \|\varphi\|^M\}$, for $i \in \Lambda$;
- $\|\mu X.\varphi(X)\|^M$ is the least fixed-point of $\Gamma_{\varphi(X)}$; and
- $\|\nu X.\varphi(X)\|^M$ is the greatest fixed-point of $\Gamma_{\varphi(X)}$.

EVALUATION GAMES. Let Λ be a set of labels, $M = \langle W, \{R_0\}_{i\in\Lambda}, V\rangle$ be a Kripke model, $w \in W$, and φ be a well-named multimodal μ-formula. We define an evaluation game $\mathcal{G}(M, w \models \varphi)$ to decide whether $M, w \models \varphi$ as we did for the unimodal μ-calculus

Again, the game $\mathcal{G}(M, w \models \varphi)$ has two players: verifier, who wants to show that $M, w \models \varphi$; and refuter, who wants to show that $M, w \not\models \varphi$. We denote verifier by V and refuter by R. The game positions are pairs $\langle v, \psi\rangle$ where v is in W and ψ is a subformula of φ. The game starts at $\langle w, \varphi\rangle$. The only difference from the games for the unimodal μ-calculus is that there are more than one accessibility relation. The players advance in the game graph as follows:

- at $\langle v, \psi_0 \wedge \psi_1\rangle$, R chooses one of $\langle v, \psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \psi_0 \vee \psi_1\rangle$, V chooses one of $\langle v, \psi_0\rangle$ and $\langle v, \psi_1\rangle$;
- at $\langle v, \Box_i\psi\rangle$, R chooses $\langle v', \psi\rangle$ with vR_iv';
- at $\langle v, \Diamond_i\psi\rangle$, V chooses $\langle v', \psi\rangle$ with vR_iv';
- at $\langle v, \eta X.\psi\rangle$, the players move to $\langle v, \psi\rangle$; and
- at $\langle v, X\rangle$, the players move to $\langle v, \eta X.\psi\rangle$ where $\eta X.\psi$ is a subformula of φ.

We summarize the possible plays in Table 4.4.

As in the unimodal case, fame semantics and Kripke semantics are equivalent:

Theorem 49. *Let $M = \langle W, R, V\rangle$ be a Kripke model, $w \in W$ and φ be a μ-sentence, then*

$$\mathsf{V} \textit{ wins } \mathcal{G}(M, w \models \varphi) \textit{ iff } M, w \models \varphi\textit{; and}$$
$$\mathsf{R} \textit{ wins } \mathcal{G}(M, w \models \varphi) \textit{ iff } M, w \not\models \varphi.$$

Proof. The proof is the same as Theorem 12. □

Table 4.4: Rules of evaluation games for multimodal μ-calculus.

Verifier		Refuter	
Position	Admissible moves	Position	Admissible moves
$\langle w, \psi_1 \vee \psi_2\rangle$	$\{\langle w, \psi_1\rangle, \langle \psi_2\rangle\}$	$\langle w, \psi_1 \wedge \psi_2\rangle$	$\{\langle w, \psi_1\rangle, \langle w, \psi_2\rangle\}$
$\langle w, \Diamond_i \psi\rangle$	$\{\langle v, \psi\rangle \mid \langle w, v\rangle \in R_i\}$	$\langle w, \Box_i \psi\rangle$	$\{\langle v, \psi\rangle \mid \langle w, v\rangle \in R_i\}$
$\langle w, P\rangle$ and $w \notin V(P)$	$\emptyset$	$\langle w, P\rangle$ and $w \in V(P)$	$\emptyset$
$\langle w, \neg P\rangle$ and $w \in V(P)$	$\emptyset$	$\langle w, \neg P\rangle$ and $w \notin V(P)$	$\emptyset$
$\langle w, \mu X.\psi_X\rangle$	$\{\langle w, \mu X.\psi_X\rangle\}$	$\langle w, \nu X.\psi_X\rangle$	$\{\langle w, \nu X.\psi_X\rangle\}$
$\langle w, X\rangle$	$\{\langle w, \psi_X\rangle\}$	$\langle w, X\rangle$	$\{\langle w, \psi_X\rangle\}$

PARITY GAMES. Remember, parity game is a tuple $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ where two players $\exists$ and $\forall$ move a token in the graph $\langle V_\exists \cup V_\forall, E\rangle$. We suppose $V_\exists$ and $V_\forall$ are disjoint sets of vertices; $E \subseteq (V_\exists \cup V_\forall)^2$ is a set of edges; and $\Omega : V_\exists \cup V_\forall \to n$ is a priority function. If a player cannot move, then the other player wins. In an infinite play ρ, the winner is determined by the following parity condition: $\exists$ wins ρ iff the longest priority which appears infinitely often in ρ is even; otherwise, $\forall$ wins ρ. $\exists$ wins the parity game $\mathcal{P}$ iff $\exists$ has a winning strategy; a winning strategy for $\exists$ is a function σ from $V_\exists$ to $V_\exists \cup V_\forall$, where, if $\exists$ follows σ, all resulting plays are winning for them. Similarly, $\forall$ wins $\mathcal{P}$ iff $\forall$ has a winning strategy.

Fix a parity game $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$. The set of winning positions for $\exists$ in $\mathcal{P}$ is the set of positions v where $\exists$ wins the parity game if the players start at v. That is, $v \in V_\exists \cup V_\forall$ is a winning position for $\exists$ iff $\exists$ wins $\mathcal{P}_v = \langle V_\exists, V_\forall, v, E, \Omega\rangle$. We will define μ-formulas W_n such that, if $\max\{\Omega(v \mid v \in V_\exists \cup V_\forall)\}$, then W_n defines the set of winning positions for $\exists$ in $\mathcal{P}$.

For technical convenience, we suppose all parity game is tree like. That is, for all $v \in V_\exists \cup V_\forall$, there is no path $v = v_0 R \cdots R v_n = v$, for all $n \in \omega$. Any parity game $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ can be unfolded into a tree-like parity game. In the unfolded game, instead of moving to a node v, the players move to a fresh copy of v. The unfolded parity game is bisimilar to the original game.

WINNING REGION FORMULAS. We define modified versions of Bradfield's winning region formulas [Bra98a] for parity games. For $n \in \omega$, define:

$W_n := \eta X_n \ldots \nu X_0.$

$$\bigvee_{0 \le i \le 3} \left[Q_i \wedge \bigvee_{0 \le j \le n} [(P_j \vee P_\exists \vee \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)) \vee (P_j \vee P_\forall \vee \Box_{r(i)}(Q_{s(i)} \to X_j))] \right].$$

Here, $r(i)$ is $i \equiv$ modulo 2 and $s(i)$ is $i+1 \equiv$ modulo 4. We explain the intended meaning of the proposition symbols below.

Let $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ be a parity game. We represent $\mathcal{P}$ a the Kripke model $\mathcal{P}^K = \langle W, R_0, R_1, V\rangle$. Define $W := V_\exists \cup V_\forall$. Given $v \in W$, $d(v_0, v)$ denotes the least n such that $v_0 R^n v$. Define

- $R_0 := \{\langle v, v'\rangle, \langle v', v\rangle \mid d(v_0, v) \text{ is even}\} \cup \{\langle v, v\rangle \mid v \in W\}$; and
- $R_1 := \{\langle v, v'\rangle, \langle v', v\rangle \mid d(v_0, v) \text{ is odd}\} \cup \{\langle v, v\rangle \mid v \in W\}$.

The frame $\langle W, R_0, R_1\rangle$ is essentially the graph of $\mathcal{P}$, with the arrows partitioned in two alternating sets, and with their reverses added.

The proposition symbols $P_\exists$ and $P_\forall$ indicate the ownership of the positions: $v \in V(P_\exists)$ iff $v \in V_\exists$ and $v \in V(P_\exists)$ iff $v \in V_\exists$. As $V_\exists \cup V_\forall = W$ and $V_\exists \cap V_\forall = \emptyset$, $\mathcal{P}^K \models P_\exists \vee P_\forall$ and $\mathcal{P}^K \models \neg(P_\exists \wedge P_\forall)$.

The proposition symbols $P_0, \dots, P_n$ indicate the parities of the positions: $v \in V(P_i)$ iff $\Omega(v) = i$. At each world, exactly one of the P_i will hold. The proposition symbols Q_0, Q_1, Q_2, Q_3 are technical devices to control the flow of the game.

We set Q_0 true at v_0 and at each step we make the next Q_i be true (looping back to Q_0). Formally, define: $v \in V(Q_i)$ iff $d(v_0, v) = i$ modulo 4.

Proposition 50. *Let $\mathcal{P} = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$ be a parity game and $\mathcal{P}^{\mathrm{K}} = \langle W, R_0, R_1, V\rangle$ be Kripke model defined above. If $\max\{\Omega(v) \mid v \in W\} \leq n$, then*

$$\mathcal{P}^K, v_0 \models W_n \textit{ iff } \exists \textit{ wins } \mathcal{P}.$$

Proof. Suppose $\mathcal{P}^{\mathrm{K}}, w_0 \models W_n$. Let σ be a winning strategy for V in the evaluation game $\mathcal{G} := \mathcal{G}(\mathcal{P}^{\mathrm{K}}, v_0 \models W_n)$. We define a winning strategy σ' for $\exists$ in $\mathcal{P}$ while playing simultaneous runs of $\mathcal{G}$ and $\mathcal{P}$.

The games $\mathcal{G}$ and $\mathcal{P}$ start at positions $\langle v_0, W_n\rangle$ and v_0, respectively. In $\mathcal{G}$, have the players move to the position

$$\left\langle v_0, \bigvee_{0\leq i\leq 3}\left[Q_i \wedge \bigvee_{0\leq j\leq n}[(P_j \vee P_\exists \vee \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)) \vee (P_j \vee P_\forall \vee \Box_{r(i)}(Q_{s(i)} \to X_j))]\right]\right\rangle.$$

Now, suppose the players are at positions

$$\left\langle v, \bigvee_{0\leq i\leq 3}\left[Q_i \wedge \bigvee_{0\leq j\leq n}[(P_j \vee P_\exists \vee \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)) \vee (P_j \vee P_\forall \vee \Box_{r(i)}(Q_{s(i)} \to X_j))]\right]\right\rangle.$$

in $\mathcal{G}$ and v in $\mathcal{P}$. As σ is winning for V in $\mathcal{G}$, σ does not make any immediately losing move. That is, V picks $i = d(v_0, v)$ modulo 4, $j = \Omega(v)$ and $P_\exists$ or $P_\forall$ according to v's owner. We also have $\forall$ make non-immediately losing moves. We have two possible cases.

Case 1. The players are in the position $\langle v, \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)\rangle$ in $\mathcal{G}$. Then v is a position for $\exists$ in $\mathcal{P}$. If v' is such that $\sigma(\Diamond_{r(i)}(Q_{s(i)} \wedge X_j)) = \langle v', Q_{s(i)} \wedge X_j\rangle$, then $\exists$ moves to $\sigma'(v) := v'$ in $\mathcal{P}$.

Case 2. The players are in the position $\langle v, \Box_{r(i)}(Q_{s(i)} \to X_j)\rangle$ in $\mathcal{G}$. Then v is a position for $\forall$ in $\mathcal{P}$. If $\forall$ moves to v', have R move to $\langle v', Q_{s(i)} \to X_j)\rangle$ in $\mathcal{G}$.

Now, have the players regenerate X_j in $\mathcal{G}$ and move until they get to a position of the form

$$\left\langle v, \bigvee_{0\leq i\leq 3}\left[Q_i \wedge \bigvee_{0\leq j\leq n}[(P_j \vee P_\exists \vee \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)) \vee (P_j \vee P_\forall \vee \Box_{r(i)}(Q_{s(i)} \to X_j))]\right]\right\rangle.$$

again. We are back to the initial situation. Repeat this process to define σ'.

We consider parallel runs ρ in $\mathcal{G}$ and ρ' in $\mathcal{P}$ played according to σ and σ', respectively. Suppose the players are in a position v owned by $\exists$ in $\mathcal{P}$. In $\mathcal{G}$, V is in a position $\langle v, \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)\rangle$ in $\mathcal{G}$. As σ is winning for V, there must be v' such σ makes V move to $\langle v', Q_{s(i)} \wedge X_j\rangle$. By the definition of $\mathcal{G}$, this means that vEv', so $\exists$ has a move. Therefore, if ρ' is finite, then the last position's owner is $\forall$—therefore $\exists$ wins ρ'.

If ρ' is infinite, ρ is also infinite. As ρ is won by V, the outermost infinitely often occurring fixed-point operator is some νX_{2k}. This means the greatest infinitely often occurring parity in $\mathcal{P}$ is $2k$, as the regenerated X_j depend on the parities. Therefore $\exists$ wins ρ'.

On the other hand, suppose $\exists$ wins $\mathcal{P}$ via σ'. We define σ for V in $\mathcal{G}$. At vertices of the form $\langle v, \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)\rangle$ in $\mathcal{G}$, have

$$\sigma(\langle v, \Diamond_{r(i)}(Q_{s(i)} \wedge X_j)\rangle) := \langle v', Q_{s(i)} \wedge X_j\rangle,$$

with $v' = \sigma'(v)$. On other positions, have σ be the non-immediately losing moves for V.

Consider parallel runs ρ in $\mathcal{G}$ and ρ' in $\mathcal{P}$ played according to σ and σ', respectively. If ρ is finite, then one of the players made a bad move, where one of the Q_i, P_j, $P_\exists$ or $P_\forall$ is false. By the definition of σ, V makes no such move. So it must be R's move, and so V wins.

If ρ is infinite, the greatest parity appearing infinitely often in ρ' is even. Therefore the outermost infinitely often occurring fixed-point operator in ρ is a ν-operator. ρ is winning for V. $\square$

EVALUATION GAMES AS PARITY GAMES. Given a model $M = (W, R_0, R_1, V)$, $w \in W$ and a bimodal μ-formula φ we defined the evaluation game $\mathcal{G}(M, w \models \varphi)$ for $M, w \models \varphi$. This evaluation game is equivalent to the parity game $\mathcal{G}^{\mathrm{P}}(M, w \models \varphi) = \langle V_\exists, V_\forall, v_0, E, \Omega\rangle$. The set of positions $V_\exists$ consists of the positions for V in $\mathcal{G}(M, w \models \varphi)$; and $V_\forall$ consists of the positions for $\forall$. The set of edges E consists of the transitions of V in $\mathcal{G}(M, w \models \varphi)$. The initial position v_0 is $\langle w, \varphi\rangle$. Define the parity function:

- $\Omega(\langle v, \mu X.\psi\rangle) = 2(i+\varepsilon) - 1$ if $\mu X.\psi \in \Sigma^\mu_{2i+\varepsilon} \setminus \Pi^\mu_{2i+\varepsilon}$;
- $\Omega(\langle v, \nu X.\psi\rangle) = 2i$ if $\nu X.\psi \in \Pi^\mu_{2i+\varepsilon} \setminus \Sigma^\mu_{2i+\varepsilon}$;
- $\Omega(\langle v, \psi\rangle) = 0$ for ψ not of the form $\eta X.\psi$;

where $\varepsilon \in \{0, 1\}$.

Proposition 51. *Let $M = (W, R_0, R_1, V)$ be a bimodal Kripke model, $w \in W$, and φ a bimodal μ-formula. Then:*

$$\mathsf{V} \textit{ wins } \mathcal{G}(M, w \models \varphi) \iff \exists \textit{ wins } \mathcal{G}^{\mathrm{P}}(M, w \models \varphi).$$

Proof. Denote $\mathcal{G}(M, w \models \varphi)$ by $\mathcal{G}$ and $\mathcal{G}^{\mathrm{P}}(M, w \models \varphi)$ by $\mathcal{G}^{\mathrm{P}}$. As both games are on the same board, strategies for V and R in $\mathcal{G}$ are strategies for $\exists$ and $\forall$ in $\mathcal{G}^{\mathrm{P}}$. As any position is owned by V in $\mathcal{G}$ iff it is owned by $\exists$ in $\mathcal{G}$, any finite run is winning for V iff it is winning for $\exists$.

Consider an infinite run ρ. The parity $\Omega(\langle v, \psi\rangle)$ is odd iff ψ is strictly in Σ^k for some $k \in \omega$. That is, $\psi \in \Sigma^\mu_k \setminus \Pi^\mu_k$. If the greatest infinitely often occurring parity in ρ is odd, then some $\mu X.\psi$ is the outermost infinitely often occurring fixed-point formula. Otherwise, if $\mu X.\psi \in \mathrm{Sub}(\nu \mathrm{Y}.\theta)$ and $\nu Y.\theta$ is the outermost infinitely occurring fixed-formula formula, then $\Omega(\langle v, \nu Y.\theta\rangle) \geq \Omega(\langle v, \mu X.\psi\rangle)$ is even. Similarly, if the greatest infinitely often occurring parity in ρ is even, then some $\nu X.\psi$ is the outermost infinitely often occurring fixed-point formula. Either way, ρ is winning for V in $\mathcal{G}$ iff ρ is winning for $\exists$ in $\mathcal{G}^{\mathrm{P}}$. $\square$

Furthermore, given an evaluation game $\mathcal{G}(M, w \models \varphi)$, we define the Kripke model $\mathcal{G}^{\mathrm{K}}(M, w \models \varphi)$ as $(\mathcal{G}^{\mathrm{P}}(M, w \models \varphi))^{\mathrm{K}}$.

Theorem 52. *Let $M = (W, R_0, R_1, V)$ be a bimodal Kripke model, $w \in W$, and φ a bimodal μ-formula. Then:*

$$M, w \models \varphi \textit{ iff } \mathcal{G}^{K}(M, w \models \varphi), \langle w, \varphi\rangle \models W_n,$$

as long as the greatest parity used in the evaluation game is less or equal than n.

Proof. We prove this theorem using the results above:

$$\begin{aligned} M, w \models \varphi &\text{ iff } \mathsf{V} \text{ wins } \mathcal{G}(M, w \models \varphi) \\ &\text{ iff } \exists \text{ wins } \mathcal{G}^{\mathrm{P}}(M, w \models \varphi) \\ &\text{ iff } \mathcal{G}^{\mathrm{K}}(M, w \models \varphi), \langle w, \varphi\rangle \models W_n. \end{aligned}$$

The first equivalence holds by Theorem 49, the second one follows from 51, the third one follows from 50 $\square$

Let $\mathcal{G}^{\mathrm{K}}(M, w \models \varphi)$ be the pointed Kripke model representing the parity game $\mathcal{G}^{\mathrm{P}}(M, w \models \varphi)$.

A FIXED-POINT LEMMA. Given a formula φ, define $f_\varphi(M, w) = (\mathcal{G}^{\mathrm{K}}(M, w \models \varphi), \langle w, \varphi\rangle)$. That is, f_φ maps a pointed model to its evaluation game with respect to φ (as a pointed Kripke model).

Let $(M_0, w_0) = \langle W_0, R_{0,0}, R_{0,1}, V_0, w_0\rangle$ and $(M_1, w_1) = \langle W_1, R_{1,0}, R_{1,1}, V_1, w_1\rangle$ be pointed transition systems without loops in their graphs. (M_0, w_0) is isomorphic to (M_1, w_1) iff there is a bijection $I : W_0 \to W_1$ such that:

- $I(w_0) = w_1$;
- for all $w, w' \in W_0$, $wR_{0,0}w'$ iff $I(w)R1, -I(w')$;

- for all $w, w' \in W_0$, $wR_{0,1}w'$ iff $I(w)R^{1,1}I(w')$; and
- for all $w \in W$, $w \in V_0(P)$ iff $I(w) \in V_1(P)$.

For all $n \in \omega$, let $(M \restriction n, w)$ be the submodel of (M, w) obtained by restricting W to worlds with distance less than n from w. We say (M, w) is n-isomorphic to (N, v) if and only if $(M \restriction n, w)$ is isomorphic to $(N \restriction n, v)$. For any (M, w), $(M \restriction 0, w)$ is an empty Kripke model. we assume empty Kripke models to be isomorphic.

Lemma 53. *Fix some μ-formula φ. If (M, w) and (N, v) are n-isomorphic via a function I, $f_{\varphi\wedge\varphi}(M, w)$ and $f_{\varphi\wedge\varphi}(N, v)$ are $(n + 1)$-isomorphic via the function J defined by*

$$J(\langle w, \psi\rangle) = (\langle I(w), \psi\rangle), \textit{for } w \in W^M \textit{ and } \psi \in \mathrm{Sub}(\varphi).$$

Proof. As (M, w) and (N, v) are n-isomorphic, the evaluation games $\mathcal{G}(M, w \models \varphi\wedge\varphi)$ and $\mathcal{G}(N, v \models \varphi \wedge \varphi)$ are going to be same for up to n-many plays from positions $\langle w, \triangle\psi\rangle$ to positions $\langle w', \psi\rangle$. As the first move in an evaluation game for the formula $\varphi \wedge \varphi$ is to choose between a conjunction, we can guarantee that the two games above are the same up to $n + 1$ moves. □

Lemma 54. *For all μ-formula φ, the function $f_{\varphi\wedge\varphi}$ has a fixed-point (up to isomorphism). That is, there is a model (M, w) such that $f_\varphi(M, w)$ is isomorphic to (M, w).*

Proof. Let (M_0, w_0) be an arbitrary pointed Kripke model. trivially, (M_0, w_0) and $f_{\varphi\wedge\varphi}(M_0, w_0)$ are 0-isomorphic.

For $n \in \omega$, let us define $(M_{n+1}, w_{n+1}) = f_{\varphi\wedge\varphi}(M_w, w_n)$. We already have that (M_0, w_0) and (M_1, w_1) are 0-isomorphic. By Lemma 53, (M_w, w_n) and (M_{n+1}, w_{n+1}) are $(n + 1)$-isomorphic via induction on n.

For $m > n$, (M_n, w_n) is n-isomorphic to (M_m, w_m). Therefore, we define a pointed Kripke model (M, w) which is n-isomorphic to (M_n, w_n) for all n. To define M, we identify $(M_n \restriction n, w_n)$ and $(M_{n+1} \restriction n, w_{n+1})$, as they are n-isomorphic. Take the graph of M to be the union of the graph of the $M_n \restriction n$, the valuation of M to be the union of the valuation of the $M_n \restriction n$ and w as w_0. Finally, we have that (M, w) is isomorphic to $f_{\varphi\wedge\varphi}(M, w)$. Otherwise, there world be n such that (M_n, w_n) is not n-isomorphic to (M_{n+1}, w_{n+1}). □

Theorem 55. *Let $n \in \omega$, then W_n is not equivalent to any formula in $\Sigma^\mu_n \cup \Pi^\mu_n$ over* $\mathsf{S5}_2$ *frames. Therefore the alternation hierarchy is strict over bimodal* S5.

Proof. Let n be even. We know that $\mathcal{W}_n \in \Pi^\mu_{n+1}$. For a contradiction, suppose that W_n is equivalent to some formula in Π^μ_n. Let $\varphi \in \Sigma^\mu_n$ be equivalent to $\neg W_n$.

Let (M, w) be a fixed-point of $f_{\varphi\wedge\varphi}$. Then

$$\begin{aligned} M, w \models \neg W_n &\iff M, w \models \varphi \wedge \varphi \\ &\iff f_{\varphi\wedge\varphi}(M, w) \models W_n \\ &\iff M, w \models W_n. \end{aligned}$$

This is a contradiction.

The case for n odd is symmetric. $\mathcal{W}_n \in \Pi^\mu_{n+1}$ if n is odd. Suppose W_n is equivalent to some formula in Σ^μ_n. Let φ be a Π^μ_n-formula equivalent to $\neg W_n$. If (M, w) is a fixed-point of $f_{\varphi\wedge\varphi}$, then $M, w \models \neg W_n$ iff $M, w \models W_n$. □

Question 3. *The alternation hierarchy is strict over finite unimodal frames. Is the alternation hierarchy strict over finite frames of bimodal* S5*?*

4.5 Inflationary μ-calculus

If X is not positive on $\varphi(X)$, then the least and greatest fixed-point as we defined above are not well-defined. Even so, we can define inflationary least and greatest fixed-points. We will show that the proof of the collapse to modal logic over S5 studied above does not generalize to inflationary μ-calculus.

BASIC DEFINITIONS. Instead of the (monotone) least and gratest fixed-point operators μ and ν, we use inflationary fixed-point operators **lfp** and **gfp**. The formulas of the inflationary μ-calculus are defined by the grammar

$$\varphi := P \mid X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \mathbf{lfp}X.\varphi \mid \mathbf{gfp}X.\varphi,$$

where $P \in \mathrm{Prop}$ and $X \in \mathrm{Var}$. Note that we allow fixed-point operators to bind non-positive occurrences of variables.

The inflationary μ-calculus is also interpreted over Kripke models. The semantics for propositional connectives and modalities are as in the standard μ-calculus. Given an inflationary μ-formula, define the operator $\Gamma_{\varphi(X)} : \mathcal{P}(W) \to \mathcal{P}(W)$ by

$$X \mapsto X \cup \|\varphi(X)\|^M.$$

$\Gamma_{\varphi(X)}$ is a monotone operator. Note that if X is positive in φ, then the operator $\Gamma_{\varphi(X)}$ is the same as the operator for the standard μ-calculus.

Dawar *et al.* proved that the inflationary μ-calculus is much more expressive than the standard μ-calculus. [DGK04]. Evaluation games for the inflationary μ-calculus were studied in [DGK06].

THE DE JONGH–SAMBIN THEOREM. The modal logic GL is obtained by adding the axioms $\Box P \to \Box\Box P$ and $\Box(\Box P \to P) \to \Box P$ to the basic modal logic K. This logic is complete over the class of transitive reverse well-founded frames. GL is central in provability logic, with the $\Box$ modality formalizing the provability predicate.

Alberucci and Facchini [AF09a] proved that the μ-calculus collapses to modal logic over frames of GL. This is an instance of a classical theorem of provability logic:

Theorem 56 (The de Jongh–Sambin Theorem [Smo85])**.** *Let $\varphi(X)$ be a modal formula whose only free-variable is X. Over frames of* GL*, there is a unique modal formula ψ such that ψ has no free-variables and ψ is equivalent to $\varphi(\psi)$ over frames of* GL*.*

This implies the inflationary μ-calculus collapses to modal logic over frames of GL.

A QUESTION. The collapses to modal logic of the standard μ-calculus over equivalence relations and the inflationary μ-calculus over reverse well-founded transitive frames leads us to the following question:

Question 4. *Does the inflationary μ-calculus' alternation hierarchy collapse to modal logic over equivalence relations? If not, is the inflationary μ-calculus' alternation hierarchy strict over equivalence relations?*

We show that the proof considered in Section 3.1 does not work. This proof did generalize to non-normal semantics, graded semantics, and intuitionistic semantics. Do note that while the proof did use game semantics, we only needed game semantics for modal logic.

Given a μ-formula $\psi(X)$, define $\psi^0(\bot) := \bot$ and $\psi^{n+1}(\bot) := \psi^n(\bot) \vee \psi(\psi^n(\bot))$. As in the standard case, $\psi^n(\bot)$ is the nth approximant of the least fixed-point $\mathbf{lfp}X.\psi$.

Given $n \in \omega$, define a formula φ_n by:

$$\begin{aligned}
\varphi_n(X) := P_0 &\vee [\neg X \wedge P_1 \wedge \Diamond(X \wedge P_0)] \\
&\vee [\neg X \wedge P_2 \wedge \Diamond(X \wedge P_1)] \\
&\vee [\neg X \wedge P_3 \wedge \Diamond(X \wedge P_2)] \\
&\dots \\
&\vee [\neg X \wedge P_n \wedge \Diamond(X \wedge P_{n-1})]
\end{aligned}$$

By a direct calculations, φ_n^{i+1} is equivalent to $P_0 \vee \cdots \vee P_i$. In particular $\mathbf{lfp}X.\varphi_n$ is equivalent to φ_n^{n+1} and to $P_0 \vee \cdots \vee P_n$. Furthermore, $\mu X.\varphi_n$ is not equivalent to φ_n^i for any $i \leq n$. So the proof of the Theorem 16 (or its generalized version Theorem 21) does not show that the inflationary μ-calculus collapses to modal logic over equivalence relations. We were not able to show the collapse nor via a different proof. We also were not able to show the non-collapse.

Chapter 5

The collapse to the alternation-free fragment on Kripke semantics

In this chapter, we prove that the μ-calculus collapses to its alternation-free fragment over weakly transitive frames—and frames which satisfy similar properties. Our proof is a generalization of D'Agostino and Lenzi's proof of the collapse over transitive frames [DL10]. This is joint work with Kazuyuki Tanaka [PT22].

5.1 Weakly transitive frames

A DEFINITION. We say a frame $F = \langle W, R \rangle$ is weakly transitive iff $wRvRu$ implies wRu or $w = u$. Weakly transitive are characterized by the modal logic wK4. We define wK4 by adding the axiom $P \wedge \Box P \to \Box\Box P$ to the basic modal logic K. Esakia [Esa04] showed that wK4 is complete over weakly transitive frames and over derivational topological semantics. He also showed that the finite model property holds for wK4.

A LEMMA. Let $\varphi(X)$ be a μ-formula. Define by recursion $\varphi^0(X) = X$ and $\varphi^{n+1}(X) = \varphi(\varphi^n(X))$, for all $n \in \mathbb{N}$. Note that, since we suppose all formulas are well-named, $\varphi^{n+1}(X)$ is actually $\varphi'(\varphi^n(X))$, where φ' is obtained by substituting the variables of φ for fresh variables.

Proposition 57. *Let $\varphi(X)$ be a μ-formula. Over transitive frames,*

$$\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi(\bot) \text{ and } \Box\nu X.\varphi(X) \equiv \Box\varphi(\top).$$

Proof. We first prove $\Diamond\mu X.\varphi \equiv \Diamond\varphi(\bot)$. Fix a transitive Kripke model $M = \langle W, R, V \rangle$ and a world $w \in W$. We suppose $W = \{w\} \cup \{v \mid wRv\}$, as only w worlds accessible from w affect the valuation of μ-formulas at w and R is transitive. Since X is positive in $\varphi(X)$, $w \in \|\Diamond\varphi(\bot)\|$ implies $w \in \|\Diamond\mu X.\varphi(X)\|$.

Suppose now that $w \in \|\Diamond\mu X.\varphi\|$. Then $v \in \|\mu X.\varphi\|$ for some v accessible from w. As $\|\mu X.\varphi\|$ is not empty, $\|\varphi(\bot)\|$ is also not empty. Let $v' \in \|\varphi(\bot)\|$. Then wRv' and so $w \in \|\Diamond\varphi(\bot)\|$.

The second equivalence follow from the first equivalence and a direct calculation:

$$\Box\nu X.\varphi(X) \equiv \neg\Diamond\mu X.\neg\varphi(\neg X) \equiv \neg\Diamond\neg\varphi(\neg\bot) \equiv \Box\varphi(\top). \qquad \Box$$

The key lemma for the collapse over weakly transitive frames is:

Lemma 58. *Let $\varphi(X)$ be a μ-formula. Over weakly transitive frames,*

$$\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^2(\bot) \text{ and } \Box \nu X.\varphi(X) \equiv \Box \varphi^2(\top).$$

Proof. We first prove $\Diamond \mu X.\varphi \equiv \Diamond \varphi^2(\bot)$. Fix a Kripke model $M = \langle W, R, V\rangle$ with R weakly transitive and a world $w \in W$. By weak transitivity, we can suppose $W = \{w\} \cup \{v \mid wRv\}$, as only w worlds accessible from w affect the valuation of μ-formulas at w. As X is positive in $\varphi(X)$, $w \in \|\Diamond \varphi^2(\bot)\|$ implies $w \in \|\Diamond \mu X.\varphi(X)\|$.

Suppose now that $w \in \|\Diamond \mu X.\varphi\|$. Then $v \in \|\mu X.\varphi\|$ for some v accessible from w. As $\|\mu X.\varphi\|$ is not empty, $\|\varphi(\bot)\|$ is also not empty. We now have two cases:

- If $v \in \|\varphi(\bot)\|$ for some v accessible from w, then $v \in \|\varphi^2(\bot)\|$ and we have finished the proof.
- Otherwise, $\|\varphi(\bot)\| = \{w\}$ and wRw does not hold. If $\|\varphi(\varphi(\bot))\|$ had no element of $W \setminus \{w\}$, we would have $\|\mu X.\varphi\| = \{w\}$, a contradiction.

Therefore $w \in \|\Diamond \varphi^2(\bot)\|$.

We show that $\Diamond \mu X.\varphi \equiv \Diamond \varphi^2(\bot)$ holds via a direct calculation:

$$\Box \nu X.\varphi(X) \equiv \neg \Diamond \mu X.\neg \varphi(\neg X) \equiv \neg \Diamond \neg \varphi(\neg\neg \varphi(\neg \bot)) \equiv \neg \Diamond \neg \varphi^2(\top) \equiv \Box \varphi^2(\top). \quad \Box$$

A QUESTION. The converse of Proposition 57 holds:

Proposition 59. *If $F \models \Diamond \mu X.\varphi(X) \equiv \Diamond \varphi(\bot)$ then F is transitive.*

Proof. Let $F = \langle W, R\rangle$ be a frame which satisfies $\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi(\bot)$. Suppose $wRvRu$. Consider the model $M = \langle W, R, V\rangle$ extending F with the valuation $V(P) = \{u\}$. Then $M, w \models \Diamond \mu X.P \vee \Diamond X$; and so $M, w \models \Diamond P$. Therefore wRu holds. We conclude R is transitive. $\Box$

But the converse of Proposition 58 fails:

Proposition 60. *There is F such that $F \models \Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^2(\bot)$ and F is not weakly transitive.*

Proof. Consider the frame $F = \langle \{w, v, u\}, \{\langle w, v\rangle, \langle v, u\rangle\}\rangle$. While F satisfies $\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^2(\bot)$, it is not weakly transitive. We picture F in Figure 5.1. $\Box$

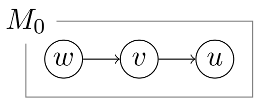


Figure 5.1: The frame F from Proposition 60.

Question 5. *Which class of frames is characterized by $\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^2(\bot)$? Which class of frames is characterized by $\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^n(\bot)$, for $n > 2$?*

5.2 Collapse over weakly transitive frames

In this section, we prove the alternation hierarchy's collapse to its alternation-free fragment over frames which satisfy

$$\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot) \text{ and } \Box\nu X.\varphi(X) \equiv \Box\varphi^n(\top)$$

for some fixed $n \in \omega$. For ease of understanding, we do the proofs for $n = 2$ and for general n. Remember that the $n = 2$ is enough to show the collapse for weakly transitive frames.

EXISTENTIAL AND WEAKLY UNIVERSAL FORMULAS. Let φ be a well-named μ-formula and X be a bound variable in φ. Let ψ be such that X occurs in ψ and $\eta X.\psi \in \mathrm{Sub}(\varphi)$. X is *existential* iff X is not in the scope of any $\Box$ in ψ, that is, X is only in the scope of diamonds. X is universal iff it not in the scope of any $\Diamond$ in ψ. X is *weakly universal* iff X is in the scope of some $\Box$ in ψ. X is weakly existential iff it is in the scope of some $\Diamond$.

Example 16. Consider the formula φ below:

$$\varphi := \nu X \nu Y \nu Z.(\Box X \vee \Diamond Y \vee \Box\Diamond Z).$$

X is universal and weakly universal in φ; Y is existential and weakly existential in φ; and Z is weakly universal and weakly existential in φ.

Lemmas 62 and 63 below will allow us to eliminate weakly universal ν-variables. Before doing so, we prove a small auxiliary lemma:

Lemma 61 (D'Agostino, Lenzi [DL10]). *Let α and β be μ-formulas. Then*

$$\beta(\nu X.\alpha(\Box\beta(X))) \equiv \nu Y.\beta(\alpha(\Box Y)).$$

Proof. Let $M = \langle W, R, V\rangle$ be a Kripke model. Fix a world $w \in W$ and μ-formulas α and β. We show that V wins the evaluation game $\mathcal{G}_{\mathrm{LHS}} = \mathcal{G}(M, w \models \beta(\nu X.\alpha(\Box\beta(X))))$ iff V wins $\mathcal{G}_{\mathrm{RHS}} = \mathcal{G}(M, w \models \nu Y.\beta(\alpha(\Box Y)))$. The evaluation games $\mathcal{G}_{\mathrm{LHS}}$ and $\mathcal{G}_{\mathrm{RHS}}$ are essentially the same game.

Let σ be a strategy for V in $\mathcal{G}_{\mathrm{LHS}}$. We define a strategy σ' for V in $\mathcal{G}_{\mathrm{RHS}}$ while simultaneously playing $\mathcal{G}_{\mathrm{LHS}}$ and $\mathcal{G}_{\mathrm{RHS}}$. The games start at positions $\langle w, \beta(\nu X.\alpha(\Box\beta(X)))\rangle$ and $\langle w, \nu Y.\beta(\alpha(\Box Y))\rangle$, respectively. The players begin by moving to the position $\langle w, \beta(\alpha(\Box Y))\rangle$ on $\mathcal{G}_{\mathrm{RHS}}$.

After the initial play, the players use analogous moves in both games for a while. For example, if they are in positions $\langle v, (\psi \vee \theta)(\nu X.\alpha(\Box\beta(X)))\rangle$ and $\langle v, (\psi \vee \theta)(\alpha(\Box Y))\rangle$ and V moves to $\langle v, \psi(\nu X.\alpha(\Box\beta(X)))\rangle$ on $\mathcal{G}_{\mathrm{LHS}}$, then V moves to $\langle v, \psi(\alpha(\Box Y))\rangle$ on $\mathcal{G}_{\mathrm{RHS}}$. Similarly, if the players are in positions $\langle v, \Box\psi(\nu X.\alpha(\Box\beta(X)))\rangle$ and $\langle v, \Box\psi(\alpha(\Box Y))\rangle$ and R moves to $\langle v', \psi(\alpha(\Box Y))\rangle$ on $\mathcal{G}_{\mathrm{RHS}}$, then R moves to $\langle v', \psi(\nu X.\alpha(\Box\beta(X)))\rangle$ on $\mathcal{G}_{\mathrm{LHS}}$.

The players continue using analogous moves until they reach positions $\langle v, \nu X.\alpha(\Box\beta(X))\rangle$ and $\langle v, \alpha(\Box Y)\rangle$, then they move to $\langle v, \alpha(\Box\beta(X))\rangle$ on $\mathcal{G}_{\mathrm{LHS}}$. Again, the players continue by using analogous moves. That happens until they reach $\langle v, \beta(X)\rangle$ and $\langle v, Y\rangle$; here, they move to $\langle v, \beta(\alpha(\Box Y))\rangle$ on $\mathcal{G}_{\mathrm{RHS}}$. And so the players keep using analogous moves or adjusting the positions on X and Y.

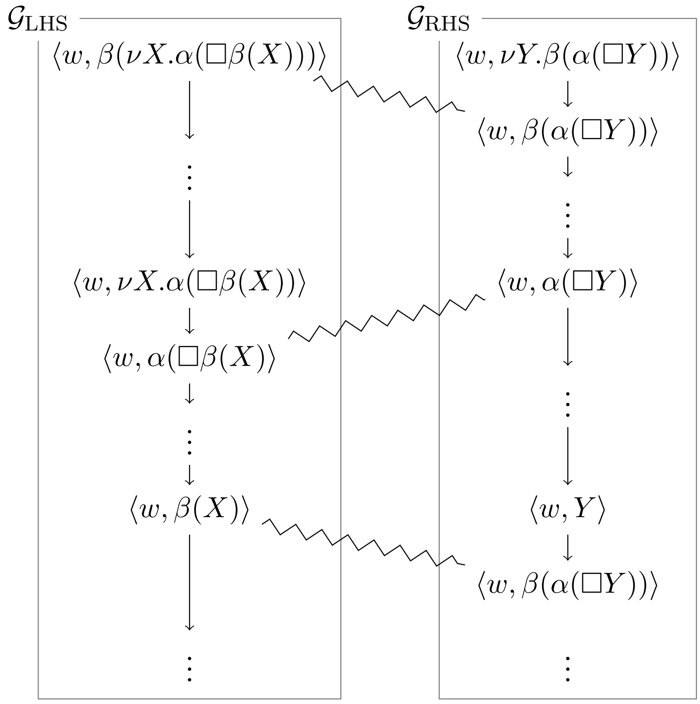


Figure 5.2: Simultaneous runs of the evaluation games $\mathcal{G}_{\mathrm{LHS}}$ and $\mathcal{G}_{\mathrm{RHS}}$ from Lemma 61. Zigzagged lines indicate equivalent states.

Suppose σ is a winning strategy for V in $\mathcal{G}_{\mathrm{LHS}}$ and σ' the strategy for $\mathcal{G}_{\mathrm{RHS}}$ defined above. Let $\rho_{\mathrm{LHS}}, \rho_{\mathrm{RHS}}$ be the parallel plays given by the procedure above. If ρ_{LHS} ends on a position $\langle v, P\rangle$, then ρ_{RHS} also ends on the position $\langle v, P\rangle$. Both are winning for V. Remember that we say X is regenerated when the players go from a position $\langle v, X\rangle$ to a position $\langle v, \eta X.\psi\rangle$. If ρ_{LHS} is an infinite play where X is regenerated infinitely often, then Y is regenerated infinitely often in ρ_{RHS}; if ρ_{LHS} in an infinite play where X is not regenerated infinitely often, then the same variables are regenerated infinitely often in ρ_{RHS}. Either way, if ρ_{LHS} is winning for V, then ρ_{RHS} is also winning for V.

Given a winning strategy σ' for V in $\mathcal{G}_{\mathrm{RHS}}$, we define σ for V in $\mathcal{G}_{\mathrm{LHS}}$ by the same method as above. $\Box$

Lemma 62. *Suppose the formula $\nu X.\varphi$ is well-named and X is weakly universal in it. Then, over weakly transitive frames,*

$$\nu X.\varphi(X) \equiv \varphi^3(\top).$$

Proof. As $\nu X\varphi$ is well-named and X is weakly universal variable, we can write

$\nu X.\varphi(X)$ as $\nu X.\alpha(\Box\beta(X))$. We have that (see Lemma 3.8 of [DL10]). We calculate:

$$\begin{aligned}
\varphi^3(\top) &\equiv \alpha(\Box\beta(\alpha(\Box\beta(\alpha(\Box\beta(\top)))))) && \text{unfold } \varphi^3(\top)\\
&\to \alpha(\Box\beta(\alpha(\Box\beta(\alpha(\top))))) && \text{monotonicity}\\
&\equiv \alpha(\Box\nu Y.\beta(\alpha(\Box Y))) && \text{Lemma 58}\\
&\equiv \alpha(\Box\beta(\nu X.\alpha(\Box\beta(X)))) && \text{Lemma 61}\\
&\equiv \nu X.\alpha(\Box\beta(X)) && \nu \text{ is a fixed-point operator}\\
&\equiv \nu X.\varphi(X) && \text{fold } \varphi
\end{aligned}$$

So $\varphi^3(\top)$ implies $\nu X.\varphi(X)$. As X is positive in φ, $\nu X.\varphi(X)$ implies $\varphi^3(\top)$. Therefore $\nu X.\varphi(X)$ and $\varphi^3(\top)$ are equivalent. ☐

Lemma 63. *Suppose the formula $\nu X.\varphi$ is well-named and X is weakly universal in it. Then, over frames which satisfy $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$,*

$$\nu X.\varphi(X) \equiv \varphi^{n+1}(\top).$$

Proof. Again, we can write $\nu X.\varphi(X)$ as $\nu X.\alpha(\Box\beta(X))$. Therefore:

$$\begin{aligned}
\varphi^{n+1}(\top) &\equiv (\alpha(\Box\beta))^n(\alpha(\Box\beta(\top))) && \text{unfold } \varphi^{n+1}(\top)\\
&\to (\alpha(\Box\beta))^n(\alpha(\Box\top)) && \text{monotonicity}\\
&\equiv \alpha(\Box[(\beta(\alpha(\Box)))^n(\top)]) && \text{reorganize parenthesis}\\
&\equiv \alpha(\Box\nu Y.\beta(\alpha(\Box Y))) && \text{hypothesis}\\
&\equiv \alpha(\Box\beta(\nu X.\alpha(\Box\beta(X)))) && \text{Lemma 61}\\
&\equiv \nu X.\alpha(\Box\beta(X)) && \nu \text{ is a fixed-point operator}\\
&\equiv \nu X.\varphi(X) && \text{fold } \varphi
\end{aligned}$$

So $\varphi^{n+1}(\top)$ implies $\nu X.\varphi(X)$. As X is positive in φ, we can show by induction that $\nu X.\varphi(X)$ implies $\varphi^{n+1}(\top)$. Therefore $\nu X.\varphi(X)$ and $\varphi^{n+1}(\top)$ are equivalent. ☐

We get the dual version of the Lemmas 62 and 63 above by direct calculations:

Lemma 64. *Suppose the formula $\mu X.\varphi$ is well-named and X is weakly existential in it. Over weakly transitive frames,*

$$\mu X.\varphi(X) \equiv \varphi^3(\bot).$$

Over frames which satisfy $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$,

$$\mu X.\varphi(X) \equiv \varphi^{n+1}(\bot).$$

BIDISJUNCTIVE FORMULAS. Just eliminating weakly universal and weakly existential fixed-point operators is not enough to show that all formulas are equivalent to alternation-free formulas over frames which satisfy the equivalence $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$. To do so, we show that any μ-formula is equivalent to a bidisjunctive formula. Bidisjunctive formulas were defined by D'Agostino and Lenzi in [DL10], generalizing Janin and Walukiewicz's disjunctive formulas [JW95].

Let Γ be a finite set of μ-formulas, define

$$\mathrm{Cover}(\Gamma) := \left(\bigwedge_{\varphi\in\Gamma} \Diamond\varphi\right) \wedge \Box \left(\bigvee_{\psi\in\Gamma} \psi\right).$$

The class of disjunctive formulas is the least class containing all literals and closed under:

- disjunctions;
- restricted fixed-points: if φ is disjunctive and there is no subformula of φ of the form $X \wedge \psi$, then $\eta X.\varphi$ is disjunctive; and
- covers: if Γ is a finite set of disjunctive formulas and σ is a conjunction of literals, then $\sigma \wedge \mathrm{Cover}(\Gamma)$ is also disjunctive.

Janin and Walukiewicz proved that every μ-formula is equivalent to a disjunctive formula using tableaus for the μ-calculus. Furthermore, their proof imply that every Π_2^μ-formula is equivalent to a disjunctive Π_2^μ-formula.

If $\mathrm{Cover}(\Gamma)$ is a subformula of φ and some variable X occurs in some formula of Γ, then φ is not well-named, as X will occur twice in φ. To solve this problem, D'Agostino and Lenzi defined the generalized cover operator $\mathrm{Cover}(\Gamma, \Delta)$. Let Γ, Δ be finite sets of μ-formulas, we define

$$\mathrm{Cover}(\Gamma, \Delta) := \left(\bigwedge_{\varphi\in\Gamma} \Diamond\varphi\right) \wedge \Box \left(\bigvee_{\psi\in\Delta} \psi\right).$$

The class of bidisjunctive formulas is the least class containing all literals and closed under:

- disjunctions;
- restricted fixed-points: if φ is bidisjunctive and there is no subformula of φ of the form $X \wedge \psi$, then $\eta X.\varphi$ is bidisjunctive; and
- covers: if Γ, Δ are finite sets of bidisjunctive formulas and σ is a conjunction of literals, then $\sigma \wedge \mathrm{Cover}(\Gamma, \Delta)$ is also bidisjunctive.

Given a set of formulas Γ, let Γ' consist of the formulas of Γ rewritten with fresh variables. Given a disjunctive formula φ, we substitute every instance of $\mathrm{Cover}(\Gamma)$ by $\mathrm{Cover}(\Gamma, \Gamma')$; and for each new fresh variable X' where X was bounded in φ, we substitute ηX by $\eta X \eta X'$. The obtained formula is going to be bidisjunctive. Furthermore, if φ was Π_2^μ, so is the new formula.

Not only every formula is equivalent to a bidisjunctive formula, but we can suppose there is a cover operator between any two alternating fixed-point operators. That is, if $\mu Y.\psi$ is a subformula of $\nu X.\varphi$, then there is $\mathrm{Cover}(\Gamma, \Delta)$ such that $\mu Y.\psi$ is a subformula of $\mathrm{Cover}(\Gamma, \Delta)$, and $\mathrm{Cover}(\Gamma, \Delta)$ is a subformula of $\nu X.\varphi$. Suppose $\mu Y.\psi$ is a subformula of $\nu X.\varphi$, but there is no cover operator between these formulas. We can substitute $\psi(\mu Y.\psi)$ for $\mu Y.\psi$ in $\nu X.\varphi$. As Y is guarded in $\mu Y.\psi$, there is a cover operator between the fixed-point operators in the newly defined formula.

WEAKLY UNIVERSAL PRENEX FORM. We say a μ-formula φ is in weakly universal prenex form iff it is of the form

$$\nu X_k \dots \nu X_1.\psi,$$

where $X_1, \dots, X_k$ are weakly universal and ψ does not contain any weakly universal occurrence of ν. Let $\varphi := \nu X_1.\psi$ be in weakly universal prenex form. By Lemmas 62 and 63, φ is equivalent to $\psi^3(\top)$ over weakly transitive frames, and to ψ^{n+1} over frames where $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$. Therefore we can eliminate weakly universal occurrences of ν.

Now, suppose φ is not in weakly universal prenex form. Then φ has a smallest subformula $\nu X.\psi$ with X weakly universal. Let θ be such that $\varphi = \theta(\nu X.\psi)$. Since $\nu X.\psi$ is equivalent to $\psi^2(\top)$, $\theta(\nu X.\psi)$ is equivalent to $\theta(\psi^2(\top))$; on $\theta(\psi^2(\top))$, we note that, for every $Y \in \mathrm{Sub}(\mu X.\psi)$, if ηY appeared in θ, we substitute it by $\eta Y \eta Y'$, where Y' is the fresh variable which appears in $\psi^2(\top)$. Now, if φ was a bidisjunctive Π^μ_2-formula, the new formula is too bidisjunctive and Π^μ_2.

COLLAPSE TO THE ALTERNATION-FREE FRAGMENT. Let $\nu X.\varphi$ be a bidisjunctive Π^μ_2-formula without weakly universal occurrences of ν. We also suppose that between any alternating fixed-point operators there is a cover operator. Suppose that there is a biggest subformula $\mu Y.\psi$ of $\nu X.\varphi$ with $X \in \mathrm{Sub}(\psi)$. By our hypothesis, there is a smallest $\mathrm{Cover}(\Gamma, \Delta) \in \mathrm{Sub}(\nu X.\varphi)$ such that $\mu Y.\psi$ is a subformula of some formula in Γ or Δ. If $\mu Y.\psi$ was a subformula of a formula in Δ then X would be weakly universal. So $\mu Y.\psi$ is a subformula of some formula in Γ We can write Γ as

$$\Gamma = \{\theta_1 \vee (\theta_2 \vee (\cdots(\theta_k \vee \mu Y.\psi)\cdots)\} \cup \Sigma,$$

where the θ_i are μ-formulas and Σ is a set of μ-formulas. Let $\mathrm{Cover}(\Gamma', \Delta)$ be

$$\Gamma' = \{\theta_1 \vee (\theta_2 \vee (\cdots(\theta_k \vee \psi^n(\bot))\cdots)\} \cup \Sigma.$$

Over frames where $\Diamond\mu X.\varphi(X) \equiv \Diamond\varphi^n(\bot)$ holds, $\mathrm{Cover}(\Gamma, \Delta)$ is equivalent to $\mathrm{Cover}(\Gamma', \Delta)$, as $\Diamond$ distributes over $\vee$:

$$\begin{aligned}
\mathrm{Cover}(\Gamma,\Delta) &\equiv \left(\bigwedge_{\varphi\in\Gamma}\Diamond\varphi\right)\wedge\Box\left(\bigvee_{\psi\in\Delta}\psi\right)\\
&\equiv \Diamond(\theta_1\vee(\theta_2\vee(\cdots(\theta_k\vee\mu Y.\psi)\cdots)\wedge\left(\bigwedge_{\varphi\in\Sigma}\Diamond\varphi\right)\wedge\Box\left(\bigvee_{\psi\in\Delta}\psi\right)\\
&\equiv (\Diamond\theta_1\vee(\Diamond\theta_2\vee(\cdots(\Diamond\theta_k\vee\Diamond\mu Y.\psi)\cdots)\wedge\left(\bigwedge_{\varphi\in\Sigma}\Diamond\varphi\right)\wedge\Box\left(\bigvee_{\psi\in\Delta}\psi\right)\\
&\equiv (\Diamond\theta_1\vee(\Diamond\theta_2\vee(\cdots(\Diamond\theta_k\vee\Diamond\psi^n(\bot))\cdots)\wedge\left(\bigwedge_{\varphi\in\Sigma}\Diamond\varphi\right)\wedge\Box\left(\bigvee_{\psi\in\Delta}\psi\right)\\
&\equiv \Diamond(\theta_1\vee(\theta_2\vee(\cdots(\theta_k\vee\psi^n(\bot))\cdots)\wedge\left(\bigwedge_{\varphi\in\Sigma}\Diamond\varphi\right)\wedge\Box\left(\bigvee_{\psi\in\Delta}\psi\right)\\
&\equiv \left(\bigwedge_{\varphi\in\Gamma'}\Diamond\varphi\right)\wedge\Box\left(\bigvee_{\psi\in\Delta}\psi\right).
\end{aligned}$$

Therefore, if we substitute $\mathrm{Cover}(\Gamma',\Delta)$ for $\mathrm{Cover}(\Gamma,\Delta)$ in φ, we get a formula φ' which is equivalent to φ. Furthermore, φ' has one less pair of alternating variables than φ; our choices of $\mu Y.\psi$ and $\mathrm{Cover}(\Gamma,\Delta)$ imply that no new alternating pair is introduced in φ'.

Theorem 65. *The alternation hierarchy collapses to the alternation-free fragment over weakly transitive frames.*

Proof. We can then suppose that φ is equivalent to a μ-formula φ' such that: if $\nu X.\varphi\in\mathrm{Sub}(\varphi')$ then there is no $\mu Y.\psi\in\mathrm{Sub}(\nu X.\varphi)$ with a free occurrence of X. As we can build φ' from $\Sigma_1^\mu\cup\Pi_1^\mu$ using only substitutions allowed in the construction of both Σ_2^μ and Π_2^μ, φ' is Δ_2^μ. That is, φ' is alternation-free and φ is equivalent to an alternation-free formula. ☐

Theorem 66. *Fix $n\in\omega$. The alternation hierarchy collapses to the alternation-free fragment over frames where $\Diamond\mu X.\varphi(X)\equiv\Diamond\varphi^n(\bot)$ holds.*

Proof. We can then suppose that φ is equivalent to a μ-formula φ' such that: if $\nu X.\varphi\in\mathrm{Sub}(\varphi')$ then there is no $\mu Y.\psi\in\mathrm{Sub}(\nu X.\varphi)$ with a free occurrence of X. As we can build φ' from $\Sigma_1^\mu\cup\Pi_1^\mu$ using only substitutions allowed in the construction of both Σ_2^μ and Π_2^μ, φ' is Δ_2^μ. That is, φ' is alternation-free and φ is equivalent to an alternation-free formula. ☐

COLLAPSE TO FIRST-ORDER LOGIC. Let F class of frames. We say the alternation hierarchy collapses to first-order logic over F when each μ-formula is equivalent to a first-order formula over frames of F. Remember that modal logic is the fragment of first-order logic invariant under bisimulations, and so is not equivalent to first-order in general.

D'Agostino and Lenzi [DL10, Therem 4.1] proved:

Theorem 67. *The alternation hierarchy collapses to first-order logic on finite weakly transitive frames.*

We can ask if the same happens on the frames studied in this section.

Question 6. *Fix $n \in \omega$. Does the alternation hierarchy collapse to first-order logic on finite frames where $\Diamond \mu X.\varphi(X) \equiv \Diamond \varphi^n(\bot)$?*

We conjecture the answer is positive, and that this can be proved using D'Agostino and Lenzi's methods.

5.3 Collapse over topological semantics

TOPOLOGICAL SEMANTICS. We now define derivative topological models $\mathcal{X} = \langle W, \tau, V \rangle$. As in a Kripke model, W is the set of possible worlds and V is the valuation function. τ is a topology on W. We define the valuation $\|\varphi\|^{\mathcal{X}}$ inductively as in relational semantics. The definition of the valuation of conjunctions, disjunctions, negations, and fixed-points is the same as in relational semantics. We think of the $\Diamond$ modality as the *Cantor derivative*:

$$w \in \|\Diamond\varphi\|^{\mathcal{X}} \text{ iff } w \text{ is a limit point of } \|\varphi\|^{\mathcal{X}},$$

where w is a limit point of a set $X \subseteq W$ for all open set U, if $x \in U$ then there is $y \in A$ such that $y \in U \setminus \{x\}$. We denote the Cantor derivative of a set A by A'. The dual operator $\Box$ of the derivative is called the co-derivative.

Interior topological semantics is also studied in the literature. We denote the box modality for interior topological semantics by $\Box_i$. Here the semantics for $\|\Box_i\varphi\|^{\mathcal{X}}$ defined to be the interior of $\|\varphi\|^{\mathcal{X}}$. The dual modality $\Box_i$ of the interior modality is the closure modality: $\|\Box\varphi\|^{\mathcal{X}}$ is the closure of $\|\varphi\|^{\mathcal{X}}$. The closure of a set X consists of the points in X and the limit points of X. Therefore, the interior modality can be defined using derivative topological semantics: $\Box_i\varphi := \varphi \wedge \Diamond\neg\varphi$. This implies derivative topological semantics is more expressive than interior topological semantics. For more information on topological semantics see [BBF21; vB07]. Internal topological semantics was first studied by McKinsey and Tarski in [MT44]—they also suggested derivative topological semantics in the same paper.

McKinsey and Tarski [MT44] proved that S4 is complete for interior topological semantics. Esakia [Esa04] showed that wK4 is complete for derivative topological semantics. Furthermore, the finite model property holds for wK4.

Let $\mathcal{X} = \langle W, \tau, V \rangle$ be a topological model and $M = \langle W, R, V \rangle$ be a Kripke model. We say $\mathcal{X}$ and M are modally equivalent iff $\|\varphi\|^{\mathcal{X}} = \|\varphi\|^{M}$ for all μ-formula φ. We say topological space is Alexandroff iff arbitrary intersections of open sets are open.

Proposition 68. *Each Alexandroff topological model is modally equivalent to an irreflexive weak transitive model.*

Proof. Let $\mathcal{X} = \langle W, \tau, V \rangle$ be an Alexandroff topological space. Given $A \subseteq$, let A' denote the set of limit points of A. Define $M = \langle W, R, V \rangle$ by taking wRv iff $w \in \{v\}'$. M is irreflexive, as $x \notin \{x\}'$.

Suppose that $wRvRu$ and $w \neq u$. Since wRv, there is $y \in U \setminus \{w\}$ such that $y \in \{v\}$, for all open set U. Therefore $w \in U$ implies $v \in U \setminus \{w\}$. Similarly, vRu implies that if $v \in U$ then $u \in U \setminus \{v\}$, for all open set U. Now, if U is an open set, then $w \in U$, $v \in U$, and $u \in U$. Therefore, if $w \in U$ then there is $y \in \{u\}$ such that $y \in U \setminus \{w\}$. That is, $w \in \{u\}'$. That is, wRu. Therefore M is weakly transitive.

We can show that $\mathcal{X}$ and M are modally equivalent by a straight proof by structural induction. $\square$

FINITE MODEL PROPERTY FOR TOPOLOGICAL SEMANTICS. Note that wK4 is a modal logic. We extend it to a logic μwK4 for the μ-calculus by adding the fixed-point axiom

$$\nu X.\theta \to \theta(\nu X.\theta),$$

and the induction rule

$$\frac{\varphi \to \theta(\varphi)}{\varphi \to \nu X.\theta}.$$

One can directly check that the fixed-point axiom and the induction rule are valid over weakly transitive frames.

We will use the following result to relate the alternation hierarchies on weakly transitive frames and derivative topological semantics:

Theorem 69 (Baltag *et al.* [BBF21])**.** *μwK4 is complete for weakly transitive frames and for derivative topological semantics. Furthermore, the μwK4 has the finite model property with respect to these semantics—if a formula is satisfiable by some weakly transitive model, then it is satisfiable by a finite weakly transitive model; and, if a formula is satisfiable by some topological model, then it is satisfiable by a finite topological model.*

This theorem is proved by using final canonical models, as the standard canonical models are not amenable to μ-calculus. We outline the basic definitions for final models below.

The canonical model $M_c = \langle W^c, R^c, V^c \rangle$ for μwK4 is defined as in modal logics without fixed-points:

- W^c consists of all maximal consistent extensions of μwK4;
- $\Gamma R^c \Delta$ iff $\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$, for all modal formula φ; and
- $\Gamma \in V^c(P)$ iff $P \in \Gamma$.

The truth lemma does not hold for μwK4 and M_c.

Let φ be a formula and Σ be a set of formulas. We say a world $\Gamma \in W^c$ is φ-final iff

- $\varphi \in \Gamma$; and
- $\varphi \in \Delta$ and $\Gamma R^c \Delta$ imply $\Delta R^c \Gamma$.

A world $\Gamma \in W^c$ is Σ-final iff it is θ-final for some $\theta \in \Sigma$. Now we are ready to define final models.

Define $\langle * \rangle\varphi := \varphi \vee \Diamond\varphi$. If $\Diamond$ satisfies wK4, then $\langle * \rangle$ satisfies S4. Let φ be a consistent formula. Let Σ be the least set which contains φ; is closed under subformulas;

and is closed under $\langle * \rangle$ and $\neg$ up to logical equivalence. Σ is a finite set. The truth lemma holds on M_c^Σ

Lemma 70 (Baltag *et al.* [BBF21])**.** *Let M_c^Σ be the restriction of M_c to Σ-final worlds, then*

$$M_c^\Sigma, \Theta \models \psi \textit{ iff } \psi \in \Theta,$$

for all $\psi \in \Sigma$.

This implies that the μwK4 is complete for topological semantics. Furthermore, as Σ is finite, if we take the quotient of M_c modulo Σ-bisimilarity, the resulting model is finite. A Σ-bisimulation is like a bisimulation—instead of requiring that bisimilar worlds satisfy the same propositions, we only require that they agree on formulas of Σ.

THE COLLAPSE OVER TOPOLOGICAL SEMANTICS. The collapse of the alternation hierarchy over weakly transitive Kripke frames implies the collapse over derivative topological semantics:

Theorem 71. *The alternation hierarchy collapses to its alternation-free fragment on derivative topological semantics.*

Proof. Fix a μ-formula φ and a topological model $\mathcal{X} = \langle W, \tau, V \rangle$. Suppose φ is not equivalent to the alternation-free formula ψ. Theorem 69 implies that there is a finite topological model M_φ where φ is not equivalent to ψ. Any finite topological model is Alexandroff, as infinite intersections of open sens are actually finite intersections. By Proposition 68, M_φ is modally equivalent to a Kripke model M'_ψ with a weakly transitive accessibility relations. So φ and ψ are not equivalent over M'_ψ, they are not equivalent over weakly transitive frames. This argument holds for any alternation-free ψ; so if the alternation hierarchy collapses to its alternation-free fragment over derivative topological semantics, then it al so collapses over weakly transitive frames. This contradicts Theorem 65. $\square$

5.4 Other semantics and open problems

In Chapter 4, we studied variations of S5 over various alternative semantics. While the strictness for multimodal S5 frames implies the strictness for multimodal S4 frames, we were not able to generalize the results of Sections 4.1, 4.2 and 4.3 to variations of S4 and wK4 frames.

Question 7. *Does the alternation hierarchy collapse to its alternation-free fragment over:*

- *non-normal weakly-transitive frames?*
- *graded semantics with weakly-transitive frames?*
- *intuitionistic weakly-transitive frames?*

As with equivalence relations, we could not determine what happens to the alternation hierarchy for the inflationary μ-calculus over transitive and weakly transitive frames.

Part II

Reflection and determinacy in second-order arithmetic

Chapter 6

Reverse mathematics

In this chapter, we review second-order arithmetic and some of its subsystems.

6.1 Second-order arithmetic

In this subsection we review some basic definitions for reverse mathematics in second-order arithmetic. The standard reference for reverse mathematics is Simpson's monograph [Sim09]. See also Dzhafarov and Mummert [DM22], Hirschfeldt [Hir15], and Stillwell [Sti18].

REVERSE MATHEMATICS. Simpson [Sim09] states the main question of reverse mathematics as

> Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?

That is, we want to find axioms which are necessary and sufficient to prove theorems of ordinary mathematics.

We work over a basic theory T_B, which is expressive enough to define the necessary concepts but not strong enough to prove the theorems under study. We then prove the equivalence between an axiom φ_a and a theorem φ_t. When proving φ_t from φ_a it is common to formalize a standard proof of φ_t—sometimes it is necessary to find a new proof. We are doing reverse mathematics when we prove the reverse implication from φ_t to φ_a.

SECOND-ORDER ARITHMETIC. The language $\mathcal{L}_2$ of second-order arithmetic consists of the symbols 0, 1, $+$, $\cdot$, $<$, $\in$ with two sorts of variables—variables for natural numbers and for sets of natural numbers. Formulas are defined by the following grammar:

$$\varphi := x \in X \mid t = t \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \forall x.\varphi \mid \exists x.\varphi \mid \forall X.\varphi \mid \exists X.\varphi,$$

where t are terms. $\forall$ and $\exists$ are called quantifiers. We distinguish number and set quantifiers. A bounded (number) quantifier is a quantifier of the forms $\exists x < t.\varphi := \exists x(x < t \wedge \varphi)$ and $\forall x < t.\varphi := \forall x(x < t \to \varphi)$.

We say a formula is Σ^0_0 iff it has only bounded number quantifiers; Σ^0_{k+1} iff it is equivalent to some formula of the form $\exists x.\varphi$ with $\varphi \in \Pi^0_k$; Π^0_k iff it is equivalent to the negation of a Σ^0_k-formula; and Δ^1_0 iff it is Σ^0_k for some k. Let $\Sigma^1_0 = \Pi^1_0 := \Delta^1_0$. A formula is Σ^1_{k+1} iff it is of the form $\exists X.\varphi$ with $\varphi \in \Pi^1_k$; and Π^1_k iff it is equivalent to the negation of a Σ^1_k-formula. A formula is Δ^i_k iff it is in Σ^1_k and Π^1_k. A formula is $\Sigma^{k,X}_k$ iff it is in Σ^i_k and its only set parameter is X. The formulas in Δ^1_0 are known as arithmetic formulas.

Second-order arithmetic Z_2 consists of the axioms for discrete ordered semirings:

- $n + 1 \neq 0$;
- $m + 1 = n + 1 \to m = n$;
- $m + 0 = m$;
- $m + (n + 1) = (m + n) + 1$;
- $m \cdot 0 = 0$;
- $m \cdot (n + 1) = m \cdot n + m$;
- $\neg(m < 0)$;
- $m < n + 1 \leftrightarrow m < n \vee m = n$;

full comprehension:

$$\exists X \forall n. n \in X \leftrightarrow \varphi(n),$$

for all formula φ where X is not free; and the induction axiom:

$$0 \in X \wedge \forall n(n \in X \to n + 1 \in X) \to \forall n. n \in X.$$

An $\mathcal{L}_2$ structure is a tuple $M = \langle |M|, \mathcal{S}_M, 0_M, 1_M, +_M, \cdot_M, <_M \rangle$; $|M|$ is the quantification domain of number variables; $\mathcal{S}_M$ is the quantification domain of set variables and is a subset of the power set of $|M|$; 0_M and 1_M are elements of $|M|$; $+_M$ and $\cdot_M$ are binary functions; and $<_M$ is a binary relation. We call structures by models when they satisfy some theory under consideration. The intended model for Z_2 is

$$\langle \omega, \mathcal{P}(\omega), 0, 1, +, \cdot, < \rangle$$

consisting of the "real" set of natural numbers, its powerset, and the standard interpretations for 0, 1, $+$, $\cdot$ and $<$.

A model M is an ω-model iff its first-order part $|M|$ is the set of natural numbers ω. If a model M is not an ω-model, then its first-order part includes non-standard natural numbers. For more on non-standard natural numbers, see [Kay91]. We reserve $\mathbb{N}$ for the set of natural numbers in second-order arithmetic and ω for the "real" set of natural numbers.

THE BIG FIVE. The following subsystems of Z_2 are called the big five:

- RCA_0 contains the axioms for discrete ordered semirings; Σ^0_1-induction:

  $$\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n + 1)) \to \forall n. \varphi(n),$$

  for all Σ^0_1-formula; and Δ^0_1-comprehension:

  $$\exists X \forall n. n \in X \leftrightarrow \varphi(n),$$

  for all Δ^0_1-formula.

- WKL_0 is obtained by adding to RCA_0 the weak König's lemma: every infinite binary tree has an infinite path.
- ACA_0 is obtained by adding to RCA_0 the scheme of arithmetical comprehension:

  $$\exists X \forall n. n \in X \leftrightarrow \varphi(n),$$

  for all Δ^1_0-formula. We can alternatively characterize ACA_0 by the existence of Turing jumps: "for all X, the Turing jump $\mathrm{TJ}(X)$ of X exists".
- ATR_0 is obtained by adding to ACA_0 the scheme of arithmetic transfinite recursion. Intuitively, ATR_0 states that the Turing jump can be iterated along any well-order starting from any set.
- Π^1_1-CA_0 is obtained by adding to RCA_0 the scheme of Π^1_1-comprehension:

  $$\exists X \forall n. n \in X \leftrightarrow \varphi(n),$$

  for all Π^1_1-formula. Alternatively, we can characterize Π^1_1-CA_0 by the existence of hyperjumps: "for all X, the hyperjump $\mathrm{HJ}(X)$ of X exists".

Let π^0_1 be a universal lightface Π^0_1-formula. See Section 6.2 for the definition of universal lightface formulas. The Turing jump of X is the set $\{m \mid \neg\pi^0_1(m, X)\}$. The hyperjump of X is the set $\{\langle m, w\rangle \mid \exists f. \pi^0_1(e, m, f, X)\}$.

Example 17. We usually take RCA_0 as our base system. While it is enough to define many concepts of ordinary mathematics, it is not strong enough to prove most classical theorems. The system ACA_0 is stronger than RCA_0; and the system Π^1_1-CA_0 is stronger than ACA_0. We can show that

- RCA_0 proves the intermediate value theorem;
- ACA_0 is equivalent to the Bolzano–Weierstrass theorem over RCA_0; and
- Π^1_1-CA_0 is equivalent to the Cantor–Bendixson theorem over RCA_0.

Therefore the Cantor–Bendixson theorem is strictly stronger that the Bolzano–Weierstrass, which is strictly stronger than the intermediate value theorem.

OTHER SUBSYSTEMS. In addition to the big five, many other subsystems of Z_2 are studied nowadays. For a general picture see the reverse mathematics zoo at `https://rmzoo.math.uconn.edu/` and `https://www.computability.org/zoo-viewer/`. Some systems on the reverse mathematics zoo are also studied in [DM22] and [Hir15].

We study in this thesis the subsystems below:

- ACA_0' is obtained by adding to RCA_0 an axiom stating that arbitrary finite iterations of the Turing jump exist: for all $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$, there is a tuple $\langle X_0, \ldots, X_n\rangle$ such that $X_0 = X$ and $X_{k+1} = \mathrm{TJ}(X_k)$ for all $k < n$.

- Π^1_1-CA'_0 is obtained by adding "for all n and X, there is a sequence of coded β-models $\langle X_0, \dots, X_n \rangle$ such that $X \in X_0$ and, for all $i < n$, $X_i \in X_{i+1}$ and $X_i \subseteq_\beta X_{i+1}$" to RCA_0. An alternative formulation is to add to RCA_0 the statement "for all n and X, the iterated hyperjump $\mathrm{HJ}^n(X)$ of X exists" to RCA_0.

- Strong Σ^1_k-DC_0 is ACA_0 plus the following scheme:

$$\exists Z \forall n \forall Y.(\eta(n, (Z)^n, Y) \to \eta(n, (Z)^n, (Z)_n)),$$

  where $\eta(n, X, Y)$ is a Σ^1_k-formula in which Z does not occur, $(Z)^n = \{\langle i, m \rangle \mid \langle i, m \rangle \in Z \wedge m < n\}$, and $(Z)_n = \{i \mid \langle i, n \rangle \in Z\}$.

- Π^1_k-CA_0 is obtained by adding to RCA_0 Π^1_1-comprehension:

$$\exists X \forall n. n \in X \leftrightarrow \varphi(n),$$

  for all Π^1_k-formula.

The following systems will be explained in detail in the sections below:

- $[\Sigma^1_1]^k$-ID states the existence of the sets inductively definable by combinations of k-many Σ^1_1-transfinite induction operators.

- Γ-Det states that every Gale–Stewart game with payoff definable by a formula in Γ is determined.

- Γ-$\mathsf{Ref}(T)$ states that all formula in Γ which is provable by the theory T is true.

While it has a natural definition, Π^1_1-CA'_0 did not appear in the literature before the preprint [PY22]. Its naming is in parallel to ACA'_0.

ω-MODELS AND β-MODELS. Let $k \in \omega$, then $\mathcal{M}$ is a (coded) β_k-model iff every Π^1_k-sentence φ with parameters in $\mathcal{M}$ is true in $\mathcal{M}$ iff it is true (in the ground model). For $k \geq 1$, Strong Σ^1_k-DC_0 is equivalent to "for all X, there is a β_k-model $\mathcal{M}$ such that $X \in \mathcal{M}$". If $i = 1, 2$, then Strong Σ^1_k-DC_0 is equivalent to Π^1_k-CA_0. Furthermore, if we assume a formalized version of $V = L$, then Strong Σ^1_k-DC_0 is equivalent to Π^1_k-CA_0 for any k. We denote the ground model by $\mathcal{N}$. Given a coded model $\mathcal{M}$ and an $\mathcal{L}_2$-sentence φ with parameters in $\mathcal{M}$, write $\mathcal{M} \models \varphi$ to mean that $\mathcal{M}$ satisfies φ. See Section VII.2 of [Sim09] for a precise definition of the satisfaction relation in second order arithmetic.

Let $\mathcal{M}, \mathcal{N}$ be coded models. The sets $(\mathcal{M})_i$ of $\mathcal{M}$ are defined by $(\mathcal{M})_i = \{\langle n, i \rangle \mid \langle n, i \rangle \in \mathcal{M}\}$. $\mathcal{M}$ is a submodel of $\mathcal{N}$ iff every set in $\mathcal{M}$ is also in $\mathcal{N}$, that is, for all $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that $(\mathcal{M})_i = (\mathcal{N})_j$. Given two coded models $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M}$ is a β_k-submodel of $\mathcal{N}$ iff for all Σ^1_k-formula φ with parameters in $\mathcal{M}$, $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$. When $\mathcal{M}$ is a β_k-submodel of $\mathcal{N}$, we write $\mathcal{M} \subseteq_{\beta_k} \mathcal{N}$. The axiom $\beta(T)$ states the existence of a coded β-model of the theory T.

6.2 Reflection principles

PROVABILITY AND TRUTH PREDICATES. Fix $i \in \{0,1\}$ and $k \in \omega$. Let $\pi^i_k(e, \bar{m}, \bar{X})$ be a Π^i_k-formula with exactly $e, \bar{m}$ as free number variables and $\bar{X}$ as free set variables. π^i_k is a universal lightface Π^i_k-formula iff for all Π^i_k-formula π' with the same free variables as π^i_k, RCA_0 proves

$$\forall e \exists e' \forall \bar{m} \forall \bar{X}. \pi^i_k(e', \bar{m}, \bar{X}) \leftrightarrow \pi'(e, \bar{m}, \bar{X}).$$

Denote the code of a formula φ by $\ulcorner \varphi \urcorner$. We define the code in a way such that $\pi^i_k(\ulcorner \varphi \urcorner)$ is equivalent to φ if φ is Π^i_k. Given a finitely axiomatizable theory T, we can define a Σ^0_1-formula $\mathrm{Pr}_T(\ulcorner \varphi \urcorner)$ stating that φ is provable in T. We can show that if φ is provable in T, then RCA_0 proves $\mathrm{Pr}_T(\ulcorner \varphi \urcorner)$. For the detail in the definition of $\mathrm{Pr}_T(\ulcorner \varphi \urcorner)$, see Section II.8 of [Sim09].

Given $n \in \omega$, we define a formula $\mathrm{Tr}_{\Pi^1_n}(\ulcorner \varphi \urcorner)$ stating that the sentence $\varphi \in \Pi^1_n$ is true. Note that Tarski proved that there is no formula Tr capturing the truth of all formulas in arithmetic.

REFLECTION PRINCIPLES. Let T be a finitely axiomatizable theory. Let Pr_T be a provability predicate for T and $\mathrm{Tr}_{\Pi^1_n}$ be a truth predicate for Π^1_n-sentences. We only consider Π^1_n-sentences with arithmetic part in Σ^0_2 or Π^0_2. If we suppose ACA'_0, we can also consider sentences with non-standard length. The reflection principle Π^1_n-$\mathsf{Ref}(T)$ is the sentence

$$\forall \varphi \in \Pi^1_n. \mathrm{Pr}_T(\ulcorner \varphi \urcorner) \to \mathrm{Tr}_{\Pi^1_n}(\ulcorner \varphi \urcorner).$$

Note that we consider all Π^1_n-sentences inside our system, including nonstandard sentences. Reflection principles for a theory T can be thought as strengthening of the consistency of T.

REFLECTION SCHEMES. We could also consider reflection *schemes*. Define Π^1_n-$\mathsf{RFN}(T)$ to be the scheme consisting of

$$\forall x. \mathrm{Pr}_T(\ulcorner \varphi(x) \urcorner) \to \varphi(x)$$

for all standard Π^1_n-formula $\varphi(x)$. Reflection schemes are equivalent to reflection principles:

Proposition 72. *Let T be a finitely axiomatizable theory extending ACA_0. Then Π^1_n-$\mathsf{Ref}(T)$ and Π^1_n-$\mathsf{RFN}(T)$ are equivalent over ACA_0.*

Proof. Work inside a model of ACA_0. First, suppose that the reflection principle Π^1_n-$\mathsf{Ref}(T)$ holds. Fix a standard formula $\varphi(x)$. Let $a \in \mathbb{N}$ be such that $\mathrm{Pr}_T(\ulcorner \varphi(a) \urcorner)$. $\varphi(a)$ is a sentence inside the model, so we can use the reflection principle, and thus $\varphi(a)$ is true.

Now, suppose the reflection scheme Π^1_n-$\mathsf{RFN}(T)$ holds. Let φ be a sentence such that $\mathrm{Pr}_T(\ulcorner \varphi \urcorner)$ holds. For a contradiction, suppose that φ is false. Let π^1_n be a universal lightface formula Π^1_n-formula. For some index $e \in \mathbb{N}$, $\mathrm{Pr}_{\mathsf{RCA}_0}(\ulcorner \varphi \leftrightarrow \pi^1_n(e) \urcorner)$ holds. The instance of Π^1_n-$\mathsf{RFN}(T)$ for π^1_n implies that $\mathrm{Pr}_{\mathsf{RCA}_0}(\ulcorner \pi^1_n(e) \urcorner)$ holds.

We take the arithmetical parts of φ and π^1_n to be Σ^0_2. Therefore, $\neg\varphi$ is of the form $\exists X \forall m_0 \exists m_1.\psi(X, m_0, m_1)$ for some $\psi \in \Sigma^0_0$. Fix X_0 such that $\forall m_0 \exists m_1.\psi(X_0, m_0, m_1)$ holds. We use ACA_0 to construct an ω-model $\mathcal{M}$ which satisfies RCA_0 and includes X_0. Since $X_0 \in \mathcal{M}$, we have $\mathcal{M} \models \mathsf{RCA}_0 + \neg\varphi$, and so $\mathcal{M} \models \neg\pi^1_n(e)$. Since the arithmetical part of π^1_n is Σ^0_2, we can show that $\mathcal{M} \models \pi^1_n(e)$ also holds. This is a contradiction. We conclude that φ is true. □

OTHER RESULTS ON REFLECTION IN SECOND-ORDER ARITHMETIC. There are two categories of reflection principles—syntactic reflection and semantic reflection. Syntactic reflection principles state that, if something is provable, then it is true. Semantic reflection principles state that, if something is true, then it is true in a smaller model. Both kinds have been studied in the context of second-order arithmetic. The author surveyed these results [Pac22]. We describe some of the surveyed results below.

The relation between reflection and induction in first-order arithmetic is well-known; Frittaion [Fri22] extended these results to second-order arithmetic. He proved that if T_0 is a finitely axiomatizable second-order arithmetic theory and T is obtained by adding full induction to T_0, then

$$T_0 + \mathsf{Ref}(T_0) \equiv T; \text{ and}$$
$$T_0 + \mathsf{Ref}(T) \equiv T_0 + \mathsf{TI}(\varepsilon_0).$$

He also proved that, if T_0 is a Π^1_{k+2} finitely axiomatizable extension of RCA_0, then

$$\Pi^1_{n+2}\text{-}\mathsf{Ref}(T_0) \equiv \Pi^1_n\text{-}\mathsf{Ind} \supseteq (\Pi^1_n\text{-}\mathsf{Ind})^- \equiv \Sigma^1_{n+1}\text{-}\mathsf{Ref}(T_0); \text{ and}$$
$$\Pi^1_{n+2}\text{-}\mathsf{Ref}(T) \equiv \Pi^1_n\text{-}\mathsf{TI}(\varepsilon_0) \supseteq \Pi^1_n\text{-}\mathsf{TI}(\varepsilon_0)^- \equiv \Sigma^1_{n+1}\text{-}\mathsf{Ref}(T),$$

whenever $n \geq k+1$. Here, $\mathsf{TI}(\varepsilon_0)$ is the scheme of transfinite induction up to ε_0; Γ-Ind is the scheme of induction for formulas in Γ; and $(\Gamma\text{-}\mathsf{Ind})^-$ is the scheme of induction on formulas in Γ without set parameters.

Paris and Harrington [Par77] proved the equivalence between the Paris-Harrington theorem and reflection for $\Pi_2(\mathsf{PA})$. In [Yok22], Yokoyama characterizes variations of the Paris-Harrington theorem as reflection theorems for subsystems of second-order arithmetic.

Pakhomov and Walsh [PW22; PW18; PW21] studied the relation between iterated reflection principles and ω-model reflection principles, and used iterated reflection principles to study the Π^1_1 proof-theoretic ordinals of theories extending ACA_0. Fix a theory T finitely axiomatizable by a Π^1_2-formula. They studied the following iterated reflection principles:

$$\Pi^1_n\text{-}\mathsf{Ref}^{\alpha}(T) := T + \{\Pi^1_n\text{-}\mathsf{Ref}(\Pi^1_n\text{-}\mathsf{Ref}^{\beta}(T)) \mid \beta < \alpha\}; \text{ and}$$
$$\Pi^1_n\text{-}\mathsf{Ref}^{\mathbf{ON}}(T) := \forall\alpha(\mathrm{WO}(\alpha) \to \Pi^1_n\text{-}\mathsf{Ref}(\Pi^1_n\text{-}\mathsf{Ref}^{\beta}(T))).$$

They showed that $\Pi^1_1\text{-}\mathsf{Ref}^{\mathbf{ON}}(T)$ is equivalent to every set being contained in an ω-model of T; and that, if T is a Π^1_{n+1} axiomatizable theory, $\Pi^1_n\text{-}\mathsf{Ref}^{\mathbf{ON}}(T)$ is equivalent to $\Pi^1_n\text{-}\omega\mathsf{Ref}^{\mathbf{ON}}(T)$. Here, $\Pi^1_n\text{-}\omega\mathsf{Ref}(T)$ formalizes "for all Π^1_n-formula $\varphi(X)$ and all set $X \subseteq \mathbb{N}$, there is a coded ω-model M such that $X \in M$, $M \models \varphi(X)$ and $M \models T$". Pakhomov and Walsh use this result to uniformly prove that $|\mathsf{ACA}_0^+|_{\Pi^1_1} = \phi_2(0)$,

$|\Sigma^1_1\text{-}\mathsf{AC}_0|_{\Pi^1_1} = |\Pi^1_2\text{-}\mathsf{Ref}^{\varepsilon_0}(\Sigma^1_1\text{-}\mathsf{AC}_0)| = \phi_{\varepsilon_0}(0)$, $|\mathsf{ATR}_0|_{\Pi^1_1} = \Gamma_0$, and $|\mathsf{ATR}|_{\Pi^1_1} = \Gamma_{\varepsilon_0}$. Here, $|T|_{\Pi^1_1}$ is the Π^1_1 proof theoretic ordinal of T.

We can also consider reflection principles for logics stronger than second-order arithmetic. ω-logic is obtained by adding the ω-rule to second-order arithmetic:

$$\frac{\varphi(0), \Gamma \quad \varphi(1), \Gamma \quad \varphi(2), \Gamma \quad \cdots}{\forall x.\varphi(x), \Gamma}.$$

Fernández-Duque [Fer15] mentions three ways to model the statement "φ is a theorem of ω-logic":

- there is a well-founded derivation tree formalizing an ω-proof of φ;
- there is a well-ordering Λ such that φ belongs to the set of theorems of ω-logic obtained by recursion along Λ;
- φ is in the least set closed under axioms and rules of ω-logic.

Denote "φ is a theorem of ω-logic" in these ways by $[P]\varphi$, $[R]\varphi$, and $[I]\varphi$, respectively. Therefore we have three varieties of reflection for ω-logic, one for each way. If X is one of P, R or I, let $[X|A]\varphi$ mean "there is an ω-logic proof of φ using A as an oracle". $\Gamma\text{-}\omega_X\mathsf{Ref}(T)$ formalizes the sentence "for all formula φ in Γ and $A \subseteq \mathbb{N}$, if $[X|A]\varphi$ holds then so does φ". We omit Γ when it is the set of all formulas, omit A when no oracle is used, and omit T when it is empty.

Cordón-Franco *et al.* [Cor+17] proved that $\Pi^1_2\text{-}\omega_R\mathsf{Ref}$ is equivalent to ATR_0. They also proved that $\Sigma^1_{n+1}\text{-}\omega_R\mathsf{Ref}(\mathsf{ACA}_0)$ is equivalent to $\mathsf{ATR}_0 + \Pi^1_n\text{-}\mathsf{TI}$, where $\Sigma^1_{n+1}\text{-}\omega_R\mathsf{Ref}(\mathsf{ACA}_0)$ is obtained by adding the axioms of ACA_0 to the ω-logic.

Fernández-Duque [Fer15] proved that $\Pi^1_3\text{-}\omega_I\mathsf{Ref}$ is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$. He also proved an analogous result for $\Pi^1_1\text{-}\mathsf{CA}_0$: $\Sigma^1_{n+1}\text{-}\omega_I\mathsf{Ref}(\mathsf{ACA}_0)$ is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0 + \Pi^1_n\text{-}\mathsf{TI}$.

Arai [Ara98] has proved the equivalence between ω-logic reflection and a transfinite induction: $\mathsf{RCA}_0 + \omega_P\mathsf{Ref}$ is equivalent to $\mathsf{RCA}_0 + \Pi^1_\omega\text{-}\mathsf{TI}_0$.

6.3 Gale-Stewart games

GALE–STEWART GAMES. Let X be a set and fix $A \subseteq X^{\mathbb{N}}$. Denote Gale–Stewart game with payoff A by $\mathcal{G}(A)$. In $\mathcal{G}(A)$, two players I and II alternate picking elements of X to build a sequence α.

$$\begin{array}{lcccccccccccc}
\mathsf{I} & x_0 & & x_2 & & x_4 & & \cdots & x_{2n} & & & \cdots & \\
\mathsf{II} & & x_1 & & x_3 & & x_5 & & \cdots & & x_{2n+1} & & \cdots
\end{array}$$
$$\alpha = \langle x_0, x_1, x_2, x_3, x_4, x_5, \ldots \rangle$$

Such sequences are called runs. The player I wins a run α of $\mathcal{G}(A)$ iff α is in the payoff A. A strategy σ for I is a function from finite sequences of even length in X to elements of X, telling I to play $\sigma(s)$ at the position $s \in X^{<\mathbb{N}}$. Similarly, a strategy τ for II is a function from finite sequences of odd length in X to elements of X. A strategy is winning iff its owner wins any run where they use the strategy. A Gale–Stewart game is determined iff there is a winning strategy for one of the players. It is not possible for both players to win. Suppose both players used winning strategies simultaneously in $\mathcal{G}(A)$. The resulting run α is such that $\alpha \in A$ and $\alpha \notin A$.

The axiom of determinacy AD states that all Gale–Stewart games are determined. AD is incompatible with the axiom of choice. On the other hand, Martin [Mar75; Mar85] proved that ZFC proves the determinacy of Gale–Stewart fames whose payoff are Borel. This is the result of a chain of increasingly stronger results:

Theorem 73. *Let Γ-Determinacy state the determinacy of sets in Γ.*

- *(Gale, Steward [GS53])* ZF *proves Σ^0_1-Determinacy.*
- *(Wolfe [Wol55])* ZF *proves Σ^0_2-Determinacy.*
- *(Davis [Dav64])* ZF *proves Σ^0_3-Determinacy.*
- *(Paris [Par72])* ZF *proves Σ^0_4-Determinacy.*
- *(Martin [Mar75; Mar85])* ZFC *proves Δ^1_1-Determinacy.*

We will consider only Gale–Stewart games with $X = \mathbb{N}$ and $X = 2(= \{0, 1\})$. The axiom Γ-Det states that every game with payoff in $\mathbb{N}^{\mathbb{N}}$ whose payoff is Γ-definable is determined; and Γ-Det* states the same for games with payoff in $2^{\mathbb{N}}$. $\mathbb{N}^{\mathbb{N}}$ is known as the Baire space, and $2^{\mathbb{N}}$ as the Cantor space.

See one of [Jec03; Kec94; Mos09] for more details on Gale–Stewart games. Another reference is Martin's unpublished book [Mar]. For determinacy on second-order arithmetic, see Sections V.8 and VI.5 of [Sim09], or the survey [Yos17].

DIFFERENCE HIERARCHIES. The difference hierarchy for Σ^0_k captures all boolean combinations of Σ^0_k sets. It is usually defined by induction: a set X is $(\Sigma^0_k)_1$ iff X is Σ^0_k; and X is $(\Sigma^0_k)_{n+1}$ iff X is the difference of a Σ^0_k set and a $(\Sigma^0_k)_n$ set: $X = Y \setminus Z$, with $Y \in \Sigma^0_k$ and $Z \in (\Sigma^0_k)_n$.

In this thesis we use a formalized version of the difference hierarchy. Fix $k \in \omega$. Let x be a number variable and f be a distinguished second-order variable. $\varphi(f)$ is a $(\Sigma^0_k)_x$-formula iff there is a Σ^0_k-formula $\psi(y, f)$ (possibly with other free variables) such that: $\psi(x, f)$ always holds; if $z < y < x$ then $\psi(z, f)$ implies $\psi(y, f)$; and $\varphi(f)$ holds iff the least $y \leq z$ such that $x = y \vee \psi(y, f)$ holds is even. Intuitively, we think of $\varphi(f)$ as the disjunction

$$\psi(0, f) \vee (\psi(2, f) \wedge \neg\psi(1, f) \vee \cdots \vee (\psi(2\lfloor x/2 \rfloor, f) \wedge \neg\psi(2\lfloor x/2 \rfloor - 1, f),$$

where $\lfloor x/2 \rfloor$ is the greatest integer n such that $n \leq x/2$.

GALE–STEWART GAMES IN SECOND-ORDER ARITHMETIC. We now show how to formalize Gale–Stewart games in second-order arithmetic. Given $k \in \omega$, Σ^0_k-Det states that every Σ^0_k-definable set in $\mathbb{N}^{\mathbb{N}}$ is determined. Formally, Σ^0_k-Det is the scheme

$$\exists\sigma\forall\tau.\varphi(\sigma \otimes \tau) \vee \exists\tau\forall\sigma.\neg\varphi(\sigma \otimes \tau),$$

where φ is a Σ^0_k-formula. Here, $\sigma \otimes \tau$ is the play obtained when I uses the strategy σ and II uses the strategy τ. Similarly, $(\Sigma^0_k)_n$-Det states that every $(\Sigma^0_k)_n$-definable set in $\mathbb{N}^{\mathbb{N}}$ is determined. Finally, $\forall n.(\Sigma^0_k)_n$-Det states that, for every set X in $\mathbb{N}^{\mathbb{N}}$, if X

Table 6.1: Existing results on the reverse mathematics of determinacy up to differences of Σ^0_2 sets.

	Determinacy in $\mathbb{N}^\mathbb{N}$	Determinacy in $2^\mathbb{N}$
WKL_0		Δ^0_1, Σ^0_1
ACA_0		$(\Sigma^0_1)_n$, for $n \in \omega$
ATR_0	Δ^0_1, Σ^0_1	Δ^0_2, Σ^0_2
Π^1_1-CA_0	$(\Sigma^0_1)_n$, for $n \in \omega$	$\mathrm{Sep}(\Sigma^0_1, \Sigma^0_2)$
Π^1_1-TR	Δ^0_2	$\mathrm{Sep}(\Delta^0_2, \Sigma^0_2)$
Σ^1_1-ID	Σ^0_2	$(\Sigma^0_1)_2$
$\{[\Sigma^1_1]^n$-$\mathsf{ID} \mid n \in \omega\}$	$\{(\Sigma^0_1)_n \mid n \in \omega\}$	$\{(\Sigma^0_1)_n \mid n \in \omega\}$

is $(\Sigma^0_k)_n$-definable for some $n \in \mathbb{N}$, then X is determined. The schema Γ-Det* state the same as Γ-Det but for sets in $2^\mathbb{N}$. They are obtained by restricting the players to playing inside $2^\mathbb{N}$.

Survey on existing results. Up to a certain point, determinacy axioms of increasing strength generate sequence of equivalent subsystems of second-order arithmetic. We describe the results up to standard finite differences of Σ^0_2 sets in Table 6.1.

The situation above boolean combinations of Σ^0_2 sets is more complicated. MedSalem and Tanaka [MT07] showed that Δ^0_3-Det is proved by Δ^1_3-CA_0 plus Σ^1_3-induction and by Π^1_2-CA_0 plus transfinite Π^1_3-induction. They also showed that even if we increase the respective inductive axioms to all $\mathcal{L}_2$-formulas, we cannot prove Σ^0_3-Det. On the other hand, we do not have a reversal: even Δ^1_1-Det does not prove Δ^1_2-CA_0. Hachtman [Hac19] proved that Σ^0_3-Det is equivalent to the existence of countable β-models of Σ^1_1-MI.

Montalbán and Shore [MS12; MS14] proved that, for every $m \geq 1$, Π^1_{m+2}-CA_0 proves $(\Pi^0_3)_m$-Det. They also showed that, if $m \geq 1$ and $X \subseteq \mathbb{N}$, then $(\Pi^0_3)_m$-Det proves the existence of a β-model $\mathcal{M}$ of Δ^1_m-CA_0 with $X \in \mathcal{M}$. These results will allow us to study the relation between determinacy for differences of Σ^0_3 sets and reflection for subsystems of Z_2.

6.4 Inductive definitions

Inductive operators. An operator is a function taking sets of natural numbers to sets of natural numbers. An operator is Σ^1_1 iff its graph is definable by a Σ^1_1-formula. We say the operator Γ is monotone iff $X \subseteq Y$ implies $\Gamma(X) \subseteq \Gamma(Y)$.

Given an operator Γ, the set Γ^∞ inductively defined by Γ is obtained by iterating Γ:

- $\Gamma^0 := \emptyset$;
- $\Gamma^{\alpha+1} := \Gamma^\alpha \cup \Gamma(\Gamma^\alpha)$, if α is a successor ordinal;
- $\Gamma^\lambda := \bigcup_{\alpha<\lambda} \Gamma^\alpha$, if λ is a limit ordinal; and
- $\Gamma^\infty := \bigcup_{\alpha \in \mathrm{Ord}} \Gamma^\alpha$.

If Γ is monotone, we can simplify the definition of successor steps to $\Gamma^{\alpha+1} := \Gamma(\Gamma^{\alpha})$.

Let Γ_0, Γ_1 be operators. To define the set $[\Gamma_0, \Gamma_1]^\infty$ inductively via Γ_0 and Γ_1, we need to mix applications of both operators. Intuitively, we start from the empty set, and apply Γ_0 until we obtain a fixed-point $\Gamma_0^{\infty,0}$:

$$\emptyset, \Gamma_0(\emptyset), \Gamma_0(\Gamma_0(\emptyset)) \cup \Gamma_0(\emptyset), \ldots$$

We then apply Γ_1 once, and generate another fixed-point $\Gamma_0^{\infty,1}$ for Γ_0:

$$\Gamma_1(\Gamma_0^{\infty,1}), \Gamma_0(\Gamma_1(\Gamma_0^{\infty,1})), \Gamma_0(\Gamma_0(\Gamma_1(\Gamma_0^{\infty,1}))) \cup \Gamma_0(\Gamma_1(\Gamma_0^{\infty,1})), \ldots$$

Repeat this process until we obtain a fixed-point for both Γ_0 and Γ_1. Formally,

- $\Gamma_1^{0,0} := \emptyset$;
- $\Gamma_0^{0,\beta} := \Gamma_1^{\beta}$;
- $\Gamma_0^{\alpha+1,\beta} := \Gamma_0^{\alpha,\beta} \cup \Gamma_0(\Gamma_0^{\alpha,\beta})$, if α is a successor ordinal;
- $\Gamma_0^{\lambda,\beta} := \bigcup_{\alpha<\lambda} \Gamma_0^{\alpha,\beta}$;
- $\Gamma_0^{\infty,\beta} := \bigcup_{\alpha\in\mathrm{Ord}} \Gamma_0^{\alpha,\beta}$;
- $\Gamma_1^{\alpha+1} := \Gamma_1^{\alpha} \cup \Gamma_1(\Gamma_0^{\infty,\alpha})$;
- $\Gamma_1^{\lambda} := \bigcup_{\alpha<\lambda} \Gamma_1^{\alpha}$; and
- $\Gamma_1^{\infty} := \bigcup_{\alpha\in\mathrm{Ord}} \Gamma_1^{\alpha}$.

In case we are combining three operators $\Gamma_0, \Gamma_1, \Gamma_2$, we apply Γ_2 whenever we get a fixed-point for Γ_0 and Γ_1, and repeat until we get a fixed-point for all three operators. The general case with k operators follows the same idea.

INDUCTIVE DEFINITIONS IN SECOND-ORDER ARITHMETIC. The scheme $[\Sigma_1^1]^k$-ID states the existence of sets inductively defined by k-many Σ_1^1-operators. MedSalem and Tanaka [MT07] defined $[\Sigma_1^1]^2$-ID as follows: $[\Sigma_1^1]$-ID asserts that for any $\Gamma_0, \Gamma_1 \in \Sigma_1^1$, there exist a pre-wellorder $W \subseteq \mathbb{N}\times\mathbb{N}$ on its field F,and V',$\langle V^m \subseteq N \times N \mid m \in F\rangle$, such that for all $m \in F$

- V^m is a pre-wellorder on its field F^m;
- $\forall y \in F^m, V_u^m = \Gamma_0^{W_{<m}}(V_{<y}^m) \cup V_{<y}^m$;
- $W_m = \Gamma_1(F^m) \cup W_{<m}$;
- $\Gamma_0^{W_{<m}}(F^m) \subseteq F^m$;
- V' is a pre-wellorder on its field F';
- $\forall y \in F', V_y' = \Gamma_0^F(V_{<y}') \cup V_{<y}'$;
- $\Gamma_0^F(F') \subseteq F$; and
- $\Gamma_1(F') \subseteq F$.

The definition above can be generalized to a definition of $[\Sigma^1_1]^k$-ID (uniform on k).

We use $[\Sigma^1_1]^k$-ID below in the proof of Lemma 98, but we need only the existence of the fixed-points. Therefore we use a simpler variation of $[\Sigma^1_1]$-ID: For $n \in \mathbb{N}$, $[\Sigma^1_1]^n$-LFP asserts that for any sequence $\langle \Gamma_i \mid i < n \rangle$, there exists a smallest set X which satisfies $\Gamma_i(X) = X$ for all $i < n$.

Lemma 74. *$\forall n.[\Sigma^1_1]^n$-ID implies $\forall n.[\Sigma^1_1]^n$-LFP over RCA_0.*

Proof. $\forall n.[\Sigma^1_1]^n$-ID gives us the least simultaneous fixed-point of the operators $\Gamma_0, \ldots, \Gamma_n$, but it also registers when each point enters the fixed-point. We need only to forget this information. $\square$

Chapter 7

μ-arithmetic

This chapter explains the connection between Part I and Part II of this thesis. We will study the μ-arithmetic—a logic obtained by adding fixed-point operators to first-order arithmetic. We also study its relation to the μ-calculus and to determinacy of Gale–Stewart games. This is joint work with Wenjuan Li and Kazuyuki Tanaka [PLT22].

7.1 Basic definitions

μ-ARITHMETIC. In this section, we define the μ-arithmetic, obtained by adding the fixed-point operators μ and ν to the first-order arithmetic. In this context, $\mu x X.\varphi$ will be the least fixed-point of the operator $\Gamma_{\varphi(x,X)} = \{x \in \omega \mid \varphi(x, X)\}$, and $\nu x X.\varphi$ is the greatest fixed-point of $\Gamma_{\varphi(x,X)}$.

FORMULAS. We add to the language of first-order arithmetic the fixed-point operators μ and ν, set variables, and the inclusion relation $\in$. Number terms represent natural numbers. They are defined by the grammar

$$t := 0 \mid 1 \mid x \mid t + t \mid t \cdot t,$$

where x is a number variable. Set terms represent sets of natural numbers, they are defined by the grammar

$$S := X \mid \mu x X.\varphi \mid \nu x X.\varphi,$$

where φ is a formula where the set variable X is positive. A set variable is positive in a formula φ iff it is under the scope of an even number of negations. The formulas of μ-arithmetic are defined by the following grammar

$$\varphi := t \in S \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \psi,$$

where t is a number term and S is a set term.[1] Note that when using the μ-operator we bind a number variable x and a set variable X. As in the μ-calculus, we say that

[1]We also call the formulas of μ-arithmetic by μ-formulas, the intended meaning will be clear in context. If necessary we say that a formula of μ-calculus is a modal μ-formula and that a formula of μ-arithmetic is an arithmetic μ-formula.

the occurrences of x or X in $\mu xX.\varphi$ are bound, and if x or X is not bound they are free. If a formula has no free *set* variable we say it is closed.

We will also consider an infinitary extension of the μ-arithmetic (and the μ-calculus). If $\{\varphi_i\}_{i\in\omega}$ is an recursive enumerable sequence of formulas with finitely many free variables, then $\bigvee_{i\in\omega}\varphi_i$ and $\bigwedge_{i\in\omega}\varphi_i$ are also formulas.

SEMANTICS. Formulas of the μ-arithmetic are interpreted over the natural numbers and its power set. 0, 1, $+$ and $\cdot$ all have their standard interpretations. Given φ where X is positive, define $\Gamma_{\varphi(x,X)} := \{x \in \omega \mid \varphi(x, X)\}$. Then $\|\mu xX.\varphi\|$ is the least fixed-point of $\Gamma_{\varphi(x,X)}$ and $\|\nu xX.\varphi\|$ is the greatest fixed-point of $\Gamma_{\varphi(x,X)}$. If $\bigvee_{i\in\omega}\varphi_i$ is a formula, then its valuation $\|\bigvee_{i\in\omega}\varphi_i\|$ is defined as $\bigcup_{i\in\omega}\|\varphi_i\|$.

Example 18. The following formula defines the set of even numbers in the μ-calculus:

$$\mu xX.(x = 0 \vee (x - 2) \in X).$$

By negation, we get that the set of odd numbers are defined by:

$$\nu xX.(x \neq 0 \wedge (x - 2) \in X).$$

THE ALTERNATION HIERARCHY. The μ-arithmetic's alternation hierarchy is defined analogously to the μ-calculus'. For each $\alpha < \omega_1^{\mathrm{ck}}$, define:

- Σ_0^μ is the set of all set variables and formulas without fixed-point operators.
- $\Sigma_{\alpha+1}^\mu$ is generated from $\Sigma_\alpha^\mu \cup \Pi_\alpha^\mu$ by closing it under $\vee$, $\wedge$, $\in$ and $\mu xX.\varphi$ for X-positive $\varphi \in \Sigma_{\alpha+1}^\mu$. Here $\mu xX.\varphi$ is called a $\Sigma_{\alpha+1}^\mu$ term.
- $\Pi_{\alpha+1}^\mu$ contains all the negations of formulas and set terms in $\Sigma_{\alpha+1}^\mu$.
- If λ is a limit ordinal, then Σ_λ^μ is generated from $\bigcup_{\alpha<\lambda}\Sigma_\alpha^\mu$ and closed under $\bigvee_{i<\omega}$.
- Π_λ^μ contains all the negations of Σ_λ^μ formulas and terms.

The ordinal ω_1^{ck} is the Church–Kleene ordinal, the least non-computable ordinal.

We say a μ-formula is Σ_α^μ-definable iff it is equivalent to a Σ_α^μ-formula. And we say a μ-term is Σ_α^μ-definable iff it is equal to some Σ_α^μ μ-term. If $\varphi \in \Sigma_0^\mu$, we say it is arithmetical. We could possibly define Emerson–Lei and Niwińsky versions for the μ-arithmetic alternation hierarchy, but in this case all three hierarchies are equal, for details see Section 4 of [Bra98b]. We could also check the validity of formulas of the μ-arithmetic by games—known as model checking games.

μ-CALCULUS OVER COMPUTABLE MODELS. We say a Kripke model $M = \langle W, R, V\rangle$ is computable iff W is a computable subset of ω, R is a computable binary relation over ω, and $V : \mathrm{Prop} \to \{0, 1\}$ is computable.

We extend the μ-calculus' alternation hierarchy to the transfinite by defining: If λ is a limit ordinal, then Σ_λ^μ is generated from $\bigcup_{\alpha<\lambda}\Sigma_\alpha^\mu$ and closed under $\bigvee_{i<\omega}$. The definition of the successor levels of the alternation hierarchy is as in the finite case. As in the μ-arithmetic, $\|\bigvee_{i\in\omega}\varphi_i\|^M$ is $\bigcup_{i\in\omega}\|\varphi_i\|^M$

The sets of natural numbers definable by arithmetic μ-formulas and the sets definable by modal μ-formulas over computable Kripke models are the same. This justifies the overload of meanings for the symbols Σ_α^μ .

Theorem 75 (Bradfield [Bra98b]). *Let $\varphi(z)$ be a Σ_n^μ-formula of μ-arithmetic. There is an computable Kripke model M and a Σ_n^μ modal μ-formula $\overline{\varphi}$ such that $\varphi(s)$ iff $s \in \|\overline{\varphi}\|^M$.*

Theorem 76 (Bradfield [Bra98b]). *For each modal μ-calculus formula $\varphi \in \Sigma_n^\mu$ and for each computable Kripke model M, $\|\varphi\|^M$ is Σ_n^μ-definable set of natural numbers.*

THE WEAK ALTERNATION HIERARCHY. We define the weak alternation hierarchy for the μ-arithmetic as follows:

- $\Sigma_0^{W\mu}$ is the set of all the first-order formulas and all set variables.
- $\Sigma_{\alpha+1}^{W\mu}$ is generated from $\Sigma_\alpha^{W\mu} \cup \Pi_\alpha^{W\mu}$ by closing it under $\vee$, $\wedge$ and the following substitution rules: (a) If $\varphi(X)$ is Σ_1^μ and if ψ is a $\Sigma_{\alpha+1}^{W\mu}$ term without free set variables, then $\varphi(X\backslash\psi)$ is also $\Sigma_{\alpha+1}^{W\mu}$; (b) if φ is a Σ_1^μ, φ' is a subformula of φ and ψ is a $\Sigma_{\alpha+1}^{W\mu}$ term without free set variables,then $\varphi(\varphi'\backslash\psi)$ is also $\Sigma_{\alpha+1}^{W\mu}$. In these substitution rules, φ can be either a formula or a term.
- If λ is a limit ordinal, then $\Sigma_\lambda^{W\mu}$ is generated from $\bigcup_{\alpha<\lambda} \Sigma_\alpha^{W\mu}$ and closed under $\bigvee_{i<\omega}$.
- $\Pi_\alpha^{W\mu}$ contains all the negations of formulas and set terms in $\Sigma_\alpha^{W\mu}$.
- $\Pi_\lambda^{W\mu}$ contains all the negations of $\Sigma_\lambda^{W\mu}$ formulas and terms.

Observe that we abuse the notation of substitution in this definition. This is necessary in the transfinite levels of the weak hierarchy, as there are no weak μ-term strictly in the limit levels.

We also have the equivalence between weak μ-arithmetic and the weak μ-calculus over computable frames:

Theorem 77 (P., Li, Tanaka [PLT22]). *Let $\varphi(z)$ be a $\Sigma_\alpha^{W\mu}$-formula of μ-arithmetic. There is a computable Kripke model M and a $\Sigma_\alpha^{W\mu}$ modal μ-formula $\overline{\varphi}$ such that $\varphi(s)$ iff $s \in \|\overline{\varphi}\|^M$.*

Theorem 78 (P., Li, Tanaka *et al* [PLT22]). *For each modal μ-calculus formula $\varphi \in \Sigma_\alpha^{W\mu}$ and for each computable Kripke model M, $\|\varphi\|^M \subseteq \omega$ is $\Sigma_\alpha^{W\mu}$-definable set of integers.*

7.2 μ-definable sets of natural numbers

THE GAME QUANTIFIER. We define the game quantifier $\Game$ by

$$\Game\alpha.P(\alpha,\vec{x}) = \{\vec{x} \mid \text{I wins the Gale–Stewart game with payoff } P(\alpha,\vec{x})\} \subseteq \omega^k.$$

where $P \subseteq \omega^\omega \times \omega^k$ for some $k \in \omega$. $\Game\alpha.P(\alpha,\vec{x})$ describes the parameters which make the player I win the game $P(\alpha,\vec{x})$.

If Γ is a subset of $\mathcal{P}(\omega^k)$, define $\Game\Gamma = \{S \mid S = \Game\alpha.P(\alpha,\vec{x}) \text{ for some } P \in \Gamma\}$. Kechris and Moschovakis proved that $\Game\Sigma_1^0$ is the collection of Π_1^1-definable sets of natural numbers. Solovay proved that $\Game\Sigma_2^0$ is the collection of sets given via an inductive definition over a Σ_1^1 predicate. See [Mos09] for proofs of these results.

THE TRANSFINITE DIFFERENCE HIERARCHY. Fix $k \in \omega$. We now define the difference hierarchy for Σ^0_k-definable sets as follows. For each $\alpha < \omega_1^{ck}$, let

$$S \in (\Sigma^0_k)_\alpha \iff S = \bigcup_{\beta \in Opp(\alpha)} (A_\beta - \cup_{\zeta<\beta} A_\zeta)$$

where $\{A_\beta\}_{\beta<\alpha}$ is an effective enumeration of a sequence of sets in Σ^0_k and $Opp(\alpha)$ is the set of ordinals less that α whose parity is opposite to the parity of α. We consider the limit ordinals to be even.

For the finite levels of the difference hierarchy we can consider an alternative definition. For $n \in \omega$, let:

- $(\Sigma^0_k)_1 = \Sigma^0_1$,
- $(\Pi^0_k)_n = \neg(\Sigma^0_k)_n$, and
- $(\Sigma^0_k)_{n+1} = \Sigma^0_k \wedge (\Pi^0_k)_n$.

We are now ready to describe the connection between the difference hierarchy for Σ^0_2 sets to the alternation hierarchy of the μ-arithmetic. Bradfield *et al.* [BDQ05; BDQ10] proved that, for all $\alpha < \omega_1^{ck}$, $\Game(\Sigma^0_2)_\alpha = \Sigma^\mu_{\alpha+1}$. MedSalem and Tanaka [MT07] proved a formalized version of the Hausdorff–Kuratowski theorem: $\bigcup_{\alpha<\omega_1^{ck}}(\Sigma^0_2)_\alpha = \Delta^0_3$. Combining the two theorems above, we have that

$$\bigcup_{\alpha<\omega_1^{ck}} \Sigma^\mu_\alpha = \Game\Delta^0_3,$$

since the game quantifier commutes with unions.

The connection above also holds between the differences of Σ^0_1 sets and the weak alternation hierarchy. The author, Li and Tanaka [PLT22] proved that $\Sigma^{W\mu}_{\alpha+1} = \Game\Sigma^{\delta,1}_\alpha$, for all $\alpha < \omega_1^{ck}$. Tanaka [Tan90] proved the following version of the Hausdorff–Kuratowski theorem: $\bigcup_{\alpha<\omega_1^{ck}}(\Sigma^0_1)_\alpha = \Delta^0_2$. As a consequence the following holds:

$$\bigcup_{\alpha<\omega_1^{ck}} \Sigma^{W\mu}_\alpha = \Game\Delta^0_2.$$

7.3 μ-arithmetic and determinacy

μ-ARITHMETIC IN SECOND-ORDER ARITHMETIC. In this section we formalize the μ-arithmetic inside Z_2. These results are from [Möl02].

We define the language $\mathcal{L}_\mu$ of μ-arithmetic by adding the constructor μ to $\mathcal{L}_2$. Define the set of $\mathcal{L}_\mu$-formulas and $\mathcal{L}_\mu$-terms to be the smallest set which includes the $\mathcal{L}_2$-formulas and is closed under the usual rules for forming $\mathcal{L}_2$-formulas and the following rule: if $\varphi(x, X)$ is an X-positive formula of $\mathcal{L}_\mu$, we add a set term $\mu x X.\varphi(x, X)$, with the restriction that $\varphi(x, X)$ has no second-order quantifiers.

The term $\mu x X.\varphi(x, X)$ denotes the least fixed-point of the operator $\Gamma_\varphi(X) = \{x \mid \varphi(x, X)\}$. To formalize this idea we define for each X-positive formula $\varphi(x, X)$ the following formula:

$$\mathrm{LFP}(\varphi, I) \iff \forall x.(x \in I \leftrightarrow \varphi(x, I)) \ \wedge\ \forall Y.(\forall x.(\varphi(x, Y) \to x \in Y) \to I \subset Y$$

LFP stands for least fixed-point and $\mathrm{LFP}(\varphi, I)$ means that I is a fixed-point of $\Gamma_\varphi(X)$ and is the least such fixed-point.

μ-arithmetic is the system containing the axioms of ACA_0 (including comprehension for $\mathcal{L}_\mu$-formulas with no set quantifiers) and contains $\mathrm{LFP}(\varphi(x, X), \mu x X.\varphi(x, X))$ for each X-positive formula $\varphi \in \mathcal{L}_\mu$ with no set quantifiers. Note that we do not consider μ a set quantifier, so we can take fixed-points of formulas which include μ.

We also define

$$\begin{aligned}\mathrm{IGF}(\varphi, S, \preceq, \prec) \iff\ & (S, \preceq, \prec) \text{ is a pre-well-ordering and} \\ & \forall x, y(x \preceq y \leftrightarrow x \prec y \vee \varphi(x, \{z \mid z \prec y\})).\end{aligned}$$

Over ACA_0, if $\varphi(x, X)$ is an X-positive formula, S is a set and $\prec, \preceq$ are binary relations on S, then $\mathrm{IGF}(\varphi, S, \preceq, \prec)$ implies $\mathrm{LFP}(\varphi, S)$.

A generalized quantifier $\mathbf{Q}$ is a subset of $\mathcal{P}(\mathbb{N})$ such that

$$\begin{aligned}&\emptyset \notin \mathbf{Q} \\ &\mathbf{Q} \neq \emptyset \\ &X \subset Y \wedge X \in \mathbf{Q} \Rightarrow Y \in \mathbf{Q}.\end{aligned}$$

We abbreviate $\{x \mid \varphi(x)\} \in \mathbf{Q}$ by $\mathbf{Q}x.\varphi(x)$. We define the inverse quantifier $\overline{\mathbf{Q}}$ by $\overline{\mathbf{Q}} = \{\neg X \mid X \notin \mathbf{Q}\}$. We have that $\forall = \{\mathbb{N}\}$, $\exists = \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$, $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

The next quantifier or open game quantifier $\mathbf{Q}^\vee$ is defined by

$$\mathbf{Q}^\vee x.\varphi(x) \iff (\overline{\mathbf{Q}}x_0)(\overline{\mathbf{Q}}x_1)(\overline{\mathbf{Q}}x_2)(\overline{\mathbf{Q}}x_3)\cdots \bigvee_{n \in \omega} \varphi(\langle x_0, \ldots, x_n\rangle).$$

We will only consider the following generalized quantifiers:

$$\exists^0 = \exists; \forall^n = \overline{\exists^n}; \exists^{n+1} = (\exists^n)^\vee$$

These quantifiers are not definable in $\mathcal{L}_2$, but we define an adequate extension to $\mathcal{L}_2$ in which all the quantifiers $\exists^n$ and $\forall^n$ are definable. We define $\mathcal{L}_\Game$ by adding the quantifier symbols $\exists^n$ and $\forall^n$ for all $n \in \omega$. The $\mathcal{L}_\Game$-formulas are defined the same way $\mathcal{L}_2$-formulas are defined, with the additional rule that $\exists^n x.\varphi(x)$ and $\forall^n x.\varphi(x)$ are valid formulas if and only if φ has no second-order quantifiers (even if $\exists^n$ and $\forall^n$ occur in φ).

The theory $\Game$ame (with language $\mathcal{L}_\Game$) consists of the following axioms:

- the axioms of ACA_0, with comprehension for all $\mathcal{L}_\Game$-formulas without second-order quantifiers.
- $\exists^0 x.\varphi(x) \leftrightarrow \exists x.\varphi(x)$
- $\exists^{n+1} x.\varphi(x, \vec{y}, \vec{Y}) \leftrightarrow \forall X(\forall x.(\varphi^{\exists^n}(x, \vec{y}, X, \vec{Y}) \to x \in X) \to \langle\rangle \in X)$

- $\forall^n x.\varphi(x) \leftrightarrow \neg\exists^n x.\neg\varphi(x)$

where φ varies over formulas without second-order quantifiers.

Above, we have that φ^Q is an abbreviation for

$$Qx.s^\frown\langle x\rangle \notin X \rightarrow \varphi(s, \vec{y}, \vec{Y})$$

As φ^Q is X-positive, $\forall X(\forall x.(\varphi^{\exists^n}(x, \vec{y}, X, \vec{Y}) \rightarrow x \in X) \rightarrow \langle\rangle \in X)$ expresses that $\langle\rangle$ is in the least fixed-point operator defined by $\varphi^{\exists^n}(x, X)$.

Möllerfeld [Möl02] showed that the μ-arithmetic and ⅁ame prove the same $\mathcal{L}_2$ sentences. Heinatsch and Möllerfeld [HM10] used this result to show that the μ-arithmetic and $\forall n.(\Sigma^0_2)$-Det prove the same $\mathcal{L}_2$ sentences over ACA_0. In turn, this result was used by Kołodziejczyk and Michalewski to prove the following theorem:

Theorem 79 (Kołodziejczyk, Michalewski [KM16])**.** *The following are equivalent over* Π^1_2-CA_0*:*

- $\forall n.(\Sigma^0_2)$-Det*;*
- Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$*;*
- *the complementation theorem for non-deterministic tree automata;*
- *the decidability of the* Π^1_3 *fragment of monadic second-order arithmetic on the infinite binary tree; and*
- *the positional determinacy of parity games.*

Chapter 8

Determinacy of differences

In this chapter, we prove the following theorem.

Theorem 80. *Over* ACA_0*:*

1. ACA_0', $\forall n.(\Sigma^0_1)_n$-Det, Π^1_3-$\mathsf{Ref}(\Pi^1_1$-$\mathsf{CA}_0)$*, and* Π^1_1-CA_0' *are pairwise equivalent.*
2. $\forall n.(\Sigma^0_1)_n$-Det^*, Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$*, and* ACA_0' *are pairwise equivalent.*
3. $\forall n.(\Sigma^0_2)_n$-Det, Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$*, and* $\forall n.[\Sigma^1_1]^n$-ID *are pairwise equivalent.*

Proof. These results are Theorems 92, 95, and 96. □

Item (3) is an improvement of a theorem by Kołodziejczyk and Michalewski [KM16], who proved the equivalence of $\forall n.(\Sigma^0_2)_n$-Det and Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$ over Π^1_2-CA_0. The results of this section are joint work with Keita Yokoyama [PY22].

8.1 Folklore results on determinacy

We now prove a few folklore results. These results also appear on the preprint [PY22].

Differences of Σ^0_1 sets in Cantor space. Nemoto *et al.* [NMT07] proved that $(\Sigma^0_1)_n$-Det^* is equivalent to ACA_0, for all $n \in \omega$. We can adapt their proof to show that determinacy of arbitrary finite differences of open sets is equivalent to ACA_0'. This includes differences of non-standard many sets.

Proposition 81. *Over* ACA_0, $\forall n.(\Sigma^0_1)_n$-Det^* *implies* ACA_0'.

Proof. Fix $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$. We prove that the nth Turing jump of X exists. That is, we show the existence of a sequence $\langle X_0, \ldots, X_n\rangle$ of sets such that $X_0 = X$ and $X_{i+1} = \mathrm{TJ}(X_i)$, for $i < n$. Remember, the Turing Jump $\mathrm{TJ}(X)$ of X is the set of m such that $\neg\pi^0_1(m, X)$ holds, where π^0_1 is a fixed universal lightface Π^0_1-formula.

We consider the following two player game. I starts the game by playing $\langle m, i\rangle$, with $i \leq n$, to ask II whether $m \in X_i$. II then plays 1 to answer 'yes', or 0 to answer 'no'. If II answers 1, they must show that $m \in X_i$; if II answers 0, I must show that $m \in X_i$. Denote by V the player who is trying to show $m \in X_i$ and by R the other player. V stands for verifier; R stands for refuter.

Suppose $i > 0$. V wants to show $m \in X_i$. So they play a finite sequence X^f_{i-1} witnessing so. X^f_{i-1} is intended to be an initial segment of X_{i-1}. Then, R plays $m' \leq \mathrm{lh}(X^f_{i-1})$, to state that $X_{i-1}(m') \neq X^f_{i-1}(m')$. If $X^f_{i-1}(m') = 0$, R is stating that $m' \in X_{i-1}$, so V and R exchange roles. Otherwise, R is asking V to show that $m' \in X_{i-1}$, and the roles stay the same. Either way, V and R proceed to argue whether $m' \in X_{i-1}$.

Now suppose V and R are arguing whether $m' \in X_0$. V wins automatically iff $m' \in X_0$.

Before we consider the complexity of the payoffs of these games, let us see an example. We can think of a game as a dialogue between I and II:

I : Is $m \in X_3$?
II : Yes.
X^f_2 is an initial segment of X_2 witnessing $m \in X_3$.
I : $X^f_2(m') = 0$ is false.
X^f_1 is an initial segment of X_1 witnessing $m' \in X_2$.
II : $X^f_1(m'') = 1$ is false.
I : X^f_0 is an initial segment of X_0 witnessing $m'' \in X_1$.

Note that in this game I and II play both roles V and R.

We can formalize this dialogue as a Gale–Stewart game on the Baire space:

$$\begin{array}{lcccccc} \mathsf{I} & \langle m,3\rangle & & m' & X^f_1 & & X^f_0 \\ \mathsf{II} & & \text{yes} \;\; X^f_2 & & & m'' & \end{array}$$

Since we want plays in Cantor space, we cannot directly play natural numbers. Code a play of a natural number m by a play 0^m1 consisting of m zeroes and a one. Also code finite sequences of natural numbers as sequences of zeroes followed by a one. The players play 0s while the other player is playing some coded natural number or sequence. We therefore can write the payoffs of the game defined above above as a finite difference of open sets on Cantor space, uniformly in n. In particular, the example play above becomes:

$$\begin{array}{lcccc} \mathsf{I} & 0^{\langle m,3\rangle}1 & & 0^{m'}10^{X^f_1} & & 0^{X^f_0} \\ \mathsf{II} & & 0^1 10^{X^f_2}1 & & 0^{m''}1 & \end{array}$$

We can check that a play in the game is in the correct form using a difference of Σ^0_1-formulas. We can check that the witnesses given by V are correct with a Π^0_1-formula. Therefore the payoff of the game described above can be written as a finite difference of Σ^0_1-formulas, uniformly in n.[1]

[1] Given n, the payoff can be described by a difference of $5n$ many Σ^0_1-formulas. A better bound is possible, but not necessary.

I has no winning strategy. For a contradiction, suppose σ is a winning strategy for I. We play two copies of the game simultaneously. Have I play $\sigma(\langle\rangle)$ in both games. Now, II plays 'no' in the first game and 'yes' in the second game. I is going to play both games according to σ, and II copies the moves of I on the other game. As σ is a winning strategy for I, the run on the first game is winning for I. The the run on the second game is essentially the same run, with the roles of verifier and refuter exchanged. This implies II wins the second run. As I also used σ in the second run, I wins it. This is a contradiction.

By $\forall n.(\Sigma^0_1)_n$-Det*, there is a winning strategy τ for II. Let Z be the set $\{\langle m,i\rangle | \tau(\langle m,i\rangle) = 1\}$. By Σ^0_1-Induction, $(Z)_0 = X$ and $(Z)_{i+1} = \mathrm{TJ}((Z)_i)$ for $i < n$. □

Proposition 82. *Over* ACA_0*,* ACA'_0 *implies* $\forall n.(\Sigma^0_1)_n$-Det*.

Proof. This proof is a generalization of Theorem 3.7 of [NMT07]. □

Corollary 83. $\forall n.(\Sigma^0_1)_n$-Det* *is equivalent to a* Π^1_2 *sentence.*

Proof. By Propositions 81 and 82, $\forall n.(\Sigma^0_1)_n$-Det* is equivalent to ACA'_0 over ACA_0. Since ACA'_0 is a Π^1_2-sentence, we are done. □

DIFFERENCES OF Σ^0_2 SETS IN CANTOR SPACE. By a theorem by Nemoto *et al.* [NMT07], we do not need to explicitly consider differences of Σ^0_2 sets of Cantor space.

Theorem 84. *If* $1 < k \leq \omega$*, then* $(\Sigma^0_2)_k$-Det *and* $(\Sigma^0_2)_{k-1}$-Det* *are equivalent over* RCA_0.

We can adapt their proof to show:

Proposition 85. $\forall n.(\Sigma^0_2)_n$-Det *and* $\forall n.(\Sigma^0_2)_n$-Det* *are equivalent over* RCA_0.

Proof. Every game on Cantor space can be seen as a game on Baire space by adding a Π^0_1 condition: I and II play only zeroes and ones. Therefore a game in Cantor space which payoff is in $(\Sigma^0_2)_n$ is still a $(\Sigma^0_2)_n$ game in Baire space. So $\forall n.(\Sigma^0_2)_n$-Det implies $\forall n.(\Sigma^0_2)_n$-Det*.

By Lemma 4.2 of [NMT07], if a game on Baire space has payoff $A \in (\Sigma^0_2)_n$, then there is a game on Cantor space with payoff $A^* \in (\Sigma^0_2)_{n+2}$ such that I(II) has a winning strategy in A iff I(II) has a winning strategy in A^*. Therefore $\forall n.(\Sigma^0_2)_n$-Det* implies $\forall n.(\Sigma^0_2)_n$-Det. □

DIFFERENCES OF Σ^0_1 SETS IN BAIRE SPACE. Tanaka [Tan90] proved that $(\Sigma^0_1)_n$-Det is equivalent to Π^1_1-CA_0 over ATR_0, for all $n \in \omega$. Similar to the case of finite differences of open sets on Cantor space, where $\forall n.(\Sigma^0_1)_n$-Det* proves ACA'_0, we can use the determinacy of arbitrary finite differences of open sets in Baire space to prove Π^1_1-CA'_0.

Proposition 86. *Over* ACA_0*,* $\forall n.(\Sigma^0_1)_n$-Det *implies* Π^1_1-CA'_0.

Proof. First note that Π^1_1-CA_0 is equivalent to "for all $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, the nth iterated hyperjump $\mathrm{HJ}^n(X)$ of X exists". The proof of this equivalence is essentially the proof of Theorem VII.2.9 of [Sim09]. Fix a sequence of coded β-models $X_0 \in X_1 \in \cdots \in X_n$ with $X \in X_0$. We can define $\mathrm{HJ}(X)$ arithmetically using X_0 as a parameter, so $\mathrm{HJ}(X)$ exists. Since $X_0 \in X_1 \models \mathsf{ACA}_0$. Again, we can define $\mathrm{HJ}^2(X)$ arithmetically using X_0 and X_1 as parameters. Repeating this process boundedly many times, we can define $\mathrm{HJ}^{n+1}(X)$. On the other hand, suppose the hyperjumps $\mathrm{HJ}(X), \ldots, \mathrm{HJ}^n(X)$ exist. From $\mathrm{HJ}(X)$ we can define a β-model $X_0 \ni X$. By bounded induction, given X_i and $\mathrm{HJ}^{i+1}(X)$, we can define X_{i+1} with $X_i \in X_{i+1}$ and $\mathrm{HJ}^i(X) \in X_{i+1}$. Therefore there is a sequence $X_0 \in \cdots \in X_n$ with $X \in X_0$.

Now, given $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$, we prove the existence of the sequence $\langle \mathrm{HJ}(X), \ldots, \mathrm{HJ}^n(X)\rangle$ using $\forall n.(\Sigma^0_1)_n$-Det. The games we consider are similar to the ones in the proof of Proposition 81. I starts by playing $\langle m, i\rangle$ with $i \leq n$, asking whether $m \in \mathrm{HJ}^i(X)$. II plays 1 to answer 'yes' and 0 to answer 'no'. If II play 1, then they play the role of V (Verifier), otherwise they play the role of R (Refuter).

If $i > 0$, V must now play the characteristic function of $\mathrm{HJ}^{i-1}(X)$ and a function f witnessing that $m \in \mathrm{HJ}^i(X)$. While V is playing, R may contest V's choice of some $\xi_{\mathrm{HJ}^{i-1}(X)}(m')$. If V stated that $m' \notin \mathrm{HJ}^{i-1}(X)$, then the players exchange roles; otherwise the roles stay the same. The players then proceed to discuss whether $m' \in \mathrm{HJ}^{i-1}(X)$. In case R never contests, V wins iff $\pi^0_1(m, f, \mathrm{HJ}^{i-1}(X))$ holds. If $i = 0$, V wins iff $m \in X$.

This game can be described by a boolean combination of Π^0_1-formulas. So it is determined, $\forall n.(\Sigma^0_1)_n$-Det. As in Proposition 81, I cannot have a winning strategy. Let τ be a winning strategy for II. Define $Z = \{\langle m, i\rangle | \tau(\langle m, i\rangle) = 1\}$. Using bounded induction on i, we can show that $(Z)_0 = X$ and $(Z)_{i+1} = \mathrm{HJ}(\mathrm{HJ}^i(X))$ for $i < n$. We conclude that Π^1_1-CA_0' holds. $\square$

The reverse of Proposition 86 will follow by Theorem 92.

DIFFERENCES OF Σ^0_2 SETS IN BAIRE SPACE. MedSalem and Tanaka [MT07] proved that $(\Sigma^0_2)_k$-Det and $[\Sigma^1_1]^k$-ID are equivalent over ATR_0, for $0 < k < \omega$.

Proposition 87. *Over* ATR_0, $\forall n.(\Sigma^0_2)_n$-Det *implies* $\forall n.[\Sigma^1_1]^n$-ID.

Proof. Fix $k \in \mathbb{N}$ and $X \subseteq \mathbb{N}$. Suppose $\forall n.(\Sigma^0_2)_n$-Det is true. Let $\langle \Gamma_0, \ldots, \Gamma_{k-1}\rangle$ be a sequence of (indices of) Σ^1_1-inductive operators. We show that the set V_k inductively defined by $\langle \Gamma_0, \ldots, \Gamma_{k-1}\rangle$ exists.

MedSalem and Tanaka [MT08] prove, for all $n \in \omega$, that $[\Sigma^1_1]^n$-ID follows from $(\Sigma^0_2)_n$-Det using ATR_0 and induction on n. We sketch how to unfold their proof to show that $[\Sigma^{1,X}_1]^k$-ID follows from $(\Sigma^{0,X}_2)_{k^3}$-Det. Here, $\Sigma^{i,X}_j$ denotes the set of Σ^i_j formulas whose only set parameter is X. Let V_i be the set inductively defined by $\langle \Gamma_0, \ldots, \Gamma_{i-1}\rangle$, for $i = 1, \ldots k$. To show the existence of the set V_k , we use the set V_{k-1} and a $(\Sigma^0_2)_k$ game. Unfolding the definitions of V_{k-1}, we can show the existence of V_k using V_{k-2} and a $(\Sigma^0_2)_{k-1} \wedge (\Sigma^0_2)_k$ game. Repeatedly unfolding the V_i, we an prove the existence of V_k using a $\Sigma^0_2 \wedge (\Sigma^0_2)_2 \wedge \cdots \wedge (\Sigma^0_2)_k$ game. Furthermore, we can show that the payoff of this game is $(\Sigma^0_2)_{k^3}$.

We claim that there is a set $\tilde{\tau}$ computing winning strategies for all $(\Sigma^0_2)_{k^3}$ games with X as a parameter. Consider the following game: I chooses an index e for the payoff of a $(\Sigma^0_2)_{k^3}$ game; II answers with their choice of role in the game with index

e; then I and II play the game with index e and whoever wins the subgame wins the whole game. The payoff of this game has complexity $(\Sigma^0_2)_{2k^3}$. As we have $\forall n.(\Sigma^0_2)_n$-Det, the game above is determined. Since I cannot win, II has a winning strategy $\tilde{\tau}$; this $\tilde{\tau}$ computes winning strategies for all $(\Sigma^0_2)_{k^3}$ games.

Since we have $\forall n.(\Sigma^0_2)_n$-Det, we also have $(\Sigma^0_1)_2$-Det, and so Π^1_1-CA_0 is true. Let $\mathcal{M}$ be a β-model including X and $\tilde{\tau}$. Since $\tilde{\tau} \in \mathcal{M}$, every $(\Sigma^0_2)_{k^3}$ game with only X as a parameter is determined in $\mathcal{M}$. As $\mathcal{M} \models \mathsf{ATR}_0 + \Pi^1_\infty$-$\mathsf{Ind}$, we use the unfolded version of MedSalem and Tanaka's proof above to show that the set V_k inductively defined by $\langle \Gamma_0, \dots, \Gamma_{k-1} \rangle$ exists inside $\mathcal{M}$.

Furthermore, the statement "V_k is the set inductively defined by $\langle \Gamma_0, \dots, \Gamma_{k-1} \rangle$" is a boolean combination of Π^1_1-sentences with X, V_k as the only parameters. As $\mathcal{M}$ is a β-models, if V_k is the set inductively defined by $\langle \Gamma_0, \dots, \Gamma_{k-1} \rangle$ inside $\mathcal{M}$, then V_k is also the set inductively defined by $\langle \Gamma_0, \dots, \Gamma_{k-1} \rangle$ outside $\mathcal{M}$, as we wanted to show. Since the argument above holds for any $k \in \mathbb{N}$, $X \subseteq \mathbb{N}$ and $\langle \Gamma_0, \dots, \Gamma_{k-1} \rangle$, we have that $\forall n.[\Sigma^1_1]^n$-ID holds. □

The reverse of Proposition 86 will follow by Theorem 96.

8.2 Sequences of coded β-models

SOME DEFINITIONS. Given $e \in \omega$, define the formula $\psi_e(i, n)$ stating that there are sequences with length n of increasing coded β_i-models where the last model is a β_e-submodel of the ground model $\mathcal{N}$:

$$\begin{aligned} X \in Y_0 \in \; \cdots \in \; & Y_n, \\ Y_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} & Y_n \subseteq_{\beta_e} \mathcal{N}. \end{aligned}$$

Here, i stands for 'internal' and e for 'external'. Formally, for each $e \in \omega$, we define $\psi_e(i, n)$ by

$$\forall X \exists Y_0, \dots, Y_n. \begin{cases} X \in Y_0 \wedge \\ Y_k \in Y_{k+1} \wedge \\ Y_k \models \mathsf{ACA}_0 \wedge \\ Y_k \subseteq_{\beta_i} Y_{k+1} \wedge \\ Y_n \subseteq_{\beta_e} \mathcal{N}. \end{cases}$$

Each $\psi_e(i, n)$ is a Π^1_{e+2}-formula. For all $e \in \omega$, $\psi_e(i, n)$ is downwards closed: if $\psi_e(i, n)$ holds and $n' \leq n, i' \leq i, e' \leq e$ then $\psi_{e'}(i', n')$ also holds.

SEQUENCES OF CODED β-MODELS AND REFLECTION. We can now show the theorem:

Theorem 88. *If $e \leq i$ then $\forall n.\psi_e(i, n)$ is equivalent to Π^1_{e+2}-$\mathsf{Ref}(\text{Strong } \Sigma^1_i\text{-}\mathsf{DC}_0)$ over ACA_0.*

We divide the proof of Theorem 88 into two parts. We use reflection principles to show of the existence of sequences of coded models of arbitrary length. All the proofs from reflection principles in this thesis will follow the same template.

Lemma 89. *Over ACA_0, Π^1_{e+2}-$\mathsf{Ref}(\text{Strong } \Sigma^1_i\text{-}\mathsf{DC}_0)$ proves $\forall n.\psi_e(i, n)$ when $e \leq i$.*

Proof. In this proof we denote $\mathrm{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0$ by T_i. We have $\mathrm{Pr}_{T_i}(\ulcorner\psi_i(i,0)\urcorner)$ and $\mathrm{Pr}_{T_i}(\ulcorner\forall n.\psi_i(i,n) \to \psi_i(i,n+1)\urcorner)$. So $\mathrm{Pr}_{T_i}(\ulcorner\psi_i(i,n)\urcorner)$ holds for each $n \in \mathbb{N}$. By reflection, $\psi_i(i,n)$ holds for any $n \in \mathbb{N}$. Thus $\forall n.\psi_i(i,n)$ holds. As ψ is downwards closed, $\forall n.\psi_e(i,n)$ holds when $e \leq i$. $\square$

We now use sequences of coded models to prove reflection principles. This direction requires a bit of work. For a contradiction, we suppose the reflection fails. That is, there is $\theta(X) \in \Sigma^1_{e+1}$ such that $\mathrm{Pr}_{\mathrm{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0}(\ulcorner\forall X.\theta(X)\urcorner)$ holds but $\forall X.\theta(X)$ is false. We define an $\mathcal{L}_1$ theory T with a new predicate symbol A. The predicate A is intended to code a sequence of models. We show that $\forall n.\psi_e(i,n)$ implies T is consistent, and so T has a model M. A well-chosen submodel M_ω of M will satisfy both $\theta(X)$ and $\neg\theta(X)$, giving us a contradiction.

Lemma 90. *Over* $\Pi^1_{e+1}\text{-}\mathsf{CA}_0$, $\forall n.\psi_e(i,n)$ *proves* $\Pi^1_{e+2}\text{-}\mathsf{Ref}(\mathrm{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0)$ *when* $e \leq i$.

Proof. Assume that $\Pi^1_{e+2}\text{-}\mathsf{Ref}(\mathrm{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0)$ is false. That is, there is an $\mathcal{L}_2$-formula $\theta(X) \in \Sigma^1_{e+1}$ such that $\mathrm{Pr}_{\mathrm{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0}(\ulcorner\forall X.\theta(X)\urcorner)$ holds but $\forall X.\theta(X)$ is false. Therefore, there is X_0 such that $\neg\theta(X_0)$ holds.

Let $\mathcal{L}'_1$ be the language $\mathcal{L}_1$ of first-order arithmetic plus a unary predicate symbol A. Denote the domain of an $\mathcal{L}'_1$ structure by M. In the definition below, $\mathcal{M}_j$ is $(M, \{(A_j)_i \mid i \in M\})$. Define the $\mathcal{L}'_1$-theory T by the following axioms:

1. M is a discrete ordered semiring;
2. $\mathcal{M}_j \models \mathsf{ACA}_0$ for all $j \in \mathbb{N}$;
3. $(\mathcal{M}_{j+1})_0 = \mathcal{M}_j$, formally:
$$\forall m.m \in M_j \leftrightarrow (M_{j+1})_0;$$
4. $\mathcal{M}_j \subseteq_{\beta_i} \mathcal{M}_{j+1}$ for all $j \in \mathbb{N}$:
$$\forall e_0 \forall s.(\exists m.m \in (M_j)_s) \to (\mathcal{M}_j \models \pi^1_i(e_0,(M_j)_s) \leftrightarrow \mathcal{M}_{j+1} \models \pi^1_i(e_0,(M_j)_s)),$$
where π^1_i is a universal lightface Π^1_i-formula;
5. $\mathcal{M}_j$ satisfies $\neg\theta(X_0)$; and
6. $X_0 = (A_0)_0$.

Fix a finite subtheory T' of T. Let j_0 be the greatest index j of a coded model $\mathcal{M}_j$ occurring in formulas of T'. $\psi_e(i,j_0)$ implies that there is a sequence

$$X_0 \in Y_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} Y_{j_0} \subseteq_{\beta_e} \mathcal{N}$$

of j_0 many coded models. Setting $A_j = Y_j$ for $j \leq j_0$ and $A_j = \emptyset$ for $j > j_0$, we have a model witnessing the consistency of T'. By compactness, T is also consistent, so there is a model $\mathcal{M} = (M, A)$ of T.

Now consider the model $\mathcal{M}_\omega = (M, \{(A_j)_i \mid i \in M, j \in \mathbb{N}\})$. $\mathcal{M}_\omega$ satisfies $\mathrm{Strong}\ \Sigma^1_i\text{-}\mathsf{DC}_0$ as it is closed under taking β_i models: if $Y \in \mathcal{M}_\omega$ then Y is in some β_i-model $\mathcal{M}_j$ which is also in $\mathcal{M}_\omega$. So $\mathcal{M}_\omega$ is a model of $\theta(X_0)$. On the other hand, each $\mathcal{M}_j$ is a β_i-submodel of $\mathcal{M}_\omega$ since $\mathcal{M}_j \subseteq_{\beta_i} \mathcal{M}_{j+1}$ for all $j \in \mathbb{N}$. As $\neg\theta(X_0)$ is

Π^1_{e+1}, it can be written as $\forall Y \theta'(X_0, Y)$ with $\theta' \in \Sigma^1_e$. Let $Y_0 \in \mathcal{M}_\omega$, then there is $j \in \mathbb{N}$ such that $Y_0 \in \mathcal{M}_j$. Since $\neg\theta(X_0)$ holds in $\mathcal{M}_j$, $\theta'(X_0, Y_0)$ also holds in $\mathcal{M}_j$. As $e \leq i$, $\theta' \in \Sigma^1_e$ and $\mathcal{M}_j \subset_{\beta_i} \mathcal{M}_\omega$, we have that $\theta'(X_0, Y_0)$ also holds in $\mathcal{M}_\omega$. Since the argument above holds for arbitrary $Y_0 \in \mathcal{M}_\omega$, we have that $\forall Y \theta'(X_0, Y)$ holds in $\mathcal{M}_\omega$. That is, $\neg\theta(X_0)$ holds in $\mathcal{M}_\omega$. Therefore both $\forall X.\theta(X)$ and $\exists X.\neg\theta(X)$ hold in $\mathcal{M}$, a contradiction. $\square$

The following two particular cases of Theorem 88 will be used later:

Corollary 91. *Over* ACA_0,

1. $\forall n.\psi_1(1, n)$ *is equivalent to* $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0)$*; and*

2. $\forall n.\psi_1(2, n)$ *is equivalent to* $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_2\text{-}\mathsf{CA}_0)$.

Proof. For (1), set $e = 1$ and $i = 1$. For (2), set $e = 1$ and $i = 2$. $\square$

DETERMINACY AND REFLECTION FOR Π^1_1-CA_0. We can get a theorem from the corollary above:

Theorem 92. $\forall n.(\Sigma^0_1)_n$-Det, $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0)$, *and* $\Pi^1_1\text{-}\mathsf{CA}'_0$ *are pairwise equivalent over* ACA_0.

Proof. By Proposition 86, $\forall n.(\Sigma^0_1)_n$-Det implies $\forall n.\psi_1(1, n)$. Also note that $\Pi^1_1\text{-}\mathsf{CA}'_0$ and $\forall n.\psi_1(1, n)$ are equivalent. Furthermore, $\Pi^1_1\text{-}\mathsf{CA}_0$ proves Σ^0_1-Det and $(\Sigma^0_1)_n$-$\mathsf{Det} \to (\Sigma^0_1)_{n+1}$-$\mathsf{Det}$ for all $n \in \omega$. So $\Pi^1_3\text{-}\mathsf{Ref}(\Pi^1_1\text{-}\mathsf{CA}_0)$ implies $\forall n.(\Sigma^0_1)_n$-Det. $\square$

A QUESTION. During the RIMS 2021 Proof Theory Workshop, Toshiyasu Arai asked the following question:

Question 8. *How do we characterize the existence of sequences of coded models of transfinite length.*

8.3 The Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$ case

SEQUENCES OF CODED ω-MODELS. We modify the ψ_e to get a similar result for ACA_0. Let $\psi'(n)$ be defined by

$$\forall X \exists Y_0, \ldots, Y_n. \begin{cases} X \in Y_0 \wedge \\ (Y_k)' \in Y_{k+1} \wedge \\ Y_k \models \mathsf{RCA}_0. \end{cases}$$

Lemma 93. *Over* ACA_0, ACA'_0 *is equivalent to* $\forall n.\psi'(n)$.

Proof sketch. Fix X. First suppose ACA'_0. Given $n \in \mathbb{N}$, we can compute the first n jumps of X. For $k \leq n$, let Y_k be the collection of sets computable from $\mathrm{TJ}^n(X)$. Then $Y_0, \ldots, Y_n$ satisfy $\psi(n)$. Now, suppose $\forall n.\psi'(n)$ holds. We can extract $\mathrm{TJ}^n(X)$ from Y_n if $Y_0, \ldots, Y_n$ witness $\psi'(n)$. $\square$

Similar to Theorem 88, we have:

Theorem 94. *Over* ACA_0, $\forall n.\psi'(n)$ *is equivalent to* Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$.

Proof. As ACA_0 proves that the Turing jump of any set exists, Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$ proves ACA_0' and $\forall n.\psi'(n)$.

Now, let $\theta(X) \in \Sigma^1_1$ be an $\mathcal{L}_2$-formula such that $\mathrm{Pr}_{\mathsf{ACA}_0}(\ulcorner \forall X.\theta(X)\urcorner)$ and there is X_0 such that $\neg\theta(X_0)$ holds. Let the language $\mathcal{L}_1'$ be as in the proof of Lemma 90 and define an $\mathcal{L}_1'$-theory T by:

1. M is a discrete ordered semiring;
2. $\mathcal{M}_j \models \mathsf{RCA}_0$ for all $j \in \mathbb{N}$;
3. $\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$ for all $j \in \mathbb{N}$;
4. $(\mathcal{M}_j)' \in \mathcal{M}_{j+1}$ for all $j \in \mathbb{N}$;
5. $\mathcal{M}_j \models \neg\theta(X_0)$ for all $j \in \mathbb{N}$; and
6. $X_0 = (A_0)_0$

Again, $\mathcal{M}_j$ is $(M, \{(A_j)_i \mid i \in M\})$.

Now, $\forall n.\psi'(n)$ supplies a model for any finite subtheory of T. By compactness, there is a model $\mathcal{M} = (M, A)$ of T. Define the coded model $\mathcal{M}_\omega$ by $(M, \{(A_j)_i \mid i \in M, j \in \mathbb{N}\})$. Since $\mathcal{M}_\omega$ is closed under Turing jumps, it is a model of ACA_0, and thus $\mathcal{M}_\omega \models \theta(X_0)$. But by the definition of T, $\mathcal{M}_\omega \models \neg\theta(X_0)$. Therefore if $\mathrm{Pr}_{\mathsf{ACA}_0}(\ulcorner \forall X.\theta(X)\urcorner)$ holds, so must $\forall X.\theta(X)$ do. □

DETERMINACY AND REFLECTION FOR ACA_0. We can then use Proposition 81 and Corollary 83 to show:

Theorem 95. $\forall n.(\Sigma^0_1)_n$-Det^*, Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$, *and* ACA_0' *are pairwise equivalent over* ACA_0.

Proof. By Proposition 81, $\forall n.(\Sigma^0_1)_n$-Det^* and ACA_0'. Now, ACA_0' and $\forall n.\psi'(n)$ are the same statement. So ACA_0' and Π^1_2-$\mathsf{Ref}(\mathsf{ACA}_0)$ are equivalent by Theorem 94. □

8.4 Determinacy and reflection for Π^1_2-CA_0

We now give a new proof of:

Theorem 96. $\forall n.(\Sigma^0_2)_n$-Det, Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$, *and* $\forall n.[\Sigma^1_1]^n$-ID *are pairwise equivalent over* ACA_0.

Proof. By Lemma 88 above and Lemmas 97 and 98 below. □

FROM REFLECTION TO DETERMINACY. As above, using reflection principles to prove $\forall n.(\Sigma^0_2)_n$-Det is a straight proof:

Lemma 97. *Over* ACA_0, Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$ *proves* $\forall n.(\Sigma^0_2)_n$-Det.

Proof. Π^1_2-CA_0 proves Σ^0_2-Det and $(\Sigma^0_2)_n$-$\mathsf{Det} \to (\Sigma^0_2)_{n+1}$-$\mathsf{Det}$, for all $n \in \omega$. Formalizing these proofs inside ACA_0, we have that $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\ulcorner\Sigma^0_2\text{-}\mathsf{Det}\urcorner)$ and $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\ulcorner(\Sigma^0_2)_n\text{-}\mathsf{Det} \to (\Sigma^0_2)_n\text{-}\mathsf{Det}\urcorner)$. By Σ^0_1-induction, we have $\forall n.\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\ulcorner(\Sigma^0_2)_n\text{-}\mathsf{Det}\urcorner)$. In particular, for any $n \in \mathbb{N}$, $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\ulcorner(\Sigma^0_2)_n\text{-}\mathsf{Det}\urcorner)$. So Π^1_3-$\mathsf{Ref}(\Pi^1_2$-$\mathsf{CA}_0)$ implies $(\Sigma^0_2)_n$-Det. □

WARM-UP: $[\Sigma^1_1]^3$-LFP IMPLIES $\psi_1(2,2)$. We now need to prove:

Lemma 98. *Over* ACA_0, $\forall n.[\Sigma^1_1]^n$-ID *proves* $\forall n.\psi_1(2,n)$.

As a warm up, we show that $[\Sigma^1_1]^3$-LFP implies $\psi_1(2,2)$, that is, we can use three Σ^1_1 operators $\Gamma_0, \Gamma_1, \Gamma_2$ to define coded models M_0, M_1 such that:

$$M_0 \subseteq_{\beta_2} M_1 \subseteq_\beta \mathcal{N}.$$

The full version of Lemma 98 is on the next section. As above, $\mathcal{N}$ is the fixed ground model.

A first *rough* idea of what Γ_0, Γ_1 and Γ_2 do is:

- Γ_0 makes M_0 and M_1 β-submodels of the ground model.
- Γ_1 makes M_0 a β_2-submodel of M_1.
- Γ_2 puts M_0 inside M_1 as an element.

If we can define all of these operators, we are done. The hardest operator to define correctly is Γ_2. In order to do so, we will define auxiliary sets of 'recipes' for making the sets in M_0 and M_1, and a copy M_0^c of M_0 at convenient stages. M_0^c is a technical artifice used to guarantee $M_0 \in M_1$.

We will have three kinds of recipes:

- Recipes for applications of comprehension will have the form $\langle \mathtt{comp}, e, \bar{j}\rangle$ where e is an index number and $\bar{j}$ are set parameters.
- Recipes where we copy an element of M_1 to M_0 will have the form $\langle \mathtt{subm}, e\rangle$ where e is the index of some set in M_1 such that $M_1 \models \forall Z.\theta((M_1)_e, Z)$, $M_0 \not\models \exists Y\forall Z\theta(Y,Z)$ and θ is some arithmetical formula.
- Recipes for putting M_0^c inside M_1 are of the form $\langle \mathtt{elem}, e\rangle$ where e is some index.

Each of comp, subm and elem are arbitrarily chosen pairwise different natural numbers. Each recipe will be the label for the set that it constructs, *i.e.*, the recipe ρ instructs us how to define the set $(M_i)_\rho$.

We now describe the recipes one application of each operator constructs:

- Γ_0: if $e \in \mathbb{N}$ and $\bar{s}$ is a finite sequence of elements of M_i, then $\langle \mathtt{comp}, e, \bar{s}\rangle$ is a recipe for M_i.
- Γ_1: if θ is an arithmetic formula with parameters Y, Z, $M_0 \not\models \exists Y\forall Z\theta(Y,Z)$ and e is the least such that $M_1 \models \forall Z\theta((M_1)_e, Z)$, then $\langle \mathtt{subm}, e\rangle$ is a recipe for M_0.

- Γ_2: if e is the least such $\exists i \in (M_0)_e$ and $\neg\exists i \in (M_0^c)_e$, then $\langle \texttt{elem}, e\rangle$ is a recipe for M_1.

These recipes will guarantee that M_0 and M_1 have the closures we want.

Now we describe how Γ_0 follows the recipes to create our sets:

- If $\rho = \langle \texttt{comp}, e, \bar{j}\rangle$ is a recipe for M_i, then $n \in (M_i)_\rho$ iff $\varphi(e, n, \overline{(M_i)_s}, X)$, where φ is a universal lightface Σ^1_1-formula.
- If $\rho = \langle \texttt{subm}, e\rangle$ is a recipe for M_0, then $n \in (M_0)_\rho$ iff $n \in (M_1)_e$.
- If $\rho = \langle \texttt{elem}, e\rangle$ is a recipe for M_1, then $n \in (M_1)_\rho$ iff $n \in M_0^c$.

We will also require that the set made by each recipe is made only once. Note that Γ_0 at the same time creates and follows recipes. Γ_0 is a Σ^1_1-operator.

Γ_1 is a Σ^1_1-operator which adds new recipes for copying members of M_1 into M_0. At last, Γ_2 is a Σ^1_1-operator which copies the current M_0 into the candidate M_0^c and creates a recipe for copying the new M_0^c into M_1. This only adds elements to the old copy, so this is not problematic.

Let $X = (M_0, M_0^r, M_0^c, M_1, M_1^r)$, then:

- if X is a fixed-point of Γ_0, then $M_0 \subseteq_\beta M_1 \subseteq_\beta \mathcal{N}$;
- if X a fixed-point of Γ_1, then $M_0 \subseteq_{\beta_2} M_1$; and
- if X is a fixed-point of Γ_2, then $M_0 \in M_1$.

$\forall n.[\Sigma^1_1]^n$-LFP IMPLIES $\forall n.\psi_1(2, n)$. In this section we show that $\forall n.[\Sigma^1_1]^n$-LFP implies $\forall n.\psi_1(2, n)$. Fix $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ such that $n \geq 1$. We define a sequence of sets

$$\begin{aligned} A \in Y_0 \ \in \ \cdots \ \in \ \ Y_n, \\ Y_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} Y_n \subseteq_{\beta_e} \mathcal{N}. \end{aligned}$$

using $2n - 1$ Σ^1_1-inductive operators $\Gamma_0, \ldots, \Gamma_{2n-2}$.

The Γ_i will play roles similar to Γ_0, Γ_1 and Γ_2 above. Γ_0 will guarantee all M_i is a β-submodel of the ground model (and that all recipes are made). The Γ_{2i+1} will guarantee that M_i is a β_2-submodel of M_{i+1}. The Γ_{2i+2} will guarantee that M_i is an element of M_{i+1}.

Write `comp`, `subm` and `elem` for 0, 1 and 2, respectively. A tuple ρ of natural numbers is a recipe iff there is a natural number n and a natural number j such that

- $\rho = \langle \texttt{comp}, e, j\rangle$; or
- $\rho = \langle \texttt{subm}, e\rangle$; or
- $\rho = \langle \texttt{elem}, e\rangle$.

`comp` recipes are for closure under Π^1_1-comprehension, `subm` recipes are for making each model into a β_2-submodel of the next model, and include recipes are for putting a copy of each model into the next model. From now on write M_i for $3n$, M_i^r for $3n + 1$ and M_i^c for $3n + 2$. Given a set X, we write M_i^X for $(X)_{M_i}$.

The operator Γ_0 creates the recipes of the form $\langle \texttt{comp}, e, j\rangle$, with j being the index of some non-empty set in the respective model. Γ_0 also makes all the not-yet-made recipes. As Γ_0 both creates and makes the recipes for closure under Π^1_1-comprehension, we can show that the models defined by a fixed-point of Γ_0 are coded β-models.

Formally, Γ_0 is the Σ^1_1-inductive operator defined by:

$$\begin{aligned} x \in \Gamma_0(X) \iff & [x = \langle M^r_i, \langle \texttt{comp}, e, s\rangle\rangle \wedge \forall j < \mathrm{lh}(s) \exists m.m \in (M^X_i)_{s_j}] \vee \\ & [x = \langle M_i, \langle\langle \texttt{comp}, e, s\rangle, m\rangle\rangle \wedge \pi^1_1(e, m, (M^X_i)_s, A) \wedge \langle \texttt{comp}, e, s\rangle \in M^{r,X}_i] \vee \\ & [x = \langle M_i, \langle\langle \texttt{subm}, e\rangle, m\rangle\rangle \wedge \langle e, m\rangle \in M^X_{i+1} \wedge \langle \texttt{subm}, e\rangle \in M^{r,X}_i] \vee \\ & [x = \langle M_{i+1}, \langle\langle \texttt{elem}, e\rangle, m\rangle\rangle \wedge \langle M^c_i, m\rangle \in X \wedge \langle \texttt{elem}, e\rangle \in M^{r,X}_{i+1} \\ & \wedge \neg \exists m.m \in (M^X_{i+1})_{\langle \texttt{elem}, e\rangle}], \end{aligned}$$

where π^1_1 is a universal lightface Π^1_1-formula.

Lemma 99. *If X is a fixed-point of Γ_0 and $i \in \mathbb{N}$, then M^X_i is a coded β-model and $A \in M^X_i$.*

Proof. Suppose that X is a fixed-point of Γ_0. Fix $i \in \mathbb{N}$ and $A \in M^X_i$. The hyperjump of A is $\mathrm{HJ}(A) = \{\langle n, e\rangle \mid \exists f.\pi^0_1(e, n, f, X)\}$, which is Π^1_1 relative to A (π^0_1 is a universal lightface Π^0_1-formula). So we can define a recipe ρ for $\mathrm{HJ}(A)$. Therefore $\rho \in M^{r,\Gamma_0(X)}_i$ and $\mathrm{HJ}(A) \in M^{\Gamma_0(\Gamma_0(X))}_i$. But $\Gamma_0(\Gamma_0(X)) = X$, so $\mathrm{HJ}(A) \in M^X_i$. Therefore M^X_i is closed under hyperjumps, and thus is a coded β-model.

As $\{a \mid a \in A\}$ is Π^1_1 with parameter A, we can similarly show that $A \in M^X_i$. $\square$

Γ_{2i+1} creates recipes to copy sets to M^X_i from M^X_{i+1}, so that the former can become a β_2-submodel of the latter after one application of Γ_0. If $M^X_i \subseteq_{\beta_2} M^X_{i+1}$, Γ_{2i+1} does nothing.

Formally, Γ_{2i+1} is the Σ^1_1-inductive operator defined below:

$$\begin{aligned} x \in \Gamma_{2i+1}(X) \iff & x = \langle M^r_i, \langle \texttt{subm}, e\rangle\rangle \\ & \exists e_0 \exists s[e = \mu e.M^X_{i+1} \models \pi^1_1(e_0, (M_{i+1})^X_e, (M_i)^X_s) \\ & \wedge M^X_i \not\models \exists Y \pi^1_1(e_0, Y, (M_i)^X_s)], \end{aligned}$$

where π^1_1 is a universal Π^1_1-formula.

Lemma 100. *Let $i \in \mathbb{N}$. If X is a fixed-point of Γ_{2i+1}, then $M^X_i \subseteq_{\beta_2} M^X_{i+1}$.*

Proof. Let X be a fixed-point of Γ_{2i+1} and φ be a Π^1_2 sentence with parameters in M^X_i. If $M^X_{i+1} \models \varphi$ then $M^X_i \models \varphi$, as otherwise X would not be a fixed-point of Γ_{2i+1}. $\square$

Γ_{2i+2} checks if there is any difference between M^X_i and $M^{c,X}_i$, and adds a new recipe for M_i if that is the case. Γ_{2i+2} simultaneously copies M^X_i over $M^{c,X}_i$. After one application of Γ_{2i+2} and one of Γ_0, we get that M^X_i is an element of $M^{(\Gamma_0(\Gamma_{2i+2}(X)))}_{i+1}$.

Formally, Γ_{2i+2} is the Σ^1_1-inductive operator defined below:

$$\begin{aligned} x \in \Gamma_{2i+2}(X) \iff & [x = \langle M^r_i, \langle \mathtt{elem}, e\rangle\rangle \\ & \wedge \exists m. \langle e, m\rangle \in M^X_i \wedge \langle e, m\rangle \notin M^{c,X}_i \\ & \wedge \forall e' < e \forall m. \langle e', m\rangle \in M^X_{i+1} \leftrightarrow \langle e', m\rangle \in M^{c,X}_{i+1}] \vee \\ & [x = \langle M^c_i, m\rangle \wedge \langle M_i, m\rangle \in X]. \end{aligned}$$

Lemma 101. *For any fixed-point X of Γ_0, $M^X_i \in M^{(\Gamma_0(\Gamma_{2i+2}(X)))}_{i+1}$.*

Proof. Fix X and $i \in \mathbb{N}$. Then either $M^X_i = M^{c,X}_i$ or $M^X_i \neq M^{c,X}_i$. If $M^X_i = M^{c,X}_i$, there is a recipe $\langle include, e\rangle \in M^{r,X}_{i+1}$ and as X is a fixed-point of Γ_0, $M^{c,X}_i \in M^X_{i+1}$. If $M^X_i \neq M^{c,X}_i$, then there is e such that $\langle \mathtt{elem}, e\rangle \in M^{r,\Gamma_{2i+2}(X)}_{i+1} \setminus M^{r,X}_{i+1}$. We also have $M^X_i = M^{c,\Gamma_{2i+2}(X)}_i$. Therefore $M^X_i \in M^{\Gamma_0(\Gamma_{2i+2}(X))}_{i+1}$. □

Proof of Lemma 98. Fix $k \geq 1$. Suppose $\forall n.[\Sigma^1_1]^n$-ID holds. In particular, $[\Sigma^1_1]^{2k-1}$-LFP holds. Let X be a simultaneous fixed-point of the operators $\Gamma_0, \ldots, \Gamma_{2k-2}$ defined above. By Lemmas 99, 100 and 101,

$$\begin{aligned} A \in M^X_0 \ \in \ \cdots \ \in \ & M^X_n, \\ M^X_0 \subseteq_{\beta_i} \cdots \subseteq_{\beta_i} & M^X_n \subseteq_{\beta_e} \mathcal{N}. \end{aligned}$$

□

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