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Lecture Notes on the Singular Limit of Reaction-diffusion Systems

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These Lecture Notes are Dedicated to the Memory of Professor Masayasu Mimura

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Part I

The Singular Limit of Competition-diffusion Systems Arising in Population Dynamics as the Reaction Coefficient Tends to Infinity

Chapter 1

Introduction

During the last 30 years, mathematical models have progressively earned their recognition in the field of ecology. Though still not as used and trusted as in other scientific fields (Sagoff [39], DeAngelis *et al.* [11]), they have been applied with success in biological invasion (Shigesada [40], Lewis [33]), endangered species conservation (Williams [45], Kingsland [30]), and many other problems. From the old models of Malthus (1789) and Verhulst (1838) describing the possible growth of a single homogeneous species, to the models proposed nowadays, huge progresses have been made. During this lecture, *we will present some results when looking at two spatially heterogeneous species competing for resources, in the limit when this competition tends to infinity*. In this chapter we introduce the competition-diffusion system which we will be working with.

1.1 Competition systems in population dynamics

For a given species, there are two kinds of competition. The first comes from the individuals of the species itself competing against each other, whether because of a lack of space or food. Such a competition is called *intraspecific* competition. One way to model it, called the logistic effect, is to have a growth rate that decreases with the increase of the population size. The simplest model with a logistic effect is (with dimensionless variables):

$$u_t = r(1 - u(t))u(t), \tag{1.1}$$

with $r > 0$ the intrinsic growth rate, and $u(t)$ is the density of the population depending on the time t (see Iannelli and Pugliese [27], chapter 1). The function $f(u) := r(1 - u)u$ is called the logistic growth function, and its graph is represented on Fig. 1.1, while Fig. 1.2 shows the evolution of the solution u.

The second kind of competition, called *interspecific* competition, is the competition between different species. Given two species u and v, we will model their interaction following mass-action laws: $ku(t)v(t)$ and $\alpha ku(t)v(t)$ where k and α are positive coefficients. Those simple hypotheses to model those two types of competition give a system called a *Lotka–Volterra competition system* which has been extensively studied (see [37, Chapter 3]):

$$\begin{cases} u_t = r_1(1 - u)u - kuv, \\ v_t = r_2(1 - v)v - \alpha kuv. \end{cases} \tag{1.2}$$

One can show that when k is large enough compared to r_1 and r_2, system (1.2) admits two stable equilibrium $(u, v) = (0, 1)$ and $(u, v) = (1, 0)$, and two unstable equilibrium $(u, v) = (0, 0)$ and $(u, v) = (u^*, v^*)$ with $u^*, v^* \in (0, 1)$. Starting from an initial data $(u_0, v_0) \in (0, 1)^2 \setminus \{(u^*, v^*)\}$, the solution $(u(t), v(t))$ of the Lotka–Volterra competition system may converge in long time to $(0, 1)$ or $(1, 0)$. Which species becomes extinct depends on the parameter through the separatrix, and on the initial values. See Fig. 1.3 for a phase plane of a bistable case.

1.2 Competition-diffusion systems in population dynamics

Our previous model does not take into account the movement of species in space. Among the many models proposed so far, reaction-diffusion equations are used to study the spatial segregation of competing species that move by diffusion. Consider a competing system consisting of n species living in a habitat $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$. We denote by $u_i(x, t)$ $(i = 1, 2, \ldots, n)$ their population densities at position $x \in \Omega$ and time $t \geq 0$. The time evolution of $u_i(x, t)$ $(i = 1, 2, \ldots, n)$ is described by the system

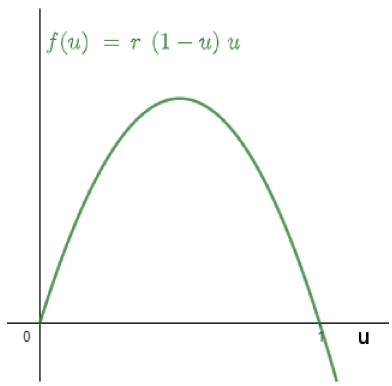

Fig. 1.1. Logistic growth function. Positive for $u < 1$, negative for $u > 1$.

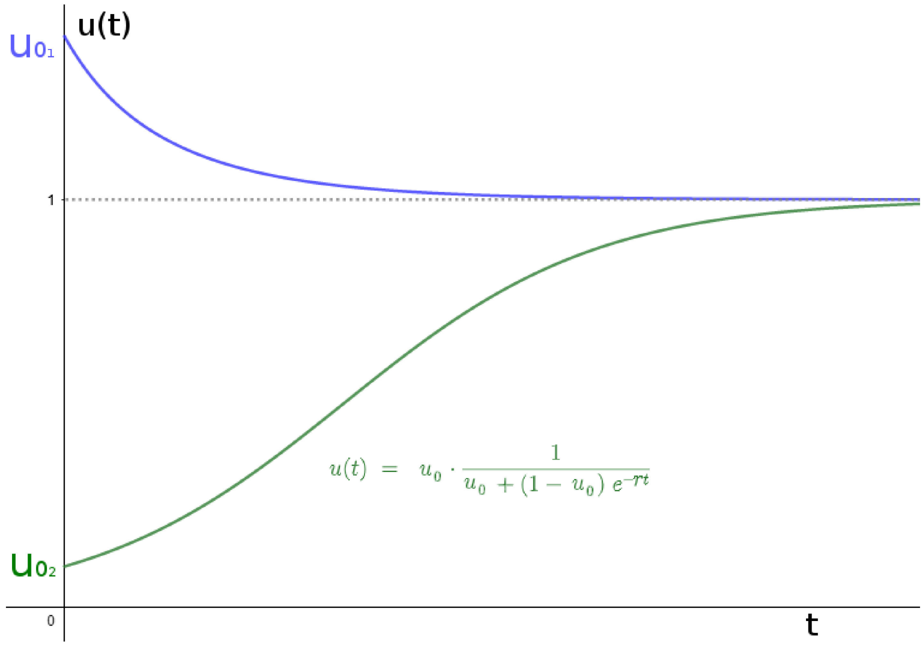

Fig. 1.2. Logistic population growth as a function of time. Note how the population tends towards a stable equilibrium.

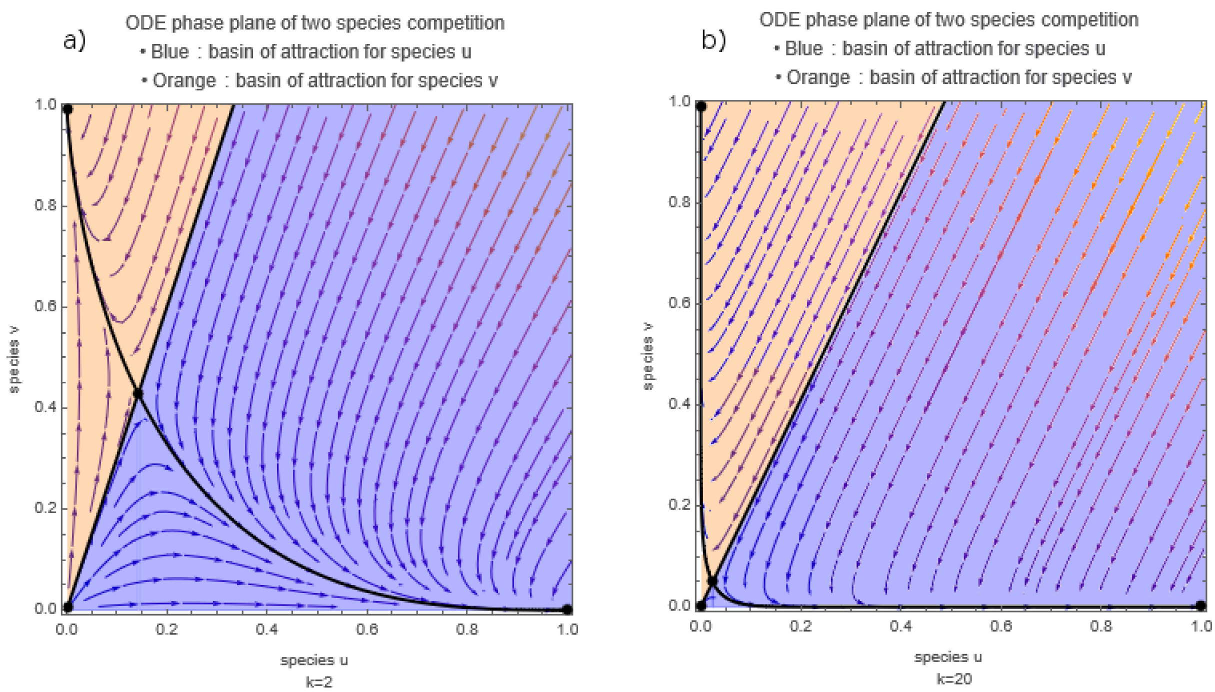

Fig. 1.3. Phase plane of a bistable case for an ODE competition model, with $f(s) = r_1 s(1-s)$, $g(s) = r_2 s(1-s)$, $r_1 = r_2 = 1$, $\alpha = 2$. In a), $k = 2$, while in b), $k = 20$. The basins of attraction illustrate how the system will evolve through time, and which steady states it will reach as $t \to \infty$. Depending on the initial values $u(0)$ and $v(0)$, the solution pair can be located in either of the two basins. This means that the steady state reached as $t \to \infty$ depends on the initial values.

$$u_{it} = d_i \Delta u_i + \left(r_i - a_i u_i - \sum_{j=1}^{n} b_{ij} u_j \right) u_i \quad (i = 1, 2, \ldots, n) \quad x \in \Omega, \quad t > 0, \tag{1.3}$$

where d_i are the diffusion rates, r_i the intrinsic growth rates, a_i the intraspecific competition rates, i.e. the competition between members of the *same* species u_i, and b_{ij} the rates of interspecific competition, i.e. the competition between members of the *different* species u_i and u_j. All rates are positive constants. We assume that Ω is bounded. We first impose zero flux boundary conditions on the boundary $\partial\Omega$,

$$\frac{\partial u_i}{\partial \nu} = 0, \quad (i = 1, 2, \dots, n) \quad x \in \partial\Omega, \quad t > 0, \tag{1.4}$$

where ν is the normal unit vector to $\partial\Omega$. Later in the lecture notes, we will also consider the case of inhomogeneous Dirichlet boundary conditions. The initial conditions are given by

$$u_i(0, x) = u_{0i}(x) \geq 0 \quad (i = 1, 2, \dots, n) \quad x \in \Omega. \tag{1.5}$$

The large time behavior of the solutions of the problem (1.3)–(1.5) has been widely analyzed in order to understand the spatio-temporal segregation of competing species. We first underline the special case where all the diffusion rates d_i are large compared to the other parameters. In this situation, the diffusion processes are dominant and it is therefore easy to find that any (non-negative) solution of the problem (1.3)–(1.5) tends to be spatially homogeneous when $t \to \infty$ (Conway *et al.* [7]). In other words, the asymptotic behavior of the solutions of the problem (1.3)–(1.5) is qualitatively the same as that of the system without diffusion corresponding to (1.3),

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \left(r_i - a_i v_i - \sum_{j=1}^{n} b_{ij} v_j \right) v_i \quad (i = 1, 2, \dots, n) \quad t > 0. \tag{1.6}$$

Thus we know that in this case (1.3) does not present any spatial segregation for the competing species. Note that (1.6) presents a temporal segregation, depending on the values of the parameters r_i, a_i and b_{ij}. We will not study this phenomenon here, but refer for example to an article by Mimura [36]. Our main interest for (1.3)–(1.5) concerns the case where at least one of the diffusion coefficients d_i is not necessarily large from the point of view of the spatial segregation of competing species. In order to analyze this case, we discuss the simplest case of (1.3) with $n = 2$, namely

$$\begin{cases} u_{1t} = d_1 \Delta u_1 + (r_1 - a_1 u_1 - b_1 u_2) u_1, & x \in \Omega,\ t > 0, \\ u_{2t} = d_2 \Delta u_2 + (r_2 - a_2 u_2 - b_2 u_1) u_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.7}$$

with boundary conditions

$$\frac{\partial u_1}{\partial \nu} = 0, \quad \frac{\partial u_2}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{1.8}$$

Let us first note that the stable attractor of (1.7), (1.8) only consists in equilibrium solutions (Hirsch [25], Matano and Mimura [35]). Thus, for the study of the asymptotic behavior of the solutions of (1.7), (1.8) we only have to pay attention to the existence and the stability of the equilibrium solutions. Along the same lines, Kishimoto & Weinberger [31] showed that if Ω is convex, then any spatially inhomogeneous equilibrium solution — when it exists — is unstable. If we assume that two species are strongly in competition, that is to say if the rate of interspecific competition is stronger than the intraspecific one, so that we require that

$$\frac{a_1}{b_2} < \frac{r_1}{r_2} < \frac{b_1}{a_2}, \tag{1.9}$$

then we find that the only stable equilibrium solutions of (1.7), (1.8) are given by $(u_1, u_2) = (r_1/a_1, 0)$ and $(u_1, u_2) = (0, r_2/a_2)$. In ecological terms, this implies that the two competing species can never coexist under strong competition. This is called Gause's competitive exclusion. On the other hand, if the domain Ω is not convex, the structure of equilibrium solutions is complicated, depending on the form of Ω [14]. In fact, if Ω takes a suitable two-dimensional dumbbell shape, there exist spatially inhomogeneous stable equilibrium solutions which exhibit spatial segregation in the sense that u_1 and u_2 take values close to $(r_1/a_1, 0)$ in one subregion and close to $(0, r_2/a_2)$ in the other. Thus the results above inform us about the asymptotic behavior of the solutions. However, from the point of view of ecological applications, it is more interesting to know the transient behavior of solutions. For this we consider the situation where the diffusion rates d_1 and d_2 are small enough or all the other rates r_i, a_i and b_i are large enough and satisfy (1.9). We rewrite (1.7) as

$$\begin{cases} u_{1t} = \varepsilon^2 \Delta u_1 + (r_1 - a_1 u_1 - b_1 u_2) u_1, & x \in \Omega,\ t > 0, \\ u_{2t} = d\varepsilon^2 \Delta u_2 + (r_2 - a_2 u_2 - b_2 u_1) u_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.10}$$

where ε is a small parameter. If the competing species are distributed according to (1.10) it is natural to define the subregions $\Omega_1(t) = \{x \in \Omega : (u_1, u_2)(x, t) \approx (r_1/a_1, 0)\}$ and $\Omega_2(t) = \{x \in \Omega : (u_1, u_2)(x, t) \approx (0, r_2/a_2)\}$. In order to study the dynamics of segregation between u_1 and u_2, we take the limit $\varepsilon \downarrow 0$ in (1.10) so that the internal layers that exist for small values of $\varepsilon > 0$ become proper interfaces, say $\Gamma(t)$, which is the boundary between the two regions $\Omega_1(t)$ and $\Omega_2(t)$. Using singular limit analysis, Ei and Yanagida [15] derived the following evolution equation to describe the motion of the interface $\Gamma(t)$,

$$V = \varepsilon L(d)(N - 1)\kappa + c, \tag{1.11}$$

where V is the normal velocity of the interface, κ its mean curvature, $L(d)$ a positive constant depending on d such that

$L(1) = 1$ and c the speed of the traveling wave solution (u_1, u_2) of the one-dimensional system corresponding to (1.7) with $d_1 = 1$ and $d_2 = d$, namely

$$\begin{cases} u_{1t} = u_{1xx} + (r_1 - a_1u_1 - b_1u_2)u_1, & x \in \mathbb{R},\ t > 0, \\ u_{2t} = du_{2xx} + (r_2 - b_2u_1 - a_2u_2)u_2, & x \in \mathbb{R},\ t > 0, \end{cases} \tag{1.12}$$

with the conditions at infinity

$$(u_1, u_2)(-\infty, t) = \left(\frac{r_1}{a_1}, 0\right) \quad \text{et} \quad (u_1, u_2)(\infty, t) = \left(0, \frac{r_2}{a_2}\right). \tag{1.13}$$

Kan-on [28] proved that the speed of the traveling wave solution of the problem (1.12), (1.13) is unique for fixed values of the rates r_i, a_i and b_i ($i = 1.2$). In particular, if a_1 is a free parameter and the other parameters are fixed and satisfy the inequalities (1.9), then there exists a unique constant $a^* > 0$ such that $c = 0$ if $a_1 = a^*$, $c > 0$ if $a_1 > a^*$, and $c < 0$ if $a_1 < a^*$. For the special case where $c = 0$, (1.11) becomes the equation of motion by mean curvature, which has been studied analytically and numerically. The interface $\Gamma(t)$ obtained from (1.11) provides information on the dynamics of spatial segregation between the two competing species.

This result clearly shows the similarity between this class of problems and the Allen–Cahn equation first studied by Keller, Sternberg and Rubinstein [29], where the boundary interface moves along its mean curvature. In the second part of this course, we will formally derive the limit problem for the Allen–Cahn equation.

In the first part of this course, we consider a situation different from the one obtained above, namely the case where only the interspecific competition rates b_1 and b_2 are very large. To study this situation, it is convenient to rewrite (1.7) as

$$\begin{cases} u_{1t} = d_1\Delta u_1 + r_1(1 - u_1)u_1 - bu_1u_2, & x \in \Omega,\ t > 0, \\ u_{2t} = d_2\Delta u_2 + r_2(1 - u_2)u_2 - \alpha bu_1u_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.14}$$

where b and α are positive constants. We assume that b is the only parameter that is large and that all other parameters are of order $O(1)$. The coefficient $\alpha > 0$ is the competition ratio between the two species u_1 and u_2. If $\alpha > 1$, then u_1 has a competitive advantage over u_2, while if $\alpha < 1$, the situation is reversed.

We will take b as a free parameter and keep the other parameters d_1, d_2, r_1, r_2 and α fixed. For values of b which are neither large nor small, we show numerically that u_1 and u_2 exhibit spatial segregation with a fairly large overlapping zone. As the value of b increases, the overlapping area becomes narrower. Thus, taking the limit $b \to \infty$, we can expect that u_1 and u_2 have disjoint supports (habitats) with a single common curve, which separates the habitats of the two competing species.

One of the purposes of these notes is to derive the limiting system of (1.7) as $b \to \infty$, which is called the *spatial segregation limit*, to describe the time evolution of the supports of u_1 and u_2. In chapter 2 we will consider the homogeneous Neumann boundary condition and in chapter 3 we will consider the inhomogeneous Dirichlet boundary condition. As will be proved, the limit system can be described by a *free boundary problem* which is a two-phase Stefan-type problem with reaction terms.

Let $\Gamma(t)$ be the interface which separates the two subregions

$$\Omega_1(t) = \{x \in \Omega : u_1(x, t) > 0,\ u_2(x, t) = 0\}$$

and

$$\Omega_2(t) = \{x \in \Omega : u_1(x, t) = 0,\ u_2(x, t) > 0\}.$$

Then u_1 and u_2 satisfy

$$\begin{cases} u_{1t} = d_1\Delta u_1 + r_1(1 - u_1)u_1, & x \in \Omega_1(t),\ t > 0 \\ u_{2t} = d_2\Delta u_2 + r_2(1 - u_2)u_2 & x \in \Omega_2(t),\ t > 0 \\ \dfrac{\partial u_1}{\partial \nu} = 0,\ \dfrac{\partial u_2}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.15}$$

On the interface,

$$u_1 = 0, \quad u_2 = 0, \quad x \in \Gamma(t) \quad \text{for } t > 0, \tag{1.16}$$

and

$$0 = -\alpha d_1 \frac{\partial u_1}{\partial \nu}(x, t) - d_2 \frac{\partial u_2}{\partial \nu}(x, t) \quad x \in \Gamma(t) \quad \text{for} \quad t > 0, \tag{1.17}$$

where ν is the normal vector to $\Gamma(t)$. The initial conditions are given by

$$u_i(x, 0) = u_{i0}(x), \quad x \in \Omega_1(0) \quad (i = 1, 2), \tag{1.18}$$

and are such that their support is separated by the line

$$\Gamma(0) = \Gamma_0. \tag{1.19}$$

The problem is to find the functions $(u_1(x,t), u_2(x,t))$ and $\Gamma(t)$ which satisfy (1.15)–(1.19). If this problem can be solved, the interface $\Gamma(t)$ determines the segregation patterns between the two strongly competing species. Note that the system (1.15)–(1.19) is quite similar to the standard two-phase Stefan problem, except for the two following points: (*i*) the partial differential equations in (1.15) for u_1 and u_2 are not heat equations, but *logistic growth equations* which are well known in theoretical ecology; (*ii*) the interface equation (1.17) is such that the latent heat coefficient is equal to zero. The coefficient α of the interspecific competition between u_1 and u_2 is contained in (1.17). These notes extend a similar study due to Evans [18] in the case of a slightly simpler system without growth terms, which he considers with more restrictive assumptions on the initial data. Let us also mention the results of Dancer and Du [9] on the limiting behavior of equilibrium solutions in higher space dimensions. For a study of the limit free boundary problem without growth terms, we refer to Cannon and Hill [5] and to a paper by Tonegawa [44] which proves regularity properties of the solution and the interface. We will finally show that our method of analyzing the spatial segregation limit can be applied to some three-component competition-diffusion systems, where the latent heat coefficient is strictly positive.

Chapter 2

Spatial Segregation Limit of a Competition-diffusion System with Neumann Boundary Conditions

In this chapter, we study the competition-diffusion system (1.14) introduced in Chapter 1, with Neumann boundary conditions, in the limit where the interspecific competition tends to infinity. More precisely, let Ω be a bounded domain of class C^1 in $\mathbb{R}^N$, $T > 0$ be an arbitrary positive time, and consider the following competition-diffusion system:

$$\begin{cases} u_t = d_1\,\Delta u + f(u) - kuv & \text{in } \Omega\times(0,T],\\ v_t = d_2\,\Delta v + g(v) - \alpha kuv & \text{in } \Omega\times(0,T],\\ u_\nu = 0,\ v_\nu = 0 & \text{on } \partial\Omega\times(0,T],\\ u(\cdot,0) = u_0^k,\ v(\cdot,0) = v_0^k & \text{on } \Omega, \end{cases} \qquad (\mathcal{P}^k)$$

where $f(s) = \mu_1 s(1-s)$, $g(s) = \mu_2 s(1-s)$, k, α, d_1, d_2, μ_1, μ_2 are all positive constants, and $u_0^k, v_0^k \in C(\overline{\Omega})$ with $0 \le u_0^k, v_0^k \le 1$. k is a positive free parameter. We assume moreover that there exist $u_0, v_0 \in C(\overline{\Omega})$ such that $u_0^k \to u_0$ and $v_0^k \to v_0$ uniformly on $C(\overline{\Omega})$. We wish to investigate the behavior of the solution pair (u^k, v^k) as k tends to infinity.

By a solution of problem $(\mathcal{P}^k)$ we mean a pair of functions (u^k, v^k) such that $u^k, v^k \in C(\overline{Q_T}) \cap C^{2,1}(\overline{\Omega}\times[\delta,T])$ for all $\delta \in (0,T)$, where $Q_T := \Omega\times(0,T)$. This chapter is based upon the articles [10] and [23].

2.1 A priori bounds and relative compactness

We start by proving a priori bounds on the solutions and their derivatives, which will be uniform with respect to the parameter k. This will enable us to use compactness arguments to obtain the convergence of the solutions as k tends to infinity, as well as to study the properties of the limit. We start with a priori bounds on u^k and v^k by applying the comparison principle.

Proposition 2.1. *Let $k > 0$, and (u^k, v^k) be a solution of $(\mathcal{P}^k)$. Then $0 \le u^k \le 1$ and $0 \le v^k \le 1$ in $\overline{Q_T}$, where $Q_T := \Omega\times(0,T)$.*

Proof. Define

$$\mathcal{L}_1(u^k) := u_t^k - d_1\,\Delta u^k - f(u^k) + ku^kv^k,$$
$$\mathcal{L}_2(v^k) := v_t^k - d_2\,\Delta v^k - g(v^k) + \alpha ku^kv^k.$$

Since $\mathcal{L}_i(0) = 0$, and $\mathcal{L}_i(1) \ge 0$ for $i = 1, 2$, the assertion follows from the comparison principle. □

As a corollary, we deduce the existence and uniqueness of the solution.

Proposition 2.2. *For any initial condition $u_0^k, v_0^k \in C(\overline{\Omega})$ with $0 \le u_0^k, v_0^k \le 1$, there exists a unique solution (u^k, v^k) of $(\mathcal{P}^k)$ with $u^k, v^k \in C(\overline{Q_T}) \cap C^{2,1}(\overline{\Omega}\times[\delta,T])$ for all $\delta \in (0,T)$.*

Proof. The result follows from Proposition 2.1 and [34, Proposition 7.3.2 p. 277]. □

Next, we obtain a bound on the interspecific competition terms, which will turn out to be useful to study the properties of the limit of the solutions (u^k, v^k) for large values of k.

Proposition 2.3. *Define $l_0 := \max_{s\in[0,1]} f(s)$. Then for all $k \ge 1$*

$$\int_0^T\int_\Omega u^kv^k \le \frac{|\Omega|}{k}(l_0T + 1).$$

Proof. Integrating the equation in $(\mathcal{P}^k)$ for u^k over $Q_T := \Omega\times(0,T)$ yields

$$\int_0^T\int_\Omega u_t^k = d_1\int_0^T\int_\Omega \Delta u^k + \int_0^T\int_\Omega f(u^k) - k\int_0^T\int_\Omega u^kv^k.$$

By Fubini's theorem $\int_0^T\int_\Omega u_t^k = \int_\Omega\int_0^T u_t^k$ so by applying the fundamental theorem of calculus and then Green first equation on $\int_\Omega \Delta u^k$, we obtain:

$$k\int_0^T\int_\Omega u^kv^k = d_1\int_0^T\int_{\partial\Omega} u_\nu^k + \int_0^T\int_\Omega f(u^k) - \int_\Omega u^k(\cdot,T) + \int_\Omega u_0^k.$$

First remark that the homogeneous Neumann boundary condition implies that $\int_{\partial\Omega} u_\nu^k = 0$. Moreover, since $0 \le u^k \le 1$ and by definition of l_0, we have

$$\int_0^T \int_\Omega f(u^k) \le T|\Omega| l_0,$$
$$-\int_\Omega u^k(\cdot, T) \le 0,$$
$$\int_\Omega u_0^k \le |\Omega|.$$

Therefore,

$$\int_0^T \int_\Omega u^k v^k \le \frac{|\Omega|}{k}(l_0 T + 1).$$

This completes the proof. □

Next, we want to obtain bounds on the sequences (u^k) and (v^k) in the function space $L^2(0, T; H^1(\Omega))$ which we will use to obtain relative compactness. Remark that by Proposition 2.1 we already know that the sequences (u^k) and (v^k) are bounded in $L^2(0, T; L^2(\Omega))$, so it only remains to prove that the sequences (∇u^k) and (∇v^k) are also bounded in $L^2(0, T; L^2(\Omega))$. This is given by the following proposition.

Proposition 2.4. *There exists a positive constant $\mathcal{C}$, which does not depend on k, such that for all $k \ge 1$,*

$$\int_0^T \int_\Omega |\nabla u^k|^2 \le \mathcal{C},$$
$$\int_0^T \int_\Omega |\nabla v^k|^2 \le \mathcal{C}.$$

Proof. We multiply the first equation in $(\mathcal{P}^k)$ by u^k and integrate it on Ω. This yields

$$\int_\Omega u^k u_t^k = d_1 \int_\Omega u^k \Delta u^k + \int_\Omega u^k f(u^k) - k \int_\Omega (u^k)^2 v^k.$$

On the one hand, $u^k u_t^k = \frac{1}{2}\partial_t (u^k)^2$, and as u^k and u_t^k are bounded in (x, t) (recall that u^k is C^1 in time), and Ω is bounded, we obtain $\int_\Omega u^k u_t^k = \frac{1}{2}\int_\Omega \partial_t (u^k)^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega (u^k)^2$. On the other hand, using Green's formula and the Neumann boundary condition, we get $\int_\Omega u^k \Delta u^k = -\int_\Omega |\nabla u^k|^2$. Finally, Proposition 2.1 gives us directly

$$\int_\Omega u^k f(u^k) \le l_0 |\Omega|,$$
$$-k \int_\Omega (u^k)^2 v^k \le 0.$$

Therefore,

$$d_1 \int_\Omega |\nabla u^k|^2 \le l_0 |\Omega| - \frac{1}{2}\frac{d}{dt}\int_\Omega (u^k)^2,$$

which we integrate in time to obtain,

$$\begin{aligned} d_1 \int_0^T \int_\Omega |\nabla u^k|^2 &\le l_0 |\Omega| T - \frac{1}{2}\int_\Omega (u^k(\cdot, T))^2 + \frac{1}{2}\int_\Omega (u_0^k)^2 \\ &\le l_0 |\Omega| T + \frac{1}{2}\int_\Omega (u_0^k)^2. \end{aligned}$$

This completes the proof of the estimate on ∇u^k. The estimate on ∇v^k can be proved similarly using the equation for v^k in $(\mathcal{P}_k)$. □

Using the previous propositions, we proved that the sequences (u^k) and (v^k) are bounded in $L^2(0, T; H^1(\Omega))$. As $L^2(0, T; H^1(\Omega))$ is a separable Hilbert space, this is sufficient to have weak convergence along a subsequence (see [3, Theorem 3.18 p. 69]). Weak convergence is enough to pass to the limit in the linear term in a weak form of equation $(\mathcal{P}^k)$, as continuous linear operator are weakly continuous (see [3]). However, the nonlinear terms are not weakly continuous, and to deal with them, we will need strong convergence in $L^1(Q_T)$. To this end, we introduce the new variable $w^k := u^k - \frac{1}{\alpha} v^k$, in order to eliminate the product term. It satisfies

$$w_t^k = d_1 \Delta u^k - \frac{d_2}{\alpha}\Delta v^k + f(u^k) - \frac{1}{\alpha} g(v^k) \text{ in } Q_T \tag{2.1}$$

together with the homogeneous Neumann boundary condition

$$w_\nu^k = 0 \text{ on } S_T := \partial\Omega \times (0, T).$$

The sequence (w^k) is bounded in $L^2(0, T; H^1(\Omega))$, which represents estimates for w^k and its spatial derivatives. In order to apply a compact embedding result that will be given later, the only thing missing is an estimate on the time derivative. We prove thus the following proposition:

Proposition 2.5. *The family (w^k_t) is bounded in $L^2(0,T;H^1(\Omega)')$, uniformly with respect to k.*

Proof. Multiplying the equation for w^k in $(\mathcal{P}_k)$ by $\xi \in L^2(0,T;H^1(\Omega))$ and integrating it on $Q_T = \Omega \times (0,T)$, we obtain after integrating by parts

$$\int_0^T \langle w^k_t, \xi\rangle = -d_1 \int_0^T \int_\Omega \nabla u^k \cdot \nabla \xi + \frac{d_2}{\alpha}\int_0^T \int_\Omega \nabla v^k \cdot \nabla \xi + \int_0^T \int_\Omega \left\{ f(u^k) - \frac{1}{\alpha} g(v^k)\right\}\xi$$

where $\langle \cdot,\cdot\rangle$ is the duality product between $H^1(\Omega)$ and $H^1(\Omega)'$. The right-hand side is clearly integrable and continuous over $\xi \in L^2(0,T;H^1(\Omega))$ by Propositions 2.1 and 2.4. Therefore $w^k_t \in L^2(0,T;H^1(\Omega)')$. Moreover, using similarly the Proposition 2.1 and 2.4, as well as the Cauchy–Schwarz inequality, we have

$$\left|\int_0^T \langle w^k_t, \xi\rangle\right| \le M\|\xi\|_{L^2(0,T;H^1(\Omega))}, \quad \forall \xi \in L^2(0,T;H^1(\Omega)),$$

where M is a positive constant independent of k or ξ. This means that

$$\|w^k_t\|_{L^2(0,T;H^1(\Omega)')} \le M,$$

which completes the proof. □

We are now ready to apply the following compactness embedding that can be found in [42, Theorem 2.1 p. 271].

Theorem 2.6. *Let X_0, X, X_1 be three Banach spaces such that $X_0 \subset X \subset X_1$, where the injections are continuous. Let $T > 0$ be a fixed number, and let α_0, α_1 be two finite numbers such that $\alpha_0, \alpha_1 > 1$. We consider the space*

$$\mathcal{Y} = \{v \in L^{\alpha_0}(0,T;X_0), v_t \in L^{\alpha_1}(0,T;X_1)\},$$

where the derivative should be understood in a weak sense. The space $\mathcal{Y}$ is endowed with the norm

$$\|v\|_{\mathcal{Y}} = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v_t\|_{L^{\alpha_1}(0,T;X_1)}.$$

It is obvious that $\mathcal{Y} \subset L^{\alpha_0}(0,T;X_0)$, with continuous injection. If moreover
- *X_i is reflexive, $i = 0, 1$,*
- *The injection $X_0 \to X$ is compact,*

then the injection $\mathcal{Y} \subset L^{\alpha_0}(0,T;X)$ is compact.

Theorem 2.7. *There exist subsequences of (u^k) and (v^k), which we denote again by (u^k) and (v^k), functions $\bar{u}, \bar{v} \in L^2(0,T;H^1(\Omega))$ such that $0 \le \bar{u}, \bar{v} \le 1$, and a function $w \in L^2(0,T;L^2(\Omega)) = L^2(Q_T)$ such that*
(i) $u^k \rightharpoonup \bar{u}$ and $v^k \rightharpoonup \bar{v}$ weakly in $L^2(0,T;H^1(\Omega))$,
(ii) $w^k \to w$ strongly in $L^2(Q_T)$ as $k \to \infty$ *and almost everywhere in Q_T,*
(iii) $u^k v^k \to 0$ strongly in $L^1(Q_T)$ as $k \to \infty$ *and almost everywhere in Q_T.*

Proof. We apply Theorem 2.6 with $X_0 = H^1(\Omega)$, $X = L^2(\Omega)$, $X_1 = H^1(\Omega)'$, and $\alpha_0 = \alpha_1 = 2$. Clearly, X_0 and X_1 are reflexive, and it is also standard that the inclusion $H^1(\Omega) \subset L^2(\Omega)$ is compact (see [3, Theorem 9.16 p. 285]), so the conclusion of the theorem applies. As by the Proposition 2.5, (w^k) is bounded in $\mathcal{Y}$, and since we already established (u^k) and (v^k) are bounded in $L^2(0,T;H^1(\Omega))$, there exists subsequences of (u^k) and (v^k), which we denote again (u^k) and (v^k), functions $\bar{u}, \bar{v} \in L^2(0,T;H^1(\Omega))$ such that $0 \le \bar{u}, \bar{v} \le 1$, and a function $w \in L^2(0,T;L^2(\Omega)) = L^2(Q_T)$ such that

$$u^k \rightharpoonup \bar{u} \text{ and } v^k \rightharpoonup \bar{v} \text{ weakly in } L^2(0,T;H^1(\Omega))$$

and

$$w^k \to w \text{ strongly in } L^2(Q_T) \text{ as } k \to \infty.$$

We can moreover assume that the convergence of (w^k) holds also almost everywhere, since convergence in an L^p space implies convergence almost everywhere along some subsequence (Theorem 4.9 in [3]). Furthermore, it also follows from Proposition 2.3 that

$$u^k v^k \to 0 \text{ as } k \to \infty \text{ in } L^1(Q_T) \text{ and a.e. in } Q_T.$$

This completes the proof. □

2.2 Characterization of the limit functions

Next we want to study the limit functions $\bar{u}$ and $\bar{v}$. We start by relating them to w.

Proposition 2.8. *The subsequences u^k and v^k are such that*

$$u^k \to w^+ = \max(0,w) \text{ and } v^k \to \alpha w^- = -\alpha \min(0,w) \text{ as } k \to \infty$$

in $L^1(Q_T)$ and a.e. in Q_T. As a consequence

$$\overline{u} = w^+ \text{ and } \overline{v} = \alpha w^- \text{ so that } w = \overline{u} - \frac{\overline{v}}{\alpha}.$$

Proof. Let $(x,t) \in Q_T$ be such that

$$w^k(x,t) = \left(u^k - \frac{v^k}{\alpha}\right)(x,t) \to w(x,t) \text{ and } (u^k v^k)(x,t) \to 0 \text{ as } k \to \infty.$$

We distinguish three cases according to the sign of w.

1. We first consider the case that $w(x,t) > 0$. Then there exists a positive constant k_0 such that

$$u^k(x,t) \geq \frac{w(x,t)}{2} > 0 \text{ for all } k \geq k_0,$$

which implies that

$$v^k(x,t) \to 0 \text{ and } u^k(x,t) \to w(x,t) = w^+(x,t) \text{ as } k \to \infty.$$

2. Next we consider the case $w(x,t) < 0$. Then there exists a positive constant k_1 such that

$$v^k(x,t) \geq -\frac{\alpha}{2} w(x,t) > 0 \text{ for all } k \geq k_1,$$

so that

$$u^k(x,t) \to 0 \text{ and } v^k(x,t) \to -\alpha w(x,t) = \alpha w^-(x,t) \text{ as } k \to \infty.$$

3. Finally we consider the case where $w(x,t) = 0$. If a subsequence of $u^k(x,t)$, which we denote again by $u^k(x,t)$, is such that $u^k(x,t) \to \mu_1 > 0$, then $v^k(x,t) \to 0$, so that $u^k(x,t) - \frac{1}{\alpha} v^k(x,t) \to \mu_1$ which contradicts the fact that $w(x,t) = 0$. Similarly it is impossible to have that $v^k(x,t) \to \mu_2 > 0$. Hence

$$u^k(x,t) \to 0 \text{ and } v^k(x,t) \to 0 \text{ as } k \to \infty.$$

The convergence in $L^1(Q_T)$ follows from the boundedness of u^k and v^k and the dominated convergence theorem. □

Note that this proposition implies that $\overline{u}$ and $\overline{v}$ have disjoint supports, see an illustration on Fig. 2.1. We now obtain a weak form satisfied by the limit functions $(\overline{u}, \overline{v})$.

Proposition 2.9. *The limit functions $(\overline{u}, \overline{v})$ are such that*

$$\int_0^T \int_\Omega \left\{\left(\overline{u} - \frac{1}{\alpha}\overline{v}\right)\varphi_t - \nabla\left(d_1\overline{u} - \frac{d_2}{\alpha}\overline{v}\right)\nabla\varphi + \left(f(\overline{u}) - \frac{1}{\alpha}g(\overline{v})\right)\varphi\right\} = \int_\Omega \left(u_0 - \frac{v_0}{\alpha}\right)\varphi(\cdot,0) \tag{2.2}$$

for all functions $\varphi \in \mathcal{F}_T := \{\psi \in C^\infty(Q_T),\ \psi(\cdot,T) = 0\}$.

Proof. Let $\varphi \in \mathcal{F}_T$. Multiplying the equation (2.1) on w^k by φ and integrating by parts (both in time and space), we obtain the identity

$$-\int_0^T \int_\Omega \left(u^k - \frac{1}{\alpha}v^k\right)\varphi_t + \int_\Omega \left(u_0^k - \frac{v_0^k}{\alpha}\right)\varphi(\cdot,0) + \int_0^T \int_\Omega \nabla\left(d_1 u^k - \frac{d_2}{\alpha}v^k\right)\cdot\nabla\varphi$$
$$- \int_0^T \int_\Omega \left(f(u^k) - \frac{1}{\alpha}g(v^k)\right)\varphi = 0.$$

The second term converges to $\int_\Omega (u_0 - \frac{v_0}{\alpha})\varphi(\cdot,0)$ by assumption on the family (u_0^k) and (v_0^k). Moreover, as

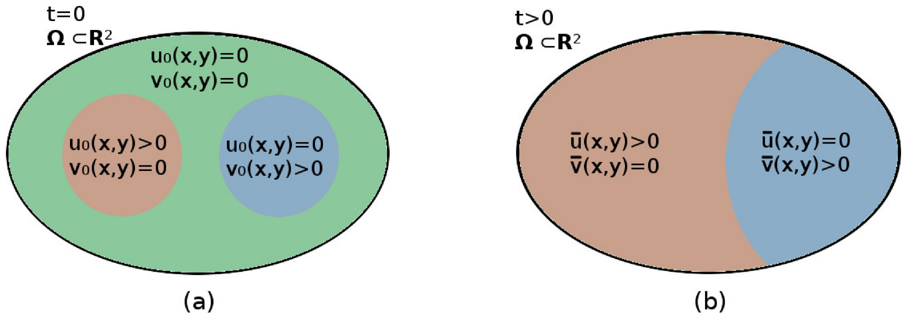

Fig. 2.1. Illustration of the supports of the limit functions $\overline{u}$ and $\overline{v}$, at time (a) $t = 0$ and (b) $t > 0$ a given time before the equilibrium. Here $\overline{u}$ has a competitive advantage on $\overline{v}$.

$$u^k \to \overline{u}, \text{ and } v^k \to \overline{v} \text{ a.e. in } Q_T \text{ as } k \to \infty,$$

and $|u^k|, |v^k| \leq 1$ for all $k \geq 1$, it follows by the dominated convergence theorem that

$$\int_0^T \int_\Omega \left(u^k - \frac{1}{\alpha} v^k\right)\varphi_t \to \int_0^T \int_\Omega \left(\overline{u} - \frac{1}{\alpha}\overline{v}\right)\varphi_t \text{ as } k \to \infty,$$

$$\int_0^T \int_\Omega \left(f(u^k) - \frac{1}{\alpha} g(v^k)\right)\varphi \to \int_0^T \int_\Omega \left(f(\overline{u}) - \frac{1}{\alpha} g(\overline{v})\right)\varphi \text{ as } k \to \infty.$$

Finally, as

$$u^k \rightharpoonup \overline{u} \text{ and } v^k \rightharpoonup \overline{v} \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ as } k \to \infty,$$

we have that

$$\int_0^T \int_\Omega \nabla\left(d_1 u^k - \frac{d_2}{\alpha} v^k\right)\nabla\varphi \to \int_0^T \int_\Omega \nabla\left(d_1 \overline{u} - \frac{d_2}{\alpha}\overline{v}\right)\nabla\varphi \text{ as } k \to \infty.$$

This ends the proof of the proposition. □

In terms of w, the previous proposition suggests that w is a solution of the following problem:

$$\begin{cases} w_t = \nabla \cdot (d(w)\nabla w) + h(w) & \text{in } Q_T, \\ w_\nu = 0 & \text{on } \partial\Omega \times (0,T], \\ w(x,0) = w_0(x) := u_0(x) - \dfrac{v_0(x)}{\alpha}, & x \in \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where

$$d(s) = \begin{cases} d_1 & \text{if } s > 0, \\ d_2 & \text{if } s < 0, \end{cases}$$

and

$$h(s) = \begin{cases} f(s) & \text{if } s > 0, \\ g(-\alpha s) & \text{if } s < 0. \end{cases}$$

Note that this problem only admits a weak solution since d is not continuous. Next, we present a possible definition of a weak solution of Problem $(\mathcal{P})$.

Definition 2.1. *A function w is a weak solution of problem $(\mathcal{P})$ on $[0,T]$ if*

$$w \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T,H^1(\Omega)) \cap C([0,T];L^2(\Omega)),$$

and for all $\varphi \in \mathcal{F}_T := \{\psi \in C^\infty(Q_T),\ \psi(\cdot,T) = 0\}$

$$\int_0^T \int_\Omega \{w\varphi_t - d(w)\nabla(w)\nabla(\varphi) + h(w)\varphi\} = \int_\Omega w_0\varphi(\cdot,0).$$

Theorem 2.10. *The function w is a weak solution of problem $(\mathcal{P})$.*

Proof. We already know that $w \in L^\infty(\Omega \times (0,T))$, and that $w \in L^2(0,T;H^1(\Omega))$. Since also $w_t \in L^2(0,T;H^1(\Omega)')$, it follows from a standard regularity result that $w \in C([0,T];L^2(\Omega))$ (see [42], Lemma 1.2 p. 260). Now observe that, by Proposition 2.8,

$$d_1\nabla\overline{u} - \frac{d_2}{\alpha}\nabla\overline{v} = d(w)\nabla w,$$

$$f(\overline{u}) - \frac{1}{\alpha} g(\overline{v}) = h(w).$$

Therefore this is a straightforward consequence of Proposition 2.9 that for all function $\varphi \in \mathcal{F}_T$,

$$\int_0^T \int_\Omega \{w\varphi_t - d(w)\nabla w \cdot \nabla\varphi + h(w)\varphi\} = \int_\Omega w_0\varphi(\cdot,0).$$

This completes the proof. □

From this, we know that problem $(\mathcal{P})$ has at least one weak solution. The following theorem gives uniqueness along with a more precise regularity result.

Theorem 2.11. *Problem $(\mathcal{P})$ has exactly one weak solution w, and w is Hölder continuous: $w \in C^{\beta,\beta/2}(\overline{\Omega} \times [0,\infty))$ for all $\beta \in (0,1)$.*

Proof. The proof of uniqueness is similar to that of Aronson, Crandall and Peletier. The regularity of w follows from DiBenedetto ([13], Theorems 1.1 and 1.3 p. 41 and 43). □

2.3 A strong form of the limit free boundary problem

Proposition 2.8 shows that $\bar{u}$ and $\bar{v}$ have disjoint supports, separated by a *moving boundary* which is the level set $\{w = 0\}$. So far we have shown a weak form of the limit problem where the moving boundary does not explicitly appear. We will now show a strong formulation of the free boundary problem, with explicit interface conditions, under a few additional regularity assumptions. However, before doing so, let us slightly rewrite the weak form of problem $(\mathcal{P})$. Let us introduce $\mathcal{D}(s) := d_1 s^+ - d_2 s^-$ where $s^+ = \max(s, 0)$ and $s^- = -\min(s, 0)$. Since $\mathcal{D}'(s) = d(s)$, we have $\nabla(\mathcal{D}(w)) = d(w)\nabla w$, and thus $(\mathcal{P})$ becomes:

$$\begin{cases} w_t = \Delta \mathcal{D}(w) + h(w) & \text{in } Q_T, \\ w_\nu = 0 & \text{on } \partial\Omega \times (0, T], \\ w(x, 0) = w_0(x) := u_0(x) - \dfrac{v_0(x)}{\alpha} & x \in \Omega. \end{cases} \qquad (\mathcal{P})$$

Theorem 2.12. *Assume that, at each time $t \in [0, T]$, there exists a closed hypersurface $\Gamma(t)$ and two subdomains $\Omega_u(t), \Omega_v(t)$ such that*

$$\overline{\Omega} = \overline{\Omega_u(t)} \cup \overline{\Omega_v(t)}, \quad \Gamma(t) = \overline{\Omega_u(t)} \cap \overline{\Omega_v(t)}, \quad \Omega_v(t) \subset\subset \Omega,$$
$$w(\cdot, t) > 0 \text{ on } \Omega_u(t) \quad w(\cdot, t) < 0 \text{ on } \Omega_v(t).$$

Assume furthermore that $t \mapsto \Gamma(t)$ is smooth enough and that $(\overline{u}, \overline{v}) := (w^+, \alpha w^-)$ are smooth up to $\Gamma(t)$. Then the functions $\overline{u}$ and $\overline{v}$ satisfy the problem

$$\begin{cases} \overline{u}_t = d_1 \Delta \overline{u} + f(\overline{u}) & \text{in } Q_u := \displaystyle\bigcup_{t \in [0,T]} \{\Omega_u(t) \times \{t\}\}, \\ \overline{v}_t = d_2 \Delta \overline{v} + g(\overline{v}) & \text{in } Q_v := \displaystyle\bigcup_{t \in [0,T]} \{\Omega_v(t) \times \{t\}\}, \\ \overline{u} = \overline{v} = 0 & \text{on } \Gamma := \displaystyle\bigcup_{t \in [0,T]} \{\Gamma(t) \times \{t\}\}, \\ d_1 \overline{u}_n = -\dfrac{d_2}{\alpha} \overline{v}_n & \text{on } \Gamma, \\ \overline{u}_\nu = 0 & \text{on } \partial\Omega \times [0, T], \\ \overline{u}(\cdot, 0) = \left[u_0 - \dfrac{v_0}{\alpha}\right]^+, \ \overline{v}(\cdot, 0) = \alpha \left[u_0 - \dfrac{v_0}{\alpha}\right]^- & \text{in } \Omega, \end{cases} \qquad (\mathcal{P})$$

where n denotes the inward pointing normal of the set Ω_v (see Fig. 2.2 for an illustration of the problem with $\Omega \subset \mathbb{R}^2$).

Proof. Recall that w satisfies Problem $(\mathcal{P})$ in the sense of Definition 2.1:

$$\int_0^T \int_\Omega \{w\varphi_t - d(w)\nabla w \nabla\varphi + h(w)\varphi\} = \int_\Omega w_0 \varphi(\cdot, 0) \quad \forall \varphi \in \mathcal{F}_T,$$

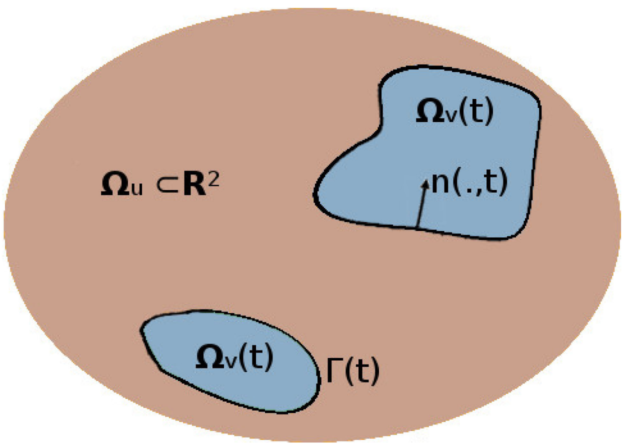

Fig. 2.2. Illustration of the free boundary problem's subdomains.

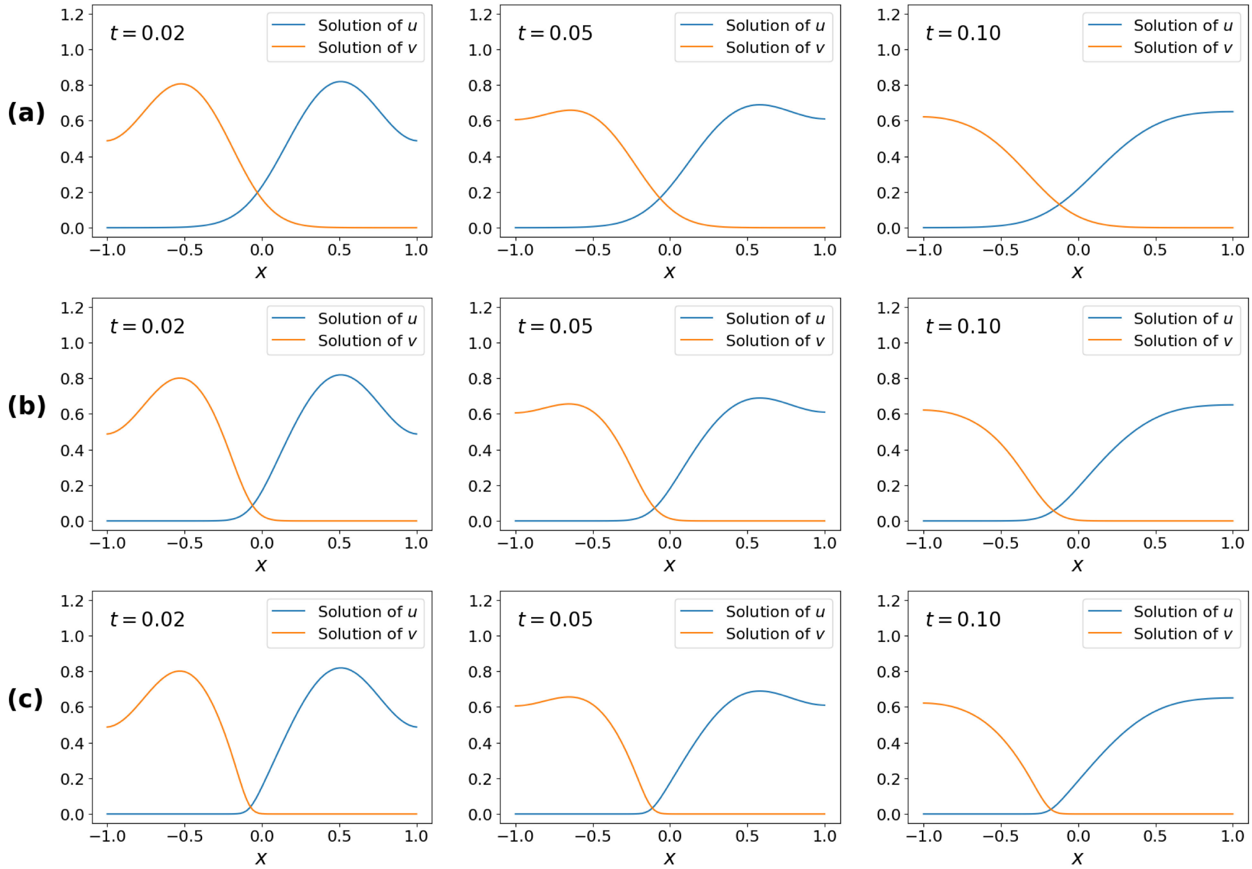

Fig. 2.3. Numerical simulation of the system $(\mathscr{P}_k)$ with $\mu_1 = \mu_2 = 1$, $\alpha = 2$, $d_1 = d_2 = 1$, $\Omega = (-1,1)$, $u_0(x) = e^{-6(x-0.5)^2}$, $v_0(x) = e^{-6(x+0.5)^2}$, for (a) $k = 100$, (b) $k = 1{,}000$, and (c) $k = 10{,}000$ at time $t = 0.02$, $t = 0.05$, and $t = 0.1$.

where $\mathcal{F}_T := \{\psi \in C^\infty(Q_T),\ \psi(\cdot,T) = 0\}$. We are going to perform integration by parts, in space and time, on the supports of u and v, respectively, which is possible because u and v are assumed to be smooth on $\overline{Q_u}$ and $\overline{Q_v}$ respectively. Let us begin with the integration by parts in space. Let $\varphi \in \mathcal{F}_T$. We take the convention that the normal to $\partial\Omega_u$ is outward while the normal to $\partial\Omega_v$ is inward (see Fig. 2.3): $n = n_u = -n_v$. Then

$$\iint_{Q_u} \nabla\overline{u} \cdot \nabla\varphi = -\iint_{Q_u} \Delta\overline{u}\varphi + \int_0^T \int_{\Gamma(t)} \partial_n\overline{u}\varphi + \int_0^T \int_{\partial\Omega} \partial_n\overline{u}\varphi,$$

$$\iint_{Q_v} \nabla\overline{v} \cdot \nabla\varphi = -\iint_{Q_v} \Delta\overline{v}\varphi - \int_0^T \int_{\Gamma(t)} \partial_n\overline{v}\varphi,$$

where we took into account our convention for the normal, and the fact that $\Omega_v(t) \subset\subset \Omega$. Thus

$$\int_0^T \int_\Omega \{-d(w)\nabla w \nabla\varphi\} = \iint_{Q_u} d_1(\Delta\overline{u})\varphi - \iint_{Q_v} \frac{d_2}{\alpha}(\Delta\overline{v})\varphi$$
$$- \int_0^T \int_{\partial\Omega} d_1(\partial_n\overline{u})\varphi - \int_0^T \int_{\Gamma(t)} \left(\frac{d_2}{\alpha}\partial_n\overline{v} + d_1\partial_n\overline{u}\right)\varphi.$$

Now, let us turn to the integration by parts in time. It requires careful attention because the boundaries of the domains $\Omega_u(t)$ and $\Omega_v(t)$ depend on time. We have

$$\frac{d}{dt}\int_{\Omega_u(t)} \overline{u}\varphi = \int_{\Omega_u(t)} (\partial_t\overline{u}\varphi + \overline{u}\partial_t\varphi) + \int_{\Gamma(t)} \overline{u}\varphi\, V_n,$$

where V_n denotes the speed of propagation of the boundary $t \mapsto \Gamma(t)$. Following our convention on the normal, we decide that when $\Omega_u(t)$ increases, then V_n is nonnegative. Remembering that $u = 0$ on Γ, we deduce that

$$\frac{d}{dt}\int_{\Omega_u(t)} \overline{u}\varphi = \int_{\Omega_u(t)} (\partial_t\overline{u}\varphi + \overline{u}\partial_t\varphi).$$

Integrating it in time, and recalling that $\varphi(\cdot,T) = 0$, we get

$$\iint_Q w\partial_t\varphi = -\iint_{Q_u} \partial_t\overline{u}\varphi + \frac{1}{\alpha}\iint_{Q_v} \partial_t\overline{v}\varphi + \int_\Omega\left(\overline{u}(\cdot,0) - \frac{\overline{v}(\cdot,0)}{\alpha}\right)\varphi(\cdot,0).$$

In the same way, we obtain

$$\frac{d}{dt}\int_{\Omega_v(t)} \overline{v}\varphi = \int_{\Omega_v(t)} (\partial_t\overline{v}\varphi + \overline{v}\partial_t\varphi).$$

Therefore, the computations yield

$$-\iint_{Q_u} (\partial_t\overline{u} - d_1\Delta\overline{u} - f(\overline{u}))\varphi + \frac{1}{\alpha}\iint_{Q_v} (\partial_t\overline{v} - d_2\Delta\overline{v} - g(\overline{v}))\varphi + \int_\Omega\left(\overline{u}(\cdot,0) - \frac{\overline{v}(\cdot,0)}{\alpha}\right)\varphi(\cdot,0)$$
$$-\int_0^T\int_{\partial\Omega} d_1(\partial_n\overline{u})\varphi - \int_0^T\int_{\Gamma(t)}\left(\frac{d_2}{\alpha}\,\partial_n\overline{v} + d_1\partial_n\overline{u}\right)\varphi = \int_\Omega w_0\varphi(\cdot,0),$$

for all $\varphi \in \mathcal{F}_T := \{\psi \in C^\infty(Q_T),\ \psi(\cdot,T) = 0\}$. Using test functions φ with suitable supports, namely $\varphi \in C_0^\infty(Q_u)$ and $\varphi \in C_0^\infty(Q_v)$, we obtain

$$\partial_t\overline{u} = d_1\Delta\overline{u} + f(\overline{u}) \quad \text{in } Q_u,$$
$$\partial_t\overline{v} = \frac{d_2}{\alpha}\,\Delta\overline{v} + g(\overline{v}) \quad \text{in } Q_v.$$

We then deduce that

$$\int_0^T\int_{\Gamma(t)}\left(\frac{d_2}{\alpha}\,\partial_n\overline{v} + d_1\partial_n\overline{u}\right)\varphi = 0 \quad \forall\varphi \in C_0^\infty(Q_T),$$

which implies that

$$\frac{d_2}{\alpha}\,\partial_n\overline{v} + d_1\partial_n\overline{u} = 0 \quad \text{on } \Gamma.$$

Then we deduce that

$$\int_0^T\int_{\partial\Omega} d_1(\partial_n\overline{u})\varphi = 0 \quad \text{for all } \varphi \in \mathcal{F}_T \text{ such that } \varphi(\cdot,0) = 0,$$

which implies that

$$\partial_n\overline{u} = 0 \quad \text{on } \partial\Omega.$$

Finally, we have

$$\int_\Omega\left(\overline{u}(\cdot,0) - \frac{\overline{v}(\cdot,0)}{\alpha}\right)\varphi(\cdot,0) = \int_\Omega w_0\varphi(\cdot,0) \quad \forall\varphi \in \mathcal{F}_T,$$

which implies that

$$\overline{u}(\cdot,0) - \frac{\overline{v}(\cdot,0)}{\alpha} = w_0.$$

As a consequence,

$$\overline{u}(\cdot,0) = \left[u_0 - \frac{v_0}{\alpha}\right]^+ = u_0, \quad \overline{v}(\cdot,0) = \alpha\left[u_0 - \frac{v_0}{\alpha}\right]^- = v_0.$$

This completes the proof of the theorem. □

2.4 Concentration of the term ku^kv^k

So far we have completely characterized the limit (u, v) of (u^k, v^k) as $k \to \infty$. We showed that the two populations segregate as k tends to infinity. We now focus on the singular limit of the interspecific term ku^kv^k as k tends to infinity. From proposition 2.3, we have

$$k\iint_{Q_T} u^kv^k \le C(T).$$

Therefore, the family (ku^kv^k) is bounded in $L^1(Q_T)$. Thus, there exists a Radon measure μ on Q_T such that $ku^kv^k \rightharpoonup \mu$ as $k \to \infty$ in the sense of the weak-* convergence of measures, along some subsequence that we still denote by ku^kv^k. We recall that the space of Radon measures on Q_T is the dual space of the space of continuous functions over $\overline{Q_T}$. We only compute μ in the case that the limit problem can be written in a strong form with a smooth interface. More precisely, we prove the following result.

Theorem 2.13. *There exists a measure μ over Q_T such that*

$$ku^k v^k \rightharpoonup \mu, \text{ in the sense of measures.}$$

If the interface Γ is smooth, then μ is localized on Γ and is given by

$$\mu(x,t) = \frac{1}{1+\alpha}[d_1 \partial_n u + d_2 \partial_n v](x,t)\delta_{\Gamma(t)}.$$

Proof. We define $\mu^k = ku^k v^k$, and take $\psi \in C_0^\infty(Q_T)$. Multiplying by ψ the equations for u^k and v^k in $(\mathcal{P}^k)$ and integrating by parts we obtain

$$\begin{aligned}\iint_{Q_T} \mu^k \psi &= \iint_{Q_T} (u^k \partial_t \psi + d_1 u^k \Delta\psi + f(u^k)\psi)\\ &= \frac{1}{\alpha}\iint_{Q_T} (v^k \partial_t \psi + d_2 v^k \Delta\psi + g(v^k)\psi).\end{aligned}$$

Therefore, letting k going to infinity, and using the convergence almost everywhere of u^k and v^k, and the dominated convergence theorem, we obtain

$$\begin{aligned}\iint_{Q_T} \mu\psi &= \iint_{Q_T} (\overline{u}\partial_t \psi + d_1 \overline{u}\Delta\psi + f(\overline{u})\psi),\\ \alpha\iint_{Q_T} \mu\psi &= \iint_{Q_T} (\overline{v}\partial_t \psi + d_2 \overline{v}\Delta\psi + g(\overline{v})\psi).\end{aligned}$$

Applying Theorem 3.12, and integrating by parts, we have

$$\begin{aligned}\iint_{Q_T} (\overline{u}\partial_t \psi + d_1 \overline{u}\Delta\psi + f(\overline{u})\psi) &= -\int_{Q_u} (\partial_t \overline{u}\psi - d_1(\Delta\overline{u})\psi - f(\overline{u})\psi) + d_1 \int_\Gamma ((\partial_n \overline{u})\psi)\\ &= d_1 \int_\Gamma ((\partial_n \overline{u})\psi).\end{aligned}$$

Similarly,

$$\begin{aligned}\iint_{Q_T} (\overline{v}\partial_t \psi + d_1 \overline{v}\Delta\psi + g(\overline{v})\psi) &= -\int_{Q_u} (\partial_t \overline{v}\psi - d_2(\Delta\overline{v})\psi - g(\overline{v})\psi) + d_2 \int_\Gamma ((\partial_n \overline{v})\psi)\\ &= d_2 \int_\Gamma ((\partial_n \overline{v})\psi).\end{aligned}$$

This yields

$$(1+\alpha)\iint_{Q_T} \mu\psi = \int_\Gamma (d_1 \partial_n \overline{u} + d_2 \partial_n \overline{v})\psi,$$

which concludes the proof. □

2.5 Illustration and interpretation of the results

Figures 2.3 and 2.4 give numerical examples for increasing values of k for the system

$$\begin{cases} u_t = d_1\,\Delta u + f(u) - kuv & \text{in } \Omega\times(0,T],\\ v_t = d_2\,\Delta v + g(v) - \alpha kuv & \text{in } \Omega\times(0,T],\\ u_\nu = 0,\ v_\nu = 0 & \text{on } \partial\Omega\times(0,T],\\ u(\cdot,0) = u_0^k,\ v(\cdot,0) = v_0^k & \text{on } \Omega,\end{cases} \qquad (\mathcal{P}^k)$$

with the initial values

$$\begin{cases} u_0(x,y) = e^{-6(x-0.5)^2}\\ v_0(x,y) = e^{-6(x+0.5)^2}\end{cases}$$

for space dimension 1 and

$$\begin{cases} u_0(x,y) = e^{-6(x+0.5)^2-6(y+0.5)^2}\\ v_0(x,y) = e^{-6(x-0.5)^2-6(y-0.5)^2}\end{cases}$$

for space dimension 2 respectively. The simulations in this section and the rest of the manuscript have been obtained using the finite element tools from Wolfram Mathematica 12.3 (see Appedix A for listings and details on meshes used).

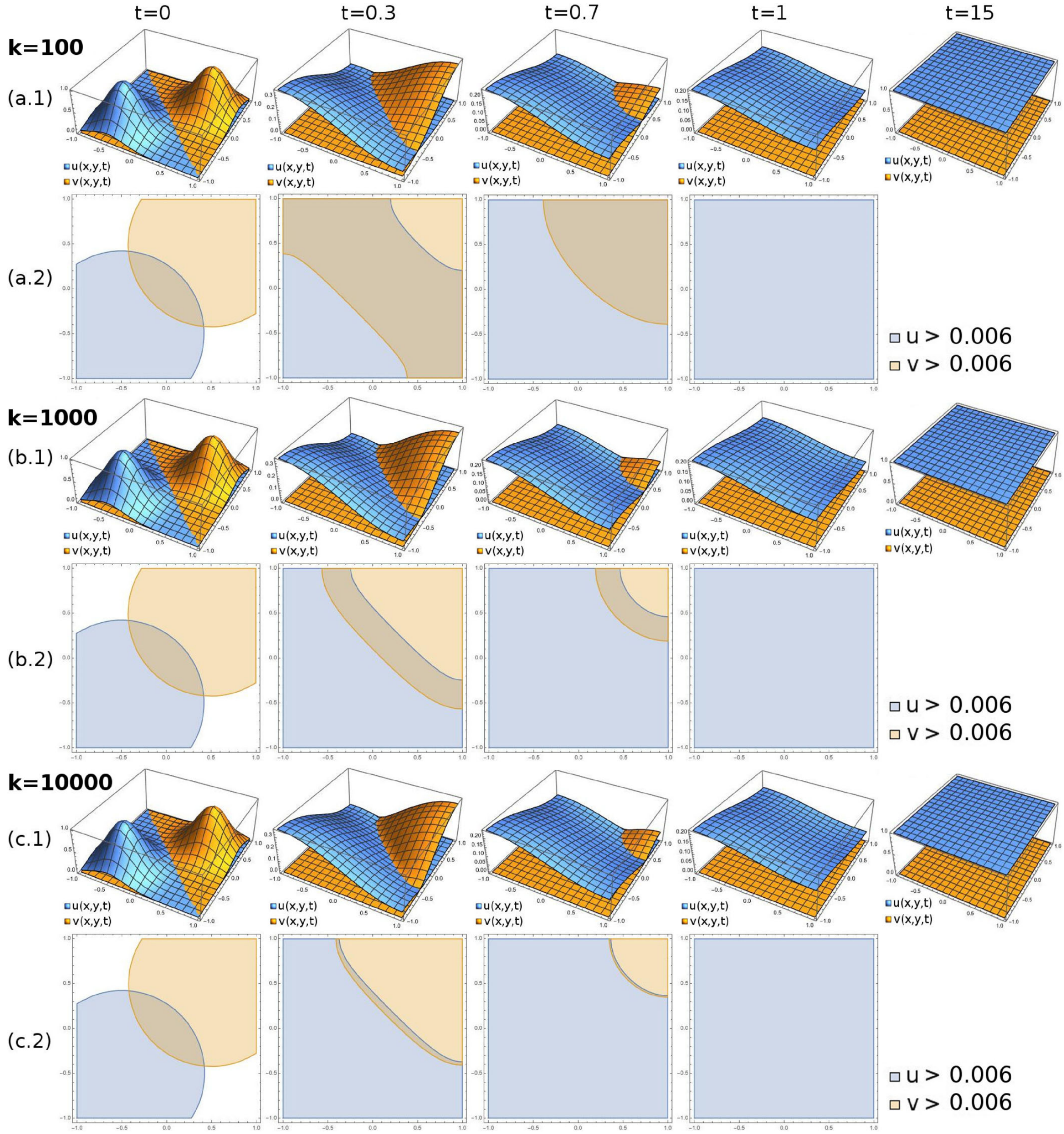

Fig. 2.4. Numerical simulation of the system $(\mathcal{P}_k)$ with $\mu_1 = \mu_2 = 1$, $\alpha = 2$, $d_1 = d_2 = 1$, $\Omega = (-1, 1)^2$, $u_0(x, y) = e^{-6(0.5+x)^2-6(0.5+y)^2}$, $v_0(x, y) = e^{-6(-0.5+x)^2-6(-0.5+y)^2}$, for (a) $k = 100$, (b) $k = 1{,}000$, and (c) $k = 10{,}000$. The choice of the value 0.006 as a delimiting value to show the presence of the species is empiric. It is only the one we found best to illustrate the phenomenon here with the parameters we chose. The graphs in (a.1), (b.1) and (c.1) show a 3D spreading of the species in Ω, with the z-axis the population density, in case of the initial values being a Gaussian distribution on opposite corners, and with Neumann boundary conditions. The graphs in (a.2), (b.2) and (c.2) show a numerical approximation of the support of each species. Since $\alpha > 1$, u has a competitive advantage which is translated by its dominance in space after some time. We see that when k tends to ∞, the supports become disjoint, though the nature of the competition, or the time taken to dominate in space does not change.

As has been proved previously, and can be seen in Figs. 2.3.c and 2.4.c, in the singular limit system, both species have disjoint support. Even in the case of $\operatorname{supp} u_0(x) \cap \operatorname{supp} v_0(x) \neq \emptyset$, after an infinitesimally small amount of time δt has passed, then $\operatorname{supp} u(x, \delta t) \cap \operatorname{supp} v(x, \delta t) = \emptyset$. See Fig. 2.5 for an example of this nearly instantaneous process. To understand why, we need to take into consideration that with k really large, there are two time-frames at play (see [38]). In the fast time frame, the system is akin to a Lotka–Volterra sytem. Indeed, since $k \gg 1$, we have, for $\delta t \ll 1$, and u, v not too small,

$$\begin{cases} u_t \approx f(u) - kuv \\ v_t \approx g(v) - \alpha kuv \end{cases} \tag{2.3}$$

from $t = 0$ to a time δt when, on the whole space, either u or v reaches 0. When k is large enough, this system is bistable: it admits two stable equilibria $(u^*, v^*) = (1, 0)$ and $(u^*, v^*) = (0, 1)$. Which solution becomes extinct depends

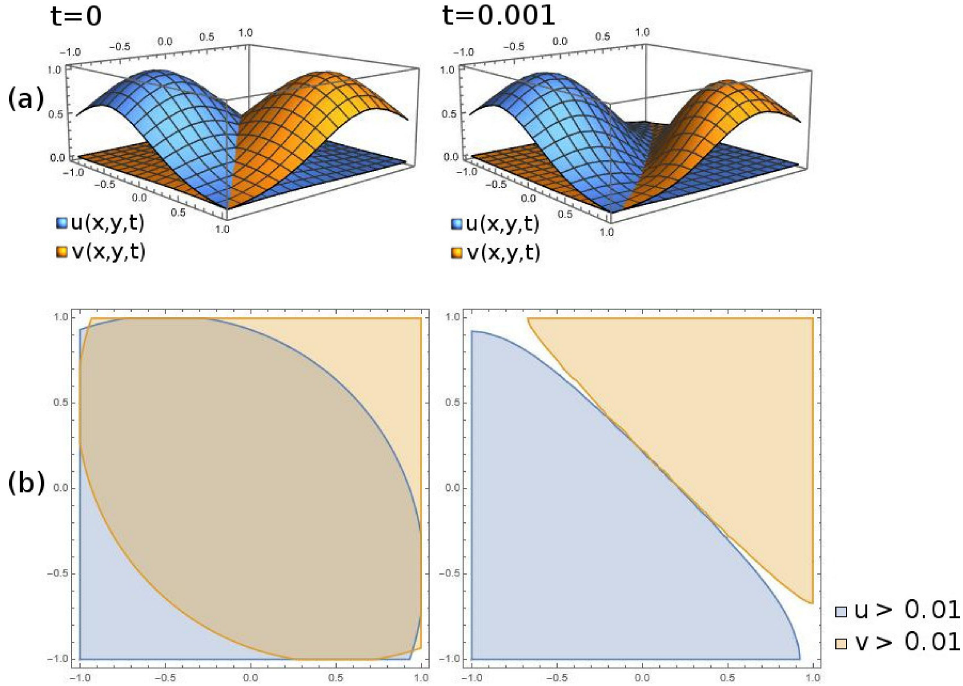

Fig. 2.5. Numerical simulation of the system with $k = 100{,}000$, $\mu_1 = \mu_2 = 1$, $\alpha = 2$, $d_1 = d_2 = 1$, $\Omega = (-1, 1)^2$. (a) shows the spreading of the species in a rectangular space when starting with partially overlapping Gaussian distribution. (b) shows the support of each species. Though it starts with supp $u_0(x) \cap$ supp $v_0(x) \neq \emptyset$, these supports become disjoint instantly after the start of the process.

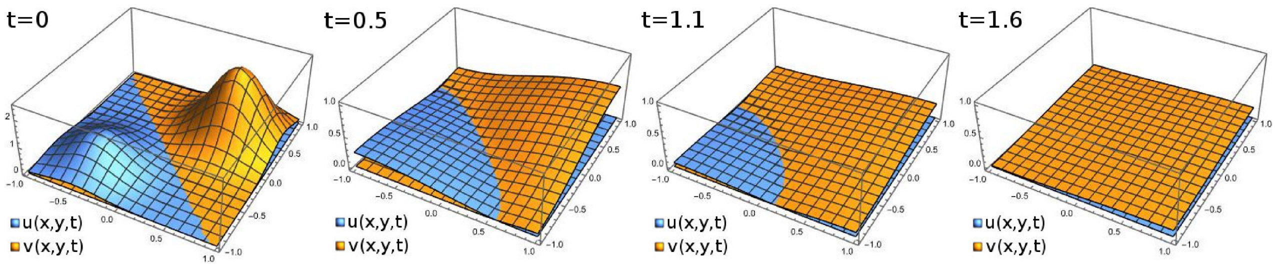

Fig. 2.6. Numerical simulation of the system with $k = 10000$, $\mu_1 = \mu_2 = 1$, $\alpha = 2$, $d_1 = d_2 = 1$, $\Omega = (-1, 1)^2$, $u_0(x, y) = e^{-6(0.5+x)^2-6(0.5+y)^2}$, $v_0(x, y) = 2.5 \times e^{-6(-0.5+x)^2-6(-0.5+y)^2}$. Since $\alpha = 2$, each u counts double in the competition with v so u has a competitive advantage. However, it is supplanted by the advantage of v having more than twice the initial values of u.

on the initial data (see Fig. 1.3). Because of this effect, when k tends to ∞, the supports of each species become disjoint as soon as $t > 0$. In the Fig. 2.4, u_0 and v_0 are symmetrical, and as predicted, we observe that the species u outcompetes the species v. However, an example with v_0 more than twice the value of u_0 gives an opposite result, see Fig. 2.6.

Chapter 3

Spatial Segregation Limit of a Competition-diffusion System with Dirichlet Boundary Conditions

In this chapter, we study the same diffusion-competition system, but now equipped with inhomogeneous Dirichlet boundary conditions. Let Ω be a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, T a strictly positive time. We consider the following problem

$$\begin{cases} u_t = d_1\,\Delta u + f(u) - kuv & \text{in } \Omega \times (0,T], \\ v_t = d_2\,\Delta v + g(v) - \alpha kuv & \text{in } \Omega \times (0,T], \\ u = m_1^k & \text{on } \partial\Omega \times (0,T], \\ v = m_2^k & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0^k,\ v(\cdot,0) = v_0^k & \text{on } \Omega. \end{cases} \qquad (\mathcal{P}^k)$$

We assume the following:

- f and g are continuously differentiable functions on $[0,\infty)$ such that $f(0) = g(0) = 0$ and $f(s) < 0$, $g(s) < 0$ for all $s > 1$.
- $m_1^k, m_2^k \in C^{2,1}(\overline{\Omega} \times \mathbb{R}^+)$, $0 \le m_1^k, m_2^k \le 1$ and $m_1^k \rightharpoonup m_1$, $m_2^k \rightharpoonup m_2$ weakly in $L^2(\partial\Omega \times (0,T))$ for all $T > 0$ as $k \to \infty$.
- The initial data u_0^k and v_0^k are defined by

$$u_0^k(x) = m_1^k(x,0), \quad v_0^k(x) = m_2^k(x,0) \quad \text{for } x \in \Omega$$

  and $u_0^k \rightharpoonup u_0$, $v_0^k \rightharpoonup v_0$ weakly in $L^2(\Omega)$ as $k \to \infty$.

By a solution of Problem $(\mathcal{P}^k)$ we mean a pair (u^k, v^k) such that $u^k, v^k \in C(\overline{Q_T}) \cap C^{2,1}(\overline{\Omega} \times (0,T])$ and satisfy pointwise the partial differential equations as well as the boundary and initial conditions in Problem $(\mathcal{P}^k)$. Throughout this part, we fix the pair of solutions (u^k, v^k), and we define the parabolic domain $Q_T := \Omega \times (0,T)$. This chapter is based upon the article [8].

3.1 A priori bounds and relative compactness

To begin with, the comparison principle gives us again bounds on the values of u^k and v^k:

Proposition 3.1. *For all $k \ge 1$, $0 \le u^k \le 1$ and $0 \le v^k \le 1$ in $\overline{Q_T}$.*

The existence and uniqueness of such a solution (u^k, v^k) of Problem $(\mathcal{P}^k)$ follows again from Proposition 7.3.2 p. 277 of Lunardi [34] for $U^k := u^k - m_1^k$ and $V^k := v^k - m_2^k$.

Following the approach of the previous part, the next step would be to obtain a bound on the interspecific competition term. However, the same computation fails, because the boundary term appearing during the integration by part does not cancel with Dirichlet boundary conditions. Integrating the equation for u^k on Q_T yields

$$k \iint_{Q_T} u^k v^k = d_1 \int_0^T \int_{\partial\Omega} u_\nu^k + \iint_{Q_T} f(u^k) + \int_\Omega u_0^k - \int_\Omega u^k(\cdot,T),$$

where we have not yet obtained an a priori estimate for the first term on the right-hand side, and it could even blow up as k tends to infinity. To cope with this difficulty, we are going to obtain a slightly weaker estimate, by multiplying with a test function vanishing on the boundary before performing the integration by parts. It will provide useful information only in subdomains ω of Ω such that $\overline{\omega} \subset\subset \Omega$. Let thus $\varphi \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $\varphi = 0$ on $\partial\Omega$ and $\varphi > 0$ in Ω. For example, one can take φ as the first eigenfunction of the operator $-\Delta$ in Ω with the homogeneous Dirichlet condition, namely the function φ such that $\|\varphi\|_{H_0^1(\Omega)} = 1$ satisfying,

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda > 0$ and $\varphi > 0$ in Ω (see Theorem 9.31 p. 311 in [3] for the existence of such a function). Then we prove the following result:

Proposition 3.2. *There exists a constant C_1 independent of k such that*

$$\iint_{Q_T} u^k v^k \varphi \le \frac{C_1}{k}.$$

Proof. Integrating the equation for u^k after multiplication by φ yields

$$k \iint_{Q_T} u^k v^k \varphi = d_1 \int_0^T \int_{\partial\Omega} \{u^k_\nu \varphi - u^k \varphi_\nu\} + d_1 \iint_{Q_T} u^k \Delta\varphi$$
$$+ \iint_{Q_T} f(u^k)\varphi + \int_\Omega u^k_0 \varphi(\cdot,0) - \int_\Omega u^k(\cdot,T)\varphi(\cdot,T).$$

Now using that $\varphi = 0$ on $\partial\Omega$, φ_ν is bounded on $\partial\Omega$, and that u^k, v^k have values in $[0,1]$, we obtain the desired result. □

As in Chapter 1, we now prove uniform estimates for $|\nabla u^k|$ and $|\nabla v^k|$.

Proposition 3.3. *There exists a positive constant C_2 independent of k such that*

$$\iint_{Q_T} |\nabla u^k|^2 \varphi, \quad \iint_{Q_T} |\nabla v^k|^2 \varphi \le C_2.$$

Proof. We multiply the parabolic equation for u^k by $u^k\varphi$ and integrate by parts. This gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^k)^2\varphi + d_1 \int_\Omega |\nabla u^k|^2 \varphi + d_1 \int_\Omega u^k \nabla u^k \cdot \nabla\varphi$$
$$= \int_\Omega f(u^k)u^k\varphi - k\int_\Omega (u^k)^2 v^k \varphi,$$

where we used that $\nabla(u^k\varphi) = \varphi\nabla u^k + u^k\nabla\varphi$, and that $\varphi = 0$ on the boundary. Integrating it in time, and integrating by parts the last term of the left-hand side we find

$$\frac{1}{2}\int_\Omega (u^k)^2(\cdot,T)\varphi(\cdot,T) + d_1 \iint_{Q_T} |\nabla u^k|^2\varphi$$
$$\le \frac{1}{2}\int_\Omega (u^k_0)^2\varphi(\cdot,0) - d_1\int_0^T\int_{\partial\Omega}(u^k)^2\varphi_\nu + d_1\iint_{Q_T}(u^k)^2\Delta\varphi + \iint_{Q_T} f(u^k)u^k\varphi.$$

Using again that $0 \le u^k, v^k \le 1$, and that φ_ν is bounded on $\partial\Omega$, we obtain the result. □

So far we have obtained $L^\infty(Q_T)$ estimates for u^k and v^k, and $L^2_{\text{loc}}(Q_T)$ estimates for $|\nabla u^k|$ and $|\nabla v^k|$. This is not sufficient to pass to the limit as k tends to infinity in the nonlinear terms and in the time-derivative term. In the first chapter we coped with this problem by finding estimates on the time derivative of the solution pair and using a compactness embedding to obtain strong L^2 convergence. Now, we obtain strong L^2-convergence by a different approach. We apply the Fréchet–Kolmogorov Theorem, as stated in the book of Brezis [3], Corollary 4.26 p. 74.

Theorem 3.4. *Let $\mathcal{F}$ be a bounded subset of $L^p(Q_T)$ with $1 \le p < \infty$. Suppose that*

- *for any ε and any subset $\omega \subset\subset Q_T$, there exists a positive constant $\delta(< \text{dist}(\omega, \partial Q_T))$ such that*
$$\|f(x+\xi,t) - f(x,t)\|_{L^p(\omega)} + \|f(x,t+\tau) - f(x,t)\|_{L^p(\omega)} < \varepsilon$$
for all ξ, τ, and $f \in \mathcal{F}$ satisfying $|\xi| + |\tau| < \delta$,.
- *for any $\varepsilon > 0$, there exists $\omega \subset\subset Q_T$ such that*
$$\|f\|_{L^p(Q_T\setminus\omega)} < \varepsilon$$
for all $f \in \mathcal{F}$.

Then $\mathcal{F}$ is relatively compact in $L^p(Q_T)$.

The Fréchet–Kolmogorov theorem is written in a way that all the variables are involved in a similar way. However, we deal with a parabolic system so that it is handy to separately derive estimates for differences of space and time translates, first the space translates and then the time translates.

Theorem 3.5. *The sequences $\{u^k\}$ and $\{v^k\}$ are relatively compact in $L^2(Q_T)$.*

Proof. To verify the first hypothesis of the Fréchet–Kolmogorov theorem it is sufficient to consider subsets of Q_T of the form $\Omega_r \times [0, T-\tau]$, where $\tau \in (0,T]$, and

$$\Omega_r := \{x \in \Omega | B(x,2r) \subset \Omega\},$$

for $r > 0$, and where $B(x,r)$ denotes the ball in $\mathbb{R}^N$ with center x and radius r. Indeed, if $\omega \subset\subset Q_T$, the continuous function $(x,t) \in \omega \mapsto \text{dist}(x,\partial\Omega)$ achieves a positive minimum on ω, and thus ω will be included in $\Omega_r \times (0,T)$ for r small enough. Reasoning similarly on time, ω will be included in a domain of the form $\Omega \times [\tau, T-\tau]$ for τ small enough. Thus we deduce that $\omega \subset \Omega_r \times [\tau, T-\tau]$ for r and τ small enough. We define in addition

$$\Omega'_r := \bigcup_{x\in\Omega_r} B(x,r).$$

We start with the estimates on the space translates. We show that for all $r > 0$, there exists a positive constant C_3 such that

$$\int_0^T \int_{\Omega_r} (u^k(x+\xi,t) - u^k(x,t))^2 \le C_3|\xi|^2,$$
$$\int_0^T \int_{\Omega_r} (v^k(x+\xi,t) - v^k(x,t))^2 \le C_3|\xi|^2,$$

for all $\xi \in \mathbb{R}^N$ such that $|\xi| \le r$, which will imply the desired result. For the proof of these inequalities, we use the estimates on $|\nabla u^k|\sqrt{\varphi}$ and $|\nabla v^k|\sqrt{\varphi}$ in $L^2(Q_T)$, as well as the Cauchy–Schwartz inequality:

$$\begin{aligned}\int_0^T \int_{\Omega_r} (u^k(x+\xi,t) - u^k(x,t))^2\,dx\,dt &= \int_0^T \int_{\Omega_r} \left(\int_0^1 \nabla u^k(x+\theta\xi,t)\cdot\xi\,d\theta\right)^2 dx\,dt\\ &\le |\xi|^2 \int_0^1 \int_0^T \int_{\Omega_r} |\nabla u^k(x+\theta\xi,t)|^2\,dx\,dt\,d\theta\\ &\le |\xi|^2 \int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2\,dx\,dt\\ &\le \frac{|\xi|^2}{\inf\limits_{y\in\Omega'_r}\varphi(y)} \int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2 \varphi(x)\,dx\,dt\\ &\le C_3|\xi|^2.\end{aligned}$$

The estimates on the space translates of v^k can be shown similarly. Next we turn to the estimates on the time translates. We prove that for each $r > 0$, there exists a positive constant C_4 such that

$$\int_0^{T-\tau} \int_{\Omega_r} (u^k(x,t+\tau) - u^k(x,t))^2\,dx\,dt \le C_4\tau,$$
$$\int_0^{T-\tau} \int_{\Omega_r} (v^k(x,t+\tau) - v^k(x,t))^2\,dx\,dt \le C_4\tau,$$

for all $\tau \in (0,T)$. We take $\mu \in C_0^\infty(\Omega'_r)$ such that $0 \le \mu(x) \le 1$ in Ω'_r and $\mu = 1$ on Ω_r, and we show that for all $\tau \in (0,T)$

$$\int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t))^2 \mu(x)\,dx\,dt \le C_4\tau,$$
$$\int_0^{T-\tau} \int_{\Omega'_r} (v^k(x,t+\tau) - v^k(x,t))^2 \mu(x)\,dx\,dt \le C_4\tau.$$

We perform the computations for u^k, the estimates for v^k being proved similarly. We have

$$\begin{aligned}&\int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t))^2 \mu(x)\,dx\,dt\\ &= \int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t))\left(\int_t^{t+\tau} u_s^k(x,s)\,ds\right)\mu(x)\,dx\,dt\\ &= \int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t))\left(\int_0^{\tau} u_t^k(x,t+s)\,ds\right)\mu(x)\,dx\,dt\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where I_1, I_2 and I_3 are obtained when replacing u_t^k in $(\int_0^\tau u_t^k(x,t+s)\,ds)$ by the equality from $(\mathscr{P}^k)$:

$$u_t^k = d_1\,\Delta u^k + f(u^k) - ku^k v^k.$$

So we have

$$I_1 := \int_0^\tau \int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t)) d_1 \Delta u^k(x,t+s)\mu(x)\,dx\,dt\,ds,$$

$$I_2 := \int_0^\tau \int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t)) f(u^k(x,t+s))\mu(x)\,dx\,dt\,ds,$$

$$I_3 := -\int_0^\tau \int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t))(ku^k v^k)(x,t+s)\mu(x)\,dx\,dt\,ds.$$

The estimate on I_1 follows from the estimate on $|\nabla u^k|^2$. Indeed, since μ vanishes on $\partial\Omega'_r$, we have

$$\begin{aligned} I_1 = &-d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega'_r} \nabla(u^k(x,t+\tau) - u^k(x,t)) \cdot \nabla u^k(x,t+s)\mu(x)\,dx\,dt\,ds \\ &- d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega'_r} (u^k(x,t+\tau) - u^k(x,t))\nabla u^k(x,t+s) \cdot \nabla\mu(x)\,dx\,dt\,ds. \end{aligned}$$

The first term of I_1 can be bounded using Cauchy–Schwartz inequality and the fact that, by definition, μ is bounded:

$$\begin{aligned} &\left| d_1 \int_0^\tau \int_0^{T-\tau} \int_{\Omega'_r} \nabla(u^k(x,t+\tau) - u^k(x,t)) \cdot \nabla u^k(x,t+s)\mu(x)\,dx\,dt\,ds \right| \\ &\le 2|d_1|\|\mu\|_\infty \int_0^\tau \left(\int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2\,dx\,dt \right)^{1/2} \left(\int_0^{T-\tau} \int_{\Omega'_r} |\nabla u^k(x,t+s)\mu(x)|^2\,dx\,dt \right)^{1/2} ds \\ &\le 2|d_1|\|\mu\|_\infty \tau \int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2\,dx\,dt. \end{aligned}$$

As for the second term of I_1, we just use that $\nabla\mu$ is bounded on Ω'_r and $0 \le u^k \le 1$. Finally we introduce φ the same way we did previously.

$$\begin{aligned} |I_1| &\le C_5\tau \int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2\,dx\,dt + C_6\tau \int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|\,dx\,dt \\ &\le \frac{C_5\tau}{\inf_{y\in\Omega'_r} \varphi(y)} \int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2 \varphi(x)\,dx\,dt \\ &\quad + \frac{C_7\tau}{\left(\inf_{y\in\Omega'_r} \varphi(y)\right)^{\frac{1}{2}}} \left(\int_0^T \int_{\Omega'_r} |\nabla u^k(x,t)|^2 \varphi(x)\,dx\,dt \right)^{\frac{1}{2}} \\ &\le C_8\tau. \end{aligned}$$

The estimates on the terms I_2 comes easily from the fact that u^k, v^k take their values in $[0,1]$, and that on the term I_3 from the bound on the interspecific competition term:

$$|I_2| \le C_9\tau,$$
$$|I_3| \le \frac{C_{10}\tau}{\inf_{y\in\Omega'_r} \varphi(y)} \int_0^T \int_{\Omega'_r} ku^k v^k \varphi \le C_{11}\tau.$$

This completes the proof of the estimates for differences of time translates of u^k. The estimate for the function v^k follows in a similar way.

The proof of the second hypothesis of the Fréchet–Kolmogorov theorem easily follows from Proposition 3.1. Indeed, since $\{u^k\}$ and $\{v^k\}$ are bounded by 1, for any $\varepsilon > 0$ fixed, there exists $r_0 > 0$ and $\tau_0 > 0$ such that for $0 \le r \le r_0$ and $0 \le \tau \le \tau_0$

$$\int_{T-\tau}^T \int_\Omega (u^k)^2, \quad \int_0^T \int_{\Omega\setminus\Omega_r} (u^k)^2 \le \varepsilon,$$

along with similar inequalities for v^k.

We are thus able to apply the Fréchet–Kolmogorov theorem and we deduce that $\{u^k\}$ and $\{v^k\}$ are relatively compact in $L^2(Q_T)$. □

We can now state the following convergence result.

Corollary 3.6. *There exist subsequences $\{u^{k_n}\}$, $\{v^{k_n}\}$, functions $u \in L^\infty(Q_T)$ and $v \in L^\infty(Q_T)$ such that*

$$u^{k_n} \to u, \quad v^{k_n} \to v \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,$$

as $k_n \to \infty$.

3.2 Charaterization of the limit problem

As with the Neumann boundary conditions we have the following result.

Proposition 3.7. *$uv = 0$ a.e. in Q_T.*

Proof. The proof is simpler than in the case of the Neumann conditions because we already know the strong convergence of (u^{k_n}) and (v^{k_n}). Indeed, recall that

$$\iint_{Q_T} u^{k_n} v^{k_n} \varphi \le \frac{C_1}{k_n} \underset{n\to\infty}{\longrightarrow} 0.$$

Since (u^{k_n}) and (v^{k_n}) strongly converge in $L^2(Q_T)$ we can pass to the limit in the left-hand side and deduce that

$$\iint_{Q_T} uv\varphi = 0.$$

Now since, $u^{k_n}, v^{k_n} \ge 0$ and they converge almost everywhere to u and v, we deduce that $u, v \ge 0$ almost everywhere. As by assumption φ is also strictly positive in the interior of Q_T it yields

$$uv = 0 \text{ a.e. in } Q_T.$$

This completes the proof. □

Next we set $w^k := u^k - \frac{v^k}{\alpha}$, and $w := u - \frac{v}{\alpha}$. We deduce from the convergence results above that

$$w^{k_n} \to w \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T$$

as $k_n \to \infty$ and furthermore that

$$u = w^+, \quad \text{and} \quad v = \alpha w^-,$$

where $s^+ = \max\{s, 0\}$ and $s^- = \max\{-s, 0\}$.

We will now prove that w is the unique weak solution of a limiting free boundary problem in the same way as in the previous chapter but with Dirichlet boundary conditions.

Proposition 3.8. *The function pair (u, v) defined above is such that*

$$\begin{aligned} -\iint_{Q_T}\left(u - \frac{v}{\alpha}\right)\psi_t - \int_\Omega\left(u_0 - \frac{v_0}{\alpha}\right)\psi(\cdot,0) = &-\int_0^T\int_{\partial\Omega}\left(d_1 m_1 - \frac{d_2 m_2}{\alpha}\right)\psi_\nu \\ &+ \iint_{Q_T}\left\{\left(d_1 u - \frac{d_2 v}{\alpha}\right)\Delta\psi + \left(f(u) - \frac{g(v)}{\alpha}\right)\psi\right\} \end{aligned} \tag{3.1}$$

for all $\psi \in \mathcal{F}_T$ where

$$\mathcal{F}_T := \{\psi \in C^{2,1}(\overline{Q_T}) \mid \psi(x,T) = 0 \text{ in } \Omega \text{ and } \psi = 0 \text{ on } \partial\Omega \times [0,T]\}.$$

Proof. We take the difference of the partial differential equations for u^k and v^k/α, multiply by $\psi \in \mathcal{F}_T$ and integrate by parts in time and space, which yields

$$\begin{aligned} -\iint_{Q_T}\left(u^k - \frac{v^k}{\alpha}\right)\psi_t - \int_\Omega\left(u_0^k - \frac{v_0^k}{\alpha}\right)\psi(\cdot,0) = &-\int_0^T\int_{\partial\Omega}\left(d_1 m_1^k - \frac{d_2 m_2^k}{\alpha}\right)\psi_\nu \\ &+ \iint_{Q_T}\left\{\left(d_1 u^k - \frac{d_2 v^k}{\alpha}\right)\Delta\psi + \left(f(u^k) - \frac{g(v^k)}{\alpha}\right)\psi\right\}. \end{aligned}$$

We can then pass to the limit as $k^n \to \infty$ in all the terms by using the dominated convergence theorem (as $0 \le u^k, v^k \le 1$). This yields the desired result. □

We rewrite this equation in terms of w, and of the following quantities

$$d(s) := \begin{cases} d_1 & \text{if } s > 0, \\ d_2 & \text{if } s < 0, \end{cases}$$

$$\mathcal{D}(s) := \begin{cases} d_1 s & \text{if } s \ge 0, \\ d_2 s & \text{if } s < 0, \end{cases}$$

$$h(s) := \begin{cases} f(s) & \text{if } s > 0, \\ -\dfrac{g(-\alpha s)}{\alpha} & \text{if } s < 0. \end{cases}$$

Thus the equation for w reads,

$$(\mathcal{P})\begin{cases} w_t = \Delta\mathcal{D}(w) + h(w) & \text{in } Q_T,\\ \mathcal{D}(w) = d_1 m_1 - \dfrac{d_2 m_2}{\alpha} & \text{on } \partial\Omega\times(0,T),\\ w(x,0) = u_0(x) - \dfrac{v_0(x)}{\alpha} & \text{in } \Omega. \end{cases}$$

Definition 3.1. *A function w is a weak solution of Problem ($\mathcal{P}$) if it satisfies:*

- $w \in L^\infty(\Omega\times(0,T))$,
- $\iint_{Q_T}(w\psi_t + \mathcal{D}(w)\Delta\psi + h(w)\psi) = \int_0^T\int_{\partial\Omega}(d_1m_1 - \frac{d_2m_2}{\alpha})\psi_\nu - \int_\Omega w_0\psi(\cdot,0)$ *for all* $T>0$ *and* $\psi\in\mathcal{F}_T$.

Theorem 3.9. *The function w defined above is a weak solution of problem ($\mathcal{P}$).*

Proof. This follows from (3.1) and from the definitions of w, D and h. □

3.3 Numerical example

As can be seen in Fig. 3.1, just as in the previous chapter with Neumann boundary conditions and as it has just been proved in this case, when $k\to\infty$, the supports of the species become disjoint. The difference in this case is that when the system reaches its steady state (when $t\to\infty$), both species coexist. Here it can be explained by the Dirichlet conditions acting as sources for each species. Indeed, in this system, we took the following Dirichlet conditions:

$$\begin{cases} u(x,-1,t) = -0.5\times x + 0.5, \quad u(-1,y,t) = -0.5\times y + 0.5,\\ u(x,1,t) = u(1,y,t) = 0.\\ \\ v(x,1,t) = 0.5\times x + 0.5, \qquad v(1,y,t) = 0.5\times y + 0.5,\\ v(x,-1,t) = v(-1,y,t) = 0. \end{cases} \tag{3.2}$$

It means that we force the species u to exist on the boundaries $x=-1$, $y=-1$, and the species v to exist on the boundaries $x=1$, $y=1$. Furthermore, as we explained in the previous chapter with Neumann boundaries, though u has a competitive advantage, it can be offset by a larger "initial value" for the species v at the space where the species interact. Since the Dirichlet boundaries act as source for each species in their respective corner, the further one species invades the other initial territory, the greater this "initial value" will be. At some point, the competitive advantage of u is not enough, and it cannot over-compete the species v.

3.4 Uniqueness of the weak solution of the limit problem

The goal of this part is to show the uniqueness of weak solution of problem ($\mathcal{P}$). Our proof relies on the following proposition.

Proposition 3.10. *Let w_1 and w_2 be two solutions of Problem ($\mathcal{P}$) with initial data $w_{0,1}$ and $w_{0,2}$ respectively. Then*

$$\begin{aligned}&\iint_{Q_T}|w_1(x,t)-w_2(x,t)|\,dx\,dt\\ &\quad\le T\int_\Omega|w_{0,1}(x)-w_{0,2}(x)|\,dx + \iint_{Q_T}(T-t)|h(w_1)-h(w_2)|\,dx\,dt.\end{aligned}$$

The proof of this result is based on properties of the solution of the adjoint problem

$$\begin{cases} \psi_t + \sigma(x,t)\Delta\psi = \eta(x,t), & (x,t)\in Q_T,\\ \psi = 0 & \text{on } \partial\Omega\times(0,T),\\ \psi(x,T) = 0, & x\in\Omega. \end{cases} \tag{$\mathcal{A}$}$$

We first show the following lemma.

Lemma 3.11. *Let $T>0$, $\eta\in C_0^\infty(Q_T)$ be such that $|\eta|\le 1$ and let $\sigma\in C^\infty(Q_T)$ be such that there exists a positive constant σ_* with*

$$\sigma(x,t)\ge\sigma_*>0 \quad \textit{in } Q_T.$$

Then there exists a unique solution $\psi\in C^{2,1}(\overline{Q_T})$ of Problem ($\mathcal{A}$). It satisfies

$$|\psi|\le T-t \quad \textit{in } Q_T$$

and

$$\iint_{Q_T}(\Delta\psi)^2 \le \frac{T|\Omega|}{\sigma_*^2}.$$

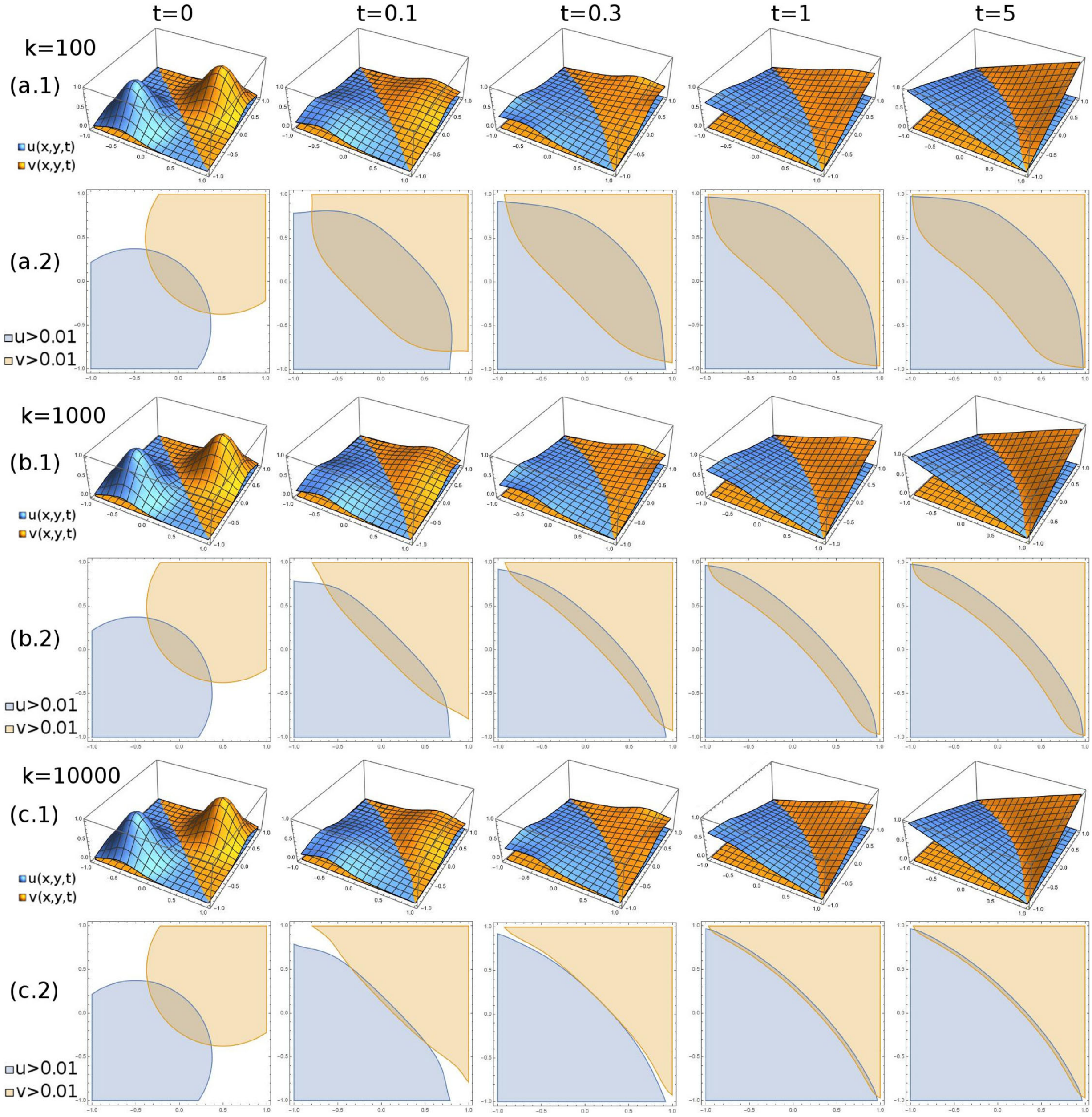

Fig. 3.1. Numerical simulations of the system with $\mu_1 = \mu_2 = 1$, $\alpha = 2$, $d_1 = d_2 = 1$, $\Omega = (-1, 1)^2$, $u_0(x, y) = e^{-6(0.5+x)^2-6(0.5+y)^2}$, $v_0(x, y) = e^{-6(-0.5+x)^2-6(-0.5+y)^2}$. For (a) $k = 100$, (b) $k = 1{,}000$, and (c) $k = 10{,}000$. The graphs in (a.1), (b.1) and (c.1) show a 3D spreading of the species in Ω, with the z-axis the population density, in case of the initial values being a Gaussian distribution on opposite sides. The boundary conditions are non-homogeneous Dirichlet boundary conditions, see (3.2). The graphs in (a.2), (b.2) and (c.2) show a numerical approximation of the support of each species. Since $\alpha > 1$, u has a competitive advantage which is translated by the dominance of its support after some time. We see that when k tends to ∞, the supports become disjoint, though the nature of the competition, or the time needed for the support of u to extend does not change.

Proof. Note that problem ($\mathcal{A}$) is a backward parabolic equation, along with a final condition at the time T. We first rewrite the equation as a classical forward (in time) problem by a change of variable, $\tau = T - t$, $\sigma(x, t) = \overline{\sigma}(x, \tau)$, $\eta(x, t) = \overline{\eta}(x, \tau)$ and $\psi(x, t) = \overline{\psi}(x, \tau)$. Then

$$\overline{\psi}_\tau = -\psi_t, \quad \Delta\overline{\psi} = \Delta\psi$$

and ψ satisfies the forward (in time) problem

$$\begin{cases} \overline{\psi}_\tau = \overline{\sigma}(x, \tau)\Delta\overline{\psi} - \overline{\eta}(x, \tau), & (x, \tau) \in Q_T, \\ \overline{\psi} = 0 & \text{on } \partial\Omega \times (0, T), \\ \overline{\psi}(x, 0) = 0, & x \in \Omega. \end{cases} \tag{$\mathcal{F}$}$$

The existence and uniqueness of the classical solution of this problem can be found in classical literature. For example, it follows from the book [32] that problem $(\mathcal{F})$ has a unique classical solution $\overline{\psi}$, which in turn yields a unique classical solution of the adjoint problem $(\mathcal{A})$.

The first estimate of the lemma follows from the maximum principle. Indeed, since $|\eta| \le 1$, the functions τ and $-\tau$ are upper and lower solutions of problem $(\mathcal{F})$. Thus

$$-\tau \le \overline{\psi} \le \tau \quad \text{in } Q_T$$

or equivalently

$$-(T-t) \le \psi \le T-t \quad \text{in } Q_T.$$

We now turn to the estimate on $\Delta\psi$. We multiply the parabolic equation in problem $(\mathcal{A})$ by $\Delta\psi$ and integrate by parts on Q_T. So,

$$\iint_{Q_T} \{\psi_t \Delta\psi + (\Delta\psi)^2\sigma\} = \iint_{Q_T} \eta\Delta\psi$$

which implies that

$$-\iint_{Q_T} (\nabla\psi)_t \cdot \nabla\psi + \iint_{Q_T} (\Delta\psi)^2\sigma = \iint_{Q_T} \eta\Delta\psi.$$

Here we have used that $\psi_t = 0$ on $\partial\Omega \times (0,T)$. Thus

$$\frac{1}{2}\int_\Omega |\nabla\psi(\cdot,0)|^2 - \frac{1}{2}\int_\Omega |\nabla\psi(\cdot,T)|^2 + \iint_{Q_T} (\Delta\psi)^2\sigma \le \iint_{Q_T} \eta\Delta\psi.$$

As $\psi(\cdot,T) \equiv 0$, also $\nabla\psi(\cdot,T) \equiv 0$, and we deduce that

$$\iint_{Q_T} (\Delta\psi)^2\sigma \le \iint_{Q_T} \eta\Delta\psi.$$

We claim that it implies the desired estimate. Indeed, using Young's inequality

$$\begin{aligned}\iint_{Q_T} (\Delta\psi)^2\sigma &\le \iint_{Q_T} \eta\Delta\psi \\ &\le \frac{\sigma_*}{2}\iint_{Q_T} (\Delta\psi)^2 + \frac{1}{2\sigma_*}\iint_{Q_T} \eta^2 \\ &\le \frac{1}{2}\iint_{Q_T} (\Delta\psi)^2\sigma + \frac{1}{2\sigma_*}\iint_{Q_T} \eta^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\iint_{Q_T} (\Delta\psi)^2\sigma &\le \frac{1}{2\sigma_*}\iint_{Q_T} \eta^2 \\ &\le \frac{T|\Omega|}{2\sigma_*},\end{aligned}$$

where we have used that $|\eta| \le 1$ for the last inequality. Hence

$$\iint_{Q_T} (\Delta\psi)^2 \le \frac{1}{\sigma_*}\iint_{Q_T} (\Delta\psi)^2\sigma \le \frac{T|\Omega|}{\sigma_*^2}.$$

This completes the proof. □

The next step is to prove the inequality for the difference of w_1 and w_2.

Proof of Proposition 3.10. Let w_1 and w_2 be two solutions of problem $(\mathcal{P})$ with initial data $w_{0,1}$ and $w_{0,2}$ respectively. Set $\tilde{w} := w_1 - w_2$, $\tilde{w}_0 := w_{0,1} - w_{0,2}$, $z := h(w_1) - h(w_2)$ and define for all $(x,t) \in Q_T$

$$q(x,t) := \begin{cases} \dfrac{\mathcal{D}(w_1(x,t)) - \mathcal{D}(w_2(x,t))}{w_1(x,t) - w_2(x,t)} & \text{if } w_1(x,t) \ne w_2(x,t), \\ \min\{d_1, d_2\} & \text{otherwise.}\end{cases}$$

Note that it easily follows from the definition of $\mathcal{D}$ that

$$\min\{d_1,d_2\} \le q(x,t) \le \max\{d_1,d_2\} \text{ in } Q_T.$$

The definition of a weak solution for problem $(\mathcal{P})$ yields, for all $\psi \in \mathcal{F}_T$,

$$\iint_{Q_T} \{\tilde{w}(\psi_t + q\Delta\psi) + z\psi\}\,dx\,dt = -\int_\Omega \tilde{w}_0\psi(x,0)\,dx.$$

Lemma 3.11 allows us to replace $\psi_t + q\Delta\psi$ by any function $\eta \in C_0^\infty(Q_T)$ such that $|\eta| \leq 1$, while ensuring $|\psi| \leq T - t$. Choosing a suitable function η would then yield the result. However, we are not quite able to apply Lemma 3.11, since q is not smooth enough. Yet using mollifiers, one can find a smooth sequence q_n converging to q in, say, $L^2(Q_T)$ and verifying the same bounds as q:

$$\|q_n - q\|_{L^2(Q_T)} \leq \frac{1}{n},$$

$$\min\{d_1, d_2\} \leq q_n(x,t) \leq \max\{d_1, d_2\} \quad \text{in } Q_T$$

(extend q to $\min\{d_1, d_2\}$ outside Q_T and then use mollifiers, the convergence in $L^2(Q_T)$ is a classical result that can be found in the fourth chapter of [3], while the bounds on q_n come easily from the fact that the integral of a mollifier is one). Then fix $\eta \in C_0^\infty(Q_T)$ with $|\eta| \leq 1$ and let ψ_n be the solution of problem $(\mathcal{A})$ with the function σ replaced by q_n (with $\sigma_* = \min(d_1, d_2)$ then). Setting $\psi = \psi_n$ in the equation for ψ gives

$$\iint_{Q_T} [\tilde{w}\{(\psi_n)_t + q\Delta\psi_n\} + z\psi_n]\,dx\,dt = -\int_\Omega \tilde{w}_0\psi_n(x,0)\,dx,$$

and hence since

$$(\psi_n)_t + q_n(x,t)\Delta\psi_n = \eta(x,t),$$

we have

$$\left|\iint_{Q_T} \tilde{w}\{(q - q_n)\Delta\psi_n + \eta\}\,dx\,dt\right| \leq \iint_{Q_T} |z\psi_n|\,dx\,dt + \int_\Omega |\tilde{w}_0\psi_n(x,0)|\,dx$$
$$\leq \iint_{Q_T} (T-t)|z|\,dx\,dt + T\int_\Omega |\tilde{w}_0|\,dx.$$

Next we show that the first term on the left-hand-side of the inequality above vanishes as $n \to \infty$. Indeed, using Cauchy-Schwartz inequality,

$$\iint_{Q_T} |\tilde{w}|\,|q(x,t) - q_n(x,t)|\,|\Delta\psi_n|\,dx\,dt$$
$$\leq (\|w_1\|_{L^\infty(Q_T)} + \|w_2\|_{L^\infty(Q_T)})\left(\iint_{Q_T} (q - q_n)^2\,dx\,dt\right)^{1/2}\left(\iint_{Q_T} (\Delta\psi_n)^2\,dx\,dt\right)^{1/2}$$
$$\leq \frac{C_{11}T^{1/2}|\Omega|^{1/2}}{n\min\{d_1, d_2\}}.$$

Letting $n \to \infty$ we obtain

$$\left|\iint_{Q_T} \tilde{w}\eta\,dx\,dt\right| \leq \iint_{Q_T} (T-t)|z|\,dx\,dt + T\int_\Omega |\tilde{w}_0|\,dx$$

for each $\eta \in C_0^\infty(Q_T)$ with $|\eta| \leq 1$. Take as functions η the elements of a subsequence $\{\eta_m\}$, $(m \in \mathbb{N})$ such that $\{\eta_m\}$ converges to $\mathrm{sign}(\tilde{w})$ in $L^1(Q_T)$ as $m \to \infty$. Letting $m \to \infty$ yields

$$\iint_{Q_T} |\tilde{w}|\,dx\,dt \leq \iint_{Q_T} (T-t)|z|\,dx\,dt + T\int_\Omega |\tilde{w}_0|\,dx,$$

which completes the proof. □

With the aid of Lemma 3.10 in hand, we are now ready to prove the uniqueness of the weak solution of problem $(\mathcal{P})$.

Corollary 3.12. *There exists at most one weak solution w of Problem $(\mathcal{P})$. The function w belongs to $C^{\beta,\beta/2}(Q_T)$ for all $\beta \in (0,1)$.*

Proof. First, the Hölder continuity of weak solutions of problem $(\mathcal{P})$ in Q_T follows from [13, Theorem 1.1, p. 41].

Suppose then that w_1 and w_2 are two weak solutions of problem $(\mathcal{P})$ with initial data $w_{0,1} = w_{0,2}$ and let $M > 0$ be such that $|w_i| \leq M$ $(i = 1,2)$. Since h is locally Lipschitz continuous on $\mathbb{R}$, there exists a constant L such that

$$|h(w_1) - h(w_2)| \leq L|w_1 - w_2| \quad \text{in } Q_T.$$

Applying the inequality above with Q_T replaced by $\Omega \times (t_0, t_0 + \tau)$ with $t_0 \in [0, T)$ and $\tau \in (0, T - t_0]$ (we can reduce to this case because as w_1 and w_2 are continuous we can show that they are solutions of problem $(\mathcal{P})$ over $[t_0, T]$ with initial conditions at time t_0) gives

$$\begin{aligned}
&\int_{t_0}^{t_0+\tau}\int_\Omega |w_1 - w_2|\,dx\,dt \\
&\quad \le \tau \int_\Omega |w_1(x,t_0) - w_2(x,t_0)|\,dx + \int_{t_0}^{t_0+\tau}\int_\Omega (t_0+\tau-t)|h(w_1) - h(w_2)|\,dx\,dt \\
&\quad \le \tau \int_\Omega |w_1(x,t_0) - w_2(x,t_0)|\,dx + \tau L \int_{t_0}^{t_0+\tau}\int_\Omega |w_1 - w_2|\,dx\,dt,
\end{aligned}$$

and thus

$$(1-\tau L)\int_{t_0}^{t_0+\tau}\int_\Omega |w_1 - w_2|\,dx\,dt \le \tau \int_\Omega |w_1(x,t_0) - w_2(x,t_0)|\,dx.$$

It follows that, for all $\tau \le \min\{1/(2L), T - t_0\}$, we have

$$\int_{t_0}^{t_0+\tau}\int_\Omega |w_1 - w_2|\,dx\,dt \le 2\tau \int_\Omega |w_1(x,t_0) - w_2(x,t_0)|\,dx.$$

Let then

$$t_0 := \sup\{t \in [0,T] \mid w_1(x,s) = w_2(x,s) \text{ for } 0 \le s \le t,\ x \in \Omega\}$$

and assume that $t_0 < T$. Then by continuity of w_1 and w_2, $w_1(\cdot,t_0) = w_2(\cdot,t_0)$ so that by the inequality above

$$w_1 = w_2 \quad \text{on } \Omega \times (t_0, t_0+\tau)$$

$\tau \in [0, \min\{1/(2L), T - t_0\}]$, which contradicts the definition of t_0. Therefore problem $(\mathcal{P})$ has at most one weak solution w. □

Chapter 4

Link with a Stefan Problem

In the Chapter 2, we showed that the strong form of the limit problem with Neumann boundary conditions is:

$$\begin{cases} \overline{u}_t = d_1 \Delta \overline{u} + f(\overline{u}) & \text{in } Q_u := \bigcup_{t\in[0,T]} \{\Omega_u(t) \times \{t\}\}, \\ \overline{v}_t = d_2 \Delta \overline{v} + g(\overline{v}) & \text{in } Q_v := \bigcup_{t\in[0,T]} \{\Omega_v(t) \times \{t\}\}, \\ \overline{u} = \overline{v} = 0 & \text{on } \Gamma := \bigcup_{t\in[0,T]} \{\Gamma(t) \times \{t\}\}, \\ d_1 \overline{u}_n = -\dfrac{d_2}{\alpha} \overline{v}_n & \text{on } \Gamma, \\ \overline{u}_\nu = 0 & \text{on } \partial\Omega \times [0,T], \\ \overline{u}(x,0) = u_0(x),\ \overline{v}(x,0) = v_0(x) & \text{for } x \in \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where n denotes the inward pointing normal of the set Ω_v (see Fig. 2.2), and ν the outward pointing normal of Ω. We will see in this chapter its relationship with a Stefan problem, and show how to transform the initial problem ($\mathcal{P}^k$) so that the limit problem ($\mathcal{P}$) includes a positive *latent heat* coefficient.

4.1 The Stefan condition

A *Stefan problem* is a problem where the diffusion equation is posed in a domain bounded by a free boundary, which is determined by an extra boundary condition (Fowler [20]). Historically, it was introduced to study the freezing of water. We suppose that there is no heat convection, only conduction. In this situation, we need to keep in mind 2 distinct phenomena.

- The first one is that the evolution of the temperature in the water is assumed to follow the heat conduction law (or Fourier's law). It states that the local heat flux density is proportional to the gradient of temperature. This gives birth to the standard heat equation. However, it is only valid until the water reaches 0 °C.
- At 0 °C, a change of phase occurs. The energy released from this point on by the water does not bring the temperature further down, but instead changes its state from liquid to ice; this energy is called the *latent heat* (of solidification in this case). See Fig. 4.1.
- Once the ice has formed, then Fourier's law is valid once again in it, albeit with different heat conductivity.

Assuming that the assumptions above are still valid, we suppose that a body of water at temperature T_l is suddenly subjected to a surface temperature $T_0 < 0\,°C$, with 0 °C the freezing temperature. We also suppose that the temperature only depends on the depth z and the time t. Then at time $t > 0$, we can expect a frozen region $0 < z < s(t)$ and an liquid region $s(t) < z < L$, where the *free boundary* $s(t)$ is a function of time. A model to describe this situation is given by

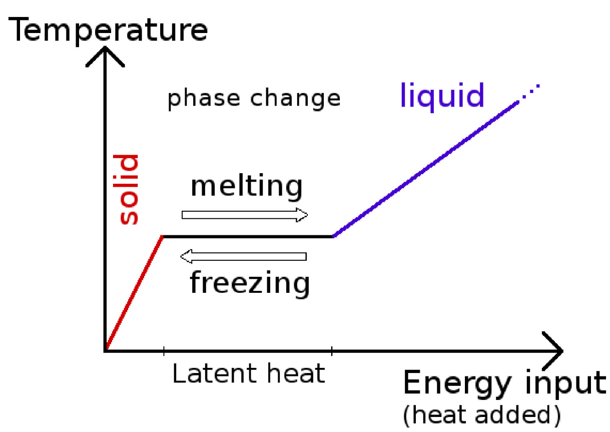

Fig. 4.1. Phase change and latent heat graph.

$$\begin{cases} T_t = \alpha_1 T_{zz}, & 0 < z < s(t),\ 0 < t \le T \\ T(0,t) = -|T_0|, & 0 < t \le T \\ T(s(t),t) = 0, & 0 < t \le T \\ \\ T_t = \alpha_2 T_{zz}, & s(t) < z < L,\ 0 < t \le T \\ T_z(L,t) = 0, & 0 < t \le T \\ \\ T(z,0) = T_I, & 0 \le z \le L \end{cases}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are the mass diffusivities in water and in ice, respectively, $T > 0$, and a Neumann boundary condition is set at $z = L$ (it is thermally insulated). Finally, what happens at the free boundary is that there is a discontinuity in the local heat flux density between $q(s^-) = -k_1 \frac{\partial T}{\partial z}|_{s^-}$ and $q(s^+) = -k_2 \frac{\partial T}{\partial z}|_{s^+}$, due to the latent heat (L, per unit mass) removed when the water freezes at $z = s$. Hence we add the energy balance condition:

$$\rho L \frac{ds}{dt} = -\left(k_1 \frac{\partial T}{\partial z}\bigg|_{s^-} - k_2 \frac{\partial T}{\partial z}\bigg|_{s^+} \right), \tag{4.1}$$

which is called the *Stefan condition*. It is a condition that directly translates into the description of the speed of the free boundary. For a full solution of this problem we refer to Fowler [20].

In Problem ($\mathcal{P}$), there is no Stefan condition on the moving boundary, or more precisely, the latent heat equivalent in this model is equal to zero.

4.2 The limit boundary problem with a positive latent heat

If our problem had a positive latent heat, then Problem ($\mathcal{P}$) would be:

$$\begin{cases} \overline{u}_t = d_1 \Delta \overline{u} + f(\overline{u}) & \text{in } Q_u := \bigcup_{t\in[0,T]} \{\Omega_u(t) \times \{t\}\}, \\ \overline{v}_t = d_2 \Delta \overline{v} + g(\overline{v}) & \text{in } Q_v := \bigcup_{t\in[0,T]} \{\Omega_v(t) \times \{t\}\}, \\ \overline{u} = \overline{v} = 0 & \text{on } \Gamma := \bigcup_{t\in[0,T]} \{\Gamma(t) \times \{t\}\}, \\ \lambda V_n = -d_1 \overline{u}_n - \dfrac{d_2}{\alpha} \overline{v}_n & \text{on } \Gamma \\ \overline{u}_\nu = 0 & \text{on } \partial\Omega \times [0,T] \\ \overline{u}(x,0) = u_0(x),\ \overline{v}(x,0) = v_0(x) & \text{for } x \in \Omega, \end{cases} \tag{$\mathcal{P}'$}$$

with $\lambda > 0$ and V_n the normal velocity of displacement of the interface. Our question is: can we modify problem ($\mathcal{P}_k$) into a reaction diffusion system ($\mathcal{P}'_k$) such that the limit of the solution of ($\mathcal{P}'_k$) is the solution of ($\mathcal{P}'$)?

4.2.1 An example of such a reaction diffusion system

In the article of Hilhorst *et al.* [21] we can find a first example:

In the half-strip $S_T = \{(x,t) : 0 < x < \infty,\ 0 < t < T\}$ we consider the following reaction-diffusion system

$$\begin{cases} u_t = u_{xx} - kF(u,v), & \text{in } S_T, \\ v_t = -kF(u,v), & \text{in } S_T, \\ u(0,t) = u_0 > 0, & t > 0, \\ u(x,0) = 0,\ v(x,0) = v_0 > 0, & x > 0, \end{cases} \tag{4.2}$$

where the function F is smooth enough and non-decreasing in u and v.

The corresponding limit problem is the simplest one-dimensional one-phase Stefan problem, similar to the one given in the previous example,

$$\begin{cases} u_t = u_{xx}, & t > 0,\ 0 < x < s(t), \\ u(0,t) = u_0, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds}{dt} = -\dfrac{1}{v_0} u_x(s(t),t), & t > 0, \\ s(0) = 0, \\ u(x,0) = 0, & x > 0. \end{cases}$$

As $k \to \infty$, the solution (u^k, v^k) of the problem (4.2) converges to $(u, 0)$ in the set $\{t > 0,\ 0 < x < s(t)\}$ and to $(0, v_0)$ in the set $\{t > 0,\ s(t) < x\}$.

In this one dimensional example, the equivalent of the latent heat is v_0, and ds/dt corresponds to the normal velocity of displacement of the interface V_n.

4.2.2 The new system coupled with an ODE

From the one-dimensional example given by system (4.2) comes the idea to couple an ODE with a PDE system, to approximate a Stefan problem with positive latent heat. We consider the system

$$\begin{cases} u_t = d_1 \Delta u + f(u) - \dfrac{s_1 uv}{\varepsilon} - \dfrac{\lambda s_1 (1-w)u}{\varepsilon}, & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + g(v) - \dfrac{s_2 uv}{\varepsilon} - \dfrac{\lambda s_2 wv}{\varepsilon}, & x \in \Omega,\ t > 0, \\ w_t = \dfrac{(1-w)u}{\varepsilon} - \dfrac{wv}{\varepsilon}, & x \in \Omega,\ t > 0, \\ u_\nu = v_\nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0^\varepsilon(x), & x \in \Omega, \\ v(x,0) = v_0^\varepsilon(x), & x \in \Omega, \\ w(x,0) = w_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{4.3}$$

where ν denotes the outward normal vector to $\partial\Omega$, and s_1, s_2, λ and ε are positive constants. We note that in the case where $\lambda = 0$ (no latent heat), we lose the coupling with w, and we recover the previously studied system (with $s_1 = 1$, $s_2 = \alpha$ and $1/\varepsilon = k$). This new system is thus a perturbation of the previous 2 component system.

Here the initial data depends on ε. We further make the following hypotheses:

$$u_0^\varepsilon, v_0^\varepsilon \in C(\overline{\Omega}), \quad w_0^\varepsilon \in L^\infty(\Omega)$$
$$0 \le u_0^\varepsilon, v_0^\varepsilon, w_0^\varepsilon \le 1, \quad \text{in } \Omega$$
$$u_0^\varepsilon \rightharpoonup u_0,\ v_0^\varepsilon \rightharpoonup v_0,\ w_0^\varepsilon \rightharpoonup w_0,\ \text{weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0,$$

for some functions $u_0, v_0, w_0 \in L^\infty(\Omega)$.

4.3 A priori bounds and relative compactness

By a solution of Problem (4.3) in Q_T $(T > 0)$ we mean a triplet of functions $(u, v, w) \in C([0,T]; C(\overline{\Omega}) \times C(\overline{\Omega}) \times L^\infty(\Omega))$ such that

$$u, v \in C^1((0,T]; C(\overline{\Omega})) \cap C((0,T]; W^{2,p}(\Omega)),\ w \in C^1([0,T]; L^\infty(\Omega))$$

for each $p \in (1, \infty)$ and such that (u, v, w) satisfies Problem (4.3).

Lemma 4.1. *There exists a positive number* $T = T(\|u_0^\varepsilon\|_{C(\overline{\Omega})}, \|v_0^\varepsilon\|_{C(\overline{\Omega})}, \|w_0^\varepsilon\|_{L^\infty(\Omega)})$ *such that (4.3) possesses a unique solution* $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ *in* Q_T.

A proof can be found in Hilhorst *et al.* [22].

Lemma 4.2. *Let* $(u_0^\varepsilon, v_0^\varepsilon, w_0^\varepsilon)$ *be a solution of (4.3) in* Q_T. *Then*

$$0 \le u^\varepsilon(x,t), v^\varepsilon(x,t), w^\varepsilon(x,t) \le 1$$

for $(x,t) \in Q_T$.

Proof. We deduce from the maximum principle that $u^\varepsilon, v^\varepsilon \ge 0$. Let now $x \in \Omega$ be such that $w_0^\varepsilon(x)$ is defined. If $w^\varepsilon(x, \bar{t}) = 0$ at a time $t = \bar{t}$, then $w_t^\varepsilon(x, \bar{t}) = u(x, \bar{t})/\varepsilon \ge 0$. The condition $w_0^\varepsilon(x) \ge 0$ implies that $w^\varepsilon(x,t) \ge 0$ for all $t > 0$. A similar argument implies that $0 \le w^\varepsilon(x,t) \le 1$ for all $t > 0$. Finally, a second application of the maximum principle yields $u^\varepsilon, v^\varepsilon \le 1$. □

Lemma 4.3. *For any positive number T, there exist positive constants* C_i $(i = 1, \ldots 5)$ *independent of* ε *and* λ *such that*

$$\iint_{Q_T} (s_1 + s_2) u^\varepsilon v^\varepsilon \le C_1 \varepsilon,$$
$$\iint_{Q_T} \lambda s_1 (1 - w^\varepsilon) u^\varepsilon \le C_2 \varepsilon,$$
$$\iint_{Q_T} \lambda s_2 w^\varepsilon v^\varepsilon \le C_3 \varepsilon,$$

$$\iint_{Q_T} d_1 |\nabla u^\varepsilon|^2 \le C_4,$$
$$\iint_{Q_T} d_2 |\nabla v^\varepsilon|^2 \le C_5.$$

Proof. As we did in the previous chapters, integrating the equation for u^ε in Q_T yields

$$\iint_{Q_T}\left(\frac{s_1 u^\varepsilon v^\varepsilon}{\varepsilon} + \frac{\lambda s_1(1-w^\varepsilon)u^\varepsilon}{\varepsilon}\right) = \int_\Omega (u_0^\varepsilon(\cdot) - u^\varepsilon(\cdot, T)) + \iint_{Q_T} f(u^\varepsilon) \le (\alpha + TM_f)|\Omega|,$$

which implies the first and second estimate. The third one can be shown similarly by integrating the equation of v^ε. Next we multiply the equation of u^ε by u^ε and integrate by parts on Ω. This yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^\varepsilon)^2 + d_1 \int_\Omega |\nabla u^\varepsilon|^2 + \int_\Omega \left(\frac{s_1 (u^\varepsilon)^2 v^\varepsilon}{\varepsilon} + \frac{\lambda s_1 (1-w^\varepsilon)(u^\varepsilon)^2}{\varepsilon}\right) \le |\Omega|\alpha M_f.$$

Hence we can integrate on $(0, T)$ to deduce the fourth estimate. The last one can be proved similarly. □

Lemma 4.4. *Let T be any positive number and set*

$$\Omega_\xi := \{x \in \Omega | x + r\xi \in \Omega \ for\ 0 \le r \le 1\}$$

with $\xi \in \mathbb{R}^N$. Then there exists positive constants C_6 and C_7 such that

$$\int_0^T \int_{\Omega_\xi} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x,t))^2 \, dx\, dt \le \frac{C_4}{d_1} |\xi|^2,$$
$$\int_0^T \int_{\Omega_\xi} (v^\varepsilon(x+\xi, t) - v^\varepsilon(x,t))^2 \, dx\, dt \le \frac{C_5}{d_2} |\xi|^2,$$
$$\int_0^{T-\tau} \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t))^2 \, dx\, dt \le C_6 \tau,$$
$$\int_0^{T-\tau} \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t))^2 \, dx\, dt \le C_7 \tau.$$

Proof. The first and second inequalities follow immediately from the estimates for $|\nabla u^\varepsilon|^2$ and $|\nabla v^\varepsilon|^2$. Indeed, we have:

$$\int_0^T \int_{\Omega_\xi} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x,t))^2 \, dx\, dt = \int_0^T \int_{\Omega_\xi} \left\{\int_0^1 \nabla u^\varepsilon(x + r\xi, t)\cdot \xi \, dr\right\}^2 dx\, dt$$
$$\le \frac{C_4}{d_1} |\xi|^2.$$

The second can be shown similarly. For the last two inequalities, we have:

$$\begin{aligned}
&\int_0^{T-\tau} \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t))^2 \, dx\, dt \\
&= \int_0^{T-\tau} \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t)) \int_0^\tau u_t^\varepsilon(x, t+r)\, dr\, dx\, dt \\
&= \int_0^{T-\tau} \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t)) \int_0^\tau \Big\{ d_1 \Delta u^\varepsilon(x, t+r) + f(u^\varepsilon(x, t+r)) \\
&\quad - \frac{s_1 u^\varepsilon(x, t+r) v^\varepsilon(x, t+r) + \lambda s_1 (1 - w^\varepsilon(x, t+r)) u^\varepsilon(x, t+r)}{\varepsilon} \Big\}\, dr\, dx\, dt.
\end{aligned}$$

For an upper bound of the first term, we have

$$\begin{aligned}
&\left| \int_0^{T-\tau} \int_\Omega (u^\varepsilon(x, t+\tau) - u^\varepsilon(x,t)) \int_0^\tau d_1 \Delta u^\varepsilon(x, t+r)\, dr\, dx\, dt \right| \\
&= d_1 \left| \int_0^\tau \int_0^{T-\tau} \int_\Omega (\nabla u^\varepsilon(x, t+\tau) - \nabla u^\varepsilon(x,t)) \cdot \nabla u^\varepsilon(x, t+r)\, dx\, dt\, dr \right| \\
&\le 2 d_1 \tau \int_0^T \int_\Omega |\nabla u^\varepsilon(x,t)|^2 \, dx\, dt \\
&\le 2 C_4 \tau.
\end{aligned}$$

Similarly,

$$\left|\int_0^{T-\tau}\int_\Omega (u^\varepsilon(x,t+\tau)-u^\varepsilon(x,t))\int_0^\tau f(u^\varepsilon(x,t+r))\,dr\,dx\,dt\right| \le 2M_f T|\Omega|\tau.$$

Finally, we have

$$\begin{aligned}&\int_0^{T-\tau}\int_\Omega \Bigg\{|u^\varepsilon(x,t+\tau)-u^\varepsilon(x,t)|\\ &\int_0^\tau \frac{s_1u^\varepsilon(x,t+r)v^\varepsilon(x,t+r)+\lambda s_1(1-w^\varepsilon(x,t+r))u^\varepsilon(x,t+r)}{\varepsilon}\,dr\Bigg\}\,dx\,dt\\ &\le 2\tau\int_0^T\int_\Omega \frac{s_1u^\varepsilon(x,t)v^\varepsilon(x,t)+\lambda s_1(1-w^\varepsilon(x,t))u^\varepsilon(x,t)}{\varepsilon}\,dx\,dt\\ &\le 2(C_1+C_2)\tau.\end{aligned}$$

In the end, we have shown that

$$\int_0^{T-\tau}\int_\Omega (u^\varepsilon(x,t+\tau)-u^\varepsilon(x,t))^2\,dx\,dt \le (2C_4+M_fT|\Omega|+C_1+C_2)\tau.$$

Similarly, we can prove the estimate

$$\int_0^{T-\tau}\int_\Omega (v^\varepsilon(x,t+\tau)-v^\varepsilon(x,t))^2\,dx\,dt \le (2C_5+M_gT|\Omega|+C_1+C_3)\tau,$$

which concludes the proof. □

We deduce from the previous estimates that the families $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ are bounded in $L^2(0,T;H^1(\Omega))$ and the family $\{w^\varepsilon\}$ is bounded in $L^\infty(Q_T)$. Furthermore, it follows from the Riesz–Fréchet–Kolmogorov theorem that the families $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ are relatively compact in $L^2(Q_T)$.

4.4 Characterization of the limit problem

4.4.1 Convergence to a weak form of the limit problem

With the previous results, we know that there exist subsequences $\{u^{\varepsilon_n}\}$, $\{v^{\varepsilon_n}\}$ and $\{w^{\varepsilon_n}\}$ as well as functions $u^*, v^* \in L^2(0,T;H^1(\Omega))$ and $w^* \in L^2(Q_T)$ such that

$$u^{\varepsilon_n} \to u^*, \quad v^{\varepsilon_n} \to v^*$$

strongly in $L^2(Q_T)$, weakly in $L^2(0,T;H^1(\Omega))$ and a.e. in Q_T, and

$$w^{\varepsilon_n} \rightharpoonup w^* \quad \text{weakly in } L^2(Q_T) \text{ as } \varepsilon_n \to 0.$$

Moreover,

$$0 \le u^*, v^*, w^* \le 1 \quad \text{a.e. on } \overline{Q_T}.$$

Hence we deduce from the first three estimates of Lemma 4.3 that

$$u^*v^* = (1-w^*)u^* = w^*v^* = 0 \quad \text{a.e. on } \overline{Q_T}. \tag{4.4}$$

Lemma 4.5. *Let T be an arbitrary positive number. The triplet of functions (u^*, v^*, w^*) defined above satisfy*

$$\begin{aligned}&\iint_{Q_T}\left\{\left(\frac{u^*}{s_1}-\frac{v^*}{s_2}+\lambda w^*\right)\zeta_t-\nabla\left(\frac{d_1u^*}{s_1}-\frac{d_2v^*}{s_2}\right)\cdot\nabla\zeta+\left(\frac{f(u^*)}{s_1}-\frac{g(v^*)}{s_2}\right)\zeta\right\}\\ &=-\int_\Omega\left(\frac{u_0}{s_1}-\frac{v_0}{s_2}+\lambda w_0\right)\zeta(\cdot,0)\end{aligned} \tag{4.5}$$

for all functions $\zeta \in C^\infty(\overline{Q_T})$ satisfying $\zeta(x,T)=0$.

Proof. We deduce from the reaction-diffusion system (4.3) for $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ that

$$\left(\frac{u^\varepsilon}{s_1}-\frac{v^\varepsilon}{s_2}+\lambda w^\varepsilon\right)_t = \frac{d_1\Delta u^\varepsilon}{s_1}-\frac{d_2\Delta v^\varepsilon}{s_2}+\frac{f(u^\varepsilon)}{s_1}-\frac{g(v^\varepsilon)}{s_2}.$$

Multiplying that by a test function $\zeta \in C^\infty(\overline{Q_T})$ with $\zeta(\cdot,T)=0$ and integrating by parts, we obtain the identity,

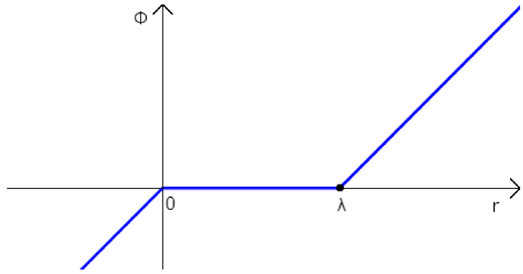

Fig. 4.2. Graph of the function $\phi(r)$.

$$\iint_{Q_T} \left\{ -\left(\frac{u^\varepsilon}{s_1} - \frac{v^\varepsilon}{s_2} + \lambda w^\varepsilon \right) \zeta_t + \nabla \left(\frac{d_1 u^\varepsilon}{s_1} - \frac{d_2 v^\varepsilon}{s_2} \right) \cdot \nabla \zeta \right\}$$
$$= \iint_{Q_T} \left(\frac{f(u^\varepsilon)}{s_1} - \frac{g(v^\varepsilon)}{s_2} \right) \zeta + \int_\Omega \left(\frac{u_0^\varepsilon}{s_1} - \frac{v_0^\varepsilon}{s_2} + \lambda w_0^\varepsilon \right) \zeta(\cdot, 0).$$

Letting $\varepsilon = \varepsilon_n \to 0$ yields the result. □

Let us now set

$$Z^* := \frac{u^*}{s_1} - \frac{v^*}{s_2} + \lambda w^*.$$

We will show that Z^* satisfies a weak form corresponding to the following parabolic boundary value problem:

$$\begin{cases} Z_t = \Delta \mathcal{D}(\phi(Z)) + h(\phi(Z)), & x \in \Omega,\ 0 < t \le T, \\ \dfrac{\partial \mathcal{D}(\phi(Z))}{\partial \nu} = 0, & x \in \partial\Omega,\ 0 < t \le T, \\ Z(x, 0) = Z_0(x), & x \in \Omega, \end{cases} \tag{4.6}$$

where

$$\mathcal{D}(r) := \begin{cases} d_1 r & \text{for } r \ge 0, \\ d_2 r & \text{for } r < 0, \end{cases}$$

$$\phi(r) := \begin{cases} r - \lambda & \text{for } r > \lambda, \\ 0 & \text{for } 0 \le r \le \lambda, \\ r & \text{for } r < 0, \end{cases}$$

$$h(r) := \begin{cases} \dfrac{f(s_1 r)}{s_1} & \text{for } r \ge 0, \\ -\dfrac{g(-s_2 r)}{s_2} & \text{for } r < 0. \end{cases}$$

The function ϕ is used to model the phase change, as can be seen by comparing the Figs. 4.1 and 4.2.

Let us now define the Heaviside function

$$H(r) := \begin{cases} 1 & \text{for } r > 0, \\ [0, 1] & \text{for } r = 0, \\ 0 & \text{for } r < 0. \end{cases}$$

$$r_+ := \max\{r, 0\}, \quad r_- := -\min\{r, 0\}.$$

Lemma 4.6. *If $w \in H(z)$, then $\phi(z + \lambda w) = z$. In particular, the limit functions u^*, v^* and w^* satisfy*

$$u^* = s_1 \phi(Z^*)_+, \; v^* = s_2 \phi(Z^*)_-, \quad \text{and} \quad w^* = \frac{Z^* - \phi(Z^*)}{\lambda}, \tag{4.7}$$

where

$$Z^* := \frac{u^*}{s_1} - \frac{v^*}{s_2} + \lambda w^*. \tag{4.8}$$

Proof. The first claim of this lemma follows from the definition of ϕ and H. We deduce that

$$w^* \in H\left(\frac{u^*}{s_1} - \frac{v^*}{s_2}\right)$$

from the equalities (4.4). Hence we have

$$\phi(Z^*) = \frac{u^*}{s_1} - \frac{v^*}{s_2},$$

which implies the result. □

Definition 4.1. *A function $Z \in L^\infty(Q_T)$ is a weak solution of the problem (4.6) with an initial datum $Z_0 \in L^\infty(\Omega)$ if*

$$\mathcal{D}(\phi(Z)) \in L^2(0,T;H^1(\Omega))$$

and

$$\iint_{Q_T} Z\zeta_t + \int_\Omega Z_0\zeta(\cdot,0) = \iint_{Q_T} \{\nabla\mathcal{D}(\phi(Z)) \cdot \nabla\zeta - h(\phi(Z))\zeta\} \tag{4.9}$$

for all functions $\zeta \in C^\infty(\overline{Q_T})$ satisfying $\zeta(\cdot,T) = 0$.

If Z is a weak solution of the problem (4.6), then $\phi(Z)$ is continuous on $\overline{\Omega} \times [\delta, T]$ for each $\delta \in (0,T]$.

Lemma 4.7. *The function Z^* defined by (4.8) is a weak solution of the problem (4.6) with an initial datum $Z_0 = u_0/s_1 - v_0/s_2 + \lambda w_0$.*

Proof. It follows from the Lemma 4.2 that $Z^* \in L^\infty(Q_T)$. We observe that (4.7) implies

$$\mathcal{D}(\phi(Z^*)) = \frac{d_1 u^*}{s_1} - \frac{d_2 v^*}{s_2}.$$

In particular, $\mathcal{D}(\phi(Z^*)) \in L^2(0,T;H^1(\Omega))$ holds true by Lemma 4.3. We also notice that

$$h(\phi(Z^*)) = \frac{f(u^*)}{s_1} - \frac{g(v^*)}{s_2}.$$

Therefore (4.5) can be rewritten as (4.9) with $Z = Z^*$ and $Z_0 = u_0/s_1 - v_0/s_2 + \lambda w_0$. This completes the proof. □

Theorem 4.8. *The function Z^* defined by (4.8) is the unique weak solution of the problem (4.6) with an initial datum $Z_0 = u_0/s_1 - v_0/s_2 + \lambda w_0$. As $\varepsilon \to 0$,*

$$u^\varepsilon \to u^*, \quad v^\varepsilon \to v^* \quad \text{strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T;H^1(\Omega)),$$
$$w^\varepsilon \rightharpoonup w^* \quad \text{weakly in } L^2(Q_T).$$

The proof for the uniqueness addressed in this theorem can be found in Hilhorst *et al.* [24].

4.4.2 The strong form of the limit problem

We set

$$\begin{cases} \Omega_+(t) := \{x \in \Omega \mid \phi(Z(x,t)) > 0\}, \\ \Omega_-(t) := \{x \in \Omega \mid \phi(Z(x,t)) < 0\}, \\ \Gamma(t) := \Omega \backslash (\Omega_+(t) \cup \Omega_-(t)), \end{cases} \tag{4.10}$$

for $t \in [0,T]$, and also use the notation

$$\begin{cases} \Omega_+ := \bigcup_{0\le t\le T} \Omega_+(t) \times \{t\}, \\ \Omega_- := \bigcup_{0\le t\le T} \Omega_-(t) \times \{t\}, \\ \Gamma := \bigcup_{0\le t\le T} \Gamma(t) \times \{t\}. \end{cases} \tag{4.11}$$

We can regard $\Omega_+(t)$ and $\Omega_-(t)$ to symbolize two distinct phases, and $\Gamma(t)$ represents a phase boundary (or an *interface*) at time t.

Theorem 4.9. *Let Z be the unique weak solution of the problem (4.6) with initial datum Z_0 and let $\Omega_\pm(t)$ and $\Gamma(t)$ be the sets defined by (4.10). Suppose that (each component of) $\Gamma(t)$ is a smooth, closed and orientable hypersurface satisfying $\Gamma(t) \cap \partial\Omega = \emptyset$ for all $t \in [0,T]$. Let n be the unit normal vector on $\Gamma(t)$ oriented from $\Omega_+(t)$ to $\Omega_-(t)$. Also, assume that $\Gamma(t)$ smoothly moves with a velocity V_n in the direction of n and that the functions*

$$u := s_1\phi(Z)_+, \quad \text{and} \quad v := s_2\phi(Z)_-$$

are smooth on $\overline{\Omega_+}$ *and* $\overline{\Omega_-}$ *respectively. Then* (Γ, u, v) *satisfies*

$$\begin{cases} u_t = d_1 \Delta u + f(u) & \text{in } \Omega_+(t), \\ v_t = d_2 \Delta v + g(v) & \text{in } \Omega_-(t), \\ \lambda V_n = -\dfrac{d_1}{s_1} u_n - \dfrac{d_2}{s_2} v_n & \text{on } \Gamma(t), \\ u = 0, \ v = 0 & \text{on } \Gamma(t), \\ u_\nu = 0, \ v_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

for $t \in (0, T]$ *and*

$$\begin{cases} \Gamma(0) = \{x \in \Omega \mid \phi(Z_0(x)) = 0\}, \\ u(x,0) = s_1(\phi(Z_0(x)))_+, \quad v(x,0) = s_2(\phi(Z_0(x)))_-, \quad x \in \Omega. \end{cases}$$

Part II

The Singular Limit of an Allen–Cahn Equation with Linear or Nonlinear Diffusion

Chapter 5

Singular Limit of the Allen–Cahn Equation with Linear or Nonlinear Diffusion

5.1 Singular limit of the Allen–Cahn equation with linear diffusion

The Allen–Cahn equation

$$u_t = \Delta u + \frac{1}{\varepsilon^2} f(u), \quad (x,t) \in D \times \mathbb{R}^+$$

was introduced to understand the phase separation phenomena which appear in the construction of polycrystalline materials. Here, u stands for the order parameter which describes the state of the material, $-f$ is the derivative of a double-well potential with two distinct local minima α_+ and α_- at two different phases, and the parameter $\varepsilon > 0$ corresponds to the interface width in the phase separation process. When $\varepsilon > 0$ is small, it is expected that u converges to either of the two states $u = \alpha_+$ and $u = \alpha_-$. Thus, the limit $\varepsilon \to 0$ creates a steep interface dividing two phases; this is a phase separation phenomenon and the limiting interface is known to evolve according to mean curvature flow [2, 5]. More precisely, the problem which we study is given by

$$\begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^2} f(u), & (x,t) \in D \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial D \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in D. \end{cases} \qquad (\mathcal{P}^\varepsilon)$$

In this model, f is the growth function, and it is bistable. In other words, the ODE

$$\frac{du}{dt} = f(u)$$

has two stable equilibria, α_- and α_+, as well as one unstable equilibrium α, with $\alpha_- < \alpha < \alpha_+$. For example, $f(u) = u - u^3$ with $\alpha_- = -1$, $\alpha = 0$ and $\alpha_+ = 1$. When this equation is used to model a mixture undergoing phase separation, then α_- and α_+ are the two different phases.

5.1.1 Rough idea of how this system evolves

Since $\varepsilon \ll 1$, for a given $x \in \Omega$, by changing the time-frame to $\tau = t/\varepsilon^2$, we can approximate the Allen–Cahn equation with the solution of the ODE

$$\hat{u}_\tau(x,\tau) = f(\hat{u}(x,\tau))$$

Since f is bistable, depending on the initial condition $\hat{u}(x,0)$, $\hat{u}(x,\tau) \to \alpha_-$ or α_+ when $\tau \to \infty$ (except if $\hat{u}(x,0) = \alpha$, then $\hat{u}(x,\tau) = \alpha$, for all $\tau > 0$). See Figs. 5.1 and 5.2 for an example. Hence, a solution u^ε of the problem $(\mathcal{P}^\varepsilon)$ has a very steep transition zone between $\{u^\varepsilon\} \approx \alpha_-$ and $\{u^\varepsilon\} \approx \alpha_+$.

The singular limit of the solution u^ε as ε tends to zero is known (see for example Allen and Cahn [2]). u^ε converges to the function $\tilde{u}(x,t)$ where $\tilde{u}(t) = \alpha_+$ inside the space delimited by the interface Γ_t, and $\tilde{u}(t) = 0$ outside Γ_t. We also know that this interface moves according to the law:

$$V_n = -(N-1)\kappa \quad \text{on } \Gamma_t \qquad (5.1)$$

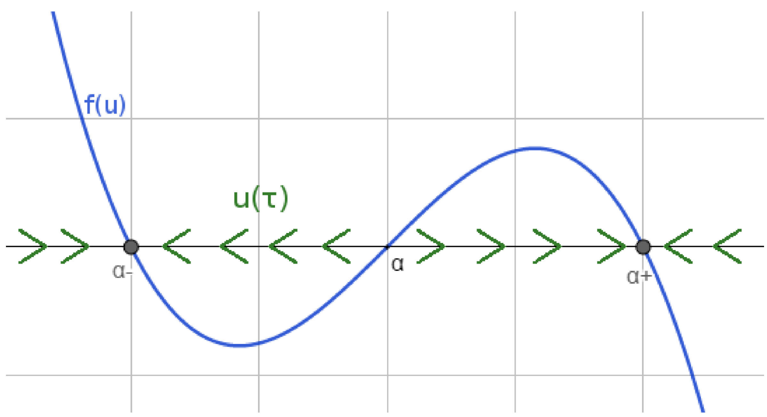

Fig. 5.1. Bistable growth function. The green arrows show the evolution of $u(\tau)$ in the case of $u_\tau = f(u)$.

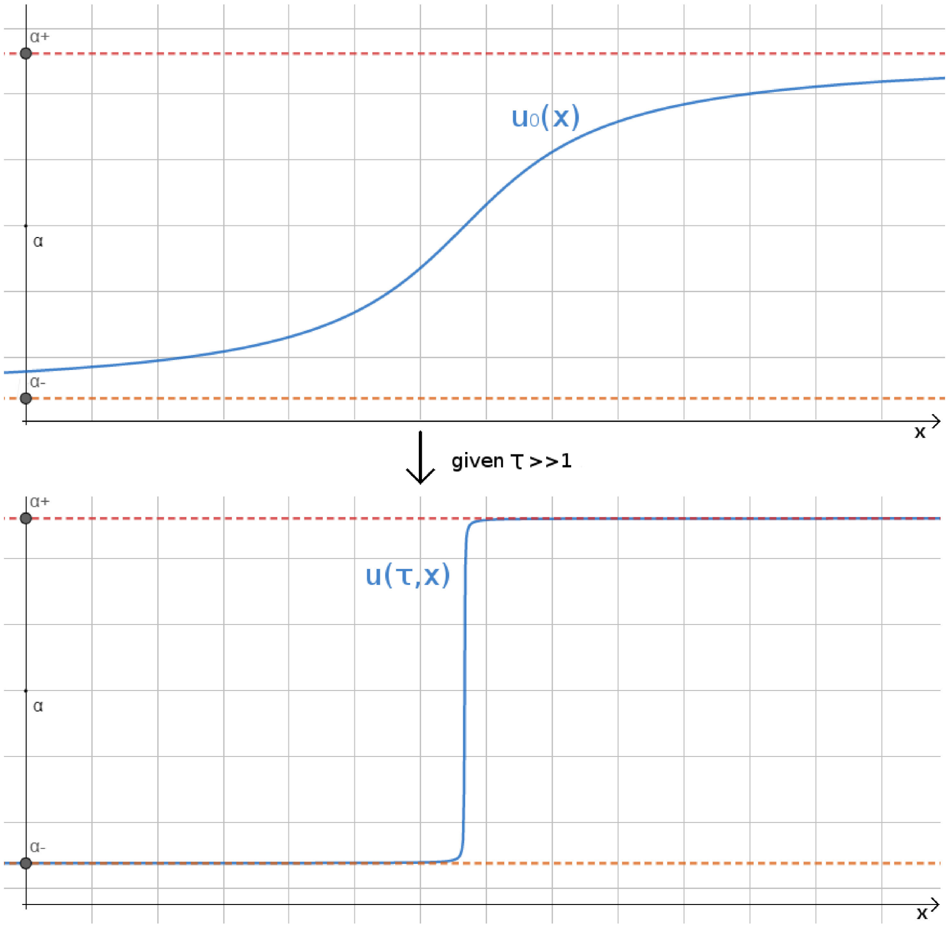

Fig. 5.2. $u_0(x)$ and $u(\tau, x)$ for a given $\tau \gg 1$ when its growth is approximated by $u_\tau \simeq u - u^3$. The transition region is called the interface.

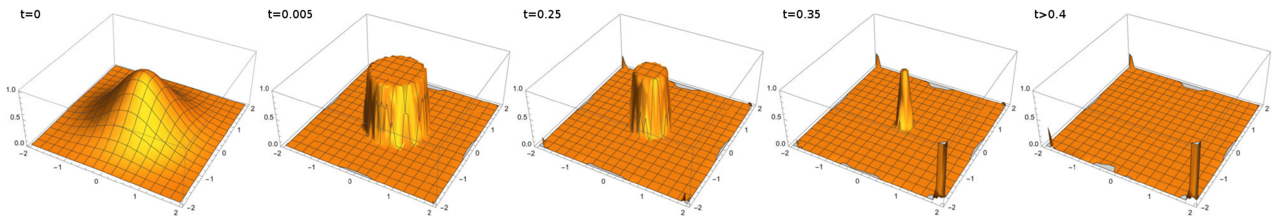

Fig. 5.3. Numerical simulations of the Allen–Cahn system in space dimension 2. The z-axis represents the density of the population u at a given time written above, and at the coordinates $(x, y) \in D = [-2, 2] \times [-2, 2]$. We have taken Neumann boundary conditions, and chosen $u_0(x, y) = e^{-x^2-y^2}$ for the initial condition, while the reaction function is defined by $f(u) = u(1-u)(u-0.5)$, $\varepsilon = 0.01$. The graphs show the evolution of the species according to the Allen–Cahn equation. We can clearly see the interface, moving towards the center before disappearing.

on some time interval $[0, T^*]$, where V_n is the normal velocity of the interface Γ_t, κ its mean curvature, and n is the unit normal vector.

5.1.2 Formal derivation of the limit problem

The main reference for this part is the paper of Alfaro, Hilhorst and Matano [1]. We suppose that the following hypotheses are satisfied:

- D is a smooth bounded domain in $\mathbb{R}^N$,
- $f \in C^2(\mathbb{R})$ has three zeros $f(\alpha_-) = f(\alpha_+) = f(\alpha) = 0$ where $\alpha_- < \alpha < \alpha_+$, and $f'(\alpha_-) < 0, f'(\alpha_+) < 0, f'(\alpha) > 0$,
- $\int_{\alpha_-}^{\alpha_+} f(s)ds = 0$,
- Γ_0 is an hypersurface of class $C^{4+\delta}$, for some $0 < \delta < 1$,
- $u_0 \in C^2(\overline{D})$, and $\nabla u_0(x) \cdot n(x) \neq 0$ if $x \in \Gamma_0$,
- $u_0 > \alpha$ in D_0^+, and $u_0 < \alpha$ in D_0^-,

where D_0^- denotes the region enclosed by Γ_0, D_0^+ the region enclosed between ∂D and Γ_0, and n is the outward normal

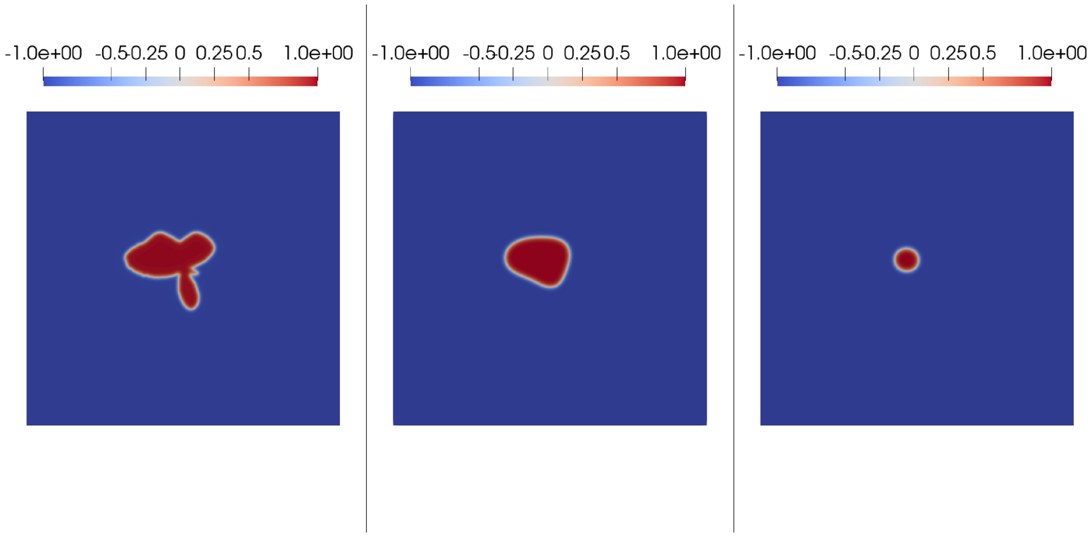

Fig. 5.4. Numerical simulations of the Allen–Cahn system in space dimension 2. The space domain is $D = [-1, 1] \times [-1, 1]$. We have taken Neumann boundary conditions, and chosen a non-convex domain D_1 such that $u_0(x, y) = 0.9$ if $(x, y) \in D_1$ and $u_0(x, y) = -0.9$, otherwise for the initial condition, while the reaction function is defined by $f(u) = u - u^3$, $\varepsilon = 0.01$. The graphs show the evolution of the species according to the Allen–Cahn equation at $t = 10^{-6}, 0.007, 0.021$. We can see the non-convex domain becomes convex and then disappears.

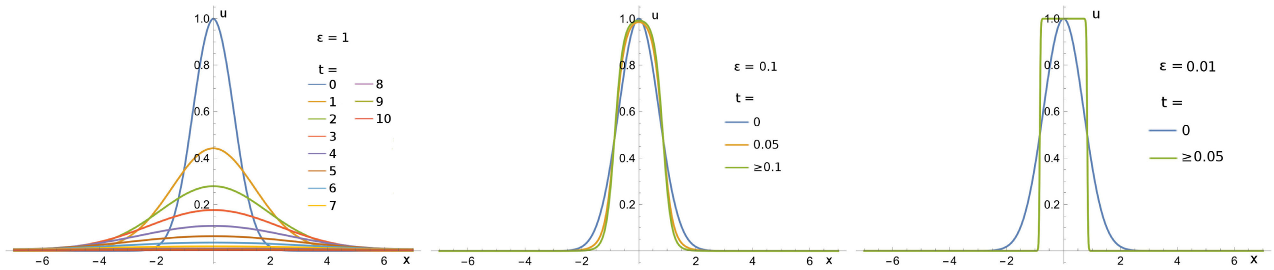

Fig. 5.5. Numerical simulations of the Allen–Cahn system in space dimension 1. We have taken Neumann boundary conditions, $u_0(x) = e^{-x^2}$ for the initial condition, and $f(u) = u(1 - u)(u - 0.5)$. The graphs show the evolution of the species according to the Allen–Cahn equation.

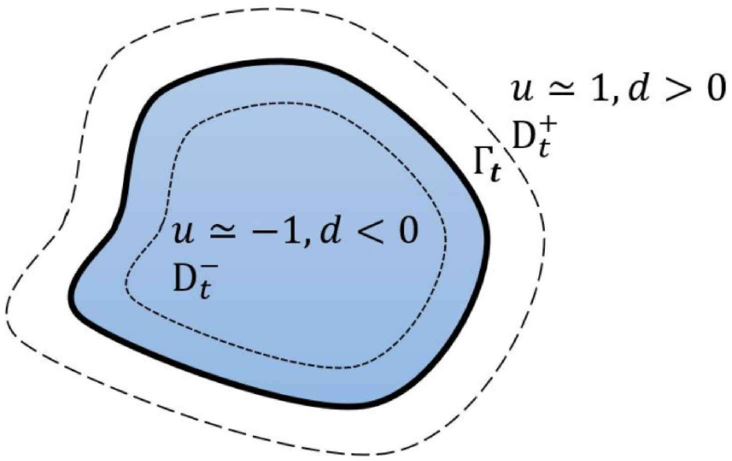

Fig. 5.6. Interface in the limit problem.

vector to D_0^-. Let also D_t^- denotes the region enclosed by the interface Γ_t, and $D_t^+ := D \setminus \overline{D_t^-}$. See Fig. 5.6.

Let $d(x, t)$ be the signed distance function to Γ_t defined by

$$d(x, t) := \begin{cases} \mathrm{dist}(x, \Gamma_t) & \text{for } x \in \overline{D_t^+} \\ -\mathrm{dist}(x, \Gamma_t) & \text{for } x \in D_t^- \end{cases}$$

We look for an approximate solution in the form

$$u^\varepsilon(x,t) = U_0(x,t,z) + \varepsilon U_1(x,t,z) + \cdots$$

near Γ_t, where $z = \frac{d}{\varepsilon}$.
To this purpose we apply the method of matched asymptotic expansions; namely we substitute this expression into (P^ε) and chose U_0, U_1 accordingly. A first calculation gives

$$\Delta u^\varepsilon = \Delta U_0 + \frac{1}{\varepsilon} U_{0z}\Delta d + \frac{1}{\varepsilon^2} U_{0zz}(\nabla d)^2$$
$$+ \varepsilon \Delta U_1 + U_{1z}\Delta d + \frac{1}{\varepsilon} U_{1zz}(\nabla d)^2 + \dots,$$
$$u_t^\varepsilon = U_{0t} + \frac{1}{\varepsilon} U_{0z} d_t + \varepsilon U_{1t} + U_{1z} d_t + \dots$$

After substitution into (P^ε), we collect the ε^{-2} terms, which yields the equation

$$U_{0zz} + f(U_0) = 0,$$

where U_0 is the unique solution of the problem

$$\begin{cases} U_{0zz} + f(U_0) = 0, \\ U_0(-\infty) = \alpha_-, \quad U_0(0) = \alpha, \quad U_0(\infty) = \alpha_+. \end{cases} \tag{5.2}$$

We note that

$$\int_{-\infty}^{+\infty} U_{0zz} U_{0z} dz + \int_{-\infty}^{+\infty} f(U_0) U_{0z} dz = 0$$

implies

$$\int_{-\infty}^{+\infty} f(U_0) U_{0z} dz = \int_{\alpha_-}^{\alpha_+} f(s) ds = 0$$

To achieve that result, remark that

$$\int_{-\infty}^{+\infty} U_{0zz} U_{0z} dz = \left[\frac{(U_{0z})^2}{2} \right]_{z=-\infty}^{z=+\infty},$$

and when $z = \pm\infty$, $U_0(z)$ reaches stable steady states so that $U_{0z}(\pm\infty) = 0$. Thus there cannot exist a solution U_0 unless $\int_{\alpha_-}^{\alpha_+} f(s) ds = 0$. Moreover one can prove the following results:

Lemma 5.1. *There exist positive constants C and λ such that*

$$0 < \alpha_+ - U_0(z) \le Ce^{-\lambda|z|} \quad \text{for } z \ge 0,$$
$$0 < U_0(z) - \alpha_- \le Ce^{-\lambda|z|} \quad \text{for } z \le 0.$$

In addition, U_0 is a strictly increasing function and, for $j = 1, 2$,

$$|D^j U_0(z)| \le Ce^{-\lambda|z|} \quad \textit{for } z \in \mathbb{R}. \tag{5.3}$$

Next, we consider the collection of ε^{-1} terms in the asymptotic expansion. This yields the problem

$$\begin{cases} U_{1zz} + f'(U_0)U_1 = U_{0z}(d_t - \Delta d), \\ U_1(x,t,0) = 0, \quad U_1 \in L^\infty(\mathbb{R}). \end{cases} \tag{5.4}$$

Now consider the more general problem

$$\begin{cases} \psi_{zz} + f'(U_0)\psi = A(z), \\ \psi = 0 \text{ when } z = 0, \quad \psi \in L^\infty(\mathbb{R}), \end{cases} \tag{5.5}$$

and multiply the ODE by U_{0z}. This gives

$$\int_{-\infty}^{+\infty} \psi_{zz} U_{0z} dz + \int_{-\infty}^{+\infty} f'(U_0)\psi U_{0z} dz = \int_{-\infty}^{+\infty} A(z) U_{0z} dz,$$

and an integration by parts yields

$$-\int_{-\infty}^{+\infty} \psi_z (U_{0zz} + f(U_0)) dz = \int_{-\infty}^{+\infty} A(z) U_{0z} dz = 0,$$

which implies the solvability condition

$$(d_t - \Delta d)(x,t)\int_{-\infty}^{+\infty} U_{0_z}^2 dz = 0,$$

or else

$$d_t(x,t) = \Delta d(x,t).$$

It is known that $d_t = -V_n$ on the interface Γ_t, and Δd is equal to $(N-1)\kappa$ where κ is the mean curvature of Γ_t. Thus we obtain the interface motion equation on Γ_t

$$V_n = -(N-1)\kappa. \tag{5.6}$$

5.1.3 Other resources

For more results on the deterministic Allen–Cahn equation with linear diffusion, see the papers:

- Bronsard, Kohn [4]: Deterministic, arbitrary space dimension in spherical symetry.
- De Mottoni, Schatzman [12]: Arbitrary space dimension, matched asymptotic expansions.
- Xinfu Chen [6]: Using comparison principle, sub-super solutions.
- Evans, Soner and Souganidis [19]: Convergence to viscosity solutions on an arbitrary time interval.

5.2 The limit of the Allen–Cahn equation with nonlinear diffusion

$$(P^\varepsilon)\begin{cases} u_t = \Delta\varphi(u) + \dfrac{1}{\varepsilon^2}f(u), & (x,t)\in D\times\mathbb{R}^+,\\ \dfrac{\partial\varphi(u)}{\partial\nu} = 0, & (x,t)\in\partial D\times\mathbb{R}^+,\\ u(x,0) = u_0(x), & x\in D.\end{cases} \tag{5.7}$$

In the case that the nonlinear diffusion term in the parabolic equation of (5.7) is degenerate, for instance if $\varphi'(0) = 0$, no rigorous propagation of interface result in arbitrary space dimension has been proved. However, we can prove these results in the case that this partial differential equation is uniformly parabolic, namely if $\varphi' \geq C_\varphi > 0$ and $\int_{\alpha_-}^{\alpha_+}\varphi'(s)f(s)ds = 0$. We are expecting phase separation as in the linear case, since we can approximate the system just as before:

$$\begin{aligned} u_t &= \Delta\varphi(u) + \frac{1}{\varepsilon^2}f(u)\\ &\simeq \frac{1}{\varepsilon^2}f(u),\end{aligned}$$

in the generation of the interface time interval.

We will not present here a rigorous proof of the propagation of interface. However, we will formally show that the limit problem is given by

$$\begin{cases} V_n = -\lambda_0(N-1)\kappa \quad \text{on } \Gamma_t,\\ \Gamma_t|_{t=0} = \Gamma_0,\end{cases} \tag{P_0}$$

with the same notations as before, V_n the normal velocity of Γ_t, and κ the mean curvature at each point of Γ_t,

$$\lambda_0 = \frac{\int_{\alpha_-}^{\alpha_+}\varphi'(u)\sqrt{W(u)}du}{\int_{\alpha_-}^{\alpha_+}\sqrt{W(u)}du}, \tag{5.8}$$

and the potential W is defined by

$$W(u) = \int_u^{\alpha_+} f(s)\varphi'(s)ds. \tag{5.9}$$

For complete proofs, we refer to [16].

We suppose similar hypotheses as in the linear case, with some more added because of the presence of the function φ:

- D is a smooth bounded domain in $\mathbb{R}^N$,
- $f\in C^2(\mathbb{R})$ has three zeros $f(\alpha_-) = f(\alpha_+) = f(\alpha) = 0$ where $\alpha_- < \alpha < \alpha_+$, and $f'(\alpha_-) < 0, f'(\alpha_+) < 0, f'(\alpha) > 0$,
- $\varphi\in C^4(\mathbb{R})$, $\varphi' \geq C_\varphi > 0$ and $\int_{\alpha_-}^{\alpha_+}\varphi'(s)f(s)ds = 0$,
- Γ_0 is a $C^{4+\delta}$, $0<\delta<1$, $u_0 > \alpha$ in D_0^+, $u_0 < \alpha$ in D_0^-, where D_0^- denotes the region enclosed by Γ_0, D_0^+ the region enclosed between ∂D and Γ_0,
- $u_0\in C^2(\overline{D})$, $\nabla u_0(x)\cdot n(x)\neq 0$ if $x\in\Gamma_0$, $u_0 > \alpha$ in D_0^+, $u_0 < \alpha$ in D_0^-,

where D_0^-, D_0^+, D_t^-, D_t^+, n and the distance function $d = d(x,t)$ are defined similarly as in the linear case.

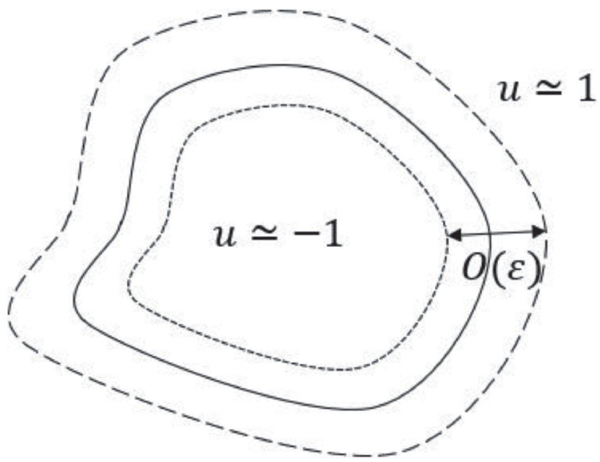

Fig. 5.7. Expected phase separation with interface of width $\mathcal{O}(\varepsilon)$.

5.2.1 Formal derivation of the limit problem

We are looking for an approximate solution in the form

$$u^{\varepsilon}(x,t) = U_0(x,t,z) + \varepsilon U_1(x,t,z) + \cdots$$

near Γ_t, where $z = d/\varepsilon$.
Once again we apply the method of matched asymptotic expansions which yields the equation

$$\varphi(U_0)_{zz} + f(U_0) = 0,$$

where U_0 is the unique solution of the problem

$$\begin{cases} (\varphi(U_0))_{zz} + f(U_0) = 0, \\ U_0(-\infty) = \alpha_-, \quad U_0(0) = \alpha, \quad U_0(\infty) = \alpha_+. \end{cases} \tag{5.10}$$

To understand this more clearly, we set

$$g(u) := f(\varphi^{-1}(u)),$$

where φ^{-1} is the inverse function of φ and define $V_0(z) := \varphi(U_0(z))$. Substituting V_0 into equation (5.10) yields

$$\begin{cases} V_{0zz} + g(V_0) = 0, \\ V_0(-\infty) = \varphi(\alpha_+), \quad V_0(0) = \varphi(\alpha), \quad V_0(\infty) = \varphi(\alpha_-). \end{cases} \tag{5.11}$$

Next, we consider the collection of ε^{-1} terms in the asymptotic expansion. This yields the following problem

$$\begin{cases} (\varphi'(U_0)U_1)_{zz} + f'(U_0)U_1 = U_{0z}d_t - (\varphi(U_0))_z \Delta d, \\ U_1(x,t,0) = 0, \quad \varphi'(U_0)U_1 \in L^{\infty}(\mathbb{R}). \end{cases} \tag{5.12}$$

To justify the existence of the solution of (5.12) we perform the change of unknown function $V_1 = \varphi'(U_0)U_1$, which yields the problem

$$\begin{cases} V_{1zz} + g'(V_0)V_1 = \dfrac{V_{0z}}{\varphi'(\varphi^{-1}(V_0))} d_t - V_{0z}\Delta d, \\ V_1(x,t,0) = 0, \quad V_1 \in L^{\infty}(\mathbb{R}). \end{cases} \tag{5.13}$$

There exists a solution V_1 provided that

$$\int_{\mathbb{R}} \left(\frac{1}{\varphi'(\varphi^{-1}(V_0))} d_t - \Delta d \right) V_{0z}^2 = 0.$$

Substituting $V_0 = \varphi(U_0)$ and $V_{0z} = \varphi'(U_0)U_{0z}$ yields

$$d_t = \frac{\int_{\mathbb{R}} V_{0z}^2}{\int_{\mathbb{R}} \frac{V_{0z}^2}{\varphi'(\varphi^{-1}(V_0))}} \Delta d = \frac{\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2}{\int_{\mathbb{R}} \varphi'(U_0)U_{0z}^2} \Delta d.$$

It is known that $d_t = -V_n$ on the interface Γ_t, and Δd is equal to $(N-1)\kappa$ where κ is the mean curvature of Γ_t. Thus we obtain the interface motion equation on Γ_t

$$V_n = -(N-1)\lambda_0 \kappa. \tag{5.14}$$

We multiply the equation (5.2) by $\varphi(U_0)_z$ and then integrate from $-\infty$ to z to deduce that

$$\lambda_0 = \frac{\int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{W(u)}du}{\int_{\alpha_-}^{\alpha_+} \sqrt{W(u)}du}, \tag{5.15}$$

where the potential W is defined by

$$W(u) = \int_u^{\alpha_+} f(s)\varphi'(s)ds. \tag{5.16}$$

If $\varphi(s) = s$, then $\lambda_0 = 1$ so that we recover the usual mean curvature equation.

5.2.2 Generation and propagation of interface

We use the notations

$$\mu = f'(\alpha), \quad t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|, \quad \eta_0 := \min(\alpha - \alpha_-, \alpha_+ - \alpha). \tag{5.17}$$

Let D_t^- denotes the region enclosed by the interface Γ_t, $D_t^+ := D \setminus \overline{D_t^-}$.

Theorem 5.2. *(i) For any given $0 < \eta < \eta_0$ there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $t \in [t^\varepsilon, T]$*

$$u^\varepsilon \in \begin{cases} [\alpha_- - \eta, \alpha_+ + \eta] & \text{for } x \in D, \\ [\alpha_- - \eta, \alpha_- + \eta] & \text{if } x \in D_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \\ [\alpha_+ - \eta, \alpha_+ + \eta] & \text{if } x \in D_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t), \end{cases} \tag{5.18}$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in D,\ \mathrm{dist}(x, \Gamma_t) < r\}$ denotes the r-neighborhood of Γ_t.

(ii) Let $\rho > 1$. Then the solution u^ε of the problem with nonlinear diffusion (P^ε) satisfies

$$\lim_{\varepsilon\to 0} \sup_{\rho t^\varepsilon \le t \le T,\ x\in D} \left| u^\varepsilon(x,t) - U_0\left(\frac{d^\varepsilon(x,t)}{\varepsilon}\right)\right| = 0, \tag{5.19}$$

where U_0 is the standing wave solution defined above and d^ε denotes the signed distance function associated with $\Gamma_t^\varepsilon := \{x \in D : u^\varepsilon(x,t) = \alpha\}$:

$$d^\varepsilon(x,t) = \begin{cases} dist(x, \Gamma_t^\varepsilon) & \text{if } x \in D_t^{\varepsilon,+}, \\ -dist(x, \Gamma_t^\varepsilon) & \text{if } x \in D_t^{\varepsilon,-}, \end{cases}$$

where $D_t^{\varepsilon,-}$ denotes the region enclosed by Γ_t^ε and $D_t^{\varepsilon,+}$ denotes the region enclosed between ∂D and Γ_t^ε.

This result extends a similar result by Alfaro and Matano which they obtained in the case of linear diffusion.

Next we extend the comparison theorem to our present case of nonlinear diffusion, which will help prove the generation of the interface.

Theorem 5.3 (Comparison Theorem). *Let $v \in C^{2,1}(\overline{D} \times \mathbb{R}^+)$ satisfy*

$$\begin{cases} v_t \ge \Delta\varphi(v) + \frac{1}{\varepsilon^2}f(v) & \text{in } D \times \mathbb{R}^+, \\ \frac{\partial\varphi(v)}{\partial\nu} = 0 & \text{in } \partial D \times \mathbb{R}^+, \\ v(x,0) \ge u_0(x) & \text{for } x \in D. \end{cases} \tag{P}$$

Then, v is a super-solution of Problem (P^ε) and we have

$$v(x,t) \ge u^\varepsilon(x,t), \quad (x,t) \in D \times \mathbb{R}^+.$$

If v satisfies the opposite inequalities in Problem (P), then v is a sub-solution of Problem (P^ε) and we have

$$v(x,t) \le u^\varepsilon(x,t), \quad (x,t) \in D \times \mathbb{R}^+.$$

Proof. Consider the inequality satisfied for the difference of a super-solution v and a solution u^ε. Apply the maximum principle to the function $w := v - u^\varepsilon$ to see that it is positive. □

In these lecture notes, we sketch the generation of interface proof but do not prove the propagation of interface property.

Theorem 5.4 (Generation of interface). *Let $\eta > 0$ be arbitrary. There exists $M_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $x \in D$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$\alpha_- - \eta \le u^\varepsilon(x, t^\varepsilon) \le \alpha_+ + \eta,$$

and

$$\begin{aligned} &\textit{if} \quad u_0(x) \geq \alpha + M_0\varepsilon \quad \textit{then} \quad u^\varepsilon(x, t^\varepsilon) \geq \alpha_+ - \eta, \\ &\textit{if} \quad u_0(x) \leq \alpha - M_0\varepsilon \quad \textit{then} \quad u^\varepsilon(x, t^\varepsilon) \leq \alpha_- + \eta. \end{aligned}$$

Idea of the proof:
Neglecting the diffusion term, we have

$$\begin{cases} \bar{u}_t = \dfrac{1}{\varepsilon^2} f(\bar{u}) \\ \bar{u}(x,0) = u_0(x) \end{cases}$$

whose solution can be given as

$$\bar{u}(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x)\right)$$

where $Y(\tau, \zeta)$ is the solution of the initial value problem for the ordinary differential equation

$$\begin{cases} Y_\tau(\tau,\zeta) = f(Y(\tau,\zeta)) \quad \text{for } \tau > 0 \\ Y(0,\zeta) = \zeta. \end{cases} \tag{5.20}$$

Here ζ ranges over the interval $(-2C_0, 2C_0)$, with $C_0 = \|u_0\|_{C^2(\overline{D})}$.

Lemma 5.5. *Let $\eta \in (0, \eta_0)$ be arbitrary. Then, there exists a positive constant $C_Y = C_Y(\eta)$ such that the following holds:*

1. *There exists a positive constant $\overline{\mu}$ such that for all $\tau > 0$ and all $\zeta \in (-2C_0, 2C_0)$, $e^{-\overline{\mu}\tau} \leq Y_\zeta(\tau,\zeta) \leq C_Y e^{\mu\tau}$.*
2. *For all $\tau > 0$ and all $\zeta \in (-2C_0, 2C_0)$,*

$$\left|\frac{Y_{\zeta\zeta}(\tau,\zeta)}{Y_\zeta(\tau,\zeta)}\right| \leq C_Y(e^{\mu\tau} - 1).$$

3. *There exists a positive constants ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$,*
   (a) *for all $\zeta \in (-2C_0, 2C_0)$*

$$\alpha_- - \eta \leq Y(\mu^{-1}|\ln\varepsilon|, \zeta) \leq \alpha_+ + \eta; \tag{5.21}$$

   (b) *if $\zeta \geq \alpha + C_Y\varepsilon$, then*

$$Y(\mu^{-1}|\ln\varepsilon|, \zeta) \geq \alpha_+ - \eta; \tag{5.22}$$

   (c) *if $\zeta \leq \alpha - C_Y\varepsilon$, then*

$$Y(\mu^{-1}|\ln\varepsilon|, \zeta) \leq \alpha_- + \eta.$$

We now construct sub- and super-solutions for the proof of the generation of interface Theorem 5.4. For simplicity, we first consider the case where

$$\frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial D. \tag{5.23}$$

In this case, we define sub- and super-solution by

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 C_2(e^{\mu t/\varepsilon^2} - 1)\right)$$

for some positive constant C_2. In the general case, where (5.23) does not necessarily hold, we need to modify $w_\varepsilon^\pm$ near the boundary ∂D. To prove that these are sub- and super-solutions, we define the operator $\mathcal{L}$ by

$$\mathcal{L}u := u_t - \Delta\varphi(u) - \frac{1}{\varepsilon^2} f(u).$$

Lemma 5.6. *Assume (5.23). Then, there exist positive constants ε_0 and $C_2, \overline{C}_2$ independent of ε such that, for all $\varepsilon \in (0, \varepsilon_0)$, $w_\varepsilon^\pm$ satisfies*

$$\begin{cases} \mathcal{L}(w_\varepsilon^-) < -\overline{C}_2 e^{-\frac{\overline{\mu}t}{\varepsilon^2}} < \overline{C}_2 e^{-\frac{\overline{\mu}t}{\varepsilon^2}} < \mathcal{L}(w_\varepsilon^+) & \textit{in } \overline{D} \times [0, t^\varepsilon] \\ \dfrac{\partial w_\varepsilon^-}{\partial \nu} = \dfrac{\partial w_\varepsilon^+}{\partial \nu} = 0 & \textit{on } \partial D \times [0, t^\varepsilon]. \end{cases} \tag{5.24}$$

Proof. We set

$$P(t) := \varepsilon^2 C_2(e^{\mu t/\varepsilon^2} - 1).$$

We only prove that w_ε^+ is the desired super-solution; the case for w_ε^- can be treated in a similar way. The assumption (5.23) implies

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \text{ on } \partial D \times \mathbb{R}^+.$$

Then, direct computation with $\tau = t/\varepsilon^2$ gives

$$\mathcal{L}(w_\varepsilon^+) = \frac{1}{\varepsilon^2} Y_\tau + P'(t) Y_\zeta - \bigg(\varphi''(w_\varepsilon^+) |\nabla u_0|^2 (Y_\zeta)^2 + \varphi'(w_\varepsilon^+) \Delta u_0 Y_\zeta + \varphi'(w_\varepsilon^+) |\nabla u_0|^2 Y_{\zeta\zeta} + \frac{1}{\varepsilon^2} f(Y) \bigg),$$

so that

$$\mathcal{L}(w_\varepsilon^+) = \frac{1}{\varepsilon^2}(Y_\tau - f(Y)) + Y_\zeta \bigg(P'(t) - \bigg(\varphi''(w_\varepsilon^+)|\nabla u_0|^2 Y_\zeta + \varphi'(w_\varepsilon^+) \Delta u_0 + \varphi'(w_\varepsilon^+) |\nabla u_0|^2 \frac{Y_{\zeta\zeta}}{Y_\zeta} \bigg) \bigg).$$

By the definition of Y, the first term on the right-hand-side vanishes. By choosing ε_0 sufficiently small, for $0 \le t \le t^\varepsilon$, we have

$$P(t) \le P(t^\varepsilon) = \varepsilon^2 C_2 (e^{\mu t^\varepsilon/\varepsilon^2} - 1) \le \varepsilon^2 C_2 (\varepsilon^{-1} - 1) < C_0.$$

Hence, $|u_0 + P(t)| < 2C_0$. Also using the bounds on u_0 and its derivatives and the bounds on φ and its derivatives, we deduce that

$$\begin{aligned} \mathcal{L} w_\varepsilon^+ &\ge Y_\zeta (C_2 \mu e^{\mu t/\varepsilon^2} - (C_0^2 C_1 C_Y e^{\mu t/\varepsilon^2} + C_0 C_1 + C_0^2 C_1 C_Y (e^{\mu t/\varepsilon^2} - 1))) \\ &= Y_\zeta ((C_2 \mu - C_0^2 C_1 C_Y - C_0^2 C_1 C_Y) e^{\mu t/\varepsilon^2} + C_0^2 C_1 C_Y - C_0 C_1). \end{aligned}$$

Hence, we deduce that for C_2 large enough, we can find a positive constant $\overline{C}_2$ independent of ε such that

$$\mathcal{L} w_\varepsilon^+ \ge \overline{C}_2 e^{-\overline{\mu} t/\varepsilon^2}.$$

Thus, by the comparison principle, w_ε^+ is a super-solution for Problem (P^ε). □

In the end we have

$$w_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le w_\varepsilon^+(x,t) \quad \text{in } \overline{D} \times [O, t^\varepsilon] \tag{5.25}$$

with

$$w_\varepsilon^\pm(x,t) = Y\bigg(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 C_2 (e^{\mu t/\varepsilon^2} - 1) \bigg).$$

Now, recalling that $t^\varepsilon = \frac{1}{\mu} \varepsilon^2 |\ln \varepsilon|$, so that $\mu \frac{t^\varepsilon}{\varepsilon^2} = |\ln \varepsilon|$, we deduce that

$$w_\varepsilon^\pm(x, t^\varepsilon) = Y\bigg(\frac{1}{\mu} \ln\bigg(\frac{1}{\varepsilon}\bigg), u_0(x) \pm C_2((\varepsilon - \varepsilon^2)) \bigg).$$

So, for ε_0 small enough, by replacing t by t^ε in 5.25 we obtain

$$-2C_0 \le u_0^\pm(x) \pm C_2((\varepsilon - \varepsilon^2)) \le 2C_0 \quad \text{in } D.$$

Hence, the first part of the Theorem 5.4 is given by (5.25) and (5.21). For the second part, we take M_0 large enough so that $M_0 \varepsilon - C_2((\varepsilon - \varepsilon^2)) \ge C_Y \varepsilon$. Then, for any $x \in D$ such that $u_0^-(x) \ge \alpha + C_Y \varepsilon$, we have

$$u_0^-(x) - C_2((\varepsilon - \varepsilon^2)) \ge \alpha + M_0 \varepsilon - C_2((\varepsilon - \varepsilon^2)) \ge \alpha + C_Y \varepsilon.$$

Combining this with (5.25) and (5.22), we deduce that

$$u^\varepsilon(x, t^\varepsilon) \ge \alpha_+ - \eta, \quad \forall x \in D \text{ with } u_0^-(x) \ge \alpha M_0 \varepsilon.$$

This yields the first inequality, and the second can be proved in a similar way. This concludes the proof of Theorem 5.4.

Appendix A: Code for the Numerical Simulations

All numerical simulations have been coded with Wolfram Mathematica 12.3. An example of code is given below; its purpose is to generate the basic components of the Fig. 2.4.

Listing A.1: Parameters

```
mu1 = 1
mu2 = 1
T = 7.5
L = 1
k = 10
a = 2
d1 = 1
d2 = 1
```

Listing A.2: Necessary functions

```
f1[u_] := mu1∗(1 − u) u
f2[v_] := mu2∗(1 − v) v
iniCondi2D1[x_, y_] := (E^(−(x + (L/2))^2 − (y + (L/2))^2))^2
iniCondi2D2[x_, y_] := (E^(−(x − (L/2))^2 − (y − (L/2))^2))^2
```

Listing A.3: System and conditions

```
eqn = {
D[u[x, y, t], t] ==
    d1∗D[u[x, y, t], {x, 2}] + d1∗D[u[x, y, t], {y, 2}] + f1[u[x, y, t]] − k∗u[x, y, t]∗v[x, y, t],
D[v[x, y, t], t] ==
    d2∗D[v[x, y, t], {x, 2}] + d2∗D[v[x, y, t], {y, 2}] + f2[v[x, y, t]] − a ∗k∗u[x, y, t]∗v[x, y, t],
u[x, y, 0] == iniCondi2D1[x, y],
v[x, y, 0] == iniCondi2D2[x, y]
}
```

If needed, we can create a finer mesh at the regions where more spatial resolution is needed. The following code was used for Fig. 2.4:

Listing A.4: Creation of a mesh 1

```
mesh = DiscretizeRegion[Rectangle[{−L, −L}, {L, L}],
  MeshRefinementFunction –>
    Function[{vertices, area},
      Block[{x, y}, {x, y} = Mean[vertices];
        If[(y > −x), area > 0.002, area > 0.01]]]]
```

whereas this one was used for the Fig. 5.3

Listing A.5: Creation of a mesh 2

```
mesh = DiscretizeRegion[Rectangle[{−L, −L}, {L, L}],
  MeshRefinementFunction –>
    Function[{vertices, area},
      Block[{x, y}, {x, y} = Mean[vertices];
        If[(x < −L + L/20 || x > L − L/20 || y > L − L/20 ||
          y < −L + L/20 || (x^{2} + y^{2} < L/2)),
        area > 0.001, area > 0.01]]]]
```

Listing A.6: Computing the solution

```
sol2 = NDSolveValue[eqn2, {u, v}, {t, 0, 1}, {x, y} \[Element] mesh,
  Method –> {"PDEDiscretization" –> "FiniteElement"}]
```

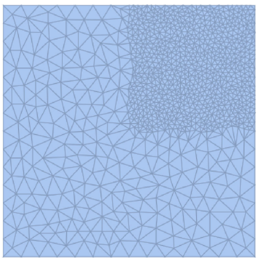

Fig. A·1. Mesh used for the numerical simulation in Fig. 2.4.

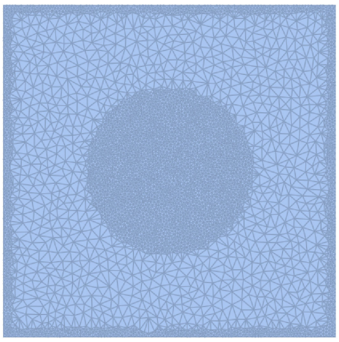

Fig. A·2. Mesh used for the numerical simulation in Fig. 5.3.

Listing A.7: Showing the graph

```
time = 0
Plot3D[{sol2 [[2]][x, y, time],
  sol2 [[1]][x, y, time]}, {x, y} \[Element] mesh, PlotRange −> All]
RegionPlot[{sol2 [[1]][x, y, time] > 0.01,
  sol2 [[2]][x, y, time] > 0.01}, {x, y} \[Element] mesh,
PlotRange −> {{−L − 0.05, L + 0.05}, {−L − 0.05, L + 0.05}}]
```

Appendix B: Definitions, Notation and Tools in Functional Analysis

B.1 Sobolev spaces

(See also Evans [17] chapter 5)

Definition B.1. *Assume $u, v \in L^1_{loc}(\Omega)$, where Ω is a subset of $\mathbb{R}^N$, and let α be a multi-index. Then v is the α-th order* weak partial derivative *of u*

$$v = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}},$$

if for all test functions $\varphi \in C_c^\infty(\Omega)$:

$$\int_U u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_U v \varphi dx.$$

If this weak derivative exists, it is unique.

Definition B.2 (Sobolev Space)**.** *Let $1 \le p \le \infty$. The space*

$$W^{k,p}(\Omega)$$

consists of all functions $u \in L^p(\Omega)$ such that for each multi-index α with $|\alpha| \le k$, $D^\alpha u$ exists in a weak sense and belongs to $L^p(\Omega)$.

Definition B.3. *For $u \in W^{k,p}(\Omega)$, we define its norm as:*

$$\|u\|_{W^{k,p}} := \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p dx \right)^{1/p} \quad (1 \le p < \infty).$$

$W^{k,p}(\Omega)$ is a Banach space with this norm. When $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$. The space $H^k(\Omega)$ is a Hilbert Space with:

Definition B.4. *For $u, v \in H^k(\Omega)$, we define the* inner product *as:*

$$(u, v)_{H^k} := \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2}$$

and the previously defined norm is the induced norm.

The dual of $H^k(\Omega)$ is denoted by $H^k(\Omega)'$.

B.2 Bochner spaces

(See also Evans [17] chapter 5)

In these lecture notes, we work on equations with functions depending on space and time, with various smoothness. Consider X a real Banach space with norm $\|.\|_X$, and introduce the following spaces:

Definition B.5. *The space*

$$L^p(0, T; X)$$

consists of all strongly measurable functions $u : [0, T] \to X$ with

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty \quad (\text{for } 1 \le p < \infty).$$

The space

$$C([0, T]; X)$$

consists of all continuous functions $u : [0, T] \to X$ with

$$\|u\|_{C([0,T];X)} := \max_{0 \le t \le T} \|u(t)\|_X < \infty.$$

Proposition B.1. *Let H be a Hilbert space. Then the space*

$$L^2(0, T; H)$$

is a Hilbert Space, with inner product

$$(u, v)_{L^2(0,T;H)} := \int_0^T (u(t), v(t))_H dt.$$

Proposition B.2. *Let E be a Banach space. If we denote* $(L^2(0, T; E))'$ *the dual space of* $L^2(0, T; E)$*, then*

$$(L^2(0, T; E))' = L^2(0, T; E').$$

See the comments in [3] Chapter 4 and Hytonen *et al.* [26], Proposition 1.3.3. for a proof.

B.3 Comparison principle for semi-linear parabolic equations

(See also Evans [17] chapter 7)

In this paper, the systems studied are reaction diffusion systems. Just like the heat equation, they belong to the family of second-order parabolic partial differential equations.

For $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ with $T > 0$, we consider a semi-linear operator of the form

$$S(u) = u_t - \sum_{i,j=1}^{n} a^{ij}(x, t)u_{x_i x_j} + F(x, t, u, \nabla u)$$

with given functions $a^{ij} = a^{ji}$ $(i, j = 1, \dots, n)$ bounded, and the non-linear term F is C^1 jointly in all of its arguments.

Definition B.6. *S is called* uniformly parabolic *if there exists a constant* θ *such that*

$$\sum_{i,j=1}^{n} a^{ij}(x, t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x, t) \in \Omega \times (0, T]$, $\xi \in \mathbb{R}^n$.

Theorem B.3 (comparison principle)**.** *Suppose that S is uniformly parabolic.*
If $u, v \in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$ *satisfy*

$$S(v) \leq S(u) \text{ in } \Omega \times (0, \infty), \quad \text{and} \quad v \geq u \text{ in } (\Omega \times \{0\}) \cup (\partial\Omega \times (0, \infty))$$

or

$$S(v) \leq S(u) \text{ in } \Omega \times (0, \infty), \quad v \geq u \text{ in } (\Omega \times \{0\}) \text{ and } \partial v/\partial n \geq \partial u/\partial n \text{ in } (\partial\Omega \times (0, \infty)),$$

then

$$v \geq u \text{ in } \Omega \times (0, \infty).$$

For this theorem, see for example the paper from Testa [43] Theorem 2.1. In these lectures, we sometimes call this theorem the comparison principle, since it is an extension of it.

B.4 Weak solution

Given a PDE of order k, a solution that is at least k times continuously differentiable is called a *classical solution*. This allows all the partial derivatives expressed in the PDE to exist and be continuous. On the other hand, there also exist functions which verify a PDE statement, are continuous, but not with enough smoothness to be classical solutions. These solutions are called *weak solutions*. For example, $u(x, t) = |x - t|$ verifies

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{B.1}$$

but is not differentiable in $x = t$. u is a weak solution of (B.1).

To define weak solutions, the differential equation has to be rewritten in a weak formulation, in which none of the derivatives of the solution appears. There is no unique way to define a weak formulation, it depends on the problem, but let us present an often used methodology. Let us consider the case of the parabolic equation:

$$\begin{cases} S(u) = f & \text{in } \Omega \times (0, T] \\ u = 0 & \text{in } \partial\Omega \times [0, T] \\ u = g & \text{in } \Omega \times \{0\}, \end{cases} \tag{B.2}$$

with S uniformly parabolic, $T > 0$, $f \in L^2(\Omega \times (0, T])$, $g \in L^2(\Omega)$. Let us now assume that u solving this system is a classical solution: $u \in C^{2,1}(\overline{\Omega} \times (0, T])) \cap C(\overline{\Omega} \times [0, T])$, and let $\varphi \in C^{2,1}(\overline{\Omega} \times [0, T])$ be an arbitrary test function which vanishes on $\partial\Omega \times [0, T]$. We will multiply by φ and integrate the PDE.

$$\int_\Omega \int_0^T u_t\varphi - \sum_{i,j=1}^n a^{ij} u_{x_i x_j}\varphi + F(x,t,u,\nabla u)\,\varphi\, dxdt = \int_\Omega \int_0^T f\varphi\, dxdt.$$

We first make the time derivative of u disappear with an integration by part of the first term:

$$\int_\Omega \int_0^T u_t\varphi\, dxdt = -\int_\Omega \int_0^T u\varphi_t\, dxdt + \int_\Omega u(x,T)\varphi(x,T) - u(x,0)\varphi(x,0)\, dx.$$

The same technique applied to the second order differential terms of $S(u)$, with the boundary terms vanishing due to φ, gives

$$-\int_\Omega \int_0^T \sum_{i,j=1}^n a^{ij} u_{x_i x_j}\varphi\, dxdt = \int_\Omega \int_0^T \sum_{i,j=1}^n a^{ij} u_{x_i}\varphi_{x_j}\, dxdt.$$

The weak form of this PDE is then:

$$\begin{aligned}\int_\Omega \int_0^T & \sum_{i,j=1}^n a^{ij} u_{x_i}\varphi_{x_j} + F(x,t,u,\nabla u)\,\varphi - u\varphi_t\, dxdt \\ &+ \int_\Omega u(x,T)\varphi(x,T) - u(x,0)\varphi(x,0)\, dx \\ &= \int_\Omega \int_0^T f\varphi\, dxdt \quad \text{in } \Omega\times(0,T], \end{aligned} \tag{B.3}$$

with as before

$$\begin{cases} u = 0 & \text{in } \partial\Omega \times [0,T] \\ u = g & \text{in } \Omega\times\{0\}. \end{cases} \tag{B.4}$$

From there, we say that a function that verifies (B.3) and (B.4) for any function test φ is a weak solution of the PDE problem (B.2).

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