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OPTIMAL CONTROL PROBLEM FOR NON-LINEAR DEGENERATE PARABOLIC VARIATION INEQUALITY: SOLVABILITY AND ATTAINABILITY ISSUES

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Abstract. We investigate the optimal control problem with respect to coefficients of the degenerate parabolic variational inequality. Since problems of this type can have the Lavrentieff effect, we consider the optimal control problem in a class of so-called H-admissible solutions. We substantiate the attainability of H-optimal pairs via optimal solutions of some nondegenerate perturbed optimal control problems.

Key words: parabolic variation inequality, optimal control problem, control in coefficients, approximation, existence result, attainability result.

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1. Introduction

The purpose of this paper is to investigate optimal control problem associated with a degenerate parabolic inequality. The control is a matrix of coefficients in the main part of elliptic operator. It is well known that degenerate control problems of this type may admit nonuniqueness of admissible solution classes, which implies non-uniqueness of optimal solutions of particular kind and the optimal control problem in the coefficients can be stated in different forms depending on the choice of the class of admissible solutions (for example W- or H-solutions if we consider the weighted Sobolev space W or its subspace H as the phase space, correspondingly) (see [1], [2] and references there). These spaces allow to enlarge the class of boundary value problems and variational inequalities which are solvable by functional-analytical methods. In fact, we consider variational inequality with some degenerate weight function which is not bounded away from zero and infinity but only satisfying some local integrability conditions. Under these assumptions the nonlinear differential operator in our inequality is not coercive in the classical sense.

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Since the range of OCPs in coefficients is very wide, including as well optimal shape design problems, optimization of certain evolution systems, some problems originating in mechanics and others, this topic has been widely studied by many authors (see [1]- [3], [6] and others).

As F. Murat showed (see [7]), in general, such problems have no solution even if the original elliptic equation is non-degenerate. It turns out that this feature is typical for the majority of problems for optimal control in coefficients. So,we have to restrict our optimization problem by introducing some additional control constrains (see, for instance, [8]). An optimal control problem for a variational inequality with the so-called anisotropic p-Laplacian in the principle part of this inequality is studied in [9] where the authors showed that the original problem is well-posed and derived existence of optimal pairs. In [10] an optimal control problem associated to Dirichlet boundary value problem for non-linear elliptic equation on a bounded domain is considered. In [6] the authors study the existence of optimal solutions in coefficients associated to a linear degenerate elliptic equa-tion with mixed boundary condition where by control variable they mean a weight coefficient in the main part of the elliptic operator. The sufficient conditions of the existence of weak solutions to one class of Neumann boundary value problems (BVP) are obtained in [11], and moreover, the authors propose a way for their approximation. In [12] the existence of H-optimal solutions for optimal control problem in coefficients for degenerate variational elliptic inequalities of monotone type in the class of so-called generalized solenoidal controls was proved. The solvability results for optimal control problems for degenerate elliptic and parabolic variation inequalities one can find in [13–16].

Taking into account a wide spectrum of application of the optimal control theory, in particular, we deal with possibilities of some types of approximation of original problems by those that are better researched and converge to the original problems in a suitable way. As for problems similar to the one studied in the given paper, in application a degenerate weight ρ occurs as the limit of a sequence of non-degenerate weights ρ for which the corresponding "approximate" optimal control problem is solvable. Thus, naturally, it arises the question: if limit points of the family of admissible solutions to the perturbed problems appear to be admissible solutions to the original problem, whether all optimal solutions are attainable in this sense? Note that for some optimal control problems the attainability and approximability questions remain in the focus of attention. In particular, similar questions were raised in [17] where the author studies the attainability issue for optimal control problem in coefficients for degenerate variational inequality of monotone type in the class of H-admissible solutions. In [2] the authors prove the existence of W-solutions to the optimal control problem and provide way for their approximation. In [18, 19] the author investigates the attainability issue for optimal control problem for degenerate linear elliptic and parabolic inequalities respectively.

Here we concentrate on the solvability of optimal control problem in coefficients for degenerate parabolic inequality in the so-called class of H-admissible solutions. Moreover, we are interested about attainability of H-optimal solutions to degene-rate problems via optimal solutions of non-degenerate problems.

2. Preliminaries and Notations

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary. For any subset $E \subset \Omega$ we denote by $|E|$ its N-dimentional Lebesgue measure $\mathcal{L}(E)$. The space $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,1}(\Omega)$. Let p be a real number such that $2 \leq p < \infty$ and let q be its conjugate, namely $p^{-1} + q^{-1} = 1$. We say that a weight function $\rho = \rho(x)$ is degenerate in $\mathbb{R}^N$ if

$$\rho(x) > 0 \text{ a.e. in } \mathbb{R}^N \text{ and } \rho + \rho^{-1/(p-1)} \in L^1_{loc}(\mathbb{R}^N), \tag{2.1}$$

and the sum $\rho + \rho^{-1/(p-1)}$ does not belong to $L^\infty(\Omega)$, in general. For a given $\Omega \in \mathbb{R}^N$ we associate to this function the weighted Sobolev space $W = W(\Omega, \rho dx)$ which is a set of functions $y \in W_0^{1,1}(\Omega)$ for which the norm

$$\|y\|_\rho = \left(\int_\Omega \left(|y|^p + \rho \sum_{i=1}^N \left| \frac{\partial y}{\partial x_i} \right|^p \right) dx \right)^{1/p} \tag{2.2}$$

is finite.

Together with W let us consider the space $H = H(\Omega, \rho dx)$ which is the closure of $C_0^\infty(\Omega)$ in W.

Note that the spaces W and H are reflexive Banach spaces with respect to the norm $\|\cdot\|_\rho$ due to the estimate

$$\int_\Omega |\nabla y|\, dx \leq \left(\int_\Omega \rho |\nabla y|_p^p \, dx \right)^{1/p} \left(\int_\Omega \rho^{-1/(p-1)} \, dx \right)^{p/(p-1)} \leq C\|y\|_\rho,$$

where $|\eta|_p = \left(\sum_{k=1}^N |\eta_k|^p \right)^{1/p}$ is a Hölder norm of order p in $\mathbb{R}^N$. It is clear that $H \subseteq W$.

Since the smooth functions are in general not dense in the weight Sobolev space W, it follows that $H \neq W$; that is, for a "typical" degenerate weight ρ the identity $W = H$ is not always valid (for the corresponding examples we refer to [3, 5]). However, if ρ is a non-degenerate weight function, that is, ρ is bounded between two positive constants, then it is easy to verify that $W = H = W_0^{1,p}(\Omega)$. We recall that the dual space of H is $H^* = W^{-1,-p/(p-1)}(\Omega, \rho^{-1/(p-1)} dx)$ (for more details see [6]).

Remark 2.1. Assume that there exists a value $\nu \in \left(\frac{N}{p}, +\infty \right) \cap \left[\frac{1}{p-1}, +\infty \right)$ such that $\rho^{-\nu} \in L^1(\Omega)$. Then the following result takes place (see [6]): relation (2.1) implies that

$$|||y|||_{\rho,\Omega} = \left[\int_\Omega \sum_{i=1}^N \left| \frac{\partial y}{\partial x_i} \right|^p \rho \, dx \right]^{1/p}$$

is a norm of the space H equivalent to (2.2) and the embedding

$$H \hookrightarrow L^p(\Omega)$$

is compact and dense.

Parabolic Variational Inequalities. Following Lions [20], let us cite some well-known results concerning solvability and solution uniqueness for non-degenerate non-linear parabolic variational inequalities which will be useful in the sequel.

Let $\mathcal{V}$ be reflexive Banach space and $\mathcal{H}$ be Hilbert space and

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*. \tag{2.3}$$

Let us consider such operator Λ that:

$$\begin{gathered} -\Lambda \text{ is an infinitesimal generating operator of a semigroup} \\ s \to G(s) \text{ in } \mathcal{V}, \mathcal{H}, \mathcal{V}^*, \text{ which is a compressive semigroup in } \mathcal{H}. \end{gathered} \tag{2.4}$$

Let us consider a non-linear operator $\mathcal{A}$ such that

$$\begin{gathered} \mathcal{A} : \mathcal{V} \to \mathcal{V}^* \text{ is a pseudomonotone operator, i.e.} \\ \text{it is bounded and if } y_k \to y \text{ weakly in } \mathcal{V}, \\ y_k, y \in \mathcal{K} \text{ and } \varlimsup_{k\to\infty} \langle \mathcal{A}(y_k), y_k - y\rangle_{\mathcal{V}} \le 0 \text{ then} \\ \varliminf_{k\to\infty} \langle \mathcal{A}(y_k), y_k - v\rangle_{\mathcal{V}} \ge \langle \mathcal{A}(y), y - v\rangle_{\mathcal{V}} \; \forall v \in \mathcal{V}, \end{gathered} \tag{2.5}$$

and

$$\begin{gathered} \mathcal{A} \text{ is a coercive operator :} \\ \text{there exists such element } v_0 \in \mathcal{K} \text{ that} \\ \frac{\langle \mathcal{A}(v), v - v_0\rangle_{\mathcal{V}}}{\|v\|_{\mathcal{V}}} \to \infty \text{ as } \|v\| \to \infty, \end{gathered} \tag{2.6}$$

where

$$\mathcal{K} \text{ is a convex closed set in } \mathcal{V}. \tag{2.7}$$

Using operators, spaces and sets discussed above, and taking $\Lambda = \frac{d}{dt}$, we can consider the following problem for variational parabolic inequalities in its "weak" statement (see for details [20]): find $u \in \mathcal{K}$ such that

$$\begin{gathered} \langle \Lambda v, v - u\rangle_{\mathcal{V}} + \langle \mathcal{A}(u), v - u\rangle_{\mathcal{V}} \ge \langle f, v - u\rangle_{\mathcal{V}} \\ \forall v \in \mathcal{K},\, v' \in \mathcal{V}^*,\, v(0) = 0, \end{gathered} \tag{2.8}$$

where $f \in \mathcal{V}^*$.

Let us consider some "consistency conditions" for Λ and $\mathcal{K}$: $\forall v \in \mathcal{K}$ there exists some "regularizing" sequence v_j which satisfies the following conditions:

$$\begin{gathered} v_j \in \mathcal{K},\, v_j' \in \mathcal{V}^*,\, v_j(0) = 0, \\ v_j \to v \text{ in } \mathcal{V},\, j \to \infty, \\ \varlimsup_{j\to\infty} \langle \Lambda v_j, v_j - v\rangle_{\mathcal{V}} \le 0. \end{gathered} \tag{2.9}$$

Theorem 2.1. *[20, Theorem 9.1] If for convex set $\mathcal{K}$ and semigroup $G(s)$ we have*

$$G(s)\mathcal{K} \subset \mathcal{K} \ \forall s \geq 0,$$

then (2.9) *takes place.*

Theorem 2.2. *[20, Theorem 9.2] Let conditions* (2.3), (2.4), (2.5), (2.6) *with $v_0 \in \mathcal{K}$ such that $v_0' \in \mathcal{V}$, $v_0(0) = 0$, and* (2.9) *are fulfilled. Then $\forall f \in \mathcal{V}^*$ there exists the solution $u \in \mathcal{K}$ for the variational evolution inequality* (2.8).

Theorem 2.3. *[20, Theorem 9.4] Let conditions of Theorem 2.2 are fulfilled. Let us assume that $\forall u, v \in \mathcal{K}$:*

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle_{\mathcal{V}} \leq 0 \Rightarrow u = v. \tag{2.10}$$

Then the inequality (2.8) *admits a unique solution.*

Smoothing. Throughout the paper ε denotes a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, while writing $\varepsilon \geq 0$, we also consider its limit $\varepsilon = 0$.

Definition 2.1. We say that a weight function ρ with properties (2.1) is approximated by non-degenerate weight functions $\{\rho^\varepsilon\}_{\varepsilon \geq 0}$ on Ω if:

$$\rho^\varepsilon(x) > 0 \text{ a.e. in } \Omega, \ \rho^\varepsilon + (\rho^\varepsilon)^{-1} \in L^\infty(\Omega), \ \forall \varepsilon > 0, \tag{2.11}$$

$$\rho^\varepsilon \to \rho, \ (\rho^\varepsilon)^{-1/(p-1)} \to \rho^{-1/(p-1)} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{2.12}$$

Remark 2.2. The family $\{\rho^\varepsilon\}_{\varepsilon > 0}$ satisfying properties (2.11)-(2.12) is called the non-degenerate perturbation of the weight function ρ.

Examples of such perturbations can be constructed using the classical smoothing. For instance, let Q be some positive compactly supported function such that $Q \in L^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} Q(x)dx = 1$, and $Q(x) = Q(-x)$. Then, for a given weight function $\rho \in L^1_{loc}(\mathbb{R}^N)$, we can take $\rho^\varepsilon = (\rho)_\varepsilon$, where

$$(\rho)_\varepsilon(x) = \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} Q\left(\frac{x - z}{\varepsilon}\right) \rho(z)\, dz = \int_{\mathbb{R}^N} Q(z)\rho(x + \varepsilon z)\, dz. \tag{2.13}$$

In this case, we say that the perturbation $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon > 0}$ of the original degenerate weight function ρ is constructed by the "direct" smoothing scheme.

Lemma 2.1. *[12] If $\rho, \rho^{-1/(p-1)} \in L^1_{loc}(\mathbb{R}^N)$ then the "direct" smoothing $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon > 0}$ possesses properties* (2.11)-(2.12).

Radon measures and convergence in variable spaces. By a nonnegative Radon measure on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. The space of all nonnegative Radon measures on Ω will be denoted by $\mathcal{M}_+(\Omega)$. If μ is a nonnegative Radon measure on Ω, we will use $L^r(\Omega, d\mu)$, $1 \le r \le \infty$, to denote the usual Lebesque space with respect to the measure μ with the corresponding norm $\|f\|_{L^r(\Omega, d\mu)} = \left(\int_\Omega |f(x)|^r \, d\mu\right)^{1/r}$.

Let $\{\mu_\varepsilon\}_{\varepsilon>0}$, μ be Radon measure such that $\mu_\varepsilon \rightharpoonup^* \mu$ in $\mathcal{M}_+(\Omega)$: that is,

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi \, d\mu_\varepsilon = \int_\Omega \varphi \, d\mu \;\; \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{2.14}$$

where $C_0^\infty(\mathbb{R}^N)$ is the space of all compactly supported continuous functions. A typical example of such measures is $d\mu_\varepsilon = \rho^\varepsilon(x)\, dx$, $d\mu = \rho(x)\, dx$, where $0 \le \rho^\varepsilon \rightharpoonup \rho$ in $L^1(\Omega)$. Let us recall the definition and main properties of convergence in the variable L^p-space.

1. A sequence $\{v_\varepsilon \in L^p(\Omega, \, d\mu_\varepsilon)\}$ is called bounded if

$$\varlimsup_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon < +\infty.$$

2. A bounded sequence $\{v_\varepsilon \in L^p(\Omega, d\mu_\varepsilon)\}$ converges weakly to $v \in L^p(\Omega, d\mu)$ if $\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon \varphi d\mu_\varepsilon = \int_\Omega v\varphi d\mu$ for any $\varphi \in C_0^\infty(\Omega)$ and we write $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$.

3. The strong convergence $v_\varepsilon \to v$ in $L^p(\Omega, d\mu_\varepsilon)$ means that $v \in L^p(\Omega, d\mu)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon z_\varepsilon \, d\mu_\varepsilon = \int_\Omega vz \, d\mu \text{ as } z_\varepsilon \rightharpoonup z \text{ in } L^q(\Omega, d\mu_\varepsilon). \tag{2.15}$$

The following convergence properties in variable spaces hold:

(a) *Compactness criterium*: if a sequence is bounded in $L^p(\Omega, d\mu_\varepsilon)$, then this sequence is compact with respect to the weak convergence.

(b) *Property of lower semicontinuity*: if $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$, then

$$\varliminf_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon \ge \int_\Omega v^p \, d\mu. \tag{2.16}$$

(c) *Criterium of strong convergence*: $v_\varepsilon \to v$ if and only if $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega |v_\varepsilon|^p \, d\mu_\varepsilon = \int_\Omega v^p \, d\mu. \tag{2.17}$$

Concluding this section, we recall some well-known results concerning the convergence in the variable space $L^p(\Omega, d\mu_\varepsilon)$.

Lemma 2.2. *If $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of the weight function $\rho(x) \ge 0$, then: (A_1) $(\rho^\varepsilon)^{-1} \to \rho^{-1}$ in $L^q(\Omega, \rho^\varepsilon dx)$. ($A_2$) $[v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, d\mu_\varepsilon)]$ $\Rightarrow [v_\varepsilon \rightharpoonup v$ in $L^1(\Omega)]$. (A_3) If a sequence $\{v_\varepsilon \in L^p(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ is bounded, then the weak convergence $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, \rho^\varepsilon dx)$ is equivalent to the weak convergence $\rho^\varepsilon v_\varepsilon \rightharpoonup \rho v$ in $L^1(\Omega)$. (A_4) If $a \in L^\infty(\Omega)$ and $v_\varepsilon \rightharpoonup v$ in $L^p(\Omega, \rho^\varepsilon dx)$, then $av_\varepsilon \rightharpoonup av$ in $L^p(\Omega, \rho^\varepsilon dx)$.*

Variable Sobolev Spaces. Let $\rho(x)$ be a degenerate weight function and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of the function ρ in the sense of Definition 2.1. We denote by $H(\Omega, \rho^\varepsilon dx)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\rho^\varepsilon}$. Since for every ε the function ρ^ε is non-degenerate, the space $H(\Omega, \rho^\varepsilon dx)$ coincides with the classical Sobolev space $W_0^{1,p}(\Omega)$.

Definition 2.2. We say that a sequence $\{y_\varepsilon \in H(\Omega, \rho^\varepsilon dx)\}_{\varepsilon>0}$ converges weakly to an element $y \in W$ as $\varepsilon \to 0$, if the following hold: (i) This sequence is bounded. (ii) $y_\varepsilon \rightharpoonup y$ in $L^p(\Omega)$. (iii) $\nabla y_\varepsilon \rightharpoonup \nabla y$ in $L^p(\Omega, \rho^\varepsilon dx)^N$.

Theorem 2.4. *[12] Let $\rho^\varepsilon = (\rho)_\varepsilon$ be a direct smoothing of a degenerate weight $\rho \in L^1_{loc}(\mathbb{R}^N)$ and let $y^\varepsilon \in H(\Omega, \rho^\varepsilon dx)$, $y^\varepsilon \rightharpoonup y$ in $L^p(\Omega)$, $\nabla y^\varepsilon \rightharpoonup v$ in $L^p(\Omega, \rho^\varepsilon dx)^N$. Then $y \in H$ and $v = \nabla y$.*

Functional spaces. For some interval S and some Banach space $\{X, \|\cdot\|_X\}$ we can consider the set of all measurable by Bochner functions $u \in (S \to X)$ $L^p(S; X)$, $1 \le p < \infty$ for which $\int_s \|u(s)\|^p ds < \infty$.

Theorem 2.5. *[21, Theorem 1.11] The set $L^p(S; X)$, $1 \le p < \infty$ which forms a linear space with natural linear operations becames a Banach space with norm*

$$\|u\|_{L^p(S;X)} = \left(\int_S \|u(s)\|^p \, ds\right)^{1/p}. \tag{2.18}$$

Remark 2.3. Taking into account the definition of $L^p(S; X)$, Theorem 2.5 and properties of Bochner's integral (see [21]), the properties of the given section are valid for $L^p(S; X)$ as well as for X.

Compensated Compactness Lemma in Variable Lebesque and Sobolev spaces. Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function ρ.

In order to discuss the attainability of H-optimal solutions we use the following result, which we can obtain applying similar suggestions to [12, 22].

Lemma 2.3. *Let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of a weight function $\rho(x) > 0$. Suppose that sequences $\{\vec{f}_\varepsilon\}_{\varepsilon>0}$ and $\{g_\varepsilon\}_{\varepsilon>0}$ are such that:*

(i) $\frac{\partial g_\varepsilon}{\partial t} - \operatorname{div}(\rho^\varepsilon \vec{f}_\varepsilon) = 0$ in the sense of distributions in $\Omega \times [0, T]$;

(ii) $\vec{f}_\varepsilon \rightharpoonup \vec{f}$ in $L^q(0, T; L^q(\Omega, \rho^\varepsilon dx)^N)$ as $\varepsilon \to 0$;

(iii) g_ε is bounded in $L^\infty(0, T; L^2(\Omega))$ and $g_\varepsilon \rightharpoonup g$ in $L^p(0, T; H(\Omega, \rho^\varepsilon dx))$ as $\varepsilon \to 0$.

If $p > \frac{2N}{N+2}$, then

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \vec{f}_\varepsilon \cdot \nabla g_\varepsilon \varphi \rho^\varepsilon \, dx dt = \int_0^T \int_\Omega \vec{f} \cdot \nabla g \varphi \rho \, dx dt, \quad \forall \varphi \in C_0^\infty(\Omega \times [0, T]). \tag{2.19}$$

3. Setting of the Optimal Control Problem

The OCP, we consider in this paper, is to minimize the descrepancy between a given distribution $z_\partial \in L^p(0,T;L^p(\Omega))$ and the solution y of the degenerate variational inequality by choosing an appropriate matrix $U \in L^\infty(\Omega;\mathbb{R}^{N\times N})$, namely we deal with the following minimization problem:

$$I(U,y) = \int_0^T \int_\Omega |y(t,x) - z_\partial(t,x)|^p \, dxdt \to \inf, \tag{3.1}$$

$$U \in M_p^{\alpha,\beta}(\Omega), y \in \hat{\mathcal{K}}, \tag{3.2}$$

$$\begin{aligned}\langle v', v-y\rangle_{L^p(0,T;W)} + \langle - \ \operatorname{div}\left(U(x)\rho(x)[(\nabla y)^{p-2}]\nabla y\right), v-y\rangle_{L^p(0,T;W)} \\ + \langle |y|^{p-2}y, v-y\rangle_{L^p(0,T;W)} \geq \langle f, v-y\rangle_{L^p(0,T;W)}\end{aligned} \tag{3.3}$$

$$v \in \hat{\mathcal{K}}, \quad v' \in L^q(0,T;L^q(\Omega)), \quad v(0,x) = 0,$$

where $f \in L^q(0,T;L^q(\Omega))$ is a fixed element, $M_p^{\alpha,\beta}(\Omega) \subset L^\infty(\Omega;\mathbb{R}^{N\times N})$ is a class of admissible controls, $\hat{\mathcal{K}} \subset L^p(0,T;W)$ is a closed convex subset and

$$[\eta^{p-2}] = \operatorname{diag}\{|\eta_1|^{p-2}, |\eta_2|^{p-2}, \dots, |\eta_N|^{p-2}\} \ \forall \eta \in \mathbb{R}^N.$$

Let α and β be constants such that $0 < \alpha \leq \beta < +\infty$. We define $M_p^{\alpha,\beta}(\Omega)$ as a set of all symmetric matrices $U(x) = \{a_{ij}(x)\}_{1\leq i,j\leq N}$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$ such that the following conditions of growth, monotonicity, and strong coercivity are fulfilled:

$$|a_{i,j}(x)| \leq \beta \text{ a.e. in } \Omega \ \forall i,j \in \{1,\dots,N\}, \tag{3.4}$$

$$\left(U(x)\left([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta\right), \zeta - \eta\right)_{\mathbb{R}^N} \geq 0 \text{ a.e. in } \Omega \ \forall \zeta, \eta \in \mathbb{R}^N, \tag{3.5}$$

$$\left(U(x)[\zeta^{p-2}]\zeta, \zeta\right)_{\mathbb{R}^N} = \sum_{i,j=1}^N a_{i,j}(x)|\zeta_j|^{p-2}\zeta_j\zeta_i \geq \alpha|\zeta|_p^p \text{ a.e. in } \Omega. \tag{3.6}$$

Remark 3.1. It is easy to see that $M_p^{\alpha,\beta}(\Omega)$ is a nonempty subset of the space $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and its typical representatives are diagonal matrices of the form $U(x) = \operatorname{diag}\{\delta_1(x), \delta_2(x), \dots, \delta_N(x)\}$, where $\alpha \leq \delta_i(x) \leq \beta$ a.e. in Ω $\forall i \in \{1,\dots,N\}$.

For every fixed control $U \in M_p^{\alpha,\beta}(\Omega)$ let us define a non-linear operator $\mathcal{A}$: $L^p(0,T;H) \to L^q(0,T;H^*)$ in the following way:

$$\begin{aligned}\langle \mathcal{A}(y), v\rangle_{L^p(0,T;H)} = \int_0^T \int_\Omega \sum_{i,j=1}^N \left(a_{i,j}(x)\left|\frac{\partial y}{\partial x_j}\right|^{p-2}\frac{\partial y}{\partial x_j}\right)\frac{\partial v}{\partial x_i}\rho \, dxdt \\ + \int_0^T \int_\Omega |y|^{p-2}yv \ dxdt.\end{aligned} \tag{3.7}$$

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Definition 3.1. We say that a matrix $U = [a_{i,j}]$ is an admissible control to degenerate problem (3.2)-(3.3) if $U \in U_{ad}$, where the set U_{ad} is defined as follows

$$U_{ad} = \{U = [\vec{a}_1, \ldots, \vec{a}_N] \in M_p^{\alpha,\beta}(\Omega) | \; |\operatorname{div}(\rho \vec{a}_i)| \le \gamma_i, \text{ a.e. in } \Omega, \forall i = \overline{1,N}\}. \tag{3.8}$$

Here $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ is a strictly positive vector.

Definition 3.2. Let K be the convex closed subset of H, $0 \in K$, $\mathcal{K} = \{v | \, v \in L^p(0,T;H), v(t) \in K \text{ a.e.}\}$ be the convex closed subset of $L^p(0,T;H)$. We say that a function $y = y(U,f) \in \mathcal{K}$ is an H-solution to degenerate variational inequality (3.2)-(3.3), if

$$\begin{gathered} \langle v', v-y\rangle_{L^p(0,T;H)} + \langle \mathcal{A}(y), v-y\rangle_{L^p(0,T;H)} \ge \langle f, v-y\rangle_{L^p(0,T;H)} \\ v \in \mathcal{K}, \quad v' \in L^q(0,T;L^q(\Omega)), \quad v(0,x) = 0. \end{gathered} \tag{3.9}$$

Definition 3.3. We say that the set Ξ_H is the set of admissible pairs to the optimal control problem (3.1)-(3.3), (3.8) if

$$\Xi_H = \{(U,y) \in U_{ad} \times L^p(0,T;H) | \, y \in \mathcal{K}, (U,y) \text{ are related by (3.9)}\}.$$

Remark 3.2. We can inroduce a W-solution and the set Ξ_W by the similar way.

Hence for given control object described by relations (3.2)-(3.3) with both fixed control constraints ($U \in U_{ad}$) and fixed cost functional (3.1), we have two different statements of the original optimal control problem, namely

$$\left\langle \inf_{(U,y)\in\Xi_W} I(U,y) \right\rangle \text{ and } \left\langle \inf_{(U,y)\in\Xi_H} I(U,y) \right\rangle.$$

As a matter of fact, there is no comparison between these problems, in general. Indeed, having assumed that $W \ne H$ for a given degenerate weight function $\rho \ge 0$, we can come to the effect which is usually called the Lavrentieff phenomenon. It means that for some $U \in U_{ad}$ and $f \in L^q(0,T;L^q(\Omega))$ an H-solution $y_H(U,f)$ to problem (3.2)-(3.3) does not coincide with its W-solution $y_W(U,f)$. In this paper we deal with H-solutions to problem (3.2)-(3.3).

Definition 3.4. We say that a pair $(U_0, y_0) \in L^\infty(\Omega; \mathbb{R}^{N\times N}) \times L^p(0,T;H)$ is an H-optimal solution to problem (3.1)-(3.3), (3.8) if $(U_0,y_0) \in \Xi_H$ and $I(U_0,y_0) = \inf_{(U,y)\in\Xi_H} I(U,y)$.

Definition 3.5. We say that a sequence $\{(U_k, y_k) \in \Xi_H\}_{k\in\mathbb{N}}$ is bounded if

$$\sup_{k\in\mathbb{N}} [\|U_k\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|y_k\|_{L^p(0,T;L^p(\Omega))} + \|\nabla y_k\|_{L^p(0,T;L^p(\Omega,\rho dx)^N)}]$$

is finite.

4. Existence of H-Optimal Solutions

In this section we show that considered optimal control problem (3.1)-(3.3) for degenerate parabolic variational inequality with monotone operator is regular in the class of H-admissible solutions. Imposing additional control constrains (3.8) and using the special version of compensated compactness lemma (Lemma 2.3) we prove that the set of H-admissible solutions for problem (3.2)-(3.3) is sequentially closed. And using the direct method of Calculus of Variations we prove the existence of H-optimal solutions for considered problem.

Theorem 4.1. *For every control $U \in M_p^{\alpha,\beta}(\Omega)$ and every $f \in L^q(0,T;L^q(\Omega))$ there exists a unique H-solution to degenerate parabolic variational inequality* (3.2)-(3.3).

Proof. Let $U \in M_p^{\alpha,\beta}(\Omega)$ be a fixed matrix. Let us consider the following elliptic operator $A_1 : H \to H^*$:

$$\langle A_1(y), v\rangle_H = \sum_{i,j=1}^{N} \int_\Omega \left(a_{i,j}(x) \left| \frac{\partial y}{\partial x_j} \right|^{p-2} \frac{\partial y}{\partial x_j} \right) \frac{\partial v}{\partial x_i} \rho\, dx + \int_\Omega |y|^{p-2} y v\, dx.$$

Then taking into account (3.6) from [12, Lemma 1] we have the next coercivity property for operator A_1:

$$\langle A_1(y), y\rangle_H \geq \min\{\alpha, 1\} \|y\|_\rho^p. \tag{4.1}$$

Hence from (3.7) and (2.18) we have that

$$\langle \mathcal{A}(y), y\rangle_{L^p(0,T;H)} \geq \min\{\alpha, 1\} \|y\|^p_{L^p(0,T;H)}, \tag{4.2}$$

where $\|y\|_{L^p(0,T;H)} = \int_0^T \|y\|_\rho^p dt$.

Let us fix an element $v_0 \in \mathcal{K}$ such that $v_0' \in L^q(0,T;L^q(\Omega))$, $v_0(0,x) = 0$ and show the coercivity property (2.6). For all $y \in \mathcal{K}$ we consider the following pairing, by estimate (4.2), we have:

$$\langle \mathcal{A}(y), y - v_0\rangle_{L^p(0,T;H)} \geq \min\{\alpha, 1\} \|y\|^p_{L^p(0,T;H)} - |\langle \mathcal{A}(y), v_0\rangle_{L^p(0,T;H)}|. \tag{4.3}$$

From [12, Lemma 1] and (3.4) it follows that

$$|\langle A_1(y), v_0\rangle_H| \leq \max\{\beta, 1\} \|v_0\|_H \|y\|_H^{p-1}.$$

Further from (3.7) and (2.18) we obtain similar estimate:

$$|\langle \mathcal{A}(y), v_0\rangle_{L^p(0,T;H)}| \leq \max\{\beta, 1\} \|v_0\|_{L^p(0,T;H)} \|y\|^{p-1}_{L^p(0,T;H)}. \tag{4.4}$$

Combining (4.3) and (4.4) we have the coercivity condition (2.6).

Taking into account the estimate (3.5) and the strict monotonicity of the term $|y|^{p-2}y$ we obtain:

$$\langle \mathcal{A}(y) - \mathcal{A}(v), y - v\rangle_{L^p(0,T;H)} \geq 2^{p-2}\|y - v\|_{L^p(0,T;L^p(\Omega))} > 0 \tag{4.5}$$

$\forall y \neq v$ a.e. in $Q = \Omega \times (0,T)$. Thus we have the strict monotonicity of operator $\mathcal{A}$.

From the semicontinuity property of operator A_1 (see [12]) we obtain the similar property for operator $\mathcal{A}$. Taking into account (3.4) and the definition of operator $\mathcal{A}$ we obtain the boundedness property for $\mathcal{A}$. Hence, from the strict monotonicity, boundedness and semicontinuity we obtain that $\mathcal{A}$ is a pseudo-monotone operator (see for details [20, Proposition 2.5]).

If we consider $\mathcal{V} = L^p(0,T;H)$, $\mathcal{H} = L^2(0,T;L^2(\Omega))$, $\mathcal{V}^* = L^q(0,T;H^*)$ we obtain condition (2.3), for $\Lambda = \frac{d}{dt}$ the condition (2.4) is valid, for operator $\mathcal{A}$ properties (2.5) and (2.6) take place. Note that for considered set $\mathcal{K}$ we have that $0 \in K$, thus we have that $G(s)\mathcal{K} \subset \mathcal{K}$ $\forall s \geq 0$ (see [20]) and from Theorem 2.1 we have conditions (2.9).

Hence, for problem (3.2)-(3.3) all conditions of Theorems 2.2 and 2.3 hold true. Therefore for every control $U \in M_p^{\alpha,\beta}(\Omega)$ and every $f \in L^q(0,T;L^q(\Omega))$ the considered problem has a unique solution. □

Let us study the topological properties of the set of H-admisible solutions $\Xi_H \subset L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;H)$. Let τ be the topology on $L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;H)$ which we define as the product of the weak-$*$ topology of the space $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and the weak topology of $L^p(0,T;H)$. In order to discuss further results we suggest that the following assumption is fulfilled:

Hypothesis A. Let for a sequence $\{u_n\}_{n\geq 1}$ that is weakly convergent in $L^p(0,T;H)$ we additionally have that $u_n \in L^\infty(0,T;L^2(\Omega))$ for all $n \in \mathbb{N}$.

Theorem 4.2. *Let $\rho(x) > 0$ be a degenerate weight function, let Hypothesis A hold true and let Hypothesis 2 from [12] hold true for $X = L^q(\Omega)$. Then for every $f \in L^q(0,T;L^q(\Omega))$ the set Ξ_H is sequentially τ-closed.*

Proof. Let $\{(U_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi_H$ be any τ-convergent sequence of admissible pairs to the problem (3.1)-(3.3), (3.8) (in view of Theorem 4.1 such choice is always possible). Let (U_0, y_0) be its τ-limit. Our aim is to prove that $(U_0, y_0) \in \Xi_H$.

Since $\{U_k = [\vec{a}_{1k}, \dots, \vec{a}_{Nk}]\}_{k\in\mathbb{N}} \subset U_{ad}$, it follows that $|\ \mathrm{div}(\rho\vec{a}_{ik})| \leq \gamma_i$ a.e. in Ω $\forall i = 1,\dots,N$ and $\forall k \in \mathbb{N}$. Let us show that $U_0 \in U_{ad}$.

Indeed, passing to the limit as $k \to \infty$ in the relations

$$\int_\Omega (\vec{a}_{ik}, \nabla\varphi)_{\mathbb{R}^N}\rho dx = -\int_\Omega \varphi\ \mathrm{div}(\rho\vec{a}_{ik})\, dx,\ \forall\varphi \in C_0^\infty(\Omega), \forall i = 1,\dots,N,$$

$$-\gamma_i\int_\Omega \varphi \leq \int_\Omega \varphi\ \mathrm{div}(\rho\vec{a}_{ik})dx \leq \gamma_i\int_\Omega \varphi\, dx, \quad \forall i = 1,\dots,N, \forall\varphi \geq 0,$$

we may suppose that $|\ \mathrm{div}(\rho\vec{a}_i^0)| \leq \gamma_i$ a.e. in Ω $\forall i \in \{1,\dots,N\}$ and

$$\mathrm{div}(\rho\vec{a}_{ik}) \rightharpoonup \mathrm{div}(\rho\vec{a}_i^0) \text{ in } L^q(\Omega) \quad \text{as } k \to \infty. \tag{4.6}$$

Thus $U_k \rightharpoonup U_0 = [\vec{a}_1^0, \dots, \vec{a}_N^0]$ weakly-$*$ in $L^\infty(\Omega; \mathbb{R}^{N\times N})$, and $U_0 \in U_{ad}$.

It remains to show that the pair (U_0, y_0) satisfies variational inequality (3.9).

Since each of the pairs (U_k, y_k) is admissible to the OCP (3.1)-(3.3), (3.8), we have

$$\begin{aligned}\langle v', v - y_k\rangle_{L^p(0,T:H)} + \langle -\ &\mathrm{div}(U_k\rho(x)[(\nabla y_k)^{p-2}]\nabla y_k) \\ &+ |y_k|^{p-2} y_k, v - y_k\rangle_{L^p(0,T;H)} \\ &\geq \langle f, v - y_k\rangle_{L^p(0,T;H)}. \end{aligned} \tag{4.7}$$

Since $U_k \to U_0$ weakly-$*$ in $L^\infty(\Omega; \mathbb{R}^{N\times N})$ and $y_k \to y_0$ weakly in $L^p(0,T;H)$ as $k \to \infty$, one gets

$$\begin{aligned} \mathrm{div}(\rho\vec{a}_{ik}) &\rightharpoonup \ \mathrm{div}(\rho\vec{a}_i^0) \text{ in } L^q(\Omega),\ \forall i = 1,\dots,N, \\ y_k &\to y_0 \text{ strongly in } L^p(0,T;L^p(\Omega)) (\text{see [22, Proposition 4.1]}), \\ \nabla y_k &\rightharpoonup \nabla y_0 \text{ in } L^p(0,T;L^p(\Omega,\rho dx)^N), \\ |y_k|^{p-2} y_k &\rightharpoonup |y_0|^{p-2} y_0 \text{ in } L^q(0,T;L^q(\Omega)) \text{ within a subsequence}, \\ \{U_k[(\nabla y_k)^{p-2}]\nabla y_k\}_{k\in\mathbb{N}} &\text{ is bounded in } L^q(0,T;L^q(\Omega,\rho dx)^N). \end{aligned}$$

Then $U_k[(\nabla y_k)^{p-2}]\nabla y_k := \vec{\xi}_k \rightharpoonup \vec{\xi}$ in $L^q(0,T;L^q(\Omega,\rho dx)^N)$ within a subsequence. Similarly to [12, Theorem 5] we obtain that function $-\ \mathrm{div}(\rho\vec{\xi}_k) + |y_k|^{p-2}y_k \in L^q(0,T;L^q(\Omega))$ having that Hypothesis 1 from [12] holds true if we set $V = H$, $X = L^q(\Omega)$ and $f, v' \in L^q(0,T;L^q(\Omega))$ $\forall k \in \mathbb{N}$, and, obviously, $\mathrm{div}(\rho\vec{\xi}_k) \in L^q(0,T;L^q(\Omega))$ $\forall k \in \mathbb{N}$. Further, the relation

$$\begin{aligned} \int_0^T\int_\Omega \mathrm{div}(\rho\vec{\xi}_k)\varphi\,dxdt &= -\int_0^T\int_\Omega \vec{\xi}_k\cdot\nabla\varphi\rho\,dxdt \\ &\to -\int_0^T\int_\Omega \vec{\xi}\cdot\nabla\varphi\rho\,dxdt = \int_0^T\int_\Omega \mathrm{div}(\rho\vec{\xi})\varphi\,dxdt \\ &\qquad \forall\varphi \in C_0^\infty(\Omega\times[0,T]), \end{aligned}$$

means that $\mathrm{div}(\rho\vec{\xi}_k) \to \mathrm{div}(\rho\vec{\xi})$ weakly in $L^q(0,T;L^q(\Omega))$ implying $\{\vec{\xi}_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{X}$, where

$$\mathcal{X} = \{\vec{f} \in L^q(0,T;L^q(\Omega,\rho dx)^N) |\ \mathrm{div}(\rho\vec{f}) \in L^q(0,T;L^q(\Omega))\},$$

that is

$$\varlimsup_{k\to\infty}(\|\vec{\xi}_k\|^q_{L^q(0,T;L^q(\Omega,\rho dx)^N)} + \|\,\mathrm{div}(\rho\vec{\xi}_k)\|^q_{L^q(0,T;L^q(\Omega))})^{1/q} < +\infty.$$

Therefore, as a result, passing to the limit in (4.7) as $k \to \infty$, we obtain

$$\begin{aligned} &\langle v', v - y_0\rangle_{L^p(0,T;H)} + \langle -\,\mathrm{div}(\rho\vec{\xi}) + |y_0|^{p-2}y_0, v - y_0\rangle_{L^p(0,T;H)} \\ &\qquad \geq \langle f, v - y_0\rangle_{L^p(0,T;H)},\ \ \forall v \in \mathcal{K}, v' \in L^q(0,T;L^q(\Omega)), v(0,x) = 0. \end{aligned} \tag{4.8}$$

It remains to prove that $\vec{\xi} = U_0[(\nabla y_0)^{p-2}]\nabla y_0$.

To do this we apply the similar suggestions to [12, Theorem 5] and [22] and by initial assumptions (see (3.5)), we have

$$\int_0^T \int_\Omega (U_k([(\nabla y_k)^{p-2}]\nabla y_k - [\vec{z}^{p-2}]\vec{z})) \cdot (\nabla y_k - \vec{z})\varphi\rho\, dxdt \geq 0 \tag{4.9}$$

for a fixed element $\vec{z}$ of $\mathbb{R}^N$.

Let us show that the sequence $\{\operatorname{div}(\rho U_k[(\vec{z})^{p-2}]\vec{z})\}_{k\in\mathbb{N}}$ is weakly convergent in $L^q(0,T;L^q(\Omega))$. Taking into account the definition of the elements $\operatorname{div}(\rho U_k[\vec{z}^{p-2}]\vec{z})$ for all $k \in N$ (see [12]) and boundedness of $\{\operatorname{div}(\rho U_k[(\nabla y_k)^{p-2}]\nabla y_k)\}_{k\in\mathbb{N}}$ in $L^q(0,T;L^q(\Omega))$ we get

$$\begin{aligned}&\operatorname{div}(\rho U_k([(\nabla y_k)^{p-2}]\nabla y_k - [\vec{z}^{p-2}]\vec{z}))\\ &\qquad\rightharpoonup \operatorname{div}(\rho\vec{\xi}) - \operatorname{div}(\rho U_0[\vec{z}^{p-2}]\vec{z}) \text{ in } L^q(0,T;L^q(\Omega)).\end{aligned} \tag{4.10}$$

Combining the property (4.10), and the fact that

$$U_k[\vec{z}^{p-2}]\vec{z} \rightharpoonup U_0[\vec{z}^{p-2}]\vec{z} \text{ in } L^q(0,T;(L^q(\Omega,\rho dx))^N)$$

it is easy to see that all suppositions of Lemma 2.3 for the sequences $\{\rho^\varepsilon \equiv (\rho)\}_{\varepsilon>0}$ are fulfilled having put in the statement of this lemma $\varepsilon = k$, $\vec{f} = U_k([(\nabla y_k)^{p-2}]\nabla y_k - [\vec{z}^{p-2}]\vec{z})$ and $g_\varepsilon = y_k$ for all $k \in \mathbb{N}$. Hence, we get

$$\int_0^T \int_\Omega (\vec{\xi} - U_0[\vec{z}^{p-2}]\vec{z}) \cdot (\nabla y_0 - \vec{z})\varphi\rho\, dxdt \geq 0, \quad \forall \vec{z} \in R^N$$

for all positive $\varphi \in C_0^\infty(\Omega \times [0,T])$. After localization, we have

$$\rho(\vec{\xi} - U_0[\vec{z}^{p-2}]\vec{z}) \cdot (\nabla y_0 - \vec{z}) \geq 0. \tag{4.11}$$

Taking into account conditions (3.4)–(3.6) and suggestions from [22] we have that the identity $\xi = \hat{A}(U_0,\nabla y_0) = U_0(x)[(\nabla y_0)^{p-2}]\nabla y_0$ holds true a.e. in $\Omega \times (0,T)$.

Thus, the above inequality takes the form

$$\begin{aligned}&\langle v', v - y\rangle_{L^p(0,T;H)} + \langle -\operatorname{div}(\rho U_0[(\nabla y_0)^{p-2}]\nabla y_0) + |y_0|^{p-2}y_0, v - y_0\rangle_{L^p(0,T;H)}\\ &\qquad \geq \langle f, v - y_0\rangle_{L^p(0,T;H(\Omega,\rho dx))}\, \forall v \in \mathcal{K}, v' \in L^q(0,T;L^q(\Omega)), v(0,x) = 0.\end{aligned}$$

Thus τ-limit pair (U_0, y_0) is admissible to the problem (3.1)-(3.3), (3.8), hence, $(U_0, y_0) \in \Xi_H$. □

Theorem 4.3. *Let $\rho(x)$ be a degenerate weight function. Then the set of H-optimal solutions to the problem* (3.1)-(3.3), (3.8) *is non-empty for every $f \in L^q(0,T;L^q(\Omega))$.*

Proof. First of all we note that in virtue of Theorem 4.1 for the given function $f \in L^q(0,T;L^q(\Omega))$ and every admissible control $U \in U_{ad}$ there exists an H-solution $y = y(U,f) \in L^p(0,T;L^p(\Omega))$ to the problem (3.2)-(3.3). Let $\{(U_k,y_k) \in \Xi_H\}_{k\in\mathbb{N}}$ be an H-minimizing sequence to the problem (3.1)-(3.3), (3.8), that is,

$$\lim_{k\to\infty} I(U_k,y_k) = \inf_{(U,y)\in\Xi_H} I(U,y) < +\infty.$$

Hence, taking into account the Definition 3.1 of U_{ad} and Definition 3.5, we may suppose that within a subsequence, there exists $(U^*,y^*) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;H)$, such that $U_k \to U^*$ weakly-$*$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$, $y_k \rightharpoonup y^*$ in $L^p(0,T;H)$. Since Ξ_H is sequentially τ-closed, the pair (U^*,y^*) is H-admissible to the problem (3.1)-(3.3), (3.8). In view of lower τ-semicontinuity of the cost functional we obtain that $I(U^*,y^*) \le \underline{\lim}_{k\to\infty} I(U_k,y_k) = \inf_{(U,y)\in\Xi_H} I(U,y)$. Hence, (U^*,y^*) is an H-optimal pair. □

5. Attainability of H-Optimal Solutions

In this section we propose an appropriate non-degenerate perturbation for the original degenerate OCP (3.1)-(3.3), (3.8) and show that H-optimal solutions of (3.1)-(3.3), (3.8) can be attained by optimal solutions of perturbed problems. In view of results obtained in the previous section we assume that the set of H-optimal solutions to the considered problem is non-empty.

Let ρ be a degenerate weight function with properties (2.1), and let $\{\rho^\varepsilon\}_{\varepsilon>0}$ be a non-degenerate perturbation of ρ in the sense of Definition 2.1.

Definition 5.1. We say that a bounded sequence

$$\{(U_\varepsilon,y_\varepsilon) \in \mathbb{Y} = L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;H(\Omega,\rho^\varepsilon dx))\}_{\varepsilon>0}$$

w-converges to $(U,y) \in L^\infty(\Omega;\mathbb{R}^{N\times N}) \times L^p(0,T;W)$ in the variable space $\mathbb{Y}$ as $\varepsilon \to 0$, if $U_\varepsilon \to U$ weakly-$*$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$, $y_\varepsilon \rightharpoonup y$ in $L^p(0,T;L^p(\Omega))$ and $\nabla y_\varepsilon \rightharpoonup \nabla y$ in $L^p(0,T;L^p(\Omega,\rho^\varepsilon dx)^N)$

Similarly to [17, Definition 8] we consider the next concept.

Definition 5.2. We say that a minimization problem

$$\left\langle \inf_{(U,y)\in\Xi_H} I(U,y) \right\rangle \tag{5.1}$$

is a weak variational limit (or variational w-limit) of the sequence

$$\left\{ \left\langle \inf_{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(U_\varepsilon,y_\varepsilon) \right\rangle ;\, \Xi_\varepsilon \in \mathbb{Y},\, \varepsilon > 0 \right\}, \tag{5.2}$$

with respect to w-convergence in the variable space $\mathbb{Y}$, if the following conditions are satisfied:

(1) if $\{\varepsilon_k\}$ is a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$, and a sequence $\{(U_k, y_k) \in \Xi_{\varepsilon_k}\}_{\varepsilon>0}$ – w-converges to a pair (U, y), then

$$(U, y) \in \Xi_H : I(U, y) \leq \varliminf_{k\to\infty} I_{\varepsilon_k}(U_k, y_k); \tag{5.3}$$

(2) for every pair $(U, y) \in \Xi_H$ and any value $\delta > 0$ there exists a realizing sequence $\{(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}\}_{\varepsilon>0}$ such that

$$(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \ \forall \varepsilon > 0, \ (\hat{U}_\varepsilon, \hat{y}_\varepsilon) \overset{w}{\to} (\hat{U}, \hat{y}), \tag{5.4}$$

$$\|U - \hat{U}\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \left(\int_0^T \|y - \hat{y}\|_\rho^p dt\right)^{1/p} \leq \delta, \tag{5.5}$$

$$I(U, y) \geq \varlimsup_{\varepsilon\to 0} I_\varepsilon(\hat{U}_\varepsilon, \hat{y}_\varepsilon) - \delta. \tag{5.6}$$

Similarly to [23] we can assume that Definition 5.2 is motivated by the following property of variational w-limits.

Theorem 5.1. *Assume that* (5.1) *is a weak variational limit of the sequence* (5.2)*, and the constrained minimization problem* (5.1) *has a solution. Suppose* $\{(U_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon\}_{\varepsilon>0}$ *is a sequence of optimal pairs to* (5.2)*. Then there exists a pair* $(U^0, y^0) \in \Xi_H$ *such that* $(U_\varepsilon^0, y_\varepsilon^0)$ *w-converges to* (U^0, y^0)*, and*

$$\inf_{(U,y)\in\Xi_H} I(U, y) = I(U^0, y^0) = \lim_{\varepsilon\to 0} \inf_{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(U_\varepsilon, y_\varepsilon).$$

Remark 5.1. Let us recall that sequential K-upper and K-lower limits of a sequence of sets $\{E_k\}_{k\in\mathbb{N}}$ are defined as follows, respectively:

$$K_s - \varlimsup E_k = \{y \in X : \exists \sigma(k) \to \infty, \exists y_k \to y, \forall k \in \mathbb{N} : y_k \in E_{\sigma(k)}\},$$
$$K_s - \varliminf E_k = \{y \in X : \exists y_k \to y, \exists k \geq k_0 \in \mathbb{N} : y_k \in E_k\}.$$

The sequence $\{E_k\}_{k\in\mathbb{N}}$ sequentially converges in the sense of Kuratovski to the set E (shortly, K_s-converges), if $E = K_s - \varliminf E_k = K_s - \varlimsup E_k$.

Let us consider the sequence $\{\mathcal{K}_\varepsilon\}_{\varepsilon>0}$ of non-empty closed and convex subsets, which sequentially converges to the set $\mathcal{K}$ in the sense of Kuratovski as $\varepsilon \to 0$ with respect to weak topology of the space $L^p(0, T; H(\Omega, \rho^\varepsilon dx))$ and the sequence $\{\tilde{\mathcal{K}}_\varepsilon\}_{\varepsilon>0}$ of non-empty closed and convex subsets, which sequentially converges to the set $\tilde{\mathcal{K}} = \{v \in L^p(0, T; H) | v' \in L^q(0, T; L^q(\Omega)), v(0, x) = 0\}$ in the sense of Kuratovski as $\varepsilon \to 0$ with respect to the topology τ_1:

$$v_\varepsilon \rightharpoonup v \text{ in } L^p(0, T; H(\Omega, \rho^\varepsilon dx)), v_\varepsilon' \rightharpoonup v' \text{ in } L^q(0, T; L^q(\Omega)), v_\varepsilon(0, x) = 0.$$

Let Hypothesis 2 from [17] hold true for $X = L^q(\Omega)$ and $V = H(\Omega, \rho^\varepsilon dx)$ $\forall \varepsilon > 0$. Taking into account Theorem 5.1, we consider the following collection of perturbed OCPs in coefficients for non-degenerate parabolic variational inequalities:

$$\text{Minimize } \left\{ I_\varepsilon(U, y) = \int_0^T \int_\Omega |y(x) - z_\partial(x)|^p \, dx dt \right\} \tag{5.7}$$

$$U \in U_{ad}^\varepsilon, y \in \mathcal{K}_\varepsilon, \tag{5.8}$$

$$\langle v', v-y\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} + \langle -\operatorname{div}(\rho^\varepsilon U[(\nabla y)^{p-2}]\nabla y) + |y|^{p-2}y, v-y\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} \geq \langle f, v-y\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\ \forall v\in\tilde{\mathcal{K}}_\varepsilon, \quad (5.9)$$

$$U_{ad}^\varepsilon = \{U=[\vec{a}_1,\dots,\vec{a}_N]\in M_p^{\alpha,\beta}(\Omega): |\operatorname{div}(\rho^\varepsilon \vec{a}_i)|\leq \gamma_i, \text{ a.e. in } \Omega, \forall i=1,\dots,N\}, \quad (5.10)$$

where the elements $z_\partial\in L^p(0,T;L^p(\Omega))$, $f\in L^q(0,T;L^q(\Omega))$ and $\gamma=(\gamma_1,\dots,\gamma_N)$ $\in\mathbb{R}^N$ are the same as for the original problem (3.1)-(3.3), (3.8). For every $\varepsilon>0$ we define Ξ_ε as a set of all admissible pairs to the problem (5.7)-(5.10), namely $(U,y)\in\Xi_\varepsilon$ if and only if the pair (U,y) satisfies (5.8)-(5.10).

Note that each of perturbed OCPs (5.7)-(5.10) is solvable provided $\{\rho^\varepsilon\}_{\varepsilon>0}$ is a non-degenerate perturbation of $\rho\geq 0$ (see [20]).

Lemma 5.1. *Let $\{\rho^\varepsilon=(\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x)\geq 0$. Let $\{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible pairs to the problem* (5.7)-(5.10) *and let Hypothesis A hold true for weakly convergent sequences in $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$. Then there exists a pair (U^*,y^*) and a subsequence*

$$\{(U_{\varepsilon_k},y_{\varepsilon_k})\}_{k\in\mathbb{N}}\subset\{(U_\varepsilon,y_\varepsilon)\in\Xi_\varepsilon\}_{\varepsilon>0}$$

such that $(U_{\varepsilon_k},y_{\varepsilon_k})$ w-converges to (U^*,y^*) as $k\to\infty$ and $(U^*,y^*)\in\Xi_H$.

Proof. Let us consider the variational inequality

$$\langle v_\varepsilon', v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} + \langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon), v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} + \langle |y_\varepsilon|^{p-2}y_\varepsilon, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} \geq \langle f, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))},\ \forall v_\varepsilon\in\tilde{\mathcal{K}}_\varepsilon. \quad (5.11)$$

As follows from (5.10) that the sequence $\{U_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega;\mathbb{R}^{N\times N})$.

Let us prove the boundedness of $\{y_\varepsilon\}_{\varepsilon>0}$ in the space $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$ by contradiction. Namely, suppose that $\|y_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}\to\infty$, $\varepsilon\to 0$. Then on the one hand

$$\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon) + |y_\varepsilon|^{p-2}y_\varepsilon, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} \leq \langle -v_\varepsilon', v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} + \langle f, v_\varepsilon-y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} \leq (\|v_\varepsilon'\|_{L^q(0,T;L^q(\Omega))} + \|f\|_{L^q(0,T;L^q(\Omega))})\|y_\varepsilon-v_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}, \quad (5.12)$$

$\forall v_\varepsilon\in\tilde{\mathcal{K}}_\varepsilon$ and $\forall\varepsilon>0$.

On the other hand, for arbitrary fixed element $v\in\tilde{\mathcal{K}}$ let us consider the sequence $\{v_\varepsilon\in\tilde{\mathcal{K}}_\varepsilon\}_{\varepsilon>0}$ such that $v_\varepsilon\to v$ in τ_1-topology (such sequence always exists provided $\tilde{\mathcal{K}} = K_s - \lim \tilde{\mathcal{K}}_\varepsilon$) and then, using the estimate (see Theorem 4.1)

$$\langle \mathcal{A}(U,y), y - v\rangle_{L^p(0,T;H)} \geq \min\{\alpha,1\}\|y\|^p_{L^p(0,T;H)} \\ - \max\{\beta,1\}\|v\|^p_{L^p(0,T;H)}\|y\|^{p-1}_{L^p(0,T;H)},\ v \in L^p(0,T;H),$$

we obtain the following relations:

$$\frac{\langle -\operatorname{div}(\rho^\varepsilon U_\varepsilon[(\nabla y_\varepsilon)^{p-2}]\nabla y_\varepsilon) + |y_\varepsilon|^{p-2}y_\varepsilon, v_\varepsilon - y_\varepsilon\rangle_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon - v_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}$$

$$\geq \|y_\varepsilon\|^{p-1}_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))} \frac{\left(\min\{\alpha,1\} - \frac{\max\{\beta,1\}\|v_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}\right)}{\left(1 + \frac{\|v_\varepsilon\|_{L^p(0,T:H(\Omega,\rho^\varepsilon dx))}}{\|y_\varepsilon\|_{L^p(0,T;H(\Omega,\rho^\varepsilon dx))}}\right)} \to \infty \text{ as } \varepsilon \to 0,$$

since the sequence $\{v_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$. The obtained contradiction with (5.12) implies that $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^p(0,T;H(\Omega,\rho^\varepsilon dx))$.

Hence, there exists a subsequence $\{\varepsilon_k\}$ of the sequence $\{\varepsilon\}$ converging to 0 and elements $U^* \in M_p^{\alpha,\beta}(\Omega)$, $y^* \in L^p(0,T;L^p(\Omega))$, $\vec{v} \in L^p(0,T;L^p(\Omega,\rho dx)^N)$ and $\vec{\xi} \in L^q(0,T;L^q(\Omega,\rho dx)^N)$ such that

$$\begin{aligned} U_{\varepsilon_k} &\to U^* \quad \text{weakly-}* \text{ in } L^\infty(\Omega;\mathbb{R}^{N\times N}),\\ y_{\varepsilon_k} &\rightharpoonup y^* \quad \text{in } L^p(0,T;L^p(\Omega)),\\ \nabla y_{\varepsilon_k} &\rightharpoonup \vec{v} \quad \text{in } L^p(0,T;L^p(\rho^{\varepsilon_k}dx)^N),\\ U_{\varepsilon_k}[(\nabla y_{\varepsilon_k})^{p-2}]\nabla y_{\varepsilon_k} := \vec{\xi}_{\varepsilon_k} &\rightharpoonup \vec{\xi} \text{ in } L^q(0,T;L^q(\Omega^{\varepsilon_k}dx)^N). \end{aligned} \tag{5.13}$$

By Theorem 2.4, taking into account properties of the Bochner integral and definitions of equivalent functions (see [21, Definition 1.6]), we have that $y^* \in L^p(0,T;H)$ and $\vec{v} = \nabla y^*$ and moreover, we have $y^* \in \mathcal{K}$.

Following arguments of the proof of [17, Lemma 11] we obtain that $U^* \in U_{ad}$.

In what follows, we consider the relation (5.11) for $(U_{\varepsilon_k}, y_{\varepsilon_k})$ and pass to the limit in it as $k \to \infty$ using the property of the strong convergence and the following relations:

$$|y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k} \rightharpoonup |y^*|^{p-2}y^* \text{ in } L^q(0,T;L^q(\Omega)) \text{ within a subsequence,} \tag{5.14}$$

$$\langle -\operatorname{div}(\rho^{\varepsilon_k}\vec{\xi}_{\varepsilon_k}), y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} \to \langle -\operatorname{div}(\rho\vec{\xi}), y^*\rangle_{L^p(0,T;H)}. \tag{5.15}$$

The latter is valid in view of Lemma 2.3 and boundedness of the sequence $\{\vec{\xi}_{\varepsilon_k}\} \subset X(\Omega,\rho^{\varepsilon_k}dx)$, which we can obtain by the similar manner as in Theorem 4.2 for

$$X(\Omega,\rho^{\varepsilon_k}dx) = \{\vec{f} \in L^q(0,T;L^q(\Omega,\rho^{\varepsilon_k}dx)^N) |\ \operatorname{div}(\rho^{\varepsilon_k}\vec{f}) \in L^q(0,T;L^q(\Omega))\},$$

with the norm

$$\|\vec{f}\|_{X(\Omega,\rho^{\varepsilon_k}dx)} = (\|\vec{f}\|^q_{L^q(0,T;L^q(\Omega,\rho^{\varepsilon_k}dx)^N)} + \|\operatorname{div}(\rho^{\varepsilon_k}\vec{f})\|^q_{L^q(0,T;L^q(\Omega))})^{1/q}.$$

Let us prove relation (5.14). We have that $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^p(0,T;L^p(\Omega))$, $\nabla y_{\varepsilon_k} \rightharpoonup \nabla y^*$ in $L^p(0,T;L^p(\Omega,\rho^{\varepsilon_k}dx)^N)$ and from [22, Proposition 4.1] we obtain that there exists an element $\tilde{y}$ such that $y_{\varepsilon_k} \to \tilde{y}$ strongly in $L^1(0,T;L^1(\Omega))$. However, it is easy to see that $y_{\varepsilon_k} \rightharpoonup y^*$ in $L^1(0,T;L^1(\Omega))$. Hence, $y^* = \tilde{y}$ a.e. on $(0,T)\times\Omega$. It means that up to a subsequence $y_{\varepsilon_k} \to y^*$ a.e. in $(0,T)\times\Omega$ and together with boundedness of $\{|y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k}\}_{k\in\mathbb{N}}$ in $L^q(0,T;L^q(\Omega))$ we have $|y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k} \rightharpoonup |y^*|^{p-2}y^*$ in $L^q(0,T;L^q(\Omega))$ (within a subsequence).

Since $v'_{\varepsilon_k} \rightharpoonup v'$ in $L^q(0,T;L^q(\Omega))$ we can obtain that

$$\langle v'_{\varepsilon_k}, v_{\varepsilon_k} - y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} \to \langle v', v - y^*\rangle_{L^p(0,T;H)} \text{ as } k\to\infty. \tag{5.16}$$

Therefore, as a result of limit passage in (5.11), taking into account (5.14), (5.15) and (5.16), we obtain

$$\begin{aligned}\langle v', v-y^*\rangle_{L^p(0,T;H)} &+ \langle -\operatorname{div}(\rho\vec{\xi}), v-y^*\rangle_{L^p(0,T;H)} + \langle |y^*|^{p-2}y^*, v\rangle_{L^p(0,T;H)} \\ &- \varlimsup_{k\to\infty}\langle |y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k}, y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} \\ &\geq \langle f, v-y^*\rangle_{L^p(0,T;H)},\ \forall v\in\tilde{\mathcal{K}}.\end{aligned} \tag{5.17}$$

In order to prove the lemma, it is left to show that $\vec{\xi} = U^*[(\nabla y^*)^{p-2}]\nabla y^*$. However it can be done in a similar manner as we did it proving Theorem 4.2.

Now, let us show that

$$\lim_{k\to\infty}\langle |y_{\varepsilon_k}|^{p-2}y_{\varepsilon_k}, y_{\varepsilon_k}\rangle_{L^p(0,T;H(\Omega,\rho^{\varepsilon_k}dx))} = \lim_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt = \int_0^T\int_\Omega |y^*|^p\,dxdt.$$

On the one hand, in view of property of lower semicontinuity, weak convergence $y_{\varepsilon_k}\to y^*$ in $L^p(0,T;L^p(\Omega))$ as $k\to\infty$, implies that:

$$\int_0^T\int_\Omega |y^*|^p\,dxdt \leq \varliminf_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt.$$

On the other hand, from (5.17), taking into account the representation of the vector-function ξ, we obtain:

$$\begin{aligned}\varlimsup_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt &\leq \langle v' - \operatorname{div}(U^*(x)\rho(x)[(\nabla y^*)^{p-2}]\nabla y^*) - f, v-y^*\rangle_{L^p(0,T;H)} \\ &+\langle |y^*|^{p-2}y^*, v\rangle_{L^p(0,T;H)},\ \forall v\in\tilde{K}.\end{aligned}$$

Having put in the last inequality $v = y^*$, we get

$$\varlimsup_{k\to\infty}\int_0^T\int_\Omega |y_{\varepsilon_k}|^p\,dxdt \leq \int_0^T\int_\Omega |y^*|^p\,dxdt.$$

Hence, summing up, the chain of inequalities

$$\int_0^T \int_\Omega |y^*|^p \, dxdt \leq \varliminf_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^p \, dxdt \leq \varlimsup_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^p \, dxdt \leq \int_0^T \int_\Omega |y^*|^p \, dxdt$$

turns into equality

$$\lim_{k\to\infty} \int_0^T \int_\Omega |y_{\varepsilon_k}|^p \, dxdt = \int_0^T \int_\Omega |y^*|^p \, dxdt$$

which implies, in view of criterium of strong convergence that $y_{\varepsilon_k} \to y^*$ strongly in $L^p(0,T;L^p(\Omega))$ as $k \to \infty$.

Therefore, variational inequality (5.17) can be represented in the form

$$\langle v', v - y^* \rangle_{L^p(0,T;H)} + \langle -\operatorname{div}(U^*(x)\rho(x)[(\nabla y^*)^{p-2}]\nabla y^*) + |y^*|^{p-2}y^*, v - y^* \rangle_{L^p(0,T;H)} \geq \langle f, v - y^* \rangle_{L^p(0,T;H)}, \ \forall v \in \tilde{\mathcal{K}}. \quad (5.18)$$

Thus, w-limit pair (U^*, y^*) is admissible to the problem (3.1)-(3.3), (3.8), hence, $(U^*, y^*) \in \Xi_H$. □

Theorem 5.2. *Let $\{\rho^\varepsilon = (\rho)_\varepsilon\}_{\varepsilon>0}$ be a "direct" smoothing of a degenerate weight function $\rho(x) > 0$. Then the minimization problem* (3.1)-(3.3), (3.8) *is a weak variational limit of the sequence* (5.7)-(5.10) *as $\varepsilon \to 0$ with respect to the w-convergencs in the variable space $\mathbb{Y}$.*

Proof. As an evident consequence of the previous lemma and the lower semi-continuity property of the cost functional (5.7) with respect to w-convergence in variable space $\mathbb{Y}$, we have the following conclusion: if $\{\varepsilon_k\}$ be a subsequence of indices $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $\{(U_k, y_k) \in \Xi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is a sequence of admissible solutions to corresponding perturbed problems (5.7)-(5.10) such that $(U_k, y_k) \to (U, y)$ with respect to w-convergence, then properties (5.3) are valid.

To discuss properties (5.4)-(5.6) similarly to suggestions from [17] and [19] we can obtain that for an admissible pair $(U, y) \in \Xi_H$ there exists a realizing sequence $\{(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \mathbb{Y}\}_{\varepsilon>0}$ such that

$$(\hat{U}_\varepsilon, \hat{y}_\varepsilon) \in \Xi_\varepsilon \ \forall \varepsilon > 0, \ \hat{U}_\varepsilon \to U \ * -\text{weakly in } L^\infty(\Omega; \mathbb{R}^{N\times N});$$
$$\operatorname{div}(\rho^\varepsilon \vec{a}_{i\varepsilon}) \rightharpoonup \operatorname{div}(\rho \vec{a}_i) \text{ in } L^q(0,T;L^q(\Omega)) \ \forall i \in \{1,\dots,N\},$$
$$\hat{y}_\varepsilon \to y \text{ strongly in } L^p(0,T;L^p(\Omega)), \ \nabla y_\varepsilon \rightharpoonup \nabla y \text{ in } L^p(0,T;L^p(\Omega;\rho^\varepsilon dx)^N).$$

From these suggestions the equality $I(U, y) = \lim_{\varepsilon\to 0} I_\varepsilon(\hat{U}_\varepsilon, \hat{y}_\varepsilon)$ follows.

Taking into account Definition 5.2 and previous suggestions of this proof we obtain the statement of the theorem. □

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