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Pattern Formation in 2D Stochastic Anisotropic Swift–Hohenberg Equation

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In this paper, we study a phenomenological model for pattern formation in electroconvection, and the effect of noise on the pattern. As such model we consider an anistropic Swift–Hohenberg equation adding an additive noise. We prove the existence of a global solution of that equation on the two dimensional torus. In addition, inserting a scaling parameter, we consider the equation on a large domain near its change of stability. We observe numerically that, under the appropriate scaling, its solutions can be approximated by a periodic wave, which is modulated by the solutions to a stochastic Ginzburg–Landau equation.

KEYWORDS: pattern formation, stochastic partial diferential equation, global existence

1. Introduction

The Swift–Hohenberg equation is a celebrated toy model for the convective instability in the Rayleigh–Bénard convection [13]. This equation has played an important role not only in the model of pattern formation in thermal convection, but also in the study of different fields including electroconvection, economics, biology, sociology, optics, etc. (see [9]).

The one-dimensional Swift–Hohenberg equation is given by

$$\partial_t u = -(1 + \partial_x^2)^2 u + \alpha u - u^3, \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is called the stress parameter. The linear part is clearly analyzed using Fourier transform. The ansatz

$$u(t, x) = e^{\lambda(k)t + ikx},$$

where $k \in \mathbb{R}$ is the wave number, yields $\lambda(k) = -(1 - k^2)^2 + \alpha$. If $\alpha > 0$, then unstable modes around $k = \pm 1$ exist and thus, in this case the convection and the pattern formation occur. Now let $\alpha = \varepsilon^2 > 0$. We expect that the solution can be described by the ansatz

$$u(t, x) = \varepsilon A(T, X)e^{ix} + c.c., \quad X = \varepsilon x, \quad T = \varepsilon^2 t,$$

where c.c. means the complex conjugate. Substituting it into the above equation, and comparing the coefficients of $\varepsilon^j e^{ikx} (j, k \in \mathbb{Z})$, we see that the so-called residual is minimized if $A(T, X)$ fulfills

$$\partial_T A = 4\partial_X^2 A + A - 3|A|^2 A. \tag{1.2}$$

Indeed, it is known that if $\varepsilon > 0$ is taken to be small enough, $u(t, x) - (A(T, X)e^{ix} + c.c.)$ becomes smaller in a suitable sense (see [14]).

We are interested in adding noise in this formulation. For the stochastic equation, it is shown in [2] that the solution u of the one dimensional stochastic Swift–Hohenberg equation;

$$\partial_t u = -(1 + \partial_x^2)^2 + \nu\varepsilon^2 u - u^3 + \varepsilon^{\frac{3}{2}}\dot{\xi}_\varepsilon, \quad t \geq 0, \tag{1.3}$$

where $\nu > 0$, and $\dot{\xi}_\varepsilon$ is the real valued space-time white noise, can be approximated by using the solution A of the stochastic Ginzburg–Landau equation:

$$\partial_T A = 4\partial_X^2 A + \nu A - 3A|A|^2 + \dot{\eta}, \quad X = \varepsilon x, \quad T = \varepsilon^2 t$$

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where $\dot{\eta}$ is a complex valued noise. The approximation is given by

$$u(t,x) \simeq \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + c.c.$$

This result is proved on the whole space $\mathbb{R}$, there is also a result by [3] on the one-dimensional torus. To our best knowledge, this approximation in the stochastic case is known only in one dimension, the results in more than two dimensions are not known. The main problem in the stochastic case in the dimension more than two is that the solution of the stochastic Ginzburg–Landau equation has a priori a negative regularity, thus we need to use a renormalization to make sense to the nonlinearity (see for example [10]), whereas the Swift–Hohenberg equation has good regularity and we do not need to use a renormalization. Rather, the linear part of Swift–Hohenberg equation can define the Wick products of Ornstein–Uhlenbeck process in a certain scale of ε. We would address this issue in the sequel paper. In the deterministic case, such approximation for more general forms of equation is already known in two dimensions, see for e.g. [11].

In this paper we thus consider a two-dimensional stochastic Swift–Hohenberg equation on the torus given by

$$\begin{cases} \partial_t u = -(1+\partial_x^2)^2 u + \partial_y^2 u + u - u^3 + \dot{\xi}, & (x,y) \in \mathbb{T}^2, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $\dot{\xi}$ is the real-valued space-time white noise. This equation is a phenomenological model for pattern formation in electroconvection, in the sense that the spectral surface is similar to the modeling of the electroconvection proposed in [12]. In this paper, as the first step, we prove the existence of a solution of this equation (1.4).

By the result of the approximation of the deterministic equation and the approximation result of [2], we can expect that the two-dimensional stochastic Swift–Hohenberg equation (1.4) also can be approximated by the two-dimensional complex stochastic Ginzburg–Landau equation. We may see by formal computations that the solutions of

$$\partial_t u = -(1+\partial_x^2)^2 u + \partial_y^2 u + \varepsilon^2 u - u^3 + \varepsilon \dot{\xi}_\varepsilon$$

defined on the domain $[-L/\varepsilon, L/\varepsilon] \times [-L/\varepsilon, L/\varepsilon]$ $(L > 0)$ with periodic boundary condition would be approximated by the solution A of

$$\partial_T A = 4\partial_X^2 A + \partial_Y^2 A + A - 3|A|^2 A + \dot{\eta},$$

on the domain $[-L, L] \times [-L, L]$, through the ansatz

$$u(t,x,y) = \varepsilon A(\varepsilon^2 t, \varepsilon x, \varepsilon y)e^{ix} + c.c.$$

We try to see whether this would be observed or not by numerical simulations.

The organization of this paper is as follows. In Sect. 2, we prepare the notation necessary for discussing the subsequent sections, and we state our main theorem. In Sect. 3, we investigate the regularity of the solution of the linear equation of the two-dimensional stochastic Swift–Hohenberg equation using the Kolmogorov test. Section 4 is dedicated to prove the existence of the solutions of equation (1.4) using the regularity of the solutions of the linear equation obtained in Sect. 3. We use the compactness method and obtain the solution as the limit of finite dimensional Galerkin approximation and its energy uniform estimates. Lastly, we will present the numerical simulations in Sect. 5.

2. Preliminaries and Main Results

2.1 Notation

In this section, we define the notation for our discussion.

(i) Our results will concern the periodic functions in $\mathbb{R}^2$ and for a fixed $L > 0$ we shall take the fundamental period in each variable to be $2L$. That is, a function f on $\mathbb{R}^2$ is said to be periodic if $f(\mathbf{x} + 2L\mathbf{k}) = f(\mathbf{x})$ for all $\mathbf{x} = (x,y) \in \mathbb{R}^2$ and $\mathbf{k} = (k,l) \in \mathbb{Z}^2$. For the analysis, a natural option would be to base the definition of the Sobolev spaces on discrete Fourier series, and those are adapted to the "torus," namely we regard the periodic functions as functions on the space $\mathbb{R}^2/((2L)\mathbb{Z})^2$ which we will call the torus and denote by $\mathbb{T}^2$. We identify $\mathbb{T}^2$ with the cube $[-L, L)^2$.

(ii) Let $\nu_2 = \frac{1}{(2L)^2} m_2$, where m_2 is two-dimensional Lebesgue measure. Then, by the identification above, ν_2 induces a measure on $\mathbb{T}^2$, but we denote it by the same ν_2 with an abuse of notation. For all $p \in [1,\infty]$, $L^p(\mathbb{T}^2)$ denotes thus $L^p([-L,L]^2)$ with this Lebesgue measure ν_2.

(iii) For measurable complex-valued functions $f, g \in L^2(\mathbb{T}^2)$, the $L^2(\mathbb{T}^2)$ inner-product is denoted by

$$(f,g) := \int_{\mathbb{T}^2} f(x)\overline{g(x)} d\nu_2 = \int_{[-L,L]^2} f(x)\overline{g(x)} d\nu_2.$$

(iv) For $\varepsilon \in [0,1]$, we set $\mathcal{L}_\varepsilon = -(1+\partial_x^2)^2 + \partial_y^2 + \varepsilon^2$ and $\lambda_{k,l,\varepsilon} = -\{(1-(\frac{\pi}{L})^2 k^2)^2 + (\frac{\pi}{L})^2 l^2 - \varepsilon^2\}$ for $k, l \in \mathbb{Z}$, which are eigenvalues of $\mathcal{L}_\varepsilon$. The dependence of the operator $\mathcal{L}_\varepsilon$ on ε is not essential for the existence of solutions, thus for the sake of simplicity we set $\varepsilon = 1$, but we use ε for the purpose of numerical simulations in Sect. 5.

(v) Let $\{e_{k,l}(x,y)\}_{k,l\in\mathbb{Z}}$ be the eigenfunctions corresponding to $\lambda_{k,l,0}$, which will simply be denoted by $\lambda_{k,l}$, i.e.,

$$\mathcal{L}_0 e_{k,l} = \lambda_{k,l} e_{k,l}, \quad e_{k,l}(x,y) = \frac{1}{2L} e^{\frac{-i\pi(k,l)}{L}\cdot(x,y)}$$

and which constitute a complete orthogonal basis in $L^2(\mathbb{T}^2)$.

(vi) For $s \in \mathbb{R}$ and $1 \le p < +\infty$, we denote by $\mathcal{W}^{s,p}(\mathbb{T}^2)$ the space of $f \in \mathcal{S}'$ satisfying

$$\|f\|_{\mathcal{W}^{s,p}(\mathbb{T}^2)} := \|(1-\mathcal{L}_0)^{\frac{s}{2}} f\|_{L^p(\mathbb{T}^2)}.$$

(vii) For $1 \le p < +\infty$, and $T > 0$, and a Banach space B with the norm $\|\cdot\|_B$, we denote by $L^p(0,T;B)$ the B-valued measurable function g on $[0,T]$ such that

$$\int_0^T \|g(t)\|_B^p dt < \infty.$$

For $\alpha \in (0,1)$, we denote by $W^{\alpha,p}(0,T;B)$ the subset of $L^p(0,T;B)$ function g such that

$$\|g\|_{W^{\alpha,p}(0,T;B)} := \left(\int_0^T \|g(t)\|_B^p dt + \int_0^T \int_0^T \frac{\|g(t)-g(s)\|_B^p}{|t-s|^{1+\alpha p}} dsdt\right)^{\frac{1}{p}} < \infty.$$

(viii) We denote by $C([0,T];B)$ the B-valued functions that are continuous on $[0,T]$. And for $\alpha > 0$, $C^\alpha([0,T];B)$ denotes the set of B-valued α-Hölder functions f such that

$$\sup_{t,s\in[0,T],t\neq s} \frac{\|f(t)-f(s)\|_B}{|t-s|^\alpha} < \infty.$$

(ix) If f and g are two quantities, we use $f \lesssim g$ to denote the statement that $f \le Cg$ for some constant $C > 0$. When this constant C depends on some parameters $a_1, \ldots\ldots, a_k$, we use $f \lesssim_{a_1,\ldots,a_k} g$ to enlighten this dependence on the parameters.

Let $\{\beta_{k,l}\}_{k,l\in\mathbb{Z}}$ be a series of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $(x,y) \in \mathbb{T}^2, t \in [0,T]$, a $L^2(\mathbb{T}^2)$ cylindrical Wiener process ξ is written by

$$\xi(t,x,y) = \sum_{k,l\in\mathbb{Z}} \beta_{k,l}(t) e_{k,l}(x,y).$$

We will see later in Sect. 5 that the ε-scaled Wiener process is defined as

$$\xi_\varepsilon(x,y,t) = \frac{1}{2L} \sum_{k,l\in\mathbb{Z}} \beta_{k,l}(\varepsilon^2 t) e^{\frac{-\varepsilon i\pi(k,l)}{L}\cdot(x,y)}.$$

Here we note the propositions which will be useful later.

Proposition 1 (Compact embedding 1)**.** *Let $B_0 \subset B \subset B_1$ be Banach spaces, B_0 and B_1 reflexive, with compact embedding of B_0 in B. Let $p \in (1,\infty)$ and $\alpha \in (0,1)$ be given. Let X be the space*

$$X = L^p(0,T;B_0) \cap W^{\alpha,p}(0,T;B_1).$$

Then the embedding of X in $L^p(0,T;B)$ is compact.

Proof. See Lemma 2.1 in [6]. □

Proposition 2 (Compact embedding 2)**.** *If $B_1 \subset \tilde{B}$ are two Banach spaces with compact embedding, and the real number $\alpha \in (0,1)$, $p > 1$ satisfy*

$$\alpha p > 1$$

then the space $W^{\alpha,p}(0,T;B_1)$ is compactly embedded into $C([0,T];\tilde{B})$.

Proof. See Theorem 2.2 in [6]. □

Proposition 3 (Gyöngy–Krylov criterion)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a complete separable metric space (E,d). Assume that, for every pair of subsequences $(n_1(k), n_2(k))$, with $n_1(k) \ge n_2(k)$ for every $k \in \mathbb{N}$, there is a subsequence $(k(h))_{h\in\mathbb{N}}$ such that the random variables $(X_{n_1(k(h))}, X_{n_2(k(h))})$ from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E \times E, d \times d)$ converge in law to a measure μ on $E \times E$ such that $\mu(D) = \mu(\{(x,y) \in E \times E; x = y\}) = 1$. Then there exists a random variable X from $(\Omega, \mathcal{F}, \mathbb{P})$ to (E,d) such that X_n converges to X in probability.*

Proof. See Lemma 9 in [5], and [7]. □

2.2 Main theorem

First of all we set $\varepsilon = 1$ and we establish the existence of a solution of the stochastic Swift–Hohenberg equation:

$$\begin{cases} \partial_t u = \mathcal{L}_1 u - u^3 + \dot{\xi}, & \text{on } [0,T] \times \mathbb{T}^2, \\ u(0) = u_0, & \text{on } \mathbb{T}^2. \end{cases} \tag{2.1}$$

To find a solution of (2.1), we use the decomposition $v = u - Z$ with Z satisfying

$$\begin{cases} \partial_t Z = \mathcal{L}_1 Z + \dot{\xi}, & \text{on } [0,T] \times \mathbb{T}^2, \\ Z(0) = 0, & \text{on } \mathbb{T}^2. \end{cases} \tag{2.2}$$

Then v satisfies the following equation formally.

$$\begin{cases} \partial_t v = \mathcal{L}_1 v - (v+Z)^3, & \text{on } [0,T] \times \mathbb{T}^2, \\ v(0) = u_0, & \text{on } \mathbb{T}^2. \end{cases} \tag{2.3}$$

This equation is a random PDE. We can thus solve the equation (2.3) as a deterministic PDE. As a result, we can get the solution of (2.1).

Our main results are as follows.

Theorem 1. *Let $T > 0$ be fixed. Let $p \geq 1$, $s \in [0, \frac{1}{8})$, and $\alpha \in (0, \frac{1}{8} - s)$. The solution Z of (2.2) has a modification in $C^\alpha([0,T]; \mathcal{W}^{s,p}(\mathbb{T}^2))$. Moreover, there exists a positive constant $M_{p,T,L}$ such that*

$$\mathbb{E}\left(\sup_{t \in [0,T]} \|Z(t)\|_{L^p(\mathbb{T}^2)} \right) \leq M_{p,T,L}.$$

Theorem 2. *Let $u_0 \in L^2(\mathbb{T}^2)$, and $\alpha > 0$. There exists a unique stochastic process v on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying (1.4) in the following weak sense, i.e., for $w \in \mathcal{W}^{1+\alpha,2}$, $t \in [0,T]$,*

$$(v_t, w) = (u_0, w) + \int_0^t (v_s, \mathcal{L}_1 w) ds - \int_0^t ((v_s + Z_s)^3, w) ds, \tag{2.4}$$

and v takes values in $L^\infty(0,T; L^2(\mathbb{T}^2)) \cap L^3(0,T; L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2})$ almost surely.

Theorem 1 is proved by using the Kolmogorov test, where convergence properties of the Gamma function are helpful in the computation. Theorem 2 is proved by using a Galerkin approximation as in [1]. Note that L^p energy estimates seem not available, thus we do not use the fixed point argument. First we consider a finite dimensional nonlinear equation. We get an energy estimate and properties of Z allow us to obtain a probabilistically uniform estimate with respect to the dimension. The Prohorov Theorem and the Skorohod Theorem imply the existence of a limit taking a subsequence on another probability space. Moreover, the Gyöngy–Krylov criterion can make the convergence on another probability space into the convergence on the original space regarding X_n as the subsequence converging to some probability measure weakly. This convergence constructs a solution of the infinite dimensional system.

3. Regularity of the Solution of the Stochastic Linear Equation

3.1 Regularity of Z

In this section, we investigate the regularity of Z by using the Kolmogorov test. We may write Z as a mild solution.

$$Z(t) = \int_0^t e^{(t-s)\mathcal{L}_1} d\xi(s) = \sum_{k,l} \int_0^t e^{(t-s)\lambda_{k,l,1}} e_{k,l}(x,y) d\beta_{k,l}(s). \tag{3.1}$$

Proposition 4. *Let $T > 0$ be fixed. Let $p \geq 1$, and $0 \leq \theta < \frac{1}{8}$. Then Z has a modification which is α-Hölder continuous on $[0,T]$ with values in $\mathcal{W}^{\theta,p}(\mathbb{T}^2)$ for $\alpha \in (0, \frac{1}{8} - \theta)$.*

Proof. Let $(x,y) \in \mathbb{T}^2$ and $\theta \geq 0$. For $t > s$, $t, s \in [0,T]$, we first calculate

$$\begin{aligned} &\mathbb{E}(|(1 - \mathcal{L}_0)^{\frac{\theta}{2}}(Z(t) - Z(s))|^2) \\ &= \mathbb{E} \sum_{k,l \in \mathbb{Z}} \left(\left(1 - \left(\frac{\pi}{L}k\right)^2\right)^2 + \left(\frac{\pi}{L}l\right)^2 + 1 \right)^\theta (Z(t,x,y) - Z(s,x,y), e_{k,l})^2 \\ &= \sum_{k,l} \mathbb{E} \left| \int_0^s \left(\left(1 - \left(\frac{\pi}{L}k\right)^2\right)^2 + \left(\frac{\pi}{L}l\right)^2 + 1 \right)^\theta (e^{(t-u)\lambda_{k,l,1}} - e^{(s-u)\lambda_{k,l,1}}) d\beta_{k,l}(u) \right|^2 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k,l}\mathbb{E}\left|\int_s^t\left(\left(1-\left(\frac{\pi}{L}k\right)^2\right)^2+\left(\frac{\pi}{L}l\right)^2+1\right)^{\theta}e^{(t-u)\lambda_{k,l,1}}d\beta_{k,l}(u)\right|^2\\
&\le\sum_{k,l}\int_0^s\left(\left(1-\left(\frac{\pi}{L}k\right)^2\right)^2+\left(\frac{\pi}{L}l\right)^2+1\right)^{2\theta}(e^{(t-u)\lambda_{k,l,1}}-e^{(s-u)\lambda_{k,l,1}})^2du\\
&+\sum_{k,l}\int_s^t\left(\left(1-\left(\frac{\pi}{L}k\right)^2\right)^2+\left(\frac{\pi}{L}l\right)^2+1\right)^{2\theta}e^{2(t-u)\lambda_{k,l,1}}du\\
&=I_1+I_2,
\end{aligned}
$$

where we have used the Itô isometry of the stochastic integral. First, we estimate I_2 dividing into $I_{2,\ge 0}$ and $I_{2,<0}$ with

$$
I_{2,\ge0}=\sum_{\substack{l\in\mathbb{Z},\\ k:1-k^2\ge0}}\int_s^t\left(\left(1-\left(\frac{\pi}{L}k\right)^2\right)^2+\left(\frac{\pi}{L}l\right)^2+1\right)^{2\theta}e^{2(t-u)\lambda_{k,l,1}}du,
$$

$$
I_{2,<0}=\sum_{\substack{l\in\mathbb{Z},\\ k:1-k^2<0}}\int_s^t\left(\left(1-\left(\frac{\pi}{L}k\right)^2\right)^2+\left(\frac{\pi}{L}l\right)^2+1\right)^{2\theta}e^{2(t-u)\lambda_{k,l,1}}du.
$$

Recall $\lambda_{k,l,1}=-\{(1-(\frac{\pi}{L}k)^2)^2+(\frac{\pi}{L}l)^2-1\}$. Note that if $k\in\mathbb{Z}$ satisfies $1-k^2\ge0$, then $k=0,\pm1$ thus $(1-(\frac{\pi}{L}k)^2)^2\le 1$. Therefore,

$$
\begin{aligned}
I_{2,\ge0}&\le\sum_{\substack{l\in\mathbb{Z},\\ k:1-k^2\ge0}}\int_s^t\left(\left(\frac{\pi}{L}l\right)^2+2\right)^{2\theta}e^{2(t-u)\lambda_{k,l,1}}du\\
&\le\frac{4L}{\pi}\int_s^t\int_0^{+\infty}\int_0^1(2+y^2)^{2\theta}e^{-(t-u)\{(1-(\frac{\pi}{L})^2x^2)^2+y^2-1\}}dxdydu\\
&\le\frac{4L}{\pi}\int_s^t\int_0^{+\infty}(2+y^2)^{2\theta}e^{-(t-u)(y^2-1)}dydu\\
&=\frac{4L}{\pi}\int_s^t e^{(t-u)}\int_0^{+\infty}(2+y^2)^{2\theta}e^{-(t-u)y^2}dydu
\end{aligned}
$$

Here, the use of the change of variable $(t-u)y^2=z\ge0$ allows us to estimate

$$
\begin{aligned}
\int_0^{+\infty}(2+y^2)^{2\theta}e^{-(t-u)y^2}dy&\lesssim_\theta\int_0^{+\infty}e^{-(t-u)y^2}dy+\int_0^{+\infty}y^{4\theta}e^{-(t-u)y^2}dy\\
&=C(\theta)(t-u)^{-\frac12}+C'(\theta)(t-u)^{-2\theta-\frac12}\Gamma\left(2\theta+\frac12\right)
\end{aligned}
$$

if $2\theta+\frac12>0$, where $\Gamma(\cdot)$ is the Gamma function, and

$$
\Gamma\left(2\theta+\frac12\right)=\int_0^{+\infty}z^{2\theta-\frac12}e^{-z}dz.
$$

Hence,

$$
I_{2,\ge0}\lesssim_{\theta,L}\int_s^t e^{(t-u)}((t-u)^{-\frac12}+(t-u)^{-2\theta-\frac12})du\lesssim_{\theta,T,L}(t-s)^{-2\theta+\frac12},
$$

if $\theta\in[0,\frac14)$. Next we estimate $I_{2,<0}$. The condition is $k^2>1$, but thanks to the symmetry, we first focus on the integral

$$
\int_{\frac{\pi}{L}}^{\infty}(1-x^2)^{4\theta}e^{-2(t-u)(1-x^2)^2}dx. \tag{3.2}
$$

Let $x^2-1=z$, and we get[1]

$$
(3.2)\le\int_1^{\infty}(1-x^2)^{4\theta}e^{-2(t-u)(1-x^2)^2}dx\lesssim_\theta\int_0^{\infty}z^{4\theta-\frac12}e^{-2(t-u)z^2}\frac{1}{2(z+1)^{\frac12}}dz.
$$

[1]considering two cases: $\frac{\pi}{L}<1$ and $x>1$, or $\frac{\pi}{L}\ge1$ or $x>1$. The former case is impossible in (3.2).

Moreover we change the variable $2(t-u)z^2 = w \geq 0$ which leads to the above RHS:

$$\begin{aligned}
&\lesssim_\theta \int_0^\infty \frac{w^{\frac{1}{2}(4\theta-\frac{1}{2})}}{(t-u)^{\frac{1}{2}(4\theta-\frac{1}{2})}} \frac{e^{-w}}{w^{\frac{1}{2}}(t-u)^{\frac{1}{2}}} dw \\
&\lesssim_\theta (t-u)^{-\frac{1}{4}-2\theta},
\end{aligned}$$

where we have again used the convergence of the Gamma function if $2\theta + \frac{1}{4} > 0$. Therefore, by symmetry,

$$\begin{aligned}
I_{2,<0} &= \sum_{\substack{l\in\mathbb{Z},\\ k:1-k^2<0}} \int_s^t \left(\left(1 - \left(\frac{\pi}{L}k\right)^2\right)^2 + \left(\frac{\pi}{L}l\right)^2 + 1 \right)^{2\theta} e^{2(t-u)\lambda_{k,l,1}} du \\
&\lesssim_L \int_s^t \int_0^{+\infty} \int_{\frac{\pi}{L}}^{+\infty} ((1-x^2)^2 + y^2 + 1)^{2\theta} e^{-2(t-u)\{(1-x^2)^2+y^2-1\}} dxdydu \\
&\lesssim_{\theta,L} \int_s^t \int_0^{+\infty} \int_{\frac{\pi}{L}}^{+\infty} (1-x^2)^{4\theta} e^{-2(t-u)\{(1-x^2)^2+y^2-1\}} dxdydu \\
&\quad + \int_s^t \int_0^{+\infty} \int_{\frac{\pi}{L}}^{+\infty} (y^{2\theta}+1) e^{-2(t-u)\{(1-x^2)^2+y^2-1\}} dxdydu \\
&\lesssim_{\theta,L} \int_s^t e^{2(t-u)} (t-u)^{-\frac{1}{4}-2\theta} \int_0^{+\infty} e^{-2(t-u)y^2} dydu \\
&\quad + \int_s^t e^{2(t-u)} \int_0^{+\infty} \int_1^{+\infty} (y^{2\theta}+1) e^{-2(t-u)\{(1-x^2)^2+y^2\}} dxdydu \\
&\lesssim_{\theta,T,L} \int_s^t (t-u)^{-\frac{3}{4}-2\theta} du + \int_s^t (t-u)^{-\frac{1}{4}} \{(t-u)^{-\frac{1}{2}} + (t-u)^{-\frac{1}{2}-\theta}\} du \lesssim_{T,\theta,L} (t-s)^{\frac{1}{4}-2\theta}
\end{aligned}$$

if $\theta < \frac{1}{8}$ which implies, $I_2 \lesssim_{\theta,T,L} (t-s)^{\frac{1}{4}-2\theta}$ if $\theta < \frac{1}{8}$. Now we estimate I_1. For $\gamma \in (0,1)$, a similar calculation as above yields, using the γ-Hölder regularity of the exp function,

$$\begin{aligned}
I_1 &\lesssim \int_0^s \int_{\mathbb{R}^2} (t-s)^{2\gamma} \left(\left(1 - \left(\frac{\pi}{L}x\right)^2\right)^2 + \left(\frac{\pi}{L}y\right)^2 + 1 \right)^{2\gamma+2\theta} e^{-(s-u)\{(1-(\frac{\pi}{L}x)^2)^2+(\frac{\pi}{L}y)^2-1\}} dxdydu \\
&\lesssim_{\gamma,\theta} (t-s)^{2\gamma} \int_0^s \int_{\mathbb{R}^2} (t-s)^{2\gamma} \left(\left(1 - \left(\frac{\pi}{L}x\right)^2\right)^{4(\gamma+\theta)} + \left(\frac{\pi}{L}y\right)^{4(\gamma+\theta)} + 1 \right) e^{-(s-u)\{(1-(\frac{\pi}{L}x)^2)^2+(\frac{\pi}{L}y)^2-1\}} dxdydu \\
&\lesssim_{\gamma,\theta,T,L} (t-s)^{2\gamma} \int_0^s (s-u)^{-\frac{3}{4}-2\gamma-2\theta} du.
\end{aligned}$$

The right hand side is finite if $\frac{1}{8} > \theta + \gamma$. Hence, for $m \in \mathbb{N}$, we obtain

$$\mathbb{E}(|(1-\mathcal{L}_0)^{\frac{\theta}{2}}(Z(t)-Z(s))|^{2m}) \leq C(m,\gamma,\theta,T)|t-s|^{\min(2\gamma,\frac{1}{4}-2\theta)\times 2m}.$$

Therefore, for $m \in \mathbb{N}$ and $1 \leq p \leq 2m$, by the Minkowski inequality,

$$\mathbb{E}(\|(1-\mathcal{L}_0)^{\frac{\theta}{2}}(Z(t)-Z(s))\|_{L^p(\mathbb{T}^2)}^{2m})^{\frac{1}{2m}} \leq C(m,\gamma,\theta,T)|t-s|^{\gamma}.$$

Set $\theta = 0$. We conclude by the Kolmogorov test that Z has a modification in $C^\alpha([0,T]; \mathcal{W}^{\theta,p}(\mathbb{T}^2))$ for any $\alpha < \frac{1}{8}$ and $p \geq 1$. In particular,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|Z(t)\|_{L^p(\mathbb{T}^2)} \right) \leq M_{p,T,L},$$

for some $M_{p,T,L} > 0$. More generally, if $\theta \in (0, \frac{1}{8})$, Z has a modification in $C^\alpha([0,T]; \mathcal{W}^{\theta,p}(\mathbb{T}^2))$ for $p \geq 1$, and $\alpha < \frac{1}{8} - \theta$. □

4. Existence of the Solution

In this section, we construct a solution of (2.3) using a compactness argument.

4.1 Approximation

We want to construct the solution of (2.3) by a Faedo Galerkin approximation. For $f \in L^2(\mathbb{T}^2)$, and $n \in \mathbb{N}$,

$$\Pi_n : L^2(\mathbb{T}^2) \to \Pi_n L^2(\mathbb{T}^2)$$

is defined by

$$\Pi_n f = \sum_{|k|+|l|\le n} (e_{k,l}, f)e_{k,l}, \tag{4.1}$$

where $\Pi_n L^2(\mathbb{T}^2)$ denotes the subspace of $L^2(\mathbb{T}^2)$ such that the $\Pi_n f$ can be represented by linear combinations of $e_{k,l}$ with $|k| + |l| \le n$. Obviously, $\Pi_n f \to f$ in $L^2(\mathbb{T}^2)$. Our goal is to find the solution of (2.3). It is defined by

$$(v(t) - u_0, w) = \int_0^t (v(\sigma), \mathcal{L}_1 w)d\sigma - \int_0^t ((v(\sigma) + Z(\sigma))^3, w)d\sigma, \quad \mathbb{P} - a.s. \tag{4.2}$$

for $w \in \mathcal{W}^{1+\alpha,2}$. To do so we use the finite dimensional solution v_n which solves

$$\begin{cases} \partial_t v_n(t) = \mathcal{L}_1 v_n(t) - \Pi_n(\Pi_n v_n(t) + \Pi_n Z(t))^3, \\ v_n(0) = \Pi_n u_0. \end{cases} \tag{4.3}$$

This solution is smooth enough to obtain the following energy estimate. Multiplying the equation (4.3) by v_n, and using periodic boundary condition, we get:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_n\|_{L^2}^2 = &-2\int_{\mathbb{T}^2} v_n\partial_x^2 v_n dxdy \\ &- \int_{\mathbb{T}^2} v_n\partial_x^4 v_n dxdy + \int_{\mathbb{T}^2} v_n\partial_y^2 v dxdy - \int_{\mathbb{T}^2} v_n(v_n + \Pi_n Z)^3 dxdy. \end{aligned}$$

The second term is estimated as:

$$\int_{\mathbb{T}^2} v_n\partial_x^2 v_n dxdy \le \int_{\mathbb{T}^2} \frac{v_n^2}{2\delta^2} dxdy + \int_{\mathbb{T}^2} \frac{\delta^2(\partial_x^2 v_n)^2}{2} dxdy, \quad (\delta > 0),$$

and the fourth term may be written as, by integration by parts,

$$\int_{\mathbb{T}^2} v_n\partial_x^4 v_n dxdy = -\int_{\mathbb{T}^2} (\partial_x v_n)(\partial_x^3 v_n)dxdy = \int_{\mathbb{T}^2} (\partial_x^2 v_n)^2 dxdy = \|\partial_x^2 v_n\|_{L^2}^2,$$

and

$$\int_{\mathbb{T}^2} v_n(v_n + \Pi_n Z)^3 dxdy \le -\frac{1}{2}\|v_n\|_{L^4}^4 + C\|\Pi_n Z\|_{L^4}^4,$$

where we have used Young's inequality. Then for any $\delta \in (0, 1)$,

$$\begin{aligned} &-2\int_{\mathbb{T}^2} v_n\partial_x^2 v_n dxdy - \int_{\mathbb{T}^2} v_n\partial_x^4 v_n dxdy + \int_{\mathbb{T}^2} v_n\partial_y^2 v_n dxdy - \int_{\mathbb{T}^2} v_n(v_n + \Pi_n Z)^3 dxdy \\ &< \frac{1}{\delta^2}\|v_n\|_{L^2}^2 + (\delta^2 - 1)\|\partial_x^2 v_n\|_{L^2}^2 - \|\partial_y v_n\|_{L^2}^2 - \frac{1}{2}\|v_n\|_{L^4}^4 + C\|\Pi_n Z\|_{L^4}^4. \end{aligned}$$

Therefore, we have,

$$\frac{d}{dt}\|v_n\|_{L^2}^2 + \|\partial_x^2 v_n\|_{L^2}^2 + \|\partial_y v_n\|_{L^2}^2 + \|v_n\|_{L^4}^4 \le C(\delta)(\|v_n\|_{L^2}^2 + \|\Pi_n Z\|_{L^4}^4),$$

which reads in other word, by the definition of the space $\mathcal{W}^{1,2}$, taking for example $\delta = \frac{1}{2}$,

$$\frac{d}{dt}\|v_n\|_{L^2}^2 + \|v_n\|_{\mathcal{W}^{1,2}}^2 + \|v_n\|_{L^4}^4 < C(\|v_n\|_{L^2}^2 + \|\Pi_n Z\|_{L^4}^4). \tag{4.4}$$

Thus, by integrating on $[0, t]$ with $t \le T$,

$$\begin{aligned} \|v_n\|_{L^2}^2 &+ \int_0^t \|v_n(s)\|_{\mathcal{W}^{1,2}}^2 ds + \int_0^t \|v_n(s)\|_{L^4}^4 ds \\ &\le C\int_0^t (\|v_n(s)\|_{L^2}^2 + \|\Pi_n Z\|_{L^4}^4)ds + \|\Pi_n u_0\|_{L^2}^2. \end{aligned} \tag{4.5}$$

Therefore, by the Gronwall inequality, we have

$$\|v_n\|_{L^2}^2(t) \le Ce^{CT}\left(\|\Pi_n u_0\|_{L^2}^2 + \int_0^T \|\Pi_n Z\|_{L^4}^4(s)ds\right). \tag{4.6}$$

On the other hand, a similar proof as in Proposition 2 infers that $\mathbb{E}\|\Pi_n Z\|_{L^4}^4 \le C_T$ where C_T is independent of n. This implies that

$$\begin{aligned} \mathbb{E}\|v_n\|_{L^2}^2 &\le C(\varepsilon)\|\Pi_n u_0\|_{L^2}^2 e^{CT} + C\int_0^T \mathbb{E}\|\Pi_n Z\|_{L^4}^4 ds \\ &\le C(T)\|\Pi_n u_0\|_{L^2}^2 + C(T). \end{aligned} \tag{4.7}$$

Thus, by (4.5)

$$\mathbb{E}\left(\int_0^T \|v_n\|^2_{\mathcal{W}^{1,2}} ds + \int_0^T \|v_n\|^4_{L^4} ds\right) \le C(T, \|\Pi_n u_0\|_{L^2}). \tag{4.8}$$

The fact $\|\Pi_n u_0\|_{L^2} \le \|u_0\|_{L^2}$ implies that $\{v_n\}_n$ is bounded (independently of n) in

$$L^2(\Omega; L^2(0,T; \mathcal{W}^{1,2})) \cap L^4(\Omega; L^4(0,T; L^4(\mathbb{T}^2))).$$

Now for any $w \in \mathcal{W}^{1,2}$, noting that $\mathcal{L}_1 = 2 - (1 - \mathcal{L}_0)$,

$$\begin{aligned} \left|\left(\frac{d}{dt} v_n(t), w\right)\right| &\le |(v_n(t), \mathcal{L}_1 w)| + |((v_n + \Pi_n Z)^3, \Pi_n w)| \\ &\le \|v_n(t)\|_{\mathcal{W}^{1,2}} \|w\|_{\mathcal{W}^{1,2}} + \|(v_n + \Pi_n Z)^3\|_{L^{\frac{4}{3}}} \|\Pi_n w\|_{L^4} \\ &\le C(\|v_n(t)\|_{\mathcal{W}^{1,2}} \|w\|_{\mathcal{W}^{1,2}} + (\|v_n\|^3_{L^4} + \|\Pi_n Z\|^3_{L^4}) \|w\|_{\mathcal{W}^{1,2}}). \end{aligned}$$

Therefore,

$$\left\|\frac{d}{dt} v_n(t)\right\|_{\mathcal{W}^{-1,2}} \le C(\|v_n(t)\|_{\mathcal{W}^{1,2}} + \|v_n\|^3_{L^4} + \|\Pi_n Z\|^3_{L^4}).$$

The uniform estimates of (4.8) and $\|\Pi_n Z\|^4_{L^4}$ in n imply

$$\mathbb{E} \int_0^T \left\|\frac{d}{dt} v_n(t)\right\|^{\frac{4}{3}}_{\mathcal{W}^{-1,2}} dt \le C(T, \|u_0\|_{L^2}), \tag{4.9}$$

which concludes that $\{v_n\}$ is bounded in $L^{\frac{4}{3}}(\Omega; W^{1,\frac{4}{3}}(0,T; \mathcal{W}^{-1,2}))$. Let us now consider $m > 0$. We multiply both sides of (4.4) by $\|v_n\|^{2m-2}_{L^2}$,

$$\frac{1}{m}\frac{d}{dt} \|v_n\|^{2m}_{L^2} + \|v_n\|^2_{\mathcal{W}^{1,2}} \|v_n\|^{2m-2}_{L^2} \le C(\|v_n\|^{2m}_{L^2} + \|\Pi_n Z\|^4_{L^4} \|v_n\|^{2m-2}_{L^2}). \tag{4.10}$$

Applying the interpolation inequality:

$$\|v_n\|_{\mathcal{W}^{\frac{2}{3},2}} \le \|v_n\|^{1/3}_{L^2} \|v_n\|^{2/3}_{\mathcal{W}^{1,2}}$$

and choosing $m = 3$ in (4.10), we want to show that $\{v_n\}$ is bounded in $L^3(\Omega; L^3(0,T; \mathcal{W}^{\frac{2}{3},2}))$.

We integrate (4.10) in time, and we get

$$\begin{aligned} &\frac{1}{3}\|v_n\|^6_{L^2} + 3\int_0^T \|v_n\|^{12}_{\mathcal{W}^{\frac{2}{3},2}} \\ &\le \frac{1}{3}\|v_0\|^6_{L^2} + C\int_0^T (\|v_n\|^6_{L^2} + \|\Pi_n Z\|^4_{L^4} \|v_n\|^4_{L^2}) \\ &\le \frac{1}{3}\|v_0\|^6_{L^2} + C\int_0^T \|v_n\|^6_{L^2} + C\int_0^T \|\Pi_n Z\|^{12}_{L^4}. \end{aligned}$$

We take the expectation and the use of the Gronwall inequality which implies that

$$\begin{aligned} \mathbb{E}(\|v_n\|^6_{L^2}) &\lesssim \|u_0\|^6_{L^2} + \mathbb{E}\left(\int_0^T \|\Pi_n Z\|^{12}_{L^4}\right) \\ &\lesssim \|u_0\|^6_{L^2} + \left(\int_{\mathbb{T}^2} \mathbb{E}(|\Pi_n Z|^{12})^4 dxdy\right)^3. \end{aligned}$$

Here we have used the Minkowski inequality, and that the integrand of the second term in the RHS may be bounded by C_T as in the proof of Proposition 4. Therefore, we obtain

$$\mathbb{E}\left(\int_0^T \|v_n\|^3_{\mathcal{W}^{\frac{2}{3},2}} ds\right) \le C(T, \|u_0\|_{L^2}).$$

We remark that embedding $L^3(0,T; \mathcal{W}^{\frac{2}{3},2}) \cap W^{1,\frac{4}{3}}(0,T; \mathcal{W}^{-1,2}) \subset L^3(0,T; L^4)$ is compact, and $L^2(0,T; \mathcal{W}^{1,2}) \cap W^{1,\frac{4}{3}}(0,T; \mathcal{W}^{-1,2}) \subset L^2(0,T; \mathcal{W}^{1-\alpha,2})$ is compact for any $\alpha > 0$ by Proposition 1. On the other hand the embedding $W^{1,\frac{4}{3}}(0,T; \mathcal{W}^{-1,2}) \subset C([0,T]; \mathcal{W}^{-(1+\alpha),2})$ is compact for any $\alpha > 0$ by Proposition 2. Meanwhile, as in the proof of Proposition 4, $\{\Pi_n Z\}$ is bounded in $C^{\alpha_0}([0,T]; \mathcal{W}^{s,p})$ for $0 \le s < \frac{1}{8}$, $\alpha_0 < \frac{1}{8} - s$, $p \ge 1$. The embedding $C([0,T]; \mathcal{W}^{\frac{1}{16},4}) \cap C^{\alpha_0}([0,T]; \mathcal{W}^{-\delta,4}(\mathbb{T}^2)) \subset C([0,T]; L^4)$ is thus compact for any $\delta > 0$. We thus conclude that the sequence $\{(v_n, \Pi_n Z)\}_n$ is tight in $L^3(0,T; L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2}) \times C([0,T]; L^4)$ for any $\alpha > 0$.

4.2 Existence of the solution

By Prohorov Theorem, the tightness of $(v_n, \Pi_n Z)$ implies that the existence of subsequence $(v_{n(k)}, \Pi_{n(k)} Z)$ and some probability measure μ such that

$$(v_{n(k)}, \Pi_{n(k)}Z) \to \mu \text{ weakly as } k \to \infty.$$

Moreover by Skorohod Theorem, there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and random variables $\{X^{n(k)}, Z^{n(k)}\}_k$, and (X, Z) such that

$$\begin{aligned}
&\mathrm{Law}(v_{n(k)}, \Pi_{n(k)}Z) = \mathrm{Law}(X^{n(k)}, Z^{n(k)}) \quad \text{for } k \geq 1, \quad \mathrm{Law}(X,Z) = \mu, \\
&\lim_{k\to\infty} (X^{n(k)}, Z^{n(k)}) = (X, Z), \quad \text{in} \\
&L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2} \cap L^2(0,T; \mathcal{W}^{1-\alpha,2}) \times C([0,T], L^4), \quad \mathbb{P}' - a.s.
\end{aligned}$$

The equivalence of probability laws leads that for $w \in \mathcal{W}^{1+\alpha,2}$ and $t \in [0,T]$,

$$(X^{n(k)}(t) - u_{n(k)}(0), w) = \int_0^t (X^{n(k)}(\sigma), \mathcal{L}_1 w) d\sigma - \int_0^t (\Pi_{n(k)}(X^{n(k)} + Z^{n(k)})^3(\sigma), w) d\sigma \tag{4.11}$$

$\mathbb{P}'$-almost surely. We prove that X is a solution of (2.3) on $(\Omega', \mathcal{F}', \mathbb{P}')$. For $t \in [0,T]$,

$$|(X^{n(k)}(t) - X(t), w)| \leq \|X^{n(k)} - X\|_{C([0,T];\mathcal{W}^{-(1+\alpha),2})} \|w\|_{\mathcal{W}^{1+\alpha,2}} \to 0$$

as $k \to \infty$. The LHS of (4.11) converges to $(X(t) - u_0, w)$. Next we observe the convergence of the second term on the RHS:

$$\begin{aligned}
\left| \int_0^t (X^{n(k)}(\sigma) - X(\sigma), \mathcal{L}_1 w) d\sigma \right| &\leq \int_0^t (X^{n(k)}(\sigma) - X(\sigma), (2 - (1 - \mathcal{L}_0))w) d\sigma \\
&\leq \int_0^T \|X^{n(k)}(\sigma) - X(\sigma)\|_{\mathcal{W}^{-1-\alpha,2}} \|2w\|_{\mathcal{W}^{1+\alpha,2}} d\sigma \\
&\quad + \int_0^T \|X^{n(k)}(\sigma) - X(\sigma)\|_{\mathcal{W}^{1-\alpha,2}} \|w\|_{\mathcal{W}^{1+\alpha,2}} d\sigma \\
&\to 0,
\end{aligned}$$

as $k \to \infty$. The convergence of the third term on the RHS can be shown as follows;

$$\begin{aligned}
&\int_0^t |(\Pi_{n(k)}(X^{n(k)}(\sigma) + Z^{n(k)}(\sigma))^3, w) - ((X(\sigma) + Z(\sigma))^3, w)| d\sigma \\
&\leq \int_0^t |((X^{n(k)}(\sigma) + Z^{n(k)}(\sigma))^3, \Pi_{n(k)}w - w)| d\sigma \\
&\quad + \int_0^t |((X^{n(k)}(\sigma) + Z^{n(k)}(\sigma))^3 - (X(\sigma) + Z(\sigma))^3, w)| d\sigma \\
&= I + J.
\end{aligned}$$

First, we estimate I.

$$\begin{aligned}
I &\leq C \int_0^T \|(X^{n(k)} + Z^{n(k)})^3\|_{L^{\frac{4}{3}}} \|\Pi_{n(k)}w - w\|_{L^4} d\sigma \\
&\leq C \int_0^T (\|X^{n(k)}(\sigma)\|_{L^4}^3 + \|Z^{n(k)}(\sigma)\|_{L^4}^3) \|\Pi_{n(k)}w - w\|_{L^4} d\sigma \\
&\leq C \int_0^T (\|X^{n(k)}(\sigma)\|_{L^4}^3 + \|Z^{n(k)}(\sigma)\|_4^3) \|\Pi_{n(k)}w - w\|_{\mathcal{W}^{1+\alpha,2}} d\sigma \\
&\to 0
\end{aligned}$$

as $k \to \infty$, for any $\alpha > 0$, where we have used the Sobolev embedding in the third inequality. Next we will see the convergence of J.

$$\begin{aligned}
J &= \int_0^t |((X^{n(k)}(\sigma) + Z^{n(k)}(\sigma))^3 - (X(\sigma) + Z(\sigma))^3, w)| d\sigma \\
&\leq C \int_0^t |((X^{n(k)}(\sigma) - X(\sigma))(|X^{n(k)}(\sigma)|^2 + |X(\sigma)|^2 + |Z^{n(k)}(\sigma)|^2 + |Z(\sigma)|^2), |w|)| d\sigma \\
&\quad + C \int_0^t |((Z^{n(k)}(\sigma) - Z(\sigma))(|X^{n(k)}(\sigma)|^2 + |X(\sigma)|^2 + |Z^{n(k)}(\sigma)|^2 + |Z(\sigma)|^2), |w|)| d\sigma. \\
&= J_1 + J_2.
\end{aligned}$$

Considering the first term J_1 we have;

$$\begin{aligned} J_1 &\leq \int_0^T \|X^{n(k)}(\sigma) - X(\sigma)\|_{L^4}(\|X^{n(k)}(\sigma)\|_{L^4}^2 + \|X(\sigma)\|_{L^4}^2 + \|Z^{n(k)}(\sigma)\|_{L^4}^2 + \|Z(\sigma)\|_{L^4}^2)d\sigma \\ &\quad \times \|w\|_{L^4} \\ &\leq C\left(\int_0^T \|X^{n(k)}(\sigma) - X(\sigma)\|_{L^4}^3 d\sigma\right)^{\frac{1}{3}} \\ &\quad \times \left(\int_0^T (\|X^{n(k)}(\sigma)\|_{L^4}^3 + \|X(\sigma)\|_{L^4}^3 + \|Z^{n(k)}(\sigma)\|_{L^4}^3 + \|Z(\sigma)\|_{L^4}^3)d\sigma\right)^{\frac{2}{3}} \|w\|_{L^4} \end{aligned}$$

The second integral is finite and the convergence $X^{n(k)} \to X$ in $L^3(0,T;L^4)$ gives the convergence of J_1. Furthermore, similarly as above, we estimate

$$\begin{aligned} J_2 &\leq \int_0^T \|Z^{n(k)}(\sigma) - Z(\sigma)\|_{L^4}(\|X^{n(k)}(\sigma)\|_{L^4}^2 + \|X(\sigma)\|_{L^4}^2 + \|Z^{n(k)}(\sigma)\|_{L^4}^2 + \|Z(\sigma)\|_{L^4}^2)d\sigma \\ &\quad \times \|w\|_{L^4} \\ &\leq C\left(\int_0^T \|Z^{n(k)}(\sigma) - Z(\sigma)\|_{L^4}^3 d\sigma\right)^{\frac{1}{3}} \\ &\left(\int_0^T (\|X^{n(k)}(\sigma)\|_{L^4}^3 + \|X(\sigma)\|_{L^4}^3 + \|Z^{n(k)}(\sigma)\|_{L^4}^3 + \|Z(\sigma)\|_{L^4}^3)d\sigma\right)^{\frac{2}{3}} \times \|w\|_{L^4} \end{aligned}$$

and use $Z^{n(k)} \to Z$ in $L^3(0,T,L^4)$. Therefore, we conclude, as $k \to \infty$,

$$(X(t) - u(0), w) = \int_0^t (X(\sigma), \mathcal{L}_1 w)d\sigma - \int_0^t ((X(\sigma) + Z(\sigma))^3, w)d\sigma. \quad \mathbb{P}' - a.s. \tag{4.12}$$

We have proved the existence of the solution on $(\Omega', \mathcal{F}', \mathbb{P}')$. To obtain the solution on the original space $(\Omega, \mathcal{F}, \mathbb{P})$, we use the Gyöngy–Krylov criterion. Regarding X_n of Proposition 3 as $(v_{n(k)}, \Pi_{n(k)}Z)$, we know that arbitrary subsequence of $(v_{n(k)}, \Pi_{n(k)}Z)$ converges to μ in law. It is thus sufficient to prove that for any $\varepsilon > 0$ (see the details in Sect. 4.4.2 of [5]),

$$\lim_{h\to\infty} \mathbb{P}(\|(v_{n_1(k(h))}, \Pi_{n_1(k(h))}) - (v_{n_2(k(h))}, \Pi_{n_2(k(h))}))\|_O > \varepsilon) = 0,$$

with

$$O := L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2}) \times C([0,T];L^4).$$

This follows from the equivalence of probability laws between $(X^{n(k)}, Z^{n(k)})$ and $(v_{n(k)}, \Pi_{n(k)}Z)$, and convergence

$$\begin{aligned} &\lim_{k\to\infty} (X^{n(k)}, Z^{n(k)}) = (X, Z), \quad \text{in} \\ &L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2}) \times C(0,T;L^4), \quad \mathbb{P}' - a.s. \end{aligned}$$

Therefore there exists a random variable (V, Z) such that $(v_{n(k)}, \Pi_{n(k)}Z) \to (V, Z)$ in $L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2}) \times C(0,T;L^4)$ in probability. Note that by taking a subsequence $\{v_{n(k(l))}\}$, the convergence in probability becomes $\mathbb{P}$-a.s. convergence. Similarly to the above discussion, we get

$$(V(t) - u(0), w) = \int_0^t (V(u), \mathcal{L}_1 w)du - \int_0^t ((V(u) + Z(u))^3, w)du. \quad \mathbb{P} - a.s. \tag{4.13}$$

Thus we have proved the existence of the solution on $(\Omega, \mathcal{F}, \mathbb{P})$.

Recall that we proved $v_{n(k)} \to V$ in $L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T, \mathcal{W}^{1-\alpha,2})$, $\mathbb{P}$ almost surely. Also,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \|v_n(t)\|_{L^2}^2\right) \leq C(T, \|u_0\|_{L^2}). \tag{4.14}$$

This inequality comes from (4.7). The inequality (4.14) implies that $\{v_{n(k)}\}$ is weak star compact in $L^2(\Omega, L^\infty(0,T;L^2))$. Thus there exist a subsequence (denoted by the same letter) and a limit v^* such that $v_{n(k)} \to v^*$ weak star in $L^2(\Omega; L^\infty(0,T;L^2))$. On the other hand, $v_{n(k(l))} \to V$ in $L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T, \mathcal{W}^{1-\alpha,2})$ $\mathbb{P}$-almost surely. In particular, $v_{n(k(l))} \to V$ weak star in $L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2})$ almost surely. We know

$$v^* \in L^\infty(0,T;L^2) \subset L^3(0,T;L^4) \cap C([0,T]; \mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T; \mathcal{W}^{1-\alpha,2}),$$

almost surely. The uniquness of weak star limit implies

$$v^* = V.$$

Therefore,

$$V \in L^3(0,T;L^4) \cap C([0,T];\mathcal{W}^{-(1+\alpha),2}) \cap L^2(0,T,\mathcal{W}^{1-\alpha,2}) \cap L^\infty(0,T:L^2),$$

$\mathbb{P}$-almost surely. Finally we show the pathwise uniqueness of solutions following the idea in [2]. Consider two solution $v_1, v_2 \in L^\infty(0,T;L^2)$, and set $d = v_1 - v_2$. Then d satisfies

$$\partial_t d = \mathcal{L}_1 d - \{(v_1+Z)^3 - (v_2+Z)^3\},$$

and

$$\frac{1}{2}\partial_t \|d\|_{L^2}^2 = (\partial_t d, d) = (\mathcal{L}_1 d, d) - \int_{\mathbb{T}^2} d\{d^3 + 3d^2(v_2+Z) + 3d(v_2+Z)^2\}dx. \tag{4.15}$$

Note that

$$-3\int_{\mathbb{T}^2} d^3(v_2+Z)dx \le \int_{\mathbb{T}^2}\left(d^4 + \frac{9}{4}d^2(v_2+Z)^2\right)dx,$$

and

$$\begin{aligned}(\mathcal{L}_1 d, d) &= 2\|d\|_{L^2}^2 - \|(1-\mathcal{L}_0)^{\frac{1}{2}} d\|_{L^2}^2 \\ &= 2\|d\|_{L^2}^2 - \|d\|_{\mathcal{W}^{1,2}}^2.\end{aligned}$$

Thus,

$$(4.15) \le \left\{2 + \frac{9}{4}\sup_{t\in[0,T]}(\|v_2\|_{L^2}^2 + \|Z\|_{L^2}^2)\right\}\|d\|_{L^2}^2,$$

which implies $d = 0$ in L^2 after an application of the Gronwall inequality. This completes the proof of Theorem 2. □

5. Numerical Simulation

In this section, we present some simulations in space dimension 2. The idea is to first perform simulations for the equation of A and convert them to u by the Ansatz, and at the same time, to perform direct simulations for the Swift–Hohenberg equation on u. We expect that with the proposed definitions of the noise term and the scaling, the patterns obtained by the two methods will be similar one to the other. And as informal observations, we perform simulations to compare the results in both the deterministic and stochastic case.

Equation of A

We recall the equation of $A(X,Y,T)$ in the deterministic case

$$\partial_T A = 4\partial_X^2 A + \partial_Y^2 A + A - 3|A|^2 A, \tag{5.1}$$

in the space domain $[-L,L]\times[-L,L]$ and time interval $[0,T_A]$. As A is complex-valued, we suppose $A = A^R + \mathbf{i}A^I$, where R stands for real and I stands for imaginary. And we separate the real part and the imaginary part of the equation into the following system

$$\begin{cases}\partial_T A^R = 4\partial_X^2 A^R + \partial_Y^2 A^R + A^R - 3|(A^R)^2 + (A^I)^2|A^R, \\ \partial_T A^I = 4\partial_X^2 A^I + \partial_Y^2 A^I + A^I - 3|(A^R)^2 + (A^I)^2|A^I,\end{cases} \tag{5.2}$$

with initial conditions and periodic boundary conditions for both A^R and A^I.

Convert A to u by the Ansatz

Once we obtain the numerical solution of A, we convert it to u by applying the following Ansatz

$$u(x,y,t) = \varepsilon A(\varepsilon x, \varepsilon y, \varepsilon^2 t)e^{\mathbf{i}x} + \varepsilon\overline{A}(\varepsilon x, \varepsilon y, \varepsilon^2 t)e^{-\mathbf{i}x} \tag{5.3}$$

where $x = X/\varepsilon$, $y = Y/\varepsilon$, $t = T/\varepsilon^2$ and $\overline{A} = A^R - \mathbf{i}A^I$. A direct computation yields

$$u(x,y,t) = 2\varepsilon(A^R(\varepsilon x, \varepsilon y, \varepsilon^2 t)\cdot\cos(x) - A^I(\varepsilon x, \varepsilon y, \varepsilon^2 t)\cdot\sin(x)). \tag{5.4}$$

Anisotropic Swift–Hohenberg equation

The anisotropic Swift–Hohenberg equation that we consider is as follows

$$\partial_t u = -(1+\partial_x^2)^2 u + \partial_y^2 u + \varepsilon^2 u - u^3. \tag{5.5}$$

In order to perform simulations, we decompose the equation into the system

$$\begin{cases} \partial_t u = -u + \partial_x^2 \mu + \partial_y^2 u + \varepsilon^2 u - u^3 \\ \mu = -\partial_x^2 u - 2u \end{cases} \tag{5.6}$$

with initial condition for u and periodic boundary condition for both u and μ.

Form of the stochastic term

In order to consider the corresponding stochastic equations of (5.1) and (5.5), we present the stochastic term. We first define the stochastic term for A, which is

$$\xi(X, Y, T) = C_L \sum_{\mathbf{k} \in \mathbb{Z}^2} \beta_{\mathbf{k}}(T) e^{\frac{-\mathbf{i}\pi\mathbf{k}}{L} \cdot \mathbf{X}}. \tag{5.7}$$

More precisely, we have the space domain $[-L, L] \times [-L, L]$ and $C_L = 1/(2L)$, $\mathbf{k} = k^R + \mathbf{i}k^I$, $\mathbf{X} = (X, Y)$ and $\beta_{\mathbf{k}} = \beta^R_{(k^R,k^I)} + \mathbf{i}\beta^I_{(k^R,k^I)}$, where $\beta^R_{(k^R,k^I)}$ and $\beta^I_{(k^R,k^I)}$ are independent real-valued Brownian motions. The corresponding stochastic equation of (5.1) is given by

$$\partial_T A = 4\partial_X^2 A + \partial_Y^2 A + A - 3|A|^2 A + \dot{\xi}, \tag{5.8}$$

with

$$\dot{\xi}(X, Y, T) = C_L \sum_{\mathbf{k} \in \mathbb{Z}^2} \dot{\beta}_{\mathbf{k}}(T) e^{\frac{-\mathbf{i}\pi\mathbf{k}}{L} \cdot \mathbf{X}}. \tag{5.9}$$

Since $\xi(X, Y, T)$ is complex-valued, we suppose $\xi(X, Y, T) = \xi^R(X, Y, T) + \mathbf{i}\xi^I(X, Y, T)$. A computation yields

$$\begin{cases} \xi^R = \sum\limits_{\substack{k^R \in \mathbb{Z} \\ k^I \in \mathbb{Z}}} \left[\beta^R_{(k^R,k^I)}(T) \cos\left(\frac{\pi(k^R X + k^I Y)}{L} \right) - \beta^I_{(k^R,k^I)}(T) \sin\left(\frac{\pi(k^R X + k^I Y)}{L} \right) \right] \\ \xi^I = \sum\limits_{\substack{k^R \in \mathbb{Z} \\ k^I \in \mathbb{Z}}} \left[\beta^R_{(k^R,k^I)}(T) \sin\left(\frac{\pi(k^R X + k^I Y)}{L} \right) + \beta^I_{(k^R,k^I)}(T) \cos\left(\frac{\pi(k^R X + k^I Y)}{L} \right) \right]. \end{cases} \tag{5.10}$$

And the corresponding stochastic system of (5.2) is given by

$$\begin{cases} \partial_T A^R = 4\partial_X^2 A^R + \partial_Y^2 A^R + A^R - 3|(A^R)^2 + (A^I)^2| A^R + \dot{\xi}^R \\ \partial_T A^I = 4\partial_X^2 A^I + \partial_Y^2 A^I + A^I - 3|(A^R)^2 + (A^I)^2| A^I + \dot{\xi}^I. \end{cases} \tag{5.11}$$

Next, we present the stochastic term for the equation of u, after some computation, we define

$$\xi_\varepsilon(x, y, t) = C_L \sum_{\mathbf{k} \in \mathbb{Z}^2} \beta_{\mathbf{k}}(\varepsilon^2 t) e^{\frac{-\mathbf{i}\pi\mathbf{k}}{L/\varepsilon} \cdot \mathbf{x}}, \tag{5.12}$$

where $\mathbf{x} = (x, y) \in [-L/\varepsilon, L/\varepsilon] \times [-L/\varepsilon, L/\varepsilon]$ with $x = X/\varepsilon$, $y = Y/\varepsilon$ and $t = T/\varepsilon^2$. We suppose that $\beta_{-\mathbf{k}} = \overline{\beta}_{\mathbf{k}}$ and as a result,

$$\begin{aligned} &\xi_\varepsilon(x, y, t) \\ &= 2C_L \sum_{\substack{k^R \in \mathbb{Z} \\ k^I \in \mathbb{Z}^+}} \left(\beta^R_{(k^R,k^I)}(\varepsilon^2 t) \cos\left(\frac{\pi(k^R x + k^I y)}{L/\varepsilon} \right) + \beta^I_{(k^R,k^I)}(\varepsilon^2 t) \sin\left(\frac{\pi(k^R x + k^I y)}{L/\varepsilon} \right) \right) \end{aligned} \tag{5.13}$$

and we remark that $\varepsilon\xi_\varepsilon$ is real-valued. The stochastic system corresponds to (5.6) is given by

$$\begin{cases} \partial_t u = -u + \partial_x^2 \mu + \partial_y^2 u + \varepsilon^2 u - u^3 + \varepsilon \cdot \dot{\xi}_\varepsilon \\ \mu = -\partial_x^2 u - 2u. \end{cases} \tag{5.14}$$

5.1 Space and time discretizations

We mainly present the numerical settings of A and the settings for u are defined correspondingly. We discretize the space domain $[-L, L] \times [-L, L]$ into a $N_X \times N_Y$ uniform mesh, so that $\Delta X = 2L/N_X$ and $\Delta Y = 2L/N_Y$. We define $p \in \{0, 1, 2, \ldots, N_X - 1\}$ and $q \in \{0, 1, 2, \ldots, N_Y - 1\}$, two indices in direction X and Y respectively. The control volume (p, q) is the volume whose barycenter satisfies

$$\mathbf{X}_{p,q} = ((p + 0.5) \cdot \Delta X, (q + 0.5) \cdot \Delta Y).$$

And we apply uniform time discretization, that is we fix the time step ΔT and define $T_n = n\Delta T$ for all $n = 0, 1, 2, \ldots$. If we consider N_T time steps, the total time interval is $\cup_{n=0}^{N_T - 1} [n\Delta T, (n + 1)\Delta T)$.

The discrete solutions of A^R and of A^I are denoted by $\{A^{R^n}_{p,q}\}$ and $\{A^{I^n}_{p,q}\}$ over control volume (p, q) in time interval $[T_n, T_{n+1})$ respectively.

In the discretization of the space derivative, we will need the values of ${A^{R}}^{n}_{-1,q}$, ${A^{R}}^{n}_{N_X,q}$, ${A^{R}}^{n}_{p,-1}$ and ${A^{R}}^{n}_{p,N_Y}$, because of the periodic boundary condition, we set

$$\begin{aligned} {A^{R}}^{n}_{-1,q} &:= {A^{R}}^{n}_{N_X-1,q} \quad \text{for all} \quad q \in \{0,1,2,\ldots,N_Y-1\} \\ {A^{R}}^{n}_{N_X,q} &:= {A^{R}}^{n}_{0,q} \quad \text{for all} \quad q \in \{0,1,2,\ldots,N_Y-1\} \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} {A^{R}}^{n}_{p,-1} &:= {A^{R}}^{n}_{p,N_Y-1} \quad \text{for all} \quad p \in \{0,1,2,\ldots,N_X-1\}, \\ {A^{R}}^{n}_{p,N_Y} &:= {A^{R}}^{n}_{p,0} \quad \text{for all} \quad p \in \{0,1,2,\ldots,N_X-1\}, \end{aligned} \tag{5.16}$$

we have the same conditions for $\{{A^{I}}^{n}_{p,q}\}$. For the approximation of u, we apply corresponding settings with notations Δx, Δy and Δt, N_t and the discrete solution is denoted by $\{u^n_{p,q}\}$.

5.2 Discretization of the noise term

Suppose β is a Brownian motion, for the numerical simulations, we approximate $\dot{\beta}$ by

$$\dot{\beta}(t) \approx \frac{\beta(t+\Delta t) - \beta(t)}{\Delta t},$$

where $\beta(t+\Delta t) - \beta(t) \sim \mathcal{N}(0, \Delta t)$ is a Gaussian random variable with mean value 0 and variance Δt.

We discretize the noise term (5.9) as follows

$$\dot{\xi}(X,Y,T) \approx C_L \sum_{\substack{k^R \in \{-m_{\mathrm{R}},\ldots,0,\ldots,m_{\mathrm{R}}\} \\ k^I \in \{-m_{\mathrm{I}},\ldots,0,\ldots,m_{\mathrm{I}}\}}} \frac{\beta_{\mathbf{k}}(T+\Delta T) - \beta_{\mathbf{k}}(T)}{\Delta T} e^{\frac{-i\pi \mathbf{k}}{L}\cdot \mathbf{X}} \tag{5.17}$$

such that

$$\begin{aligned} &\dot{\xi}^R(X,Y,T) \\ &\approx C_L \sum_{\substack{k^R \in \{-m_{\mathrm{R}},\ldots,0,\ldots,m_{\mathrm{R}}\} \\ k^I \in \{-m_{\mathrm{I}},\ldots,0,\ldots,m_{\mathrm{I}}\}}} \left[\frac{\beta^R_{(k^R,k^I)}(T+\Delta T) - \beta^R_{(k^R,k^I)}(T)}{\Delta T} \cos\left(\frac{\pi(k^R X + k^I Y)}{L}\right) \right. \\ &\quad \left. - \frac{\beta^I_{(k^R,k^I)}(T+\Delta T) - \beta^I_{(k^R,k^I)}(T)}{\Delta T} \sin\left(\frac{\pi(k^R X + k^I Y)}{L}\right) \right] \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} &\dot{\xi}^I(X,Y,T) \\ &\approx C_L \sum_{\substack{k^R \in \{-m_{\mathrm{R}},\ldots,0,\ldots,m_{\mathrm{R}}\} \\ k^I \in \{-m_{\mathrm{I}},\ldots,0,\ldots,m_{\mathrm{I}}\}}} \left[\frac{\beta^R_{(k^R,k^I)}(T+\Delta T) - \beta^R_{(k^R,k^I)}(T)}{\Delta T} \sin\left(\frac{\pi(k^R X + k^I Y)}{L}\right) \right. \\ &\quad \left. + \frac{\beta^I_{(k^R,k^I)}(T+\Delta T) - \beta^I_{(k^R,k^I)}(T)}{\Delta T} \cos\left(\frac{\pi(k^R X + k^I Y)}{L}\right) \right] \end{aligned} \tag{5.19}$$

where $\beta^R_{(k^R,k^I)}(T+\Delta T) - \beta^R_{(k^R,k^I)}(T)$ and $\beta^I_{(k^R,k^I)}(T+\Delta T) - \beta^I_{(k^R,k^I)} \sim \mathcal{N}(0, \Delta T)$. The m_{R} and m_{I} are the truncation numbers. We denote this approximation of $\dot{\xi}(X,Y,T)$ by Ξ such that $\dot{\xi}^R(X,Y,T)$ is approximated by Ξ^R and $\dot{\xi}^I(X,Y,T)$ by Ξ^I. In view of (5.13), the discretization of the noise term (5.12) is as follows

$$\begin{aligned} &\dot{\xi}_\varepsilon(x,y,t) \\ &\approx 2C_L \sum_{\substack{k^R \in \{-m_{\mathrm{R}},\ldots,0,\ldots,m_{\mathrm{R}}\} \\ k^I \in \{0,\ldots,m_{\mathrm{I}}\}}} \left[\frac{\beta^R_{(k^R,k^I)}(\varepsilon^2(t+\Delta t)) - \beta^R_{(k^R,k^I)}(\varepsilon^2 t)}{\Delta t} \cos\left(\frac{\pi(k^R x + k^I y)}{L/\varepsilon}\right) \right. \\ &\quad \left. + \frac{\beta^I_{(k^R,k^I)}(\varepsilon^2(t+\Delta t)) - \beta^I_{(k^R,k^I)}(\varepsilon^2 t)}{\Delta t} \sin\left(\frac{\pi(k^R x + k^I y)}{L/\varepsilon}\right) \right] \end{aligned} \tag{5.20}$$

where $\beta_{k^R}(\varepsilon^2(t+\Delta t)) - \beta_{k^R}(\varepsilon^2 t)$ and $\beta_{k^I}(\varepsilon^2(t+\Delta t)) - \beta_{k^I}(\varepsilon^2 t) \sim \mathcal{N}(0, \varepsilon^2 \Delta t)$. We denote this approximation term by Ξ_ε.

We present in the following directly the numerical scheme for the stochastic case. For the deterministic case, we perform the simulation by omitting the stochastic term.

5.3 Numerical schemes

We apply the finite difference scheme and first present the scheme for A of the system (5.11)

$$\begin{cases} \dfrac{A^{R^{n+1}}_{p,q} - A^{R^n}_{p,q}}{\Delta T} = 4d_X^2 A^{R^{n+1}}_{p,q} + d_Y^2 A^{R^n}_{p,q} + A^{R^{n+1}}_{p,q} - 3|(A^{R^n}_{p,q})^2 + (A^{I^n}_{p,q})^2|A^{R^n}_{p,q} + \Xi^R \\ \dfrac{A^{I^{n+1}}_{p,q} - A^{I^n}_{p,q}}{\Delta T} = 4d_X^2 A^{I^{n+1}}_{p,q} + d_Y^2 A^{I^n}_{p,q} + A^{I^{n+1}}_{p,q} - 3|(A^{R^n}_{p,q})^2 + (A^{I^n}_{p,q})^2|A^{I^n}_{p,q} + \Xi^I, \end{cases} \tag{5.21}$$

where d_X^2 and d_Y^2 are discrete operators such that

$$d_X^2 A^{R^{n+1}}_{p,q} = \frac{A^{R^{n+1}}_{p-1,q} + A^{R^{n+1}}_{p+1,q} - 2A^{R^{n+1}}_{p,q}}{(\Delta X)^2} \quad \text{and} \quad d_Y^2 A^{R^n}_{p,q} = \frac{A^{R^n}_{p,q-1} + A^{R^n}_{p,q+1} - 2A^{R^n}_{p,q}}{(\Delta Y)^2},$$

for all $p \in \{0,1,2,\dots,N_X-1\}$ and $q \in \{0,1,2,\dots,N_Y-1\}$. We refer to (5.15) and (5.16) for the periodic boundary condition. For $n = 0,1,2,\dots,N_T-1$, knowing the values of $(A^{R^n}_{p,q}, A^{I^n}_{p,q})$, we compute the values of $(A^{R^{n+1}}_{p,q}, A^{I^{n+1}}_{p,q})$, for all $p \in \{0,1,2,\dots,N_X-1\}$ and $q \in \{0,1,2,\dots,N_Y-1\}$.

And we implement the following numerical scheme for the Swift–Hohenberg equation in the form of system (5.14).

$$\begin{cases} \dfrac{u^{n+1}_{p,q} - u^n_{p,q}}{\Delta t} = -u^{n+1}_{p,q} + d_x^2 \mu^{n+1}_{p,q} + d_y^2 u^n_{p,q} + \varepsilon^2 u^n_{p,q} - (u^n_{p,q})^3 + \varepsilon \Xi_\varepsilon \\ \mu^{n+1}_{p,q} = -d_x^2 u^{n+1}_{p,q} - u^{n+1}_{p,q} - u^n_{p,q}, \end{cases} \tag{5.22}$$

with corresponding definitions of d_x^2 and d_y^2 and the periodic boundary conditions. For $n = 0,1,2,\dots,N_t-1$, knowing the values of $(u^n_{p,q}, \mu^n_{p,q})$, we compute the values of $(u^{n+1}_{p,q}, \mu^{n+1}_{p,q})$, for all $p \in \{0,1,2,\dots,N_x-1\}$ and $q \in \{0,1,2,\dots,N_y-1\}$.

5.4 Numerical settings

In the simulations, ε is the parameter to connect A and u, so we first fix the value of ε. And we perform simulations for A with the following settings.

Numerical settings for A

- The space domain to be $[-L, L] \times [-L, L]$ with $L = \pi/2$;
- we discretize the space into 100×100 uniform square;
- we fix the time step $\Delta t = 0.0001$;
- we set the initial condition $A^R(X, Y, 0) = A_0^R(X, Y)$ and $A^I = A_0^I(X, Y)$;
- we perform simulations of A and convert the numerical results to u by the Ansatz.

We perform simulations of u by (5.22) with the following settings.

Numerical settings for u

- The space domain to be $[-L/\varepsilon, L/\varepsilon] \times [-L/\varepsilon, L/\varepsilon]$ with $L = \pi/2$;
- we discretize the space into 100×100 uniform square;
- we choose time step $\Delta t = 0.001$;
- we compute the initial condition for u based on the initial condition of A^R and A^I by the Ansatz (5.3), which yields

$$u_0(x, y) = 2\epsilon(A^R(\varepsilon x, \varepsilon y, 0) \cdot \cos(x) - A^I(\varepsilon x, \varepsilon y, 0) \cdot \sin(x)); \tag{5.23}$$

- we perform simulations for u.

5.5 Results and observations

We set $\varepsilon = 0.25$ and perform numerical simulations for A^I and A^R with the initial conditions

$$A_0^R(X, Y) = \begin{cases} 1 & \text{if } Y \in (-L, 0) \\ 0 & \text{otherwise} \end{cases} \tag{5.24}$$

and

$$A_0^I(X, Y) = \begin{cases} 0 & \text{if } Y \in (-L, 0) \\ 1 & \text{otherwise.} \end{cases} \tag{5.25}$$

The result is as follows.

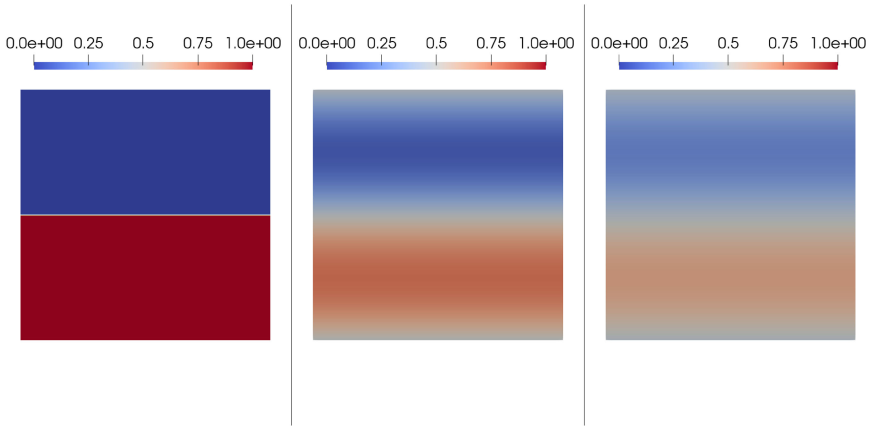

Fig. 1. Solution for A^R at time $T = 0$, 0.1 and 0.2.

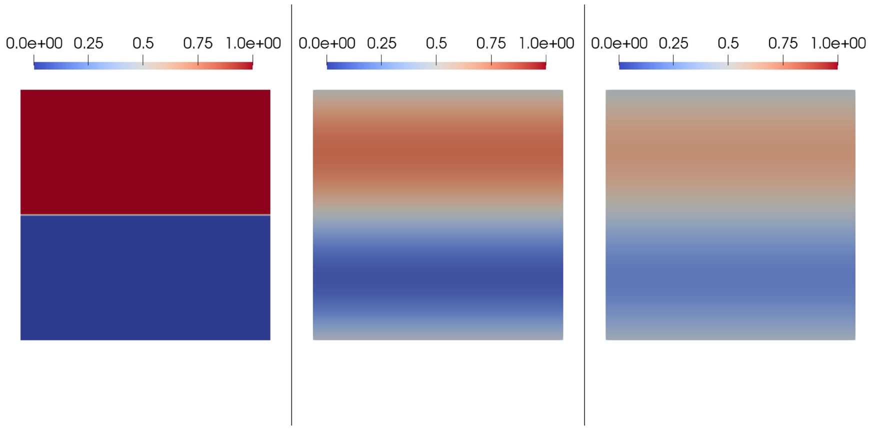

Fig. 2. Solution for A^I at time $T = 0$, 0.1 and 0.2.

Deterministic case

We convert the numerical results from A to u.

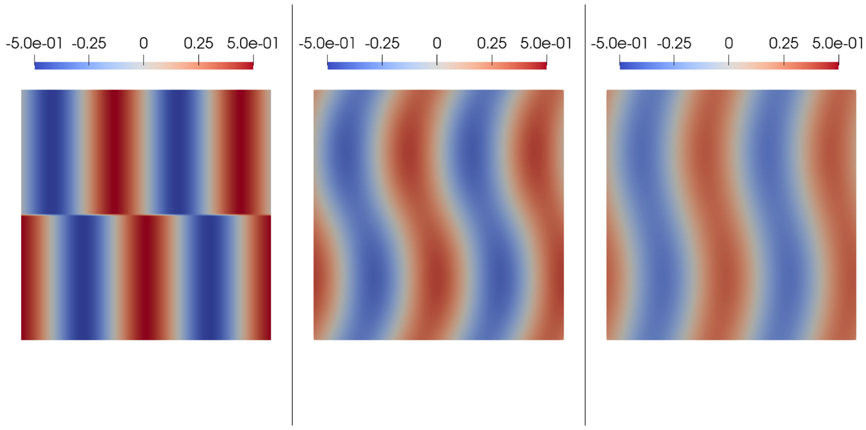

Fig. 3. u at time $t = 0$, $0.1/\varepsilon^2$ and $0.2/\varepsilon^2$ converted by the Ansatz from A of time $T = 0$, 0.1 and 0.2.

Then we perform directly the simulation for u and compare it to the simulation results of u convert by A which are presented in Fig. 3.

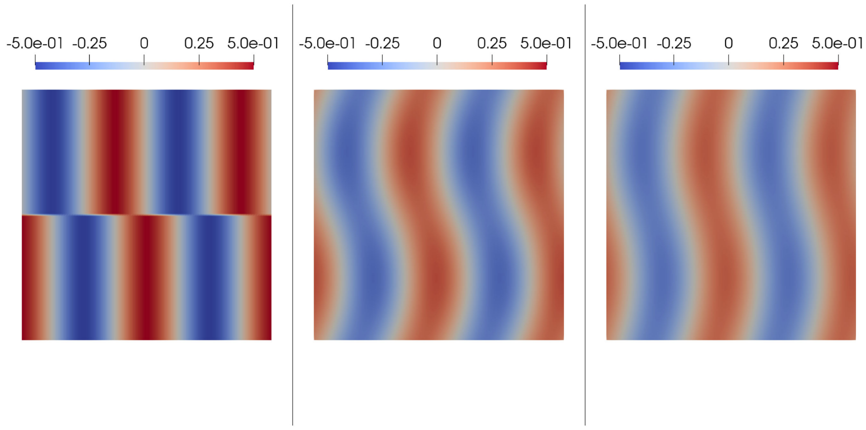

Fig. 4. u by direct simulation at time $t = 0$, $0.1/\varepsilon^2$ and $0.2/\varepsilon^2$.

Stochastic case

We set the truncation numbers $m_R = 10$ and $m_I = 10$. We first present the simulations results of A^R and A^I.

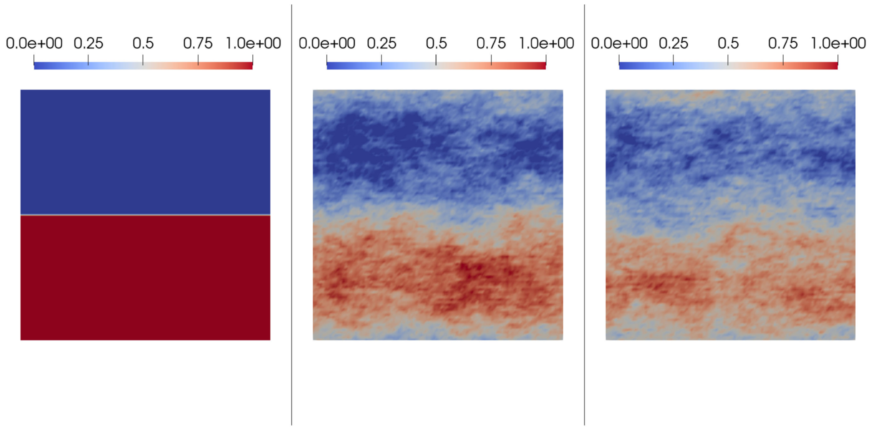

Fig. 5. Solution for A^R at time $T = 0$, 0.1 and 0.2.

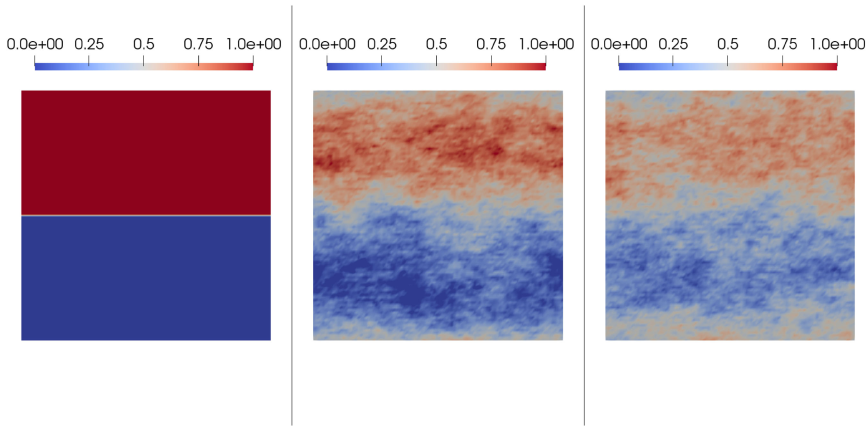

Fig. 6. Solution for A^I at time $T = 0$, 0.1 and 0.2.

We convert the numerical results from A to u.

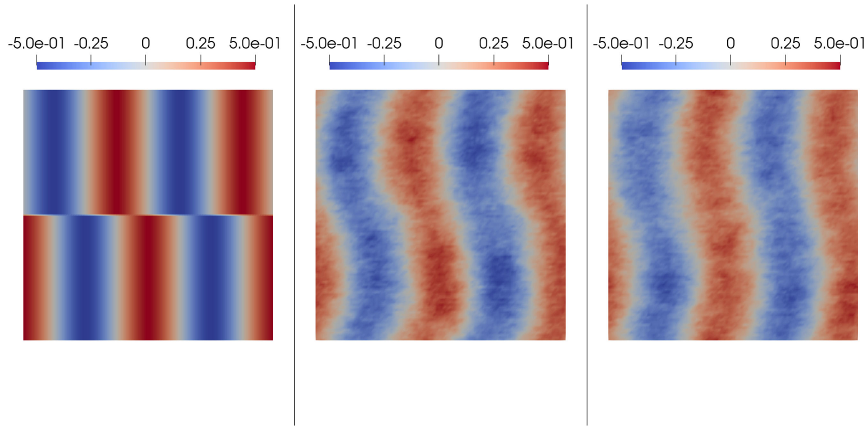

Fig. 7. u at time $t = 0$, $0.1/\varepsilon^2$ and $0.2/\varepsilon^2$ converted by the Ansatz from A of time $T = 0$, 0.1 and 0.2.

Then we perform direct simulation for u and compare it to the simulation results of u convert by A which are presented in Fig. 7.

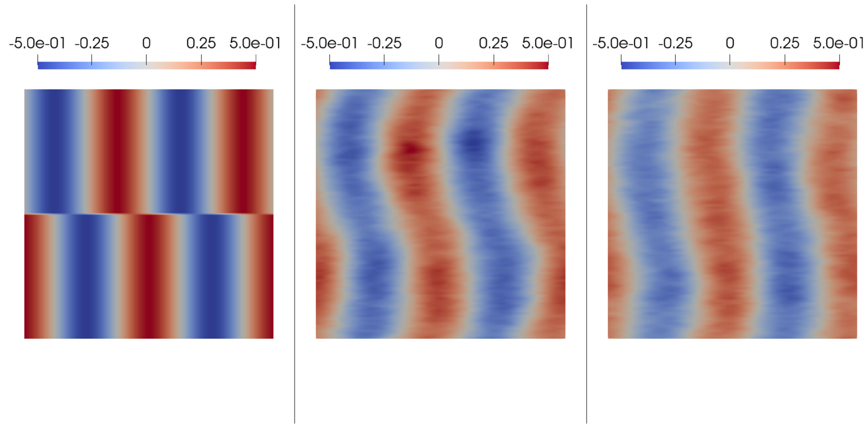

Fig. 8. u by direct simulation at time $t = 0$, $0.1/\varepsilon^2$ and $0.2/\varepsilon^2$.

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REFERENCES

[1] Barton-Smith, M., "Invariant measure for the stochastic Ginzburg Landau equation," *Nonlinear Differential Equations and Applications NoDEA*, **11**: 29–52 (2004).
[2] Bianchi, L. A., Blömker, D., and Schneider, G., "Modulation equation and SPDEs on unbounded domains," *Communications in Mathematical Physics*, **371**: 19–54 (2019).
[3] Blömker, D., Hairer, M., and Pavliotis, G. A., "Modulation equations: Stochastic bifurcation in large domains," *Communications in Mathematical Physics*, **258**: 479–512 (2005).
[4] Da Prato, G., and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press (1992).
[5] Flandoli, F., *Stochastic Navier–Stokes Equations and State Dependent Noise*, Lecture notes for Waseda University (2021).
[6] Flandoli, F., and Gatarek, D., "Martingale and stationary solutions for stochastic Navier–Stokes equations," *Probability Theory and Related Fields*, **102**: 367–391 (1995).
[7] Gyöngy, I., and Krylov, N., "Existence of strong solutions for Itô's stochastic equations via approximations," *Probability Theory and Related Fields*, **105**: 143–158 (1996).
[8] Kirrmann, P., Schneider, G., and Mielke, A., "The validity of modulation equations for extended systems with cubic nonlinearities," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, **122**: 85–91 (1992).
[9] Klapp, J., and Medina, A., *Experimental and Computational Fluid Mechanics*, Springer, Cham (2014).
[10] Mourrat, J.-C., and Weber, H., "Global well-posedness of the dynamic Φ^4 model in the plane," *Annals of Probability*, **45**:

2398–2476 (2017).
[11] Schneider, G., “Validity and limitation of the Newell–Whitehead equation,” *Mathematische Nachrichten*, **176**: 249–263 (1995).
[12] Schneider, G., and Uecker, H., “The amplitude equations for the first instability of electro-convection in nematic liquid crystals in the case of two unbounded space directions,” *Nonlinearity*, **20**: 1361–1386 (2007).
[13] Swift, J., and Hohenberg, P. C., “Hydrodynamic fluctuations at the convective instability,” *Physical Review A*, **15**: 319–328 (1977).
[14] Uecker, H., *Amplitude Equations — An Invitation to Multi-scale Analysis*, Lecture given at the International Summer School Modern Computational Science, Oldenburg, Germany (2010).